Operations on Multiple Random Variables

Previously we discussed operations on one Random Variable:

Expected Value

$$E[X] = \bar{X} = \begin{cases} \int_{-\infty}^{\infty} x f_X(x)dx & \text{Continuous} \\ \sum_{i=1}^{N} x_i P(x_i) & \text{Discrete} \end{cases}$$

Moment

$$m_n = E\left[X^n\right] = \begin{cases} \int_{-\infty}^{\infty} x^n f_X(x)dx & \text{Continuous} \\ \sum_{i=1}^{N} x_i^n P(x_i) & \text{Discrete} \end{cases}$$

Central moment

$$\mu_n = E\left[(X - \bar{X})^n\right] = \begin{cases} \int_{-\infty}^{\infty} (x - \bar{X})^n f_X(x)dx & \text{Continuous} \\ \sum_{i=1}^{N} (x_i^n - \bar{X})^n P(x_i) & \text{Discrete} \end{cases}$$

Characteristic Function

$$\Phi_X(\omega) = E\left[e^{j\omega X}\right] = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx \qquad f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_X(\omega) e^{-j\omega x} d\omega$$

Moment Generation

$$m_n = (-j)^n \left. \frac{d^n \Phi_X(\omega)}{d\omega^n} \right|_{\omega = 0}$$

Function of Random Variable

Monotone Transformation

$$f_Y(y) = f_X\left[T^{-1}(y)\right] \left| \frac{dT^{-1}(y)}{dy} \right|$$

Nonmonotone Transformation

$$f_Y(y) = \sum_n \frac{f_X(x_n)}{\left. \left| \frac{dT(x)}{dx} \right| \right|_{x = x_n}}$$

EXPECTED VALUE OF A FUNCTION OF Multiple RANDOM VARIABLES

Two Random Variables

$$\overline{g} = E\left[g(X,Y)\right] = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy & \text{Continuous} \\ \sum_{i}\sum_{k} g(x_i,y_k) P_{X,Y}(x_i,y_k) & \text{Discrete} \end{cases}$$

N Random Variables

$$\overline{g} = E\left[g(X_1,\ldots,X_n)\right] = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\ldots,x_n) f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)\,dx_1\cdots dx_n$$

Joint Moment about the Origin

$$m_{nk} = E\left[X^n Y^k\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y)dxdy$$

$m_{n0} = E[X^n]$ the n^{th} moment m_n of the one random variable X

$m_{0k} = E[Y^k]$ the k^{th} moment m_k of the one random variable Y

$n + k$ is called the order of the moments

Thus m_{02} , m_{20} and m_{11} are all 2^{ed} order moments of X and Y

The first order moments $m_{10} = E[X] = \bar{X}$ and $m_{01} = E[Y] = \bar{Y}$ are the expected values of X and Y and are the coordinates of the **center of gravity** of the function $f_{XY}(x,y)$

The second-order moment $m_{11} = E[XY]$ is called the **correlation** of X and Y and given the symbol R_{XY} hence

$$R_{XY} = m_{11} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dx\,dy$$

If the correlation can be written as $R_{XY} = E[X]E[Y]$

Then the random variables X and Y are said to be **uncorrelated**.

Statistically independence of X and Y $\rightarrow$ X and Y are **uncorrelated**

The converse is not true in general

$$R_{XY} = m_{11} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \underbrace{f_{X,Y}(x,y)}_{f_{X,Y}(x,y) = f_X(x) f_Y(y)} dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_X(x) f_Y(y)\,dx\,dy$$

$$R_{XY} = \int_{-\infty}^{\infty} x f_X(x)\,dx \int_{-\infty}^{\infty} y f_Y(y)\,dy = E[X]E[Y]$$

Uncorrelated of X and Y does not imply that X and Y are Statistically independent in general, except for the Gaussian random variables as will be shown later

If $R_{XY} = 0$ then the random variables X and Y are called **orthogonal**.

For N random variables $X_1, X_2, \ldots, X_N$ the $(n_1 + n_2 + \ldots n_N)$ order joint moments are defined by

$$m_{n_1 n_2 \ldots n_N} = E\left[X_1^{n_1} X_2^{n_2} \ldots X_N^{n_N}\right]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} X_1^{n_1} \ldots X_N^{n_N} f_{X_1,\ldots,X_N}(x_1,\ldots,x_N)\, dx_1 \ldots dx_N$$

where $n_1, n_2, \ldots n_N$ are all integers 0, 1, 2,

Joint Central Moments

We define the joint central moments for two random variables X and Y as follows

$$\mu_{nk} = E\left[(X - \bar{X})^n (Y - \bar{Y})^k\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \bar{X})^n (y - \bar{Y})^k f_{X,Y}(x,y)\,dx\,dy$$

The second-order (n + k =2) central moments are

$$\mu_{20} = E\left[(X - \bar{X})^2\right] = \sigma_X^2$$ **The variance of X**

$$\mu_{02} = E\left[(Y - \bar{Y})^2\right] = \sigma_Y^2$$ **The variance of Y**

The second-order joint moment μ_{11} is very important. It is called the covariance of X and Y and is given the symbol C_{XY}

The second-order joint moment μ_{11} is very important. It is called the **covariance** of X and Y and is given the symbol C_{XY}

$$C_{XY} = \mu_{11} = E\left[(X - \bar{X})(Y - \bar{Y})\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - \bar{X})(y - \bar{Y})f_{X,Y}(x,y)dxdy$$

By direct expansion of the product $(X-\bar{X})(Y-\bar{Y})$ we get

$$C_{XY} = \mu_{11} = E\left[(XY - X\bar{Y} - \bar{X}Y - \bar{X}\bar{Y})\right]$$

$$= E[XY] - E[X]\bar{Y} - \bar{X}E[Y] - \bar{X}\bar{Y} = E[XY] - \bar{X}\bar{Y} - \bar{X}\bar{Y} + \bar{X}\bar{Y}$$

$$\Rightarrow C_{XY} = R_{XY} - \bar{X}\bar{Y} = R_{XY} - E[X]E[Y]$$

Note on the 1-D , the variance was defined as

$$\sigma_X = \mu_2 = E[(X - \bar{X})^2] = E[X^2] - \bar{X}^2$$

Which can be written as

$$\sigma_X = \mu_2 = E[(X - \bar{X})(X - \bar{X})] = E[XX] - \bar{X}\bar{X}$$

$$C_{XY} = R_{XY} - \bar{X}\bar{Y} = R_{XY} - E[X]E[Y]$$

$$C_{XY} = \begin{cases} 0 & \text{if X and Y are independents} \\ -E[X]E[Y] & \text{if X and Y are orthogonal} \end{cases}$$

$$C_{XY} = 0 \quad \text{if} \quad \bar{X}=0 \quad \text{or} \quad \bar{Y}=0$$

The normalized second order-moments defined as $\rho = \dfrac{C_{XY}}{\sigma_X \sigma_Y}$

is known as the correlation coefficient of X and Y

It can be shown that $-1 \leq \rho \leq 1$

For N random variables $X_1, X_2, \ldots, X_N$ the $(n_1 + n_2 + \ldots n_N)$ order joint central moments are defined by

$$\mu_{n_1 n_2 \ldots n_N} = E\left[(X_1 - \bar{X}_1)^{n_1}(X_2 - \bar{X}_2)^{n_2} \ldots (X_N - \bar{X}_N)^{n_N}\right]$$

$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} (x_1 - \bar{X}_1)^{n_1} \ldots (x_N - \bar{X}_N)^{n_N} f_{X_1,\ldots,X_N}(x_1,\ldots, x_N) dx_1 \ldots dx_N$$

JOINT CHARACTERISTIC FUNTIONS

Let u first review the characteristic function for the single random variable

Let X be a random variable with probability density function $f_X(x)$

We defined the characteristic function $\Phi_X(\omega)$ as follows

$$\Phi_X(\omega) = E\left[e^{j\omega X}\right] = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx \qquad f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_X(\omega) e^{-j\omega x} d\omega$$

$\Phi_X(\omega)$ is the Fourier transform of $f_X(x)$ with the sign of ω is reverse

The moments m_n can be found from the characteristic function as follows:

$$m_n = (-j)^n \left.\frac{d^n \Phi_X(\omega)}{d\omega^n}\right|_{\omega = 0}$$

We now define the joint characteristic function of two random variables X and Y with joint probability density function $f_{XY}(x,y)$ as follows

$$\Phi_{X,Y}(\omega_1,\omega_2) = E\left[e^{j\omega_1 X + j\omega_2 Y}\right]$$ were ω_1 and ω_2 are real numbers

$$\Phi_{X,Y}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, e^{j\omega_1 x + j\omega_2 y} dxdy$$

$\Phi_{X,Y}(\omega_1,\omega_2)$ is the two-dimensional Fourier Transform of $f_{XY}(x,y)$ with reversal of sign of ω_1 and ω_2

From the inverse two-dimensional Fourier Transform we have

$$f_{X,Y}(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{X,Y}(\omega_1,\omega_2)\, e^{-j\omega_1 x - j\omega_2 y} d\omega_1 d\omega_2$$

By setting $\omega_2 = 0$ in the expression of $\Phi_{X,Y}(\omega_1,\omega_2)$ above we obtain the following

$$\Phi_{X,Y}(\omega_1,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, e^{j\omega_1 x + j(0)y}\, dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, e^{j\omega_1 x}\, dxdy = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy\right) e^{j\omega_1 x}\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega_1 x}\, dx = \Phi_X(\omega_1)$$

The characteristic function for the marginal probability density function $f_X(x)$

Similarly

$$\Phi_Y(\omega_2) = \Phi_{X,Y}(0,\omega_2)$$

The joint moments can be found from the joint characteristic function

$$m_{nk} = (-j)^{n+k} \frac{\partial^{n+k} \Phi_{X,Y}(\omega_1, \omega_2)}{\partial \omega_1^n \partial \omega_2^k} \Bigg|_{\omega_1 = 0,\ \omega_2 = 0}$$

This expression is the two-dimensional extension of the one-dimension

$$m_n = (-j)^n \frac{d^n \Phi_X(\omega)}{d\omega^n} \Bigg|_{\omega = 0}$$

The joint characteristic function for N random variables $X_1, \ldots, X_n$

$$\Phi_{X_1,\ldots,X_N}(\omega_1, \ldots, \omega_N) = E\left[e^{j\omega_1 x_1 + \ldots + j\omega_N x_N} \right]$$

The joint moments are obtained from

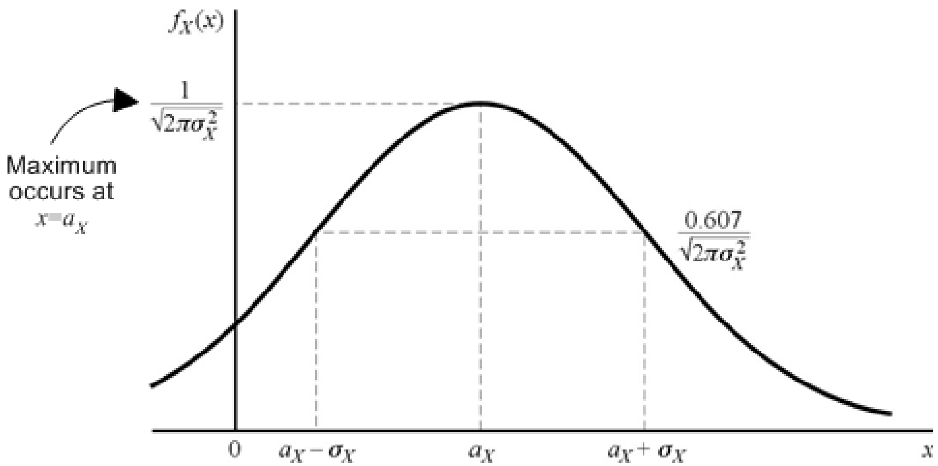
$$m_{n_1 n_2 \ldots n_N} = (-j)^R \frac{\partial^R \phi_{X1,\ldots,X_N}(\omega_1, \ldots, \omega_N)}{\partial \omega_1^{n_1} \partial \omega_2^{n_2} \ldots \partial \omega_N^{n_N}} \Bigg|_{\text{all } \omega_i = 0} \quad \text{were} \quad R = n_1 + n_2 + \cdots n_N$$

The Gaussian Random Variable

A random variable X is called Gaussian if its density function has the form

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-(x - a_X)^2/2\sigma_x^2}$$

where σ_X^2 (*the* varinace) *and* a_X (*the* mean)



The "spread" about the point $x = a_x$ is related to σ_x

JOINTLY GAUSSINA RANDOM VARIABLES

Two random variables X and Y are said to be jointly gaussian or Bivariate gaussian density if their joint density function is of the form

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$

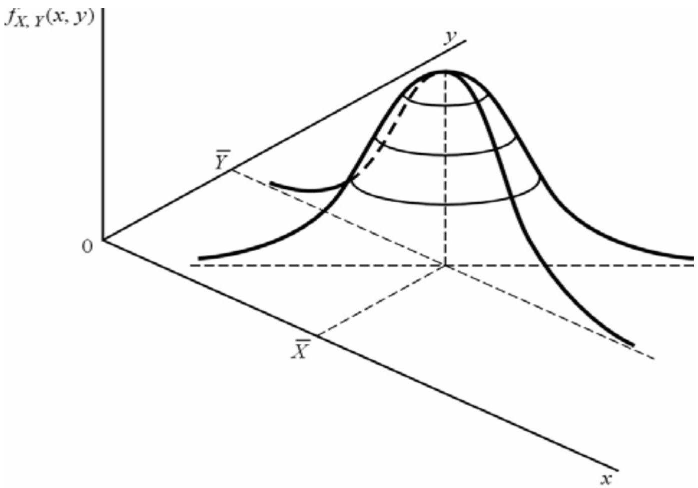
$$\cdot \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\bar{X})^2}{\sigma_X^2} - \frac{2\rho(x-\bar{X})(y-\bar{Y})}{\sigma_X\sigma_Y} + \frac{(y-\bar{Y})^2}{\sigma_Y^2}\right]\right\}$$

were

$$\bar{X} = E[X] \quad \bar{Y} = E[Y] \quad \sigma_X^2 = E\left[(X-\bar{X})^2\right] \quad \sigma_Y^2 = E\left[(Y-\bar{Y})^2\right]$$

$$\rho = E\left[(X-\bar{X})(Y-\bar{Y})\right]/\sigma_X\sigma_Y$$

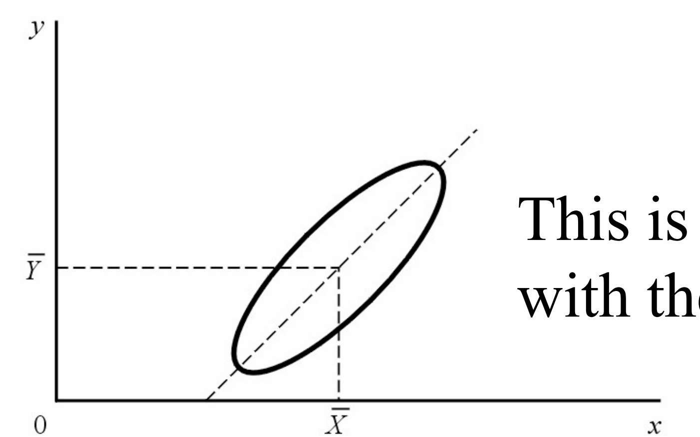
The joint Gaussian density function and its maximum is located at the point $(\bar{X},\bar{Y})$



The maximum value is obtained from

$$f_{X,Y}(x,y) \le f_{X,Y}(\bar{X},\bar{Y}) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$

The locus of constant values of $f_{X,Y}(x,y)$ will be an ellipse



This is equivalent to the intersection of $f_{X,Y}(x,y)$ with the plane parallel to the xy-plane

If $\rho = 0$ then the joint Gaussian density

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\bar{X})^2}{\sigma_X^2} - \frac{2\rho(x-\bar{X})(y-\bar{Y})}{\sigma_X\sigma_Y} + \frac{(y-\bar{Y})^2}{\sigma_Y^2}\right]\right\}$$

$$\Rightarrow \quad f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} \cdot \exp\left\{\frac{-1}{2}\left[\frac{(x-\bar{X})^2}{\sigma_X^2} + \frac{(y-\bar{Y})^2}{\sigma_Y^2}\right]\right\}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi\sigma_X^2}} \cdot \exp\left\{-\left[\frac{(x-\bar{X})^2}{2\sigma_X^2}\right]\right\}}_{f_X(x)} \bullet \underbrace{\frac{1}{\sqrt{2\pi\sigma_Y^2}} \cdot \exp\left\{-\left[\frac{(y-\bar{Y})^2}{2\sigma_Y^2}\right]\right\}}_{f_Y(y)}$$

$$= f_X(x)\ f_Y(y)$$

$$f_Y(y) = f_X\left[T^{-1}(y)\right] \left|\frac{dT^{-1}(y)}{dy}\right|$$

$$f_Y(y) = \sum_n \frac{f_X(x_n)}{\left|\frac{dT(x)}{dx}\right|_{x = x_n}}$$

$$Y_i = T_i\ (X_1, X_2, \ldots, X_N) \qquad i = 1, 2, \ldots, N$$

$$X_j = T_j^{-1}\ (Y_1, Y_2, \ldots, Y_N) \qquad j = 1, 2, \ldots, N$$

$$f_{Y_1, Y_2, \ldots, Y_N}(y_1, \ldots, y_N) = f_{X_1, X_2, \ldots, X_N}(x_1 = T_1^{-1}, \ldots, x_N = T_N^{-1})\left|J\right|$$

$$J = \begin{vmatrix} \frac{\partial T_1^{-1}}{\partial Y_1} & \cdots & \frac{\partial T_1^{-1}}{\partial Y_N} \\ \vdots & & \vdots \\ \frac{\partial T_N^{-1}}{\partial Y_1} & \cdots & \frac{\partial T_N^{-1}}{\partial Y_N} \end{vmatrix}$$

Let X and Y be independent, positive random variables with densities f_X and f_Y, and let $Z = XY$

find the density of Z ?

We find the density of Z by introducing a new random variable W, as follows:

$$Z = XY, \quad W = Y \quad (W = X \text{ would be equally good})$$

The transformation is one-to-one because we can solve for X, Y in terms of Z, W

$$X = Z/W \qquad Y = W \qquad f_{ZW}(z,w) = \left. \frac{f_{XY}(x,y)}{\left|\frac{\partial(z,w)}{\partial(x,y)}\right|} \right|_{\substack{x=\phi(z,w) \\ y=\psi(z,w)}}$$

$$\frac{\partial(z,w)}{\partial(x,y)} = \begin{vmatrix} \partial z/\partial x & \partial z/\partial y \\ \partial w/\partial x & \partial w/\partial y \end{vmatrix} = \begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix} = y$$

$$f_{zw}(z,w) = \left. \frac{f_X(x) f_Y(y)}{y} \right|_{x=z/w \quad y=w} = \frac{f_X(z/w) f_Y(w)}{w}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)\, dw = \int_0^{\infty} \frac{1}{w} f_X(z/w) f_Y(w)\, dw$$

Let X and Y be independent uniform r.v.'s over (0, 1). Find the pdf of $Z = XY$

We have

$$f_{XY}(x, y) = \begin{cases} 1 & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The range of Z is (0, 1)

Introducing auxiliary V = X

$$J(z, w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{w} & -\frac{z}{w^2} \end{vmatrix} = -\frac{1}{w}$$

$$f_{ZW}(z, w) = \left|\frac{1}{w}\right| f_{XY}\left(w, \frac{z}{w}\right)$$

We have

$$f_{XY}(x, y) = \begin{cases} 1 & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The range of Z is (0, 1)

$$f_{ZW}(z, w) = \left|\frac{1}{w}\right| f_{XY}\left(w, \frac{z}{w}\right)$$

$$f_{XY}\left(w, \frac{z}{w}\right) = \begin{cases} 1 & 0 < w < 1, 0 < z/w < 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & 0 < z < w < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{\infty} \left|\frac{1}{w}\right| f_{XY}\left(w, \frac{z}{w}\right) dw = \int_z^1 \frac{1}{w}\, dw = -\ln z \qquad 0 < z < 1$$