Ch4: Multiple Random Variables

Vector Random Variable

Previously we discussed single random variable X in some random experiment such as tossing a die

$$S = \{ 1, 2, 3, 4, 5, 6 \}$$

However in many situations we are interested in more than one random variable as in tossing the die twice or tossing two die.

Here we have two random variables say X represent the first die and Y represent the second die

The two random variables X and Y are now defined on a joint ordered two-dimension space (x,y) with a sample space S given as

S ={ (1,1), (1,2), (1,3), … (6,6)}

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

The order pair of numbers (x,y) may be considered as a specific value of a random vector.

Joint Distribution and its Properties

We defined the distribution and density for rolling a single die as

$$F_X(x) = \sum_{i=1}^{6} P(x_i)u(x - x_i)$$

$$f_X(x) = \sum_{i=1}^{6} P(x_i)\delta(x - x_i)$$

We now define the distribution function $F_{XY}(x,y)$ and the density function $f_{XY}(x,y)$ for the joint XY random variable in a similar way to the one-dimension case as

$$F_{X,Y}(x,y) = P(X \le x,\ Y \le y) = \sum_{n=1}^{6}\sum_{m=1}^{6} P(x_n, y_m)\, u(x - x_m)\, u(y - y_m)$$

$$f_{X,Y}(x,y) = \sum_{n=1}^{6}\sum_{m=1}^{6} P(x_n, y_m)\, \delta(x - x_m)\, \delta(y - y_m)$$

In general for two discrete random variables X and Y

$$F_{X,Y}(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n, y_m)\, u(x - x_m)\, u(y - y_m)$$

$$f_{X,Y}(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n, y_m)\, \delta(x - x_m)\, \delta(y - y_m)$$

If the random variables are continuous, the same behavior as the discrete except the surface is smooth and the probability mass become the continuous joint density and the relation between the distribution and density become

$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\xi_1,\xi_2)\, d\xi_1 d\xi_2 \qquad f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

When N random variables $X_1,\ldots,X_n$ $n=1,2\ldots N$ are involved, the joint density function $f_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N)$ becomes the N-fold partial derivative of the N-dimensional distribution function $F_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N)$ as follows

$$F_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots x_N) = P\{X_1 \le x_1,\ X_2 \le x_2,\ \ldots\ ,\ X_N \le x_N\}$$

$$\Rightarrow F_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N) = \int_{-\infty}^{x_N}\cdots\int_{-\infty}^{x_2}\int_{-\infty}^{x_1} f_{X_1,X_2,\ldots,X_N}(\xi_1,\xi_2,\ldots,\xi_N)d\xi_1 d\xi_2\ldots d\xi_N$$

$$\Rightarrow f_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N) = \frac{\partial^N F_{X_1,X_2,\ldots,X_N}(x,\ x_2,\ \ldots,\ x_N)}{\partial x_1 \partial x_2 \ldots \partial x_N}$$

Properties of the joint distribution

The properties for the two dimension distribution function is an extension of the one dimension distribution function

(1) $F_{X,Y}(-\infty,-\infty)=0 \quad F_{X,Y}(-\infty, y)=0 \quad F_{X,Y}(x,-\infty)=0$

(*similar to the one dimension distribution*) $F_X(-\infty)=0$

(2) $F_{X,Y}(\infty,\infty)=1$

(*similar to the one dimension distribution*) $F_X(\infty)=1$

(3) $0 \le F_{X,Y}(x,y) \le 1$

(*similar to the one dimension distribution*) $0 \le F_X(x) \le 1$

(4) $F_{X,Y}(x,y)$ is a nondecreasing function of both x and y

(*similar to the one dimension distribution*) $F_X(x)$ is nondecreasing

(5) $P\{x_1 < X \le x_2, y_1 < Y \le y_2\} = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1, y_1) \ge 0$

(*similar to the one dimension distribution*) $P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)$

(6) $F_{X,Y}(x,\infty) = F_X(x) \quad F_{X,Y}(\infty, y) = F_Y(y)$

Property (6) is an important property which will be discus in detail next

Marginal Distribution Functions

Property 6 of the joint distribution is given as

$$F_{X,Y}(x,\infty) = F_X(x) \qquad F_{X,Y}(\infty,y) = F_Y(y)$$

the property state that the distribution function of one random variable can be obtained by setting the value of the other variable to **infinity** in $F_{X,Y}(x,y)$

The functions $F_X(x)$, $F_Y(y)$ obtained in this manner are called marginal distribution functions

To justify this let us look at property 6

$$F_{X,Y}(x,\infty) = F_X(x) \qquad F_{X,Y}(\infty,y) = F_Y(y)$$

$F_{X,Y}(x,\infty) = P(X \leq x\ ,\ Y \leq\ \infty)$

since the event $\{Y \leq \infty\}$ is a sure event

$$\Rightarrow\ F_{X,Y}(x,\infty) = P(X \leq x\ ,Y \leq\ \infty) = P(\{X \leq x\} \cap \underbrace{\{Y \leq\ \infty\}}_{\text{Sure Event or } S})$$

$$= P(\{X \leq x\} \cap S) = P(X \leq x) = F_X(x)$$

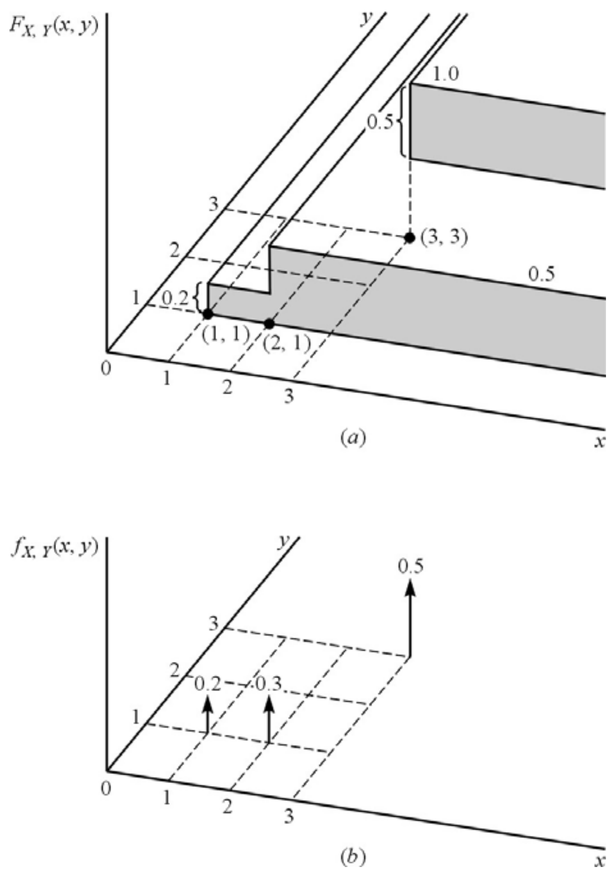
Similarly

$$\Rightarrow\ F_{X,Y}(\infty,y) = P(X \leq \infty\ ,\ Y \leq\ y) = P(Y \leq y) = F_Y(y)$$

Example: 4.2-2 Let the joint sample space S_J and joint probabilities

S_J = {(1,1), (2,1), (3,3)} P(1,1) =0.2 P(2,1) = 0.3 P(3,3) = 0.5

Derive explicit expressions for $F_{X,Y}(x,y)$ and the marginal distributions $F_X(x)$ and $F_Y(y)$.



(a)

(b)

The general expression for the joint distribution is given by

$$F_{X,Y}(x,y)=\sum_{n=1}^{N}\sum_{m=1}^{M}P(x_n,y_m)u(x-x_n)u(y-y_m)$$

If we recognize that only three probabilities are non-zero

$$F_{X,Y}(x,y)=P(1,1)u(x-1)u(y-1)+P(2,1)u(x-2)u(y-1)$$
$$+P(3,3)u(x-3)u(y-3)$$

where $P(1,1)=0.2,\ P(2,1)=0.3,$ and $P(3,3)=0.5$. If we set $y=\infty$:

$$F_X(x)=F_{X,Y}(x,\infty)$$

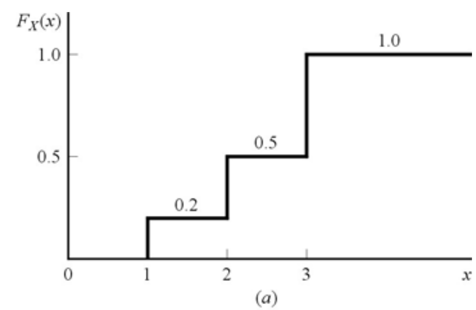
$$=P(1,1)u(x-1)\overbrace{u(\infty-1)}^{=1}+P(2,1)u(x-2)\overbrace{u(\infty-1)}^{=1}+P(3,3)u(x-3)\overbrace{u(\infty-3)}^{=1}$$

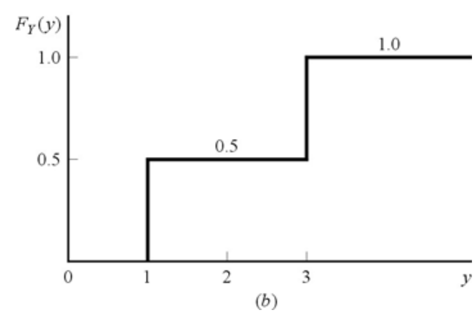
$$=0.2u(x-1)+0.3u(x-2)+0.5u(x-3)$$

If we set $x = \infty$:

$$F_Y(y) = F_{X,Y}(\infty, y)$$
$$= 0.2u(y-1) + 0.3u(y-1) + 0.5u(y-3)$$
$$= 0.5u(y-1) + 0.5u(y-3)$$

The plot of these marginal distributions are shown in the following figure





Suppose we have three random variables X, Y, Z with distribution function given as

$$F_{X,Y,Z}(x,y,z) = P\{X \le x,\ Y \le y,\ Z \le z\}$$

Then

The one dimension marginal distribution

$$F_{X,Y,Z}(x,\infty,\infty) = P(X \le x\ ,\ Y \le \infty, Z \le \infty) = P(X \le x) = F_X(x)$$

$$F_{X,Y,Z}(\infty,y,\infty) = P(X \le \infty\ ,\ Y \le y, Z \le \infty) = P(Y \le y) = F_Y(y)$$

$$F_{X,Y,Z}(\infty,\infty,z) = P(X \le \infty\ ,\ Y \le \infty, Z \le z) = P(Z \le z) = F_Z(z)$$

The two dimension marginal distribution

$$F_{X,Y,Z}(x,y,\infty) = P(X \le x\ ,\ Y \le y, Z \le \infty) = P(X \le x, Y \le y) = F_{X,Y}(x,y)$$

$$F_{X,Y,Z}(x,\infty,z) = P(X \le x\ ,\ Y \le \infty, Z \le z) = P(X \le x, Z \le z) = F_{X,Z}(x,z)$$

$$F_{X,Y,Z}(\infty,y,z) = P(X \le \infty\ ,\ Y \le y, Z \le z) = P(Y \le y, Z \le z) = F_{Y,Z}(y,z)$$

For N-dimensional joint distribution function

$$F_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots x_N) = P\{X_1 \le x_1,\ X_2 \le x_2,\ \ldots,\ X_N \le x_N\}$$

we may obtain a k-dimensional marginal distribution function for any selected group of k of the N random variables by selecting the values of the other N–k random variables to infinity.

Here k can be any integer 1, 2, 3, …, N – 1

Joint Density and its Properties

We extend the concept of density function to include multiple random variables

For two random variables X and Y , the joint probability density function denoted $f_{X,Y}(x,y)$ is defined as the second derivative of the joint distribution function wherever it exists

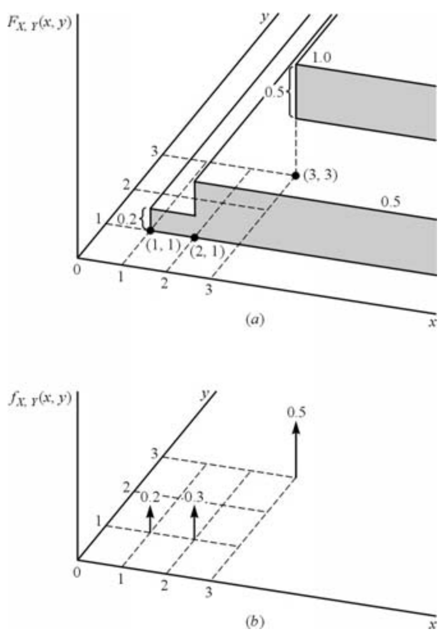
$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} \quad \Rightarrow \quad F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\xi_1,\xi_2)\,d\xi_1 d\xi_2$$

If X and Y are discrete random variables , $F_{X,Y}(x,y)$ will possess step discontinuities

$$F_{X,Y}(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n, y_m)\, u(x - x_m)\, u(y - y_m)$$

the density function $f_{X,Y}(x,y)$ will be impulses at these discontinuities and the impulse strength will be the mass probability at that discontinuities

$$f_{X,Y}(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n, y_m)\, \delta(x - x_n)\, \delta(y - y_m)$$



Therefore the joint density function for a discrete random variables is given as

$$f_{X,Y}(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n, y_m)\delta(x - x_n)\delta(y - y_m)$$

When N random variables $X_1, \ldots, X_n$ are involved, the joint density function becomes the N-fold partial derivative of the N-dimensional distribution function

$$f_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N) = \frac{\partial^N F_{X_1,X_2,\ldots,X_N}(x, x_2, \ldots, x_N)}{\partial x_1 \partial x_2 \ldots \partial x_N}$$

$$\Rightarrow \quad f_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N) = \int_{-\infty}^{x_N} \ldots \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f_{X_1,X_2,\ldots,X_N}(\xi_1,\xi_2,\ldots,\xi_N) d\xi_1 d\xi_2 \ldots d\xi_N$$

Properties of the Joint Density

(1) $$f_{X,Y}(x,y) \geq 0$$

(2) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)d_x d_y = 1$$

Properties (1) and (2) may be used as sufficient test to determine if some function can be a valid density function

(3) $$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\xi_1,\xi_2)d\xi_1 d\xi_2$$

(4) $$F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_2 d\xi_1$$

$$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_1 d\xi_2$$

Marginal Distribution

(5) $$P\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(x,y)\, d_x d_y$$

(6) $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

Marginal Densities

Example 4.3-1 (TBA)

Example 4.3-1

Let b be a positive constant. We wish to test the following function to see if it can be a valid probability density function.

$$g(x,y) = \begin{cases} be^{-x}\cos(y) & 0 \le x \le 2 \text{ and } 0 \le y \le \pi/2 \\ 0 & \text{all other } x \text{ and } y \end{cases}$$

For the allowed values of x and y the function is not negative and satisfies

(1) $f_{X,Y}(x,y) \ge 0$

The final test is

(2) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

For the function in question, we get

$$\int_0^{\pi/2}\int_0^2 be^{-x}(\cos(y)dxdy = b\int_0^2 e^{-x}\int_0^{\pi/2}\cos(y)dy$$

$$= b(1-e^{-2}) = 1$$

Thus to be valid, $b = 1/[1-\exp(-2)]$ is necessary.

Marginal Density Functions

The functions $f_X(x)$ and $f_Y(y)$ of property 6

(6) $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

are called marginal probability Density functions or just marginal density functions.

They are the density functions of a single variables X and Y and are defined as the derivatives of the marginal distribution functions:

$$f_X(x) = \frac{dF_X(x)}{dx} \qquad f_Y(y) = \frac{dF_Y(y)}{dy}$$

From property (4) Marginal Distribution

$$F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_2 d\xi_1$$

$$F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_1 d\xi_2$$

Therefore

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{d}{dx}\int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_2 d\xi_1$$

$$\Rightarrow \quad f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_2$$

Similarly

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_1 d\xi_2$$

$$\Rightarrow \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_1$$

Example (TBA)

Example 4.3-2

We will find marginal probability density functions when the joint density is given by the following

$$f_{X,Y}(x,y) = u(x)u(y)xe^{-x(y+1)}$$

Since $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$

$$f_X(x) = \int_0^{\infty} u(x)xe^{-x(y+1)}dy = u(x)xe^{-x}\int_0^{\infty} e^{-xy}dy$$

$$= u(x)xe^{-x}(1/x) = u(x)e^{-x}$$

Similarly, $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^{\infty} u(y)xe^{-x(y+1)}dx$

$$= \frac{u(y)}{(y+1)^2} \quad \text{(using Appendix C)}$$

For N random variables $X_1,\ldots,X_N$ the k-dimensional marginal density function is defined as the k-fold partial derivative of the k-dimensional distribution function

$$f_{X_1,X_2,\ldots,X_k}(x_1,x_2,\ldots,x_k) = \frac{\partial^k F_{X_1,X_2,\ldots,X_k}(x, x_2, \ldots, x_k)}{\partial x_1 \partial x_2 \ldots \partial x_k}$$

It can also be found for the joint density function by integarting out all variables except the k variables of interest $X_1,\ldots,X_k$

$$f_{X_1,X_2,\ldots,X_k}(x_1,x_2,\ldots,x_k) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N)dx_{k+1}dx_{k+2}\ldots dx_N$$

Conditional Distribution and Density

The conditional distribution function of a random variable X given some event B was defined as

$$F_X(x|B) = P\{X \le x|B\} = \frac{P\{X \le x \cap B\}}{P(B)} \qquad \text{were} \quad P(B) \neq 0$$

The corresponding conditional density function was defined through the derivative

$$f_X(x|B) = \frac{dF_X(x|B)}{dx}$$

From the definition of the conditional distribution function of a random variable X given some event B, we can write the conditional distribution function as

$$F_X(x|\underbrace{y - \Delta y < Y \le y + \Delta y}_{B}) = \frac{P\{X \le x \cap (y - \Delta y < Y \le y + \Delta y)\}}{P(y - \Delta y < Y \le y + \Delta y)}$$

$$= \frac{\int_{y-\Delta y}^{y+\Delta y} \int_{-\infty}^{x} f_{X,Y}(\xi_1, \xi_2) d\xi_1 \xi_2}{\int_{y-\Delta y}^{y+\Delta y} f_Y(\xi) d\xi}$$

We will consider two cases

First case, we assume X and Y are both discrete random variables

Second case, we assume X and Y are both continuous random variables

(1) X and Y are Discrete

$$X = \{x_i\} \qquad i = 1,\ 2,\ \dots N \qquad Y = \{y_j\} \qquad j = 1,\ 2,\ \dots M$$

$$P(x_i) \qquad i = 1,\ 2,\ \dots N \qquad P(y_j) \qquad j = 1,\ 2,\ \dots M$$

Let $P(x_i,y_j)$ is the probability of the joint occurrence of x_i and y_j .

Therefore the conditional distribution (**proof will be shown on the web**)

$$\Rightarrow \quad F_X(x|Y=y_K) = \frac{\sum_{i=1}^{N} P(x_i,y_K)u(x-x_i)}{P(y_K)} = \sum_{i=1}^{N} \frac{P(x_i,y_K)}{P(y_K)} u(x-x_i)$$

After differentiation we have the conditional density function

$$f_X(x|Y=y_K) = \frac{dF_X(x|Y=y_K)}{dx} = \sum_{i=1}^{N} \frac{P(x_i,y_K)}{P(y_K)} \delta(x-x_i)$$

Example 4.4-1

(2) X and Y are Continuous

Therefore the conditional distribution (**proof will be shown on the web**)

$$F_X(x|Y=y) = \frac{\int_{-\infty}^{x} f_{X,Y}(\xi_1, y)\,d\xi_1}{f_Y(y)} \qquad \text{For every } y \text{ such that } f_Y(y) \neq 0$$

After differentiation we have the the conditional density

$$f_X(x|Y=y) = \frac{dF_X(Y=y)}{dx} = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Example 4.4-1

STATICAL INDEPENDENCE

Two Events are Statistically Independent iff (if and only if)

$$P(A \cap B) = P(A)P(B)$$

Two Random Variables X and Y are Statistically Independent iff

$$P\{X \le x, Y \le y\} = P\{X \le x\}P\{Y \le y\}$$

for any events $\{X \le x\}$ and $\{Y \le y\}$

Then from the definition of distribution function

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

$\Rightarrow$ the density $f_{X,Y}(x,y) = \dfrac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} = f_X(x)f_Y(y)$

Example 4-5-1

For the more general case , let $X_1, \ldots, X_N$ be N- random variables

Define event A_i by $A_i = \{X_i \leq x_i\}$

We then say the N random variables are statistically independent if (from chapter 1)

$$P(A_i \cap A_j) = P(A_i)P(A_j)$$

$$P(A_i \cap A_j \cap A_k) = P(A_i)P(A_j)P(A_k)$$

$$\vdots$$

$$P(A_1 \cap A_2 \cdots \cap A_N) = P(A_1)P(A_2)\cdots P(A_N)$$

which can be written in terms of the distribution and densities functions as

$$F_{X_iX_j}(x_i,x_j) = F_{X_i}(x_i)F_{X_j}(x_j)$$

$$F_{X_iX_jX_k}(x_i,x_j,x_k) = F_{X_i}(x_i)F_{X_j}(x_j)F_{X_k}(x_k)$$

$$\vdots$$

$$F_{X_1X_2\cdots X_N}(x_1,x_2,\ldots x_N) = F_{X_1}(x_1)F_{X_2}(x_2)\ldots F_{X_N}(x_N)$$

$$f_{X_iX_j}(x_i,x_j) = f_{X_i}(x_i)f_{X_j}(x_j)$$

$$f_{X_iX_jX_k}(x_i,x_j,x_k) = f_{X_i}(x_i)f_{X_j}(x_j)f_{X_k}(x_k)$$

$$\vdots$$

$$f_{X_1X_2\cdots X_N}(x_1,x_2,\ldots x_N) = f_{X_1}(x_1)f_{X_2}(x_2)\ldots f_{X_N}(x_N)$$

Example (4.5-1)

Example 4.5-1

We wish to determine whether the random variables X and Y are statistically independent when their joint probability distribution is given by the following

$$f_{X,Y}(x,y) = u(x)u(y)xe^{-x(y+1)}$$

From the previous example, the marginal densities are

$$f_X(x) = u(x)e^{-x} \text{ and } f_Y(y) = \frac{u(y)}{(y+1)^2}, \text{ therefore}$$

$$f_X(x)\,f_Y(y) = u(x)u(y)\frac{e^{-x}}{(y+1)^2} \neq f_{X,Y}(x,y)$$

Hence the random variables X and Y are not independent.

Example 4.5-2

We wish to determine whether the random variables X and Y are statistically independent when their joint probability distribution is given by the following

$$f_{X,Y}(x,y)=\frac{1}{12}u(x)u(y)e^{-(x/4)-(y/3)}$$

First we need to determine the marginal densities

$$f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy=\int_0^{\infty}(1/12)u(x)e^{-x/4}e^{-y/3}dy=(1/4)u(x)e^{-x/4}$$

$$f_Y(y)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx=\int_0^{\infty}(1/12)u(y)e^{-y/3}e^{-x/4}dx=(1/3)u(y)e^{-y/3}$$

Since $f_X(x)\,f_Y(y)=f_{X,Y}(x,y)$, X and Y are independent.

DISTRIBUTION AND DENSITY OF A SUM OF RANDOM VARIABLES

Sum of Two Random Variables

Let W be a random variable equal the sum of two Independent RV X and Y

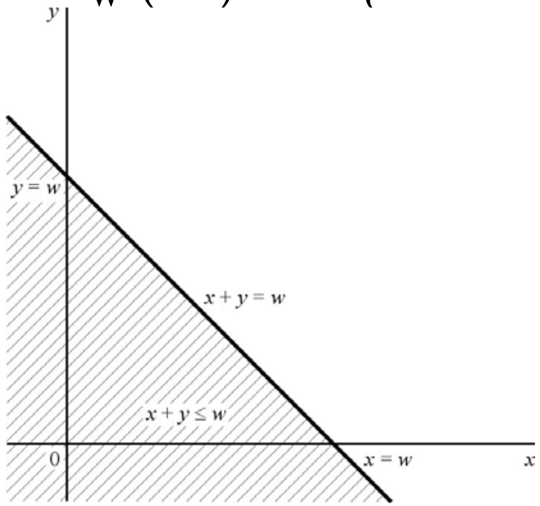
$$W = X + Y$$

This is a very practical problem that appear in signal processing were X represent an instant of a random signal and Y represent an instant of a random noise.

We will take about random signal or functions when we discus random process

The probability distribution function we seek

$$F_W(w) = P\{W \le w\} = P\{X+Y \le w\}$$



The Figure above illustrate the region in the xy-plane were x+y ≤w

$$\Rightarrow \quad F_W(w) = \int_{-\infty}^{\infty}\int_{x=-\infty}^{w-y} f_{X,Y}(x,y)dxdy$$

From Statistically Independent $\quad f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$

$$\Rightarrow \quad F_W(w) = \int_{-\infty}^{\infty} f_Y(y) \int_{x=-\infty}^{w-y} f_X(x)\, dx\, dy$$

using Leibnize rule we get

$$f_W(w) = \frac{dF_W(w)}{dw} = \underbrace{\int_{-\infty}^{\infty} f_Y(y) f_X(w-x) dy}_{\textit{Convolution Integral}} = f_Y(y) * f_X(x)$$

Therefore the density function of the sum of two statistically Independent random variables is the convolution of their individual Density functions

Example1 (TBA)

Example2 (TBA)