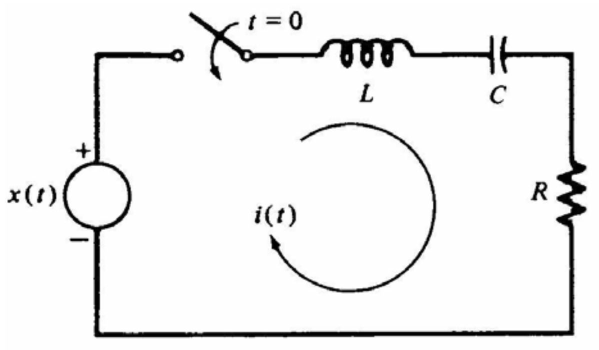
Chapter 6 Applications of the Laplace Transform

EXAMPLE 5-3



In analyzing the circuit , we first wrote down the differential equation using KVL

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(\lambda)\,d\lambda = x(t)$$

Initial condition $i(0^-) = 0$

Taking Laplace Transform for both side ⟹ $LsI(s) + RI(s) + \frac{I(s)}{sC} + \frac{v_c(0^-)}{s} = X(s)$

were $v_C(0^-) = \frac{1}{C}\int_{-\infty}^{0^-} i(\lambda)d\lambda$

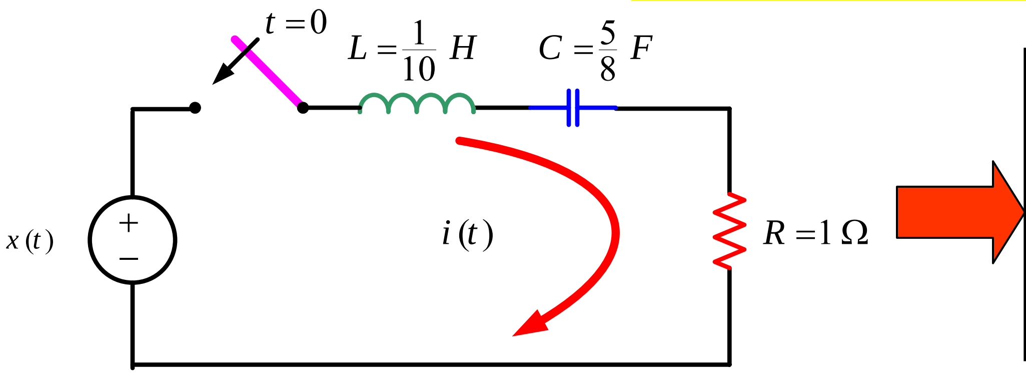
Inverse Back

Solving for $I(s)$ $I(s) = \frac{sX(s) - v_c(0^-)}{L[s^2 + (R/L)s + 1/LC]}$ ⟹ $i(t)$

In this chapter , we are going to do the

First Transform the circuit to the *s*-domain (Laplace Transform)

Transform to the *s*-domain



Each element of the circuit
Is replaced by its Laplace-transformed
Equivalent

Similar what was done using the phasor

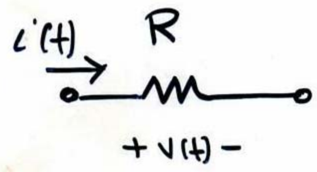
Second Solve for the s-transformed required variable (i.e $I(s)$)
using all linear circuit techniques such as:

OHM , KVL, KCL , VDR, CDR, Thavenin, source transformation , Nodal and Mesh

Third Inverse back , to obtain the time domain variable i(t)

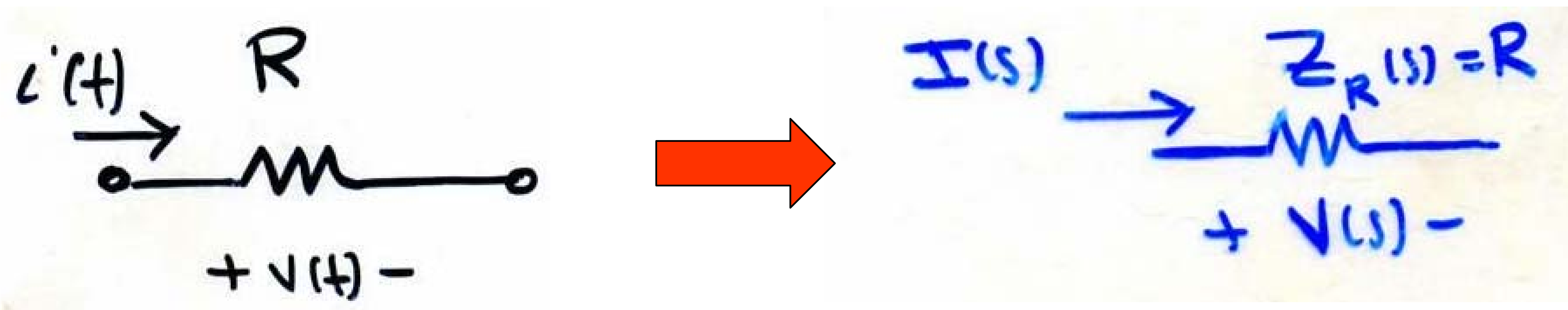
Laplace transform for the passive elements

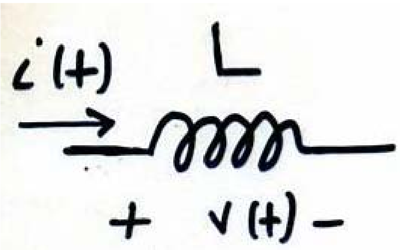
R, L, C



$v(t) = R\, i(t) \quad \Rightarrow \quad V(s) = R\, I(s)$

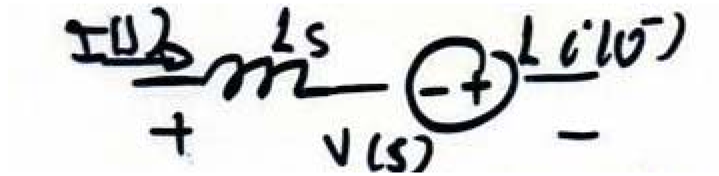
$$Z_R(s) = \frac{V(s)}{I(s)} = R$$

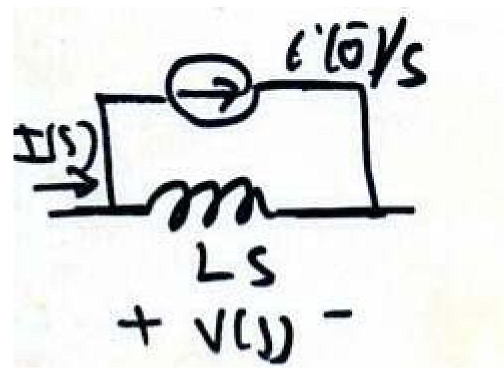




$$v(t) = L\frac{di(t)}{dt}$$

$$\Rightarrow \quad V(s) = L\left[sI(s) - i(0^-)\right]$$



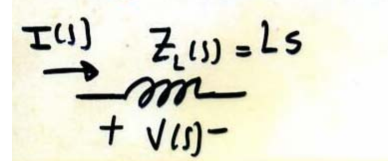


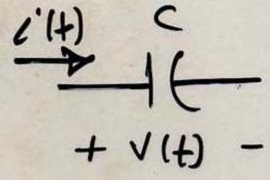
$$I(s) = \frac{1}{Ls}V(s) + \frac{i(0^-)}{s}$$

For zero initial condition $i(0^-) = 0$

$$V(s) = Ls\,I(s) \qquad Z_L(s) = \frac{V(s)}{I(s)} = Ls$$

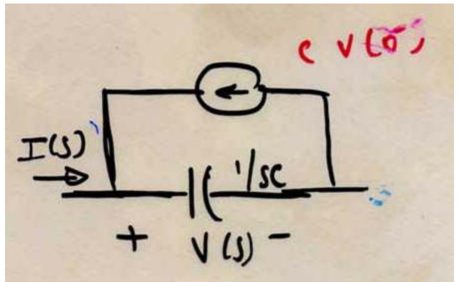
Zero initial current



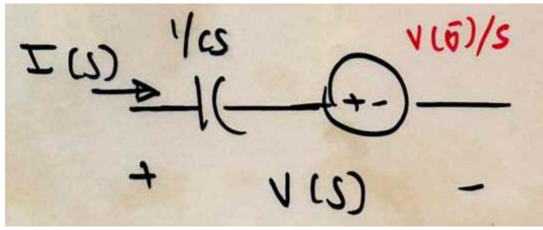


$$i(t) = C\frac{dv(t)}{dt}$$

$$\Rightarrow I(s) = C\left[sV(s) - v(0^-)\right]$$

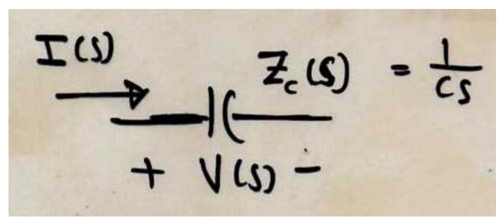


$$V(s) = \frac{1}{Cs} I(s) + \frac{v(0^-)}{s}$$

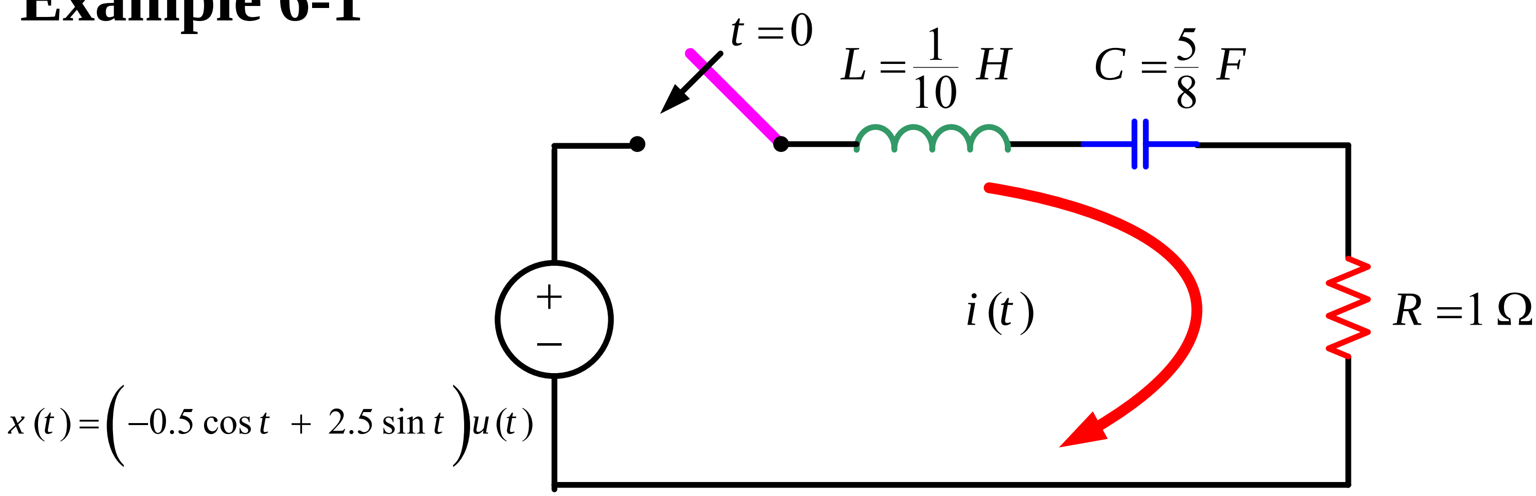


for zero initial conditions $v(0^-) = 0$

$$V(s) = \frac{1}{Cs} I(s)$$



Example 6-1



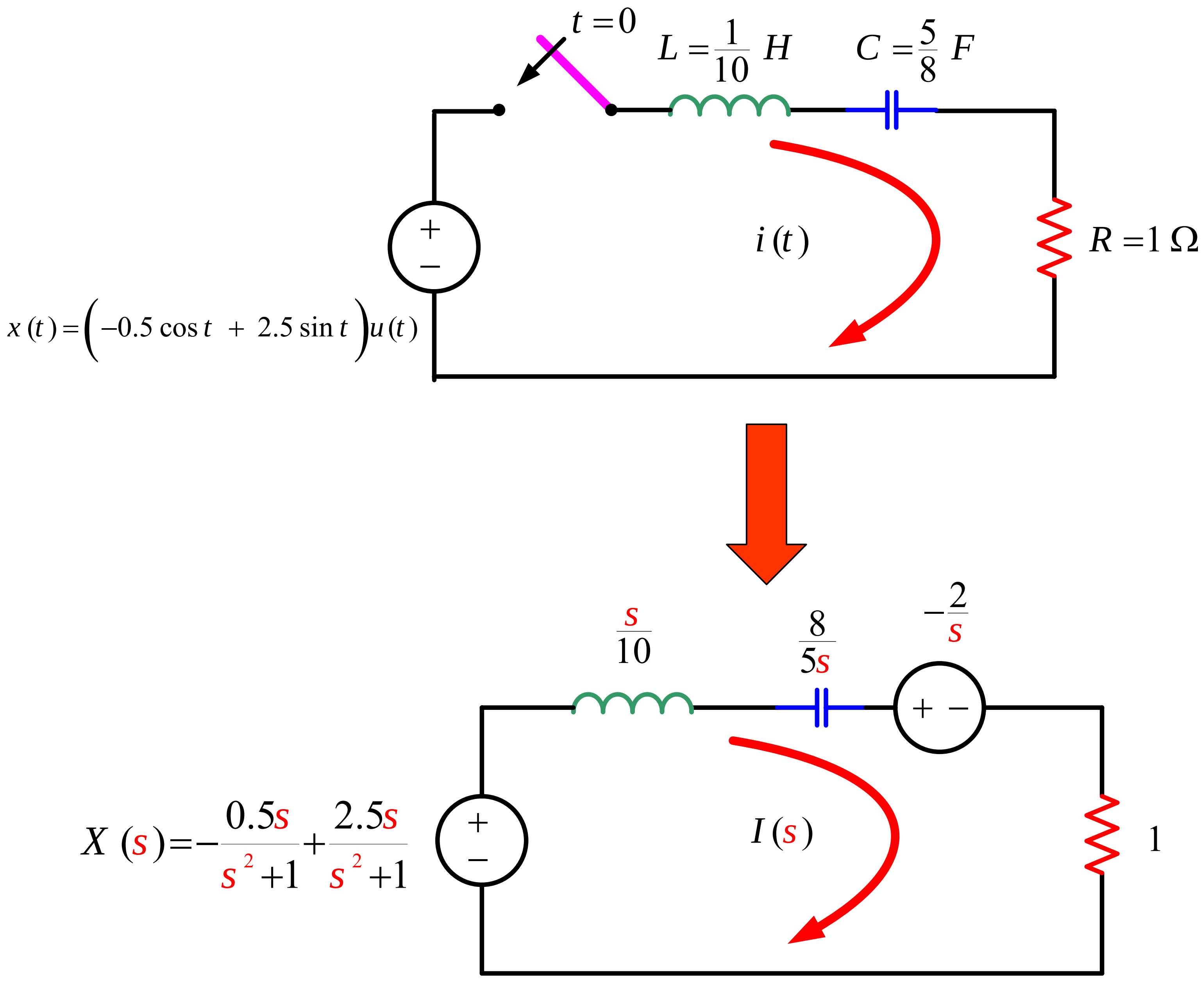
$$i_L(0^-)=\ 0\ V \qquad v_C(0^-)=\ -2\ V$$

$$x(t)\ \Rightarrow\ X(s)=-\frac{0.5s}{s^2+1}+\frac{2.5s}{s^2+1}$$

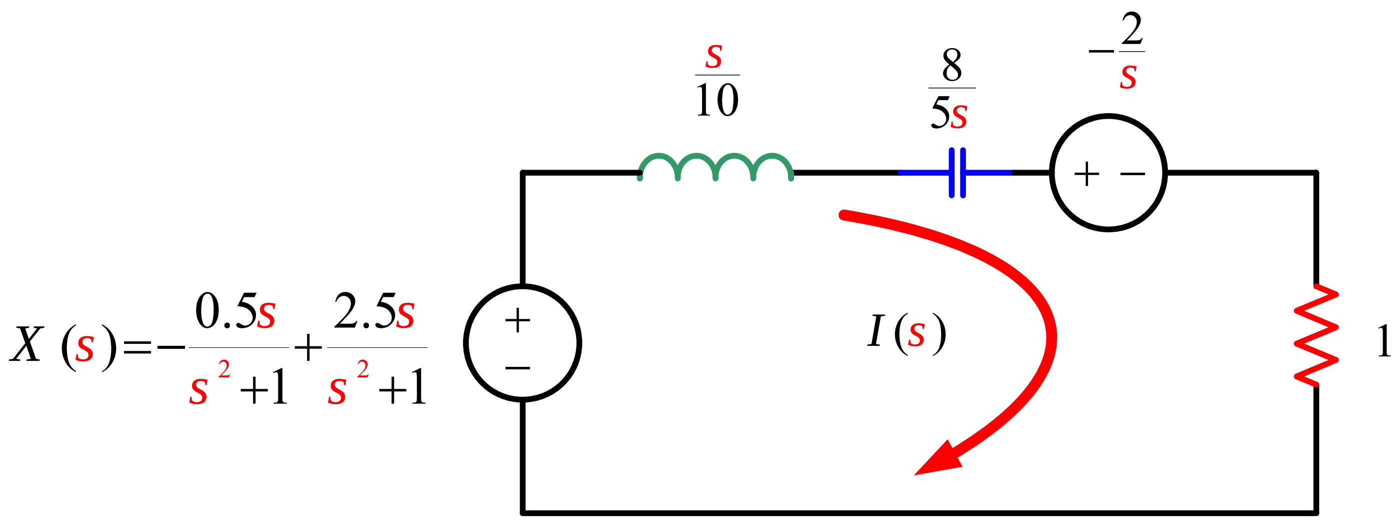
$$L=\frac{1}{10}\ H\ \Rightarrow\ Z_L=\frac{s}{10}\ \Omega$$

$$C=\frac{5}{8}\ F\ \Rightarrow\ Z_C=\frac{8}{5s}\ \Omega$$

$$R=1\ \Omega\ \Rightarrow\ Z_R=1\ \Omega$$



$t=0$
$L=\frac{1}{10}\ H$
$C=\frac{5}{8}\ F$
$i(t)$
$R=1\ \Omega$
$x(t)=\left(-0.5\cos t\ +\ 2.5\sin t\right)u(t)$
$\frac{s}{10}$
$\frac{8}{5s}$
$-\frac{2}{s}$
$I(s)$
1
$X(s)=-\frac{0.5s}{s^2+1}+\frac{2.5s}{s^2+1}$



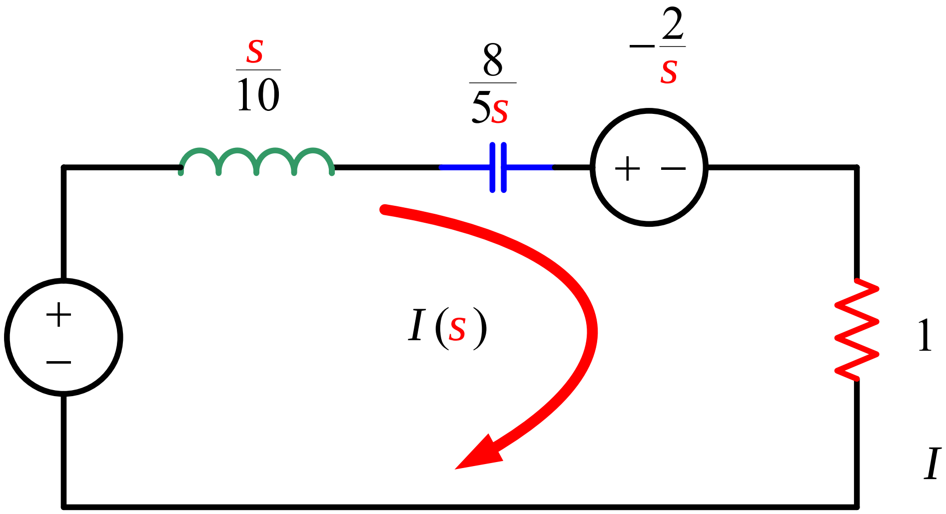
KVL

$$-X(s) + \left(\frac{s}{10}\right) I(s) + \left(\frac{8}{5s}\right) I(s) - \frac{2}{s} + (1) I(s) = 0$$

$$-\frac{0.5s}{s^2+1} + \frac{2.5s}{s^2+1} = \left(\frac{s}{10} + \frac{8}{5s} + 1\right) I(s) - \frac{2}{s}$$

$$I(s) = \frac{15s^2 + 25s + 20}{(s^2+1)(s^2+10s+16)}$$

$$X(s) = -\frac{0.5s}{s^2+1} + \frac{2.5s}{s^2+1}$$



$$I(s) = \frac{15s^2+25s+20}{(s^2+1)(s^2+10s+16)}$$

From Example 5-10 (Imaginary Roots)

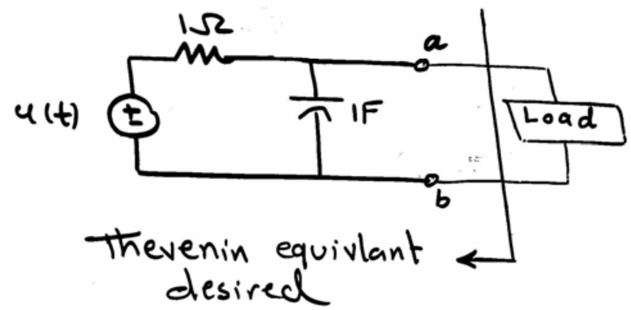
$$I(s) = \frac{15s^2+25s+20}{(s^2+1)(s^2+10s+16)} = \frac{(15s^2+25s+20)}{(s+j)(s-j)(s+2)(s+8)}$$

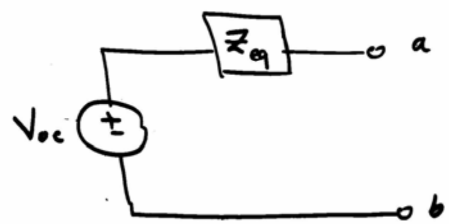
$$= \frac{A_1}{(s+j)} + \frac{A_2}{(s-j)} + \frac{A_3}{(s+2)} + \frac{A_4}{(s+8)}$$

See Example 5-10 for details

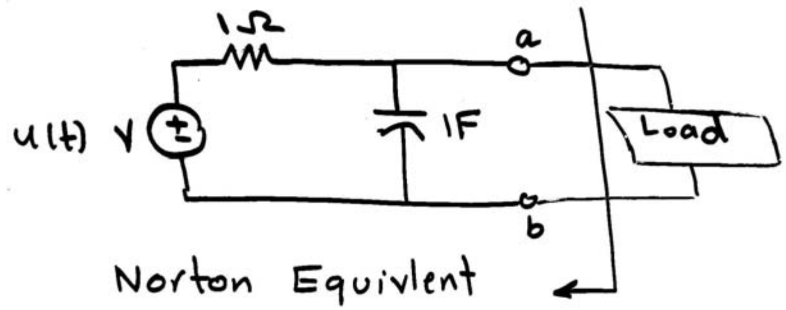
$$i(t) = \left(\cos(t) + \sin(t) + e^{-2t} + e^{-8t}\right)u(t)$$

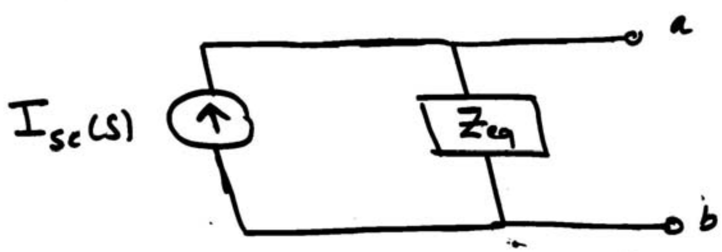
Thevenin Thm

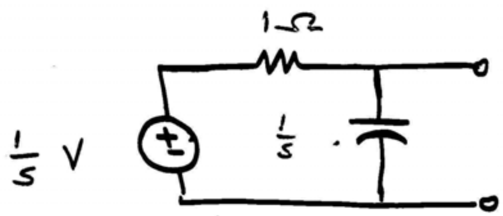




Norton Equivlent







$$V_{oc}(s) = \frac{1}{s(s+1)} \text{ V}$$

$$I_{sc}(s) = \frac{1}{s} \text{ A}$$

$$Z_{eq}(s) = (1\,\Omega) \,//\, \left(\tfrac{1}{s}\,\Omega\right)$$

$$= \frac{1}{s+1} \;\Omega$$

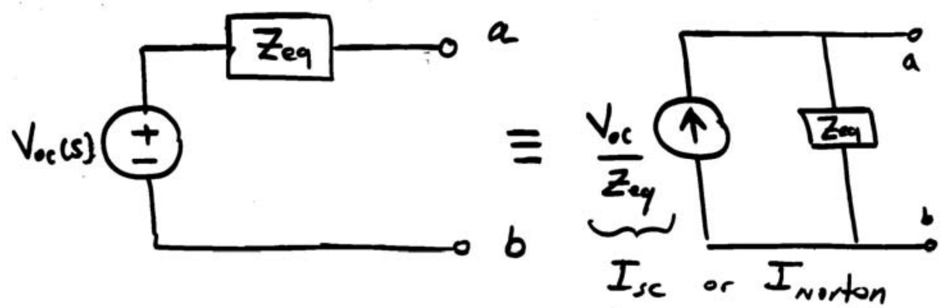
OR

$$Z_{eq}(s) = \frac{V_{oc}(s)}{I_{sc}(s)}$$

$$= \frac{\frac{1}{s(s+1)}}{\frac{1}{s}}$$

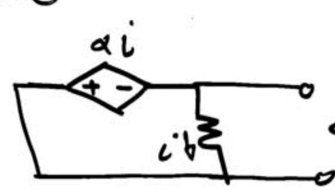
$$= \frac{1}{s+1} \;\Omega$$

Note the following :



Note: If you can not find Z_{eq} by combining impedense in parallel or seires

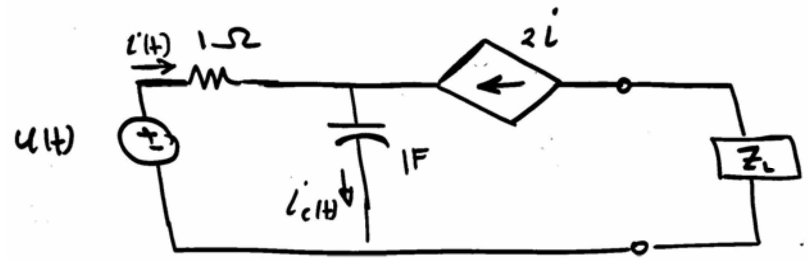
Example The circuit containe dependent source

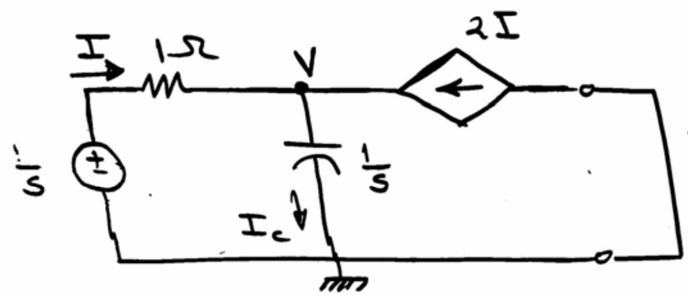


⟸ on this case you can not combine the resistor with dependent source

Solution : Apply a test voltage or current method

Find the Norton equivlent for the following CRT,





$I_{sc} = -2I$
find I

KCL at V

$$I + 2I = I_c \Rightarrow 3I = I_c$$

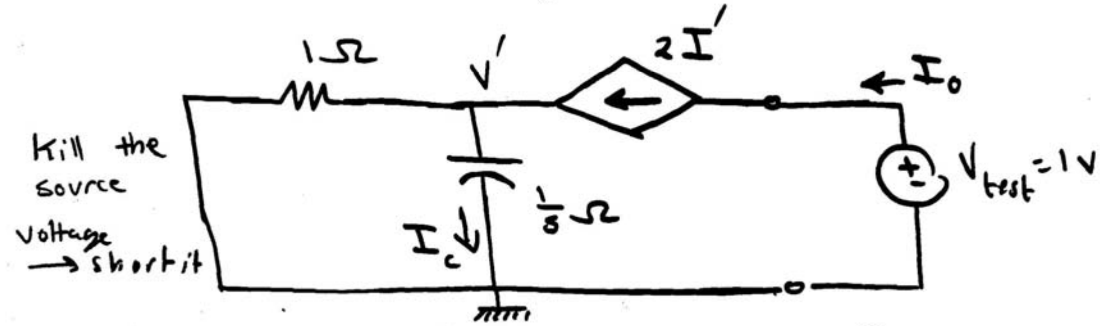
But $\quad I = \dfrac{1/s - V}{1} \qquad I_c = \dfrac{V}{1/s} = sV$

$$\Rightarrow 3\left(\tfrac{1}{s} - V\right) = 2\,(sV) \Rightarrow V(s) = \frac{3}{s\,(s+3)}$$

$$\Rightarrow I = \frac{1}{s} - \frac{3}{s\,(s+3)} = \frac{1}{(s+3)}$$

$$\Rightarrow I_{sc} = -2I = -2/(s+3)$$

To obtain the Equivlent impedence or admittance, we apply a test source method



$$Z_{eq} = \frac{V_{test}}{I_o} = \frac{1}{I_o} \Rightarrow \text{find } I_o$$

<u>KCL at V'</u> $\quad 3I' = I_c \Rightarrow 3\left(\frac{0 - V'}{1}\right) = \frac{V'}{1/s}$

$$\Rightarrow 3(-V') = sV' \Rightarrow (s+3)V' = 0$$

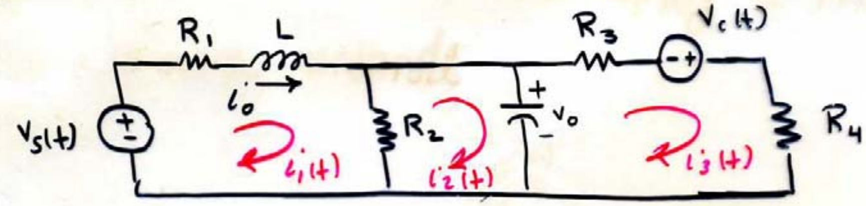
$$\Rightarrow V' = 0 \Rightarrow I' = 0 \Rightarrow I_o = 0$$

$$\Rightarrow Z_{eq} = \frac{1}{0} = \infty \quad \text{open ckt}$$

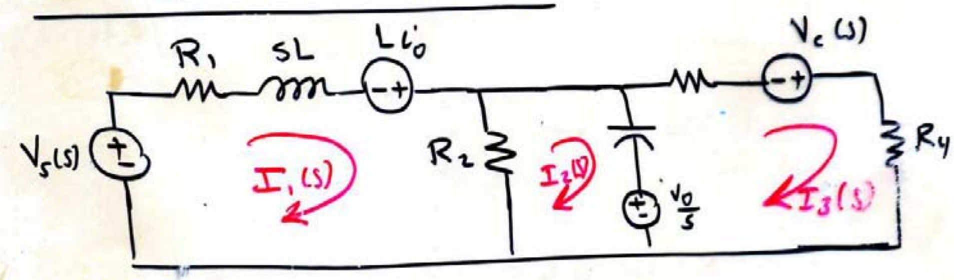
⇒ The Norton equivlent is an ideal current source

6.3 Loop and Node analysis

Ex 6-5



Taking Laplace Transform



Then writting KVL on each mesh

$$\begin{bmatrix} (R_1+R_2+sL) & -R_2 & 0 \\ -R_2 & (R_2+\frac{1}{sC}) & -\frac{1}{sC} \\ 0 & -\frac{1}{sC} & (R_3+R_4+\frac{1}{sC}) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{bmatrix} = \begin{bmatrix} Li_0 + V_s(s) \\ -\frac{v_0}{s} \\ \frac{v_0}{s} + V_c(s) \end{bmatrix}$$

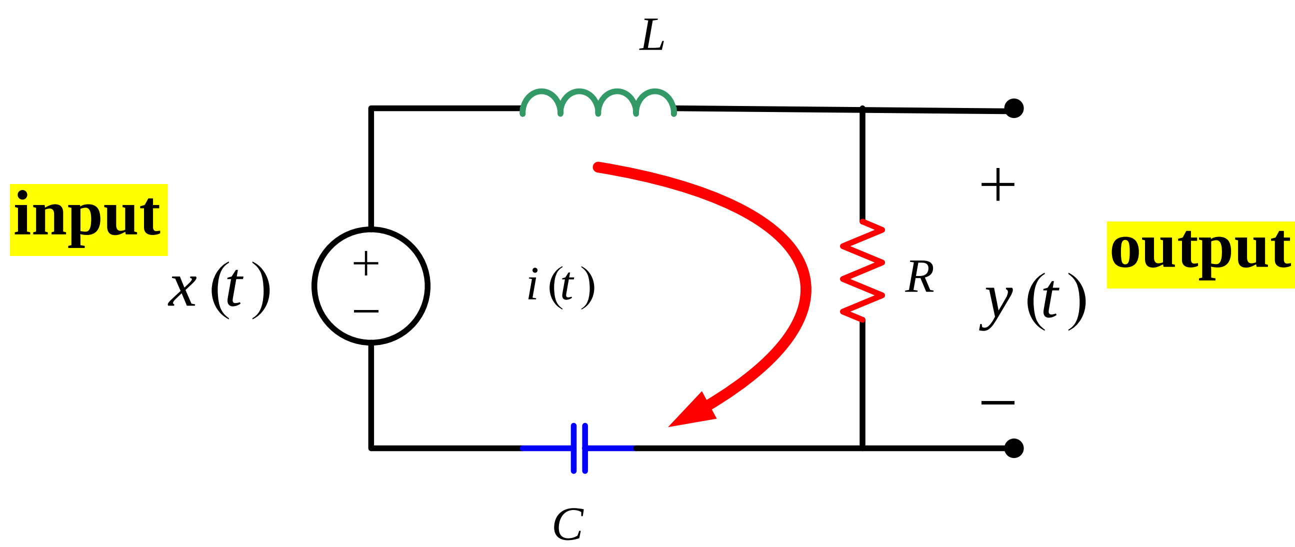
$$\bar{\bar{Z}}\,\bar{I} = \bar{V}$$

$I_1(s) = \frac{\Delta_1}{\Delta}$

$I_2(s) = \frac{\Delta_2}{\Delta}$ $\quad I_3 = \frac{\Delta_3}{\Delta}$

6-4 Transfer Functions

Consider the following circuit

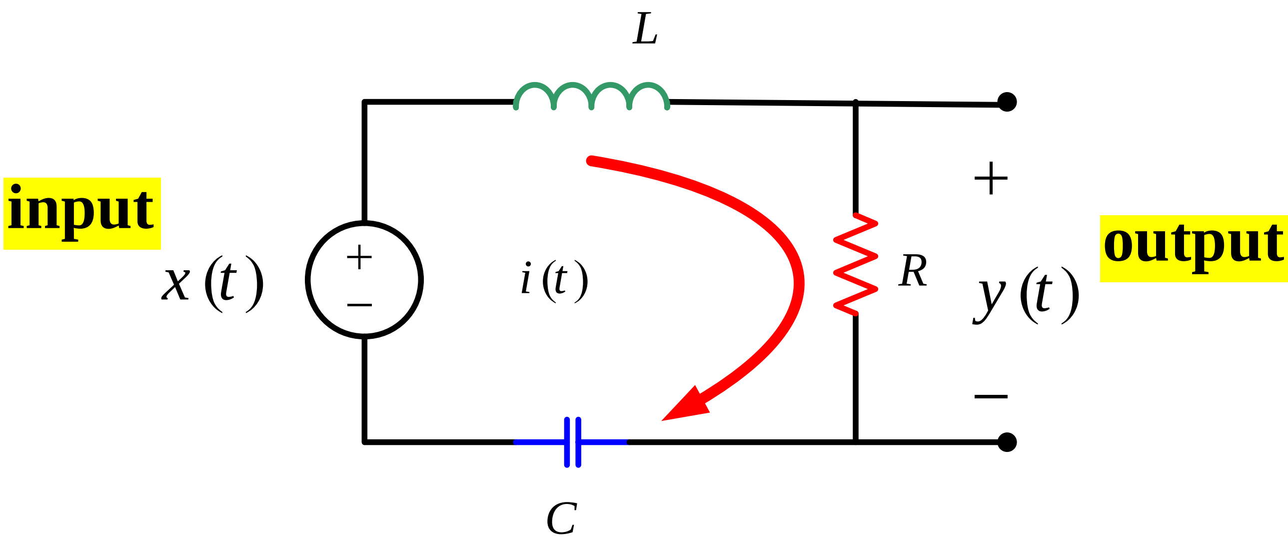


We want a relation (an equation) between the input *x*(t) and output *y*(t)

KVL

$$x(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t')dt'$$

$$\frac{dx(t)}{dt} = L\frac{di^2(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{i(t)}{C}$$

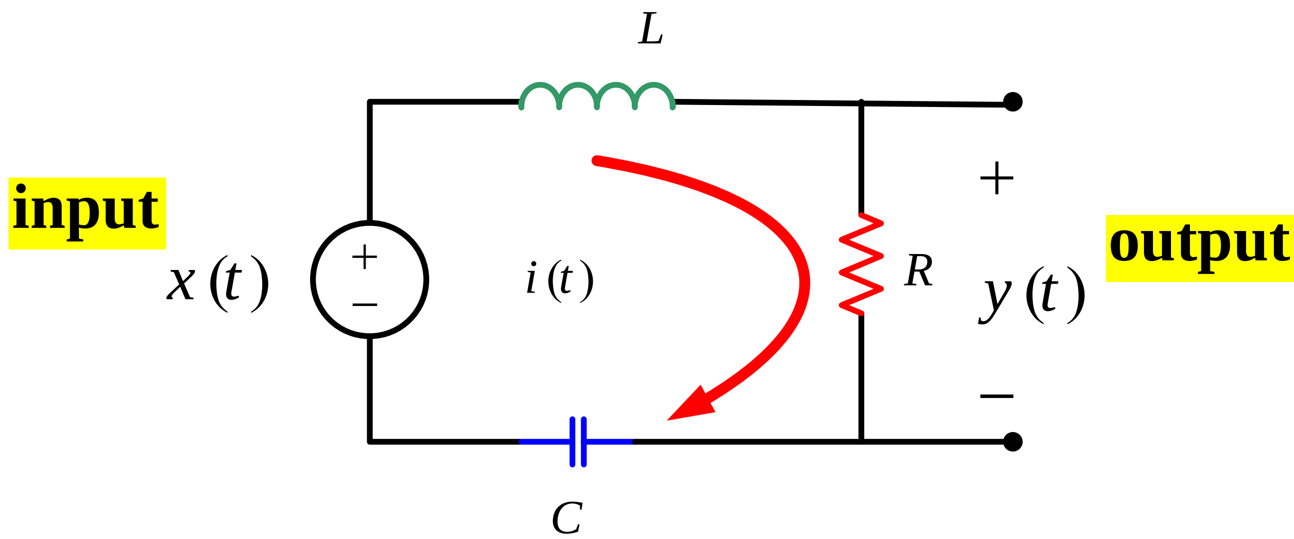


$$\frac{dx(t)}{dt} = L\frac{di^2(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{i(t)}{C}$$

Since $i(t) = \dfrac{y(t)}{R}$ $\Rightarrow$ $\dfrac{dx(t)}{dt} = \dfrac{L}{R}\dfrac{dy^2(t)}{dt^2} + R\dfrac{dy(t)}{dt} + \dfrac{y(t)}{RC}$

Writing the differential equation as

$$RC\frac{dx(t)}{dt} = LC\frac{dy^2(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t)$$



$$\underbrace{\{RC\}}\frac{dx(t)}{dt} = \underbrace{\{LC\}}\frac{dy^{2}(t)}{dt^{2}} + \underbrace{\{RC\}}\frac{dy(t)}{dt} + \underbrace{\{1\}}y(t)$$

Real coefficients, **non negative** which results from system components $R,\ L,\ C$

In general,

$$a_n \frac{dy^n(t)}{dt^n} + a_{n-1}\frac{dy^{n-1}(t)}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{dx^m(t)}{dt^m} + b_{m-1}\frac{dx^{m-1}(t)}{dt^{m-1}} + \cdots + b_0 y(t)$$

were a_n's , b_m's **are real**, **non negative** which results from system components R, L, C

Now if we take the Laplace Transform of both side (Assuming Zero initial Conditions)

$$a_n s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_0 Y(s) = b_m s^m X(s) + b_{m-1}s^{m-1}X(s) + \cdots + b_0 X(s)$$

We now define the transfer function $H(s)$,

$$H(s) \triangleq \left.\frac{Y(s)}{X(s)}\right|_{\text{all initial conditions are zero}} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

$$a_n \frac{dy^n(t)}{dt^n} + a_{n-1}\frac{dy^{n-1}(t)}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{dx^m(t)}{dt^m} + b_{m-1}\frac{dx^{m-1}(t)}{dt^{m-1}} + \cdots + b_0 x(t)$$

$$H(s) \triangleq \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0} \triangleq \frac{N(s)}{D(s)}$$

Since a_n's , b_m's **are real, non negative**

The roots of the polynomials $N(s)$, $D(s)$ are either real or occur in complex conjugate

The roots of $N(s)$ are referred to as the zero of $H(s)$ ($H(s) = 0$)

The roots of $D(s)$ are referred to as the pole of $H(s)$ ($H(s) = \pm\infty$)

$$H(s) \triangleq \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \triangleq \frac{N(s)}{D(s)}$$

The Degree of $N(s)$ (which is related to input) must be less than or Equal of $D(s)$ (which is related to output) for the system to be Bounded-input, bounded-output (**BIBO**)

Example : $$H(s) = \frac{4s^3 + 2s^2 + s + 1}{s^2 + 6s + 8}$$

Using polynomial division , we obtain $$H(s) = 4s + 2 + \frac{-19s + 17}{s^2 + 6s + 8}$$

Now assume the input $x(t) = u(t)$ (**bounded input**) $\Rightarrow X(s) = \frac{1}{s}$

$$Y(s) = X(s)H(s) = 4 + \frac{2}{s} + \frac{1}{s}\left(\frac{-19s + 17}{s^2 + 6s + 8}\right)$$

$$y(t) = \underbrace{4\delta(t)}_{\text{unbounded } (\to \infty)} + 2 + L^{-1}\left(\frac{-19s + 17}{s(s^2 + 6s + 8)}\right)$$

We see that for finite bounded Input (i.e x(t) =u(t))

We get an infinite (unbounded) output

$m \le n$ for **BIBO**

$$H(s) \triangleq \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \triangleq \frac{N(s)}{D(s)}$$

The poles of $H(s)$ must have real parts which are negative

➡ The poles must lie in the left half of the s-plan

Components of System Response

Consider the following differential equation (Input / Output),

$$a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t)$$

Taking Laplace Transform of both side,

$$a_1[sY(s) - y(0^-)] + a_0 Y(s) = b_0 X(s)$$

$$[a_1 s \;\; -a_0]\, Y(s) = b_0 X(s) + a_1 y(0^-)$$

$$Y(s) = \frac{b_0 X(s) + a_1 y(0^-)}{[a_1 s \;\; -a_0]}$$

$$Y(s) = \frac{b_0}{[a_1 s \;\; -a_0]} X(s) + \frac{a_1 y(0^-)}{[a_1 s \;\; -a_0]}$$

$$a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t)$$

$$Y(s) = \frac{b_0}{[a_1 s \quad - a_0]} X(s) + \frac{a_1 y(0^-)}{[a_1 s \quad - a_0]}$$

If initial conditions are zeros ⇒ $$Y(s) = \frac{b_0}{[a_1 s \quad - a_0]} X(s)$$

$$H(s) \triangleq \left. \frac{Y(s)}{X(s)} \right|_{\text{all initial conditions are zero}} = \frac{N(s)}{D(s)} = \frac{b_0}{[a_1 s \quad - a_0]}$$

Transfer Function All initial conditions are zeros

A polynomial related to initial conditions

$$Y(s) = H(s)X(s) + \frac{a_1 y(0^-)}{D(s)} = H(s)X(s) + \frac{C(s)}{D(s)}$$

$$a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t)$$

$$Y(s) = \frac{b_0}{[a_1 s - a_0]} X(s) + \frac{a_1 y(0^-)}{[a_1 s - a_0]}$$

Transfer Function All initial conditions are zeros

A polynomial related to initial conditions

$$Y(s) = H(s)X(s) + \frac{a_1 y(0^-)}{D(s)} = H(s)X(s) + \frac{C(s)}{D(s)}$$

$$y(t) = L^{-1}[H(s)X(s)] + L^{-1}\left[\frac{C(s)}{D(s)}\right]$$

$$y(t) = \underbrace{y_{\mathbf{ZSR}}(t)}_{\text{Zero State Response}} + \underbrace{y_{\mathbf{ZIR}}(t)}_{\text{Zero State Response}}$$

Zero State Response **(Steady State)**

Zero State Response **(Steady State)**

Example 6-7