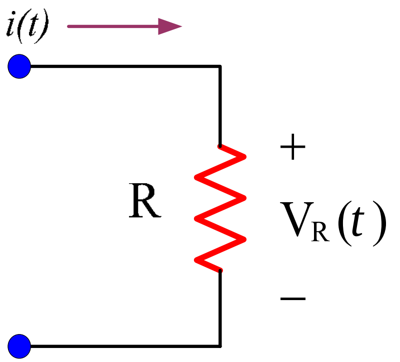
Dr. Adil S. Balghonaim

EE 207 Class Notes

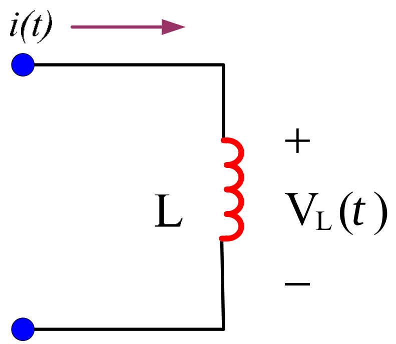
Chapter 2

Chapter 2: System Modeling and Analysis in Time domain

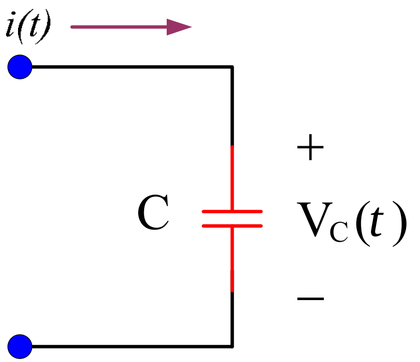
Consider The following Input/Output relations



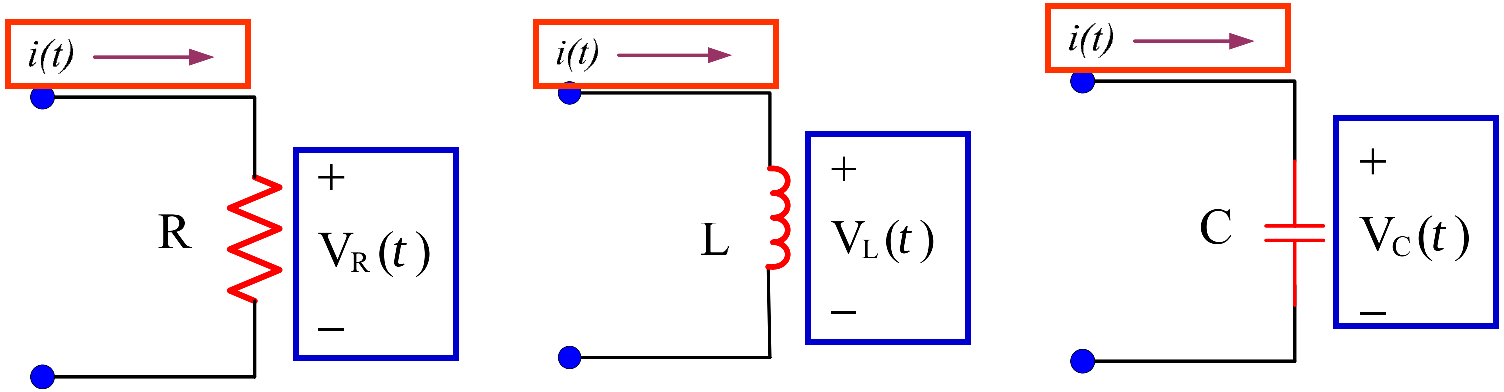
$$V_R(t) = R\,i(t)$$



$$V_L(t) = L\frac{di(t)}{dt}$$



$$V_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau$$



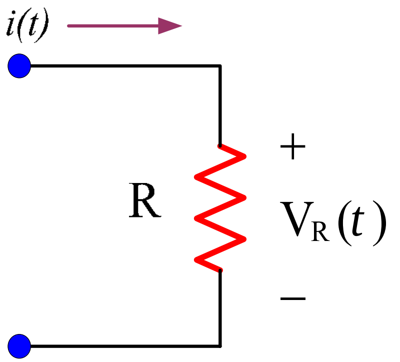
We can think or consider i(t) as the input or excitation which is usually known

We can think of $V_R(t)$, $V_C(t)$, $V_L(t)$ as the output or response

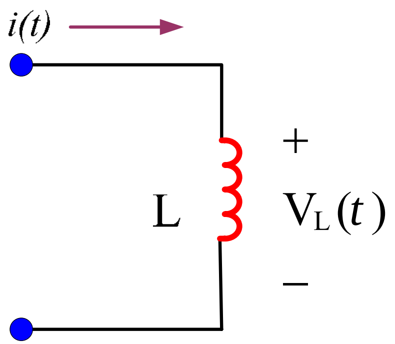
In general we can represent the simple relation between the input and output as:

$$y(t) = H[\, x(t) \,]$$

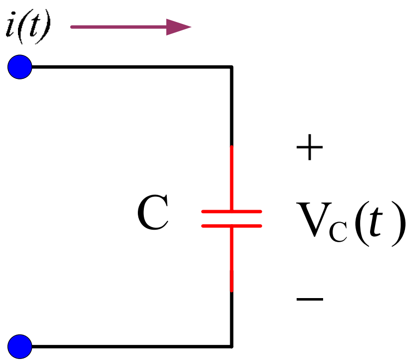
Were **H[ ]** is an operator that map the **function** x(t) to another **function** y(t) .(Function to Function mapping)



$$V_R(t) = R\,i(t) \quad \Rightarrow \quad H_R[\ ] = R[\ ]$$



$$V_L(t) = L\frac{di(t)}{dt} \quad \Rightarrow \quad H_L[\ ] = L\frac{d}{dt}[\ ]$$



$$V_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau \quad \Rightarrow \quad H_C[\ ] = \frac{1}{C}\int_{-\infty}^{t} [\ ]\,dt'$$

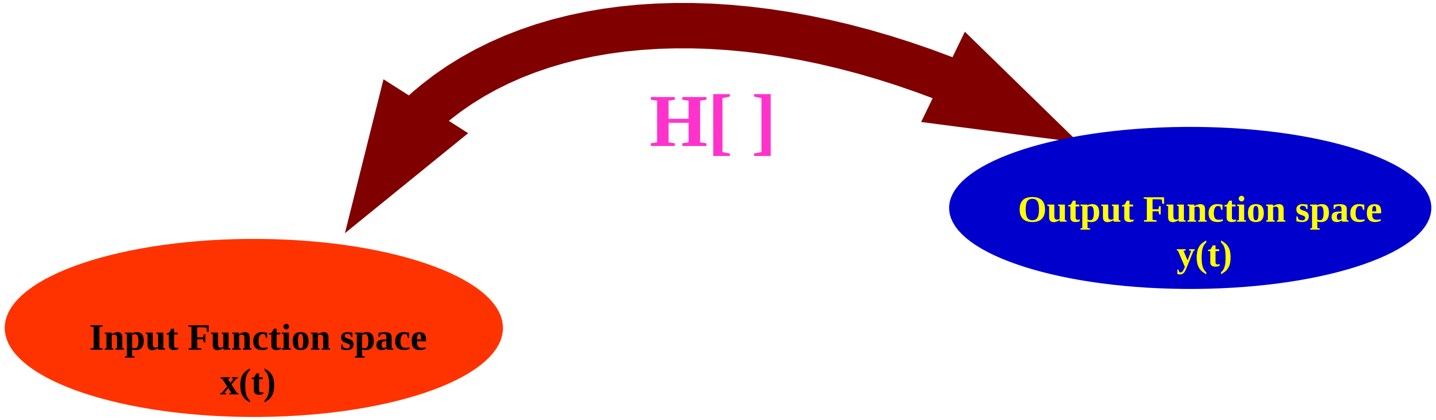
Example

Let the operator $H[\ ] = \frac{d}{dt}[\ ]$ ⇒ Differential Operator

Let the input $x(t) = 2\sin(4\pi t)$ then the output $y(t)$ be

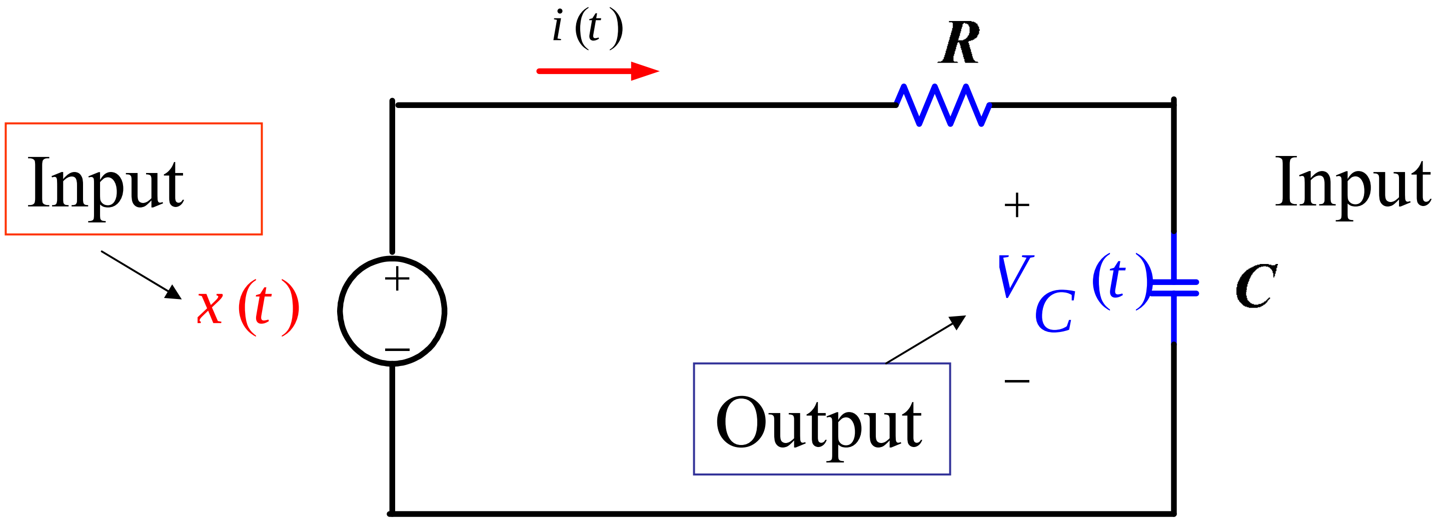
$$y(t) = H[x(t)] = \frac{d}{dt}[2\sin(4\pi t)] = 8\cos(4\pi t)$$

Function $2\sin(4\pi t)$ ⇒ **mapped** ⇒ **Function** $8\cos(4\pi t)$



Note operator map **function** x(t) to another **function** y(t)

<u>In comparison</u> to functions , it maps **Domain** (numbers) to **Range** (domain)



$$x(t) = Ri + V_c(t) \qquad i(t) = C\frac{dV_c(t)}{dt}$$

$$x(t) = RC\frac{dV_c(t)}{dt} + V_c(t)$$

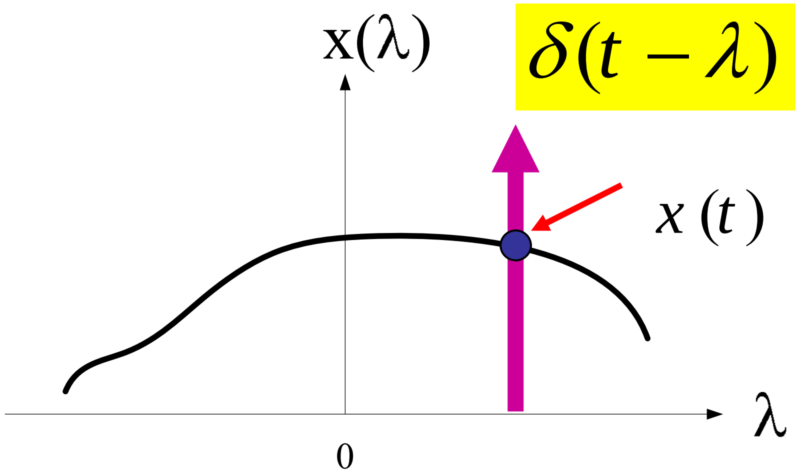
The operator or relation H can be defined as

- Linear / Non linear
- Time Invariant / Time Variant
- Continuous-Time / Discrete-Time
- Causal / Non Causal

From Chapter 1, we have

(II)

$$\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda = x(t)$$

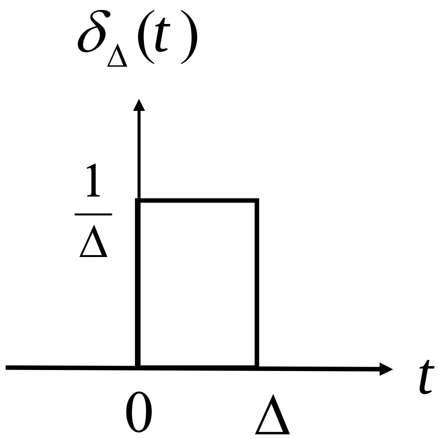


This property is known as **"convolution"** (التفاف الإلتواء)

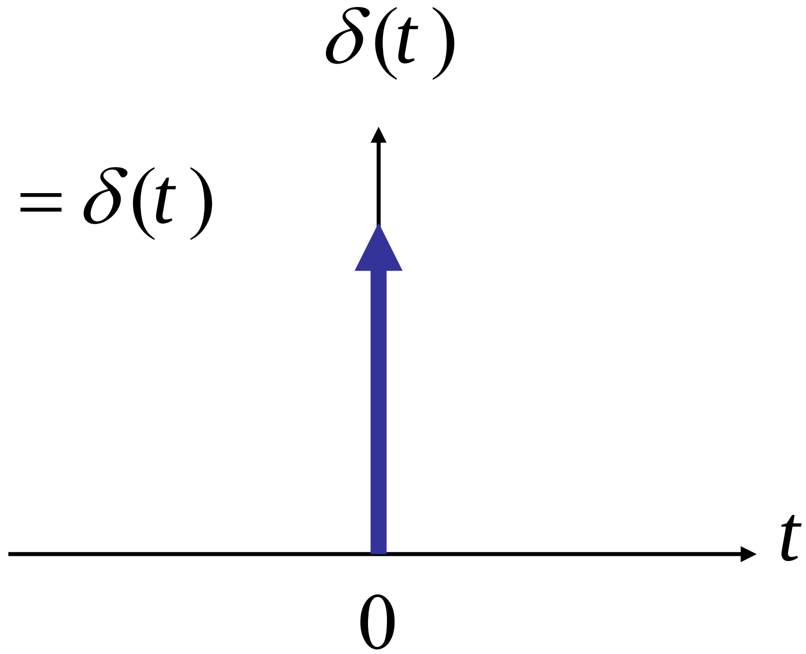
$$\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)dt = x(t)$$

Proof

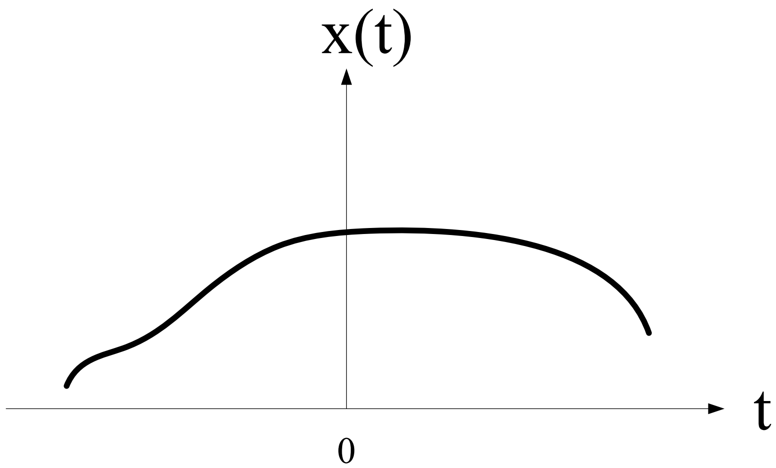
Define the pulse of with Δ as

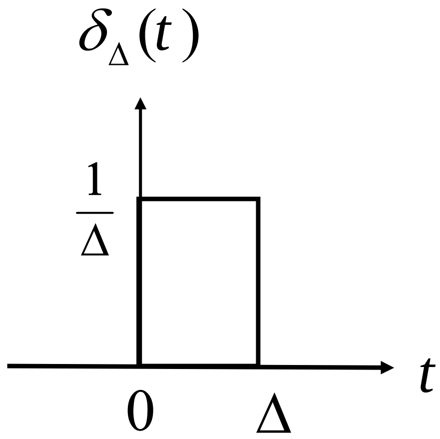


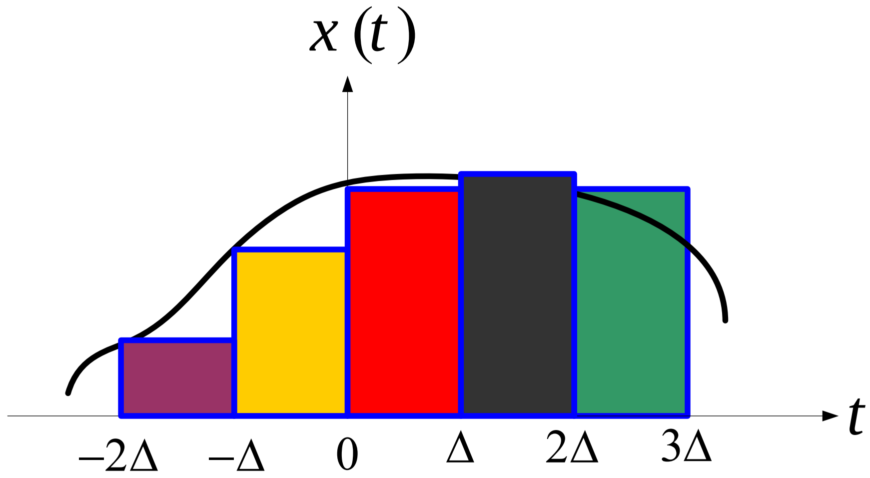
$$\lim_{\Delta \to 0} \delta_{\Delta}(t) = \delta(t)$$

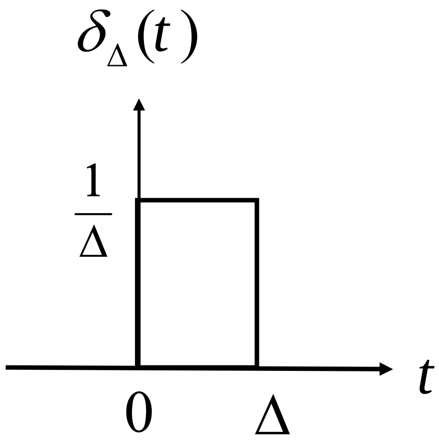


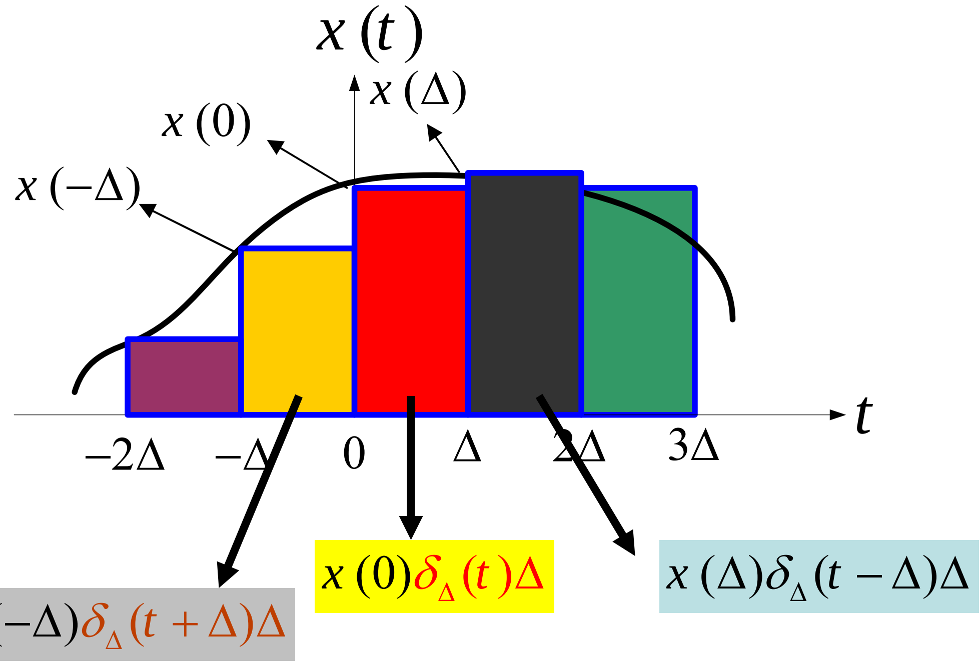
We now can approximate the function $x(t)$ In terms of the pulse function $\delta_\Delta(t)$





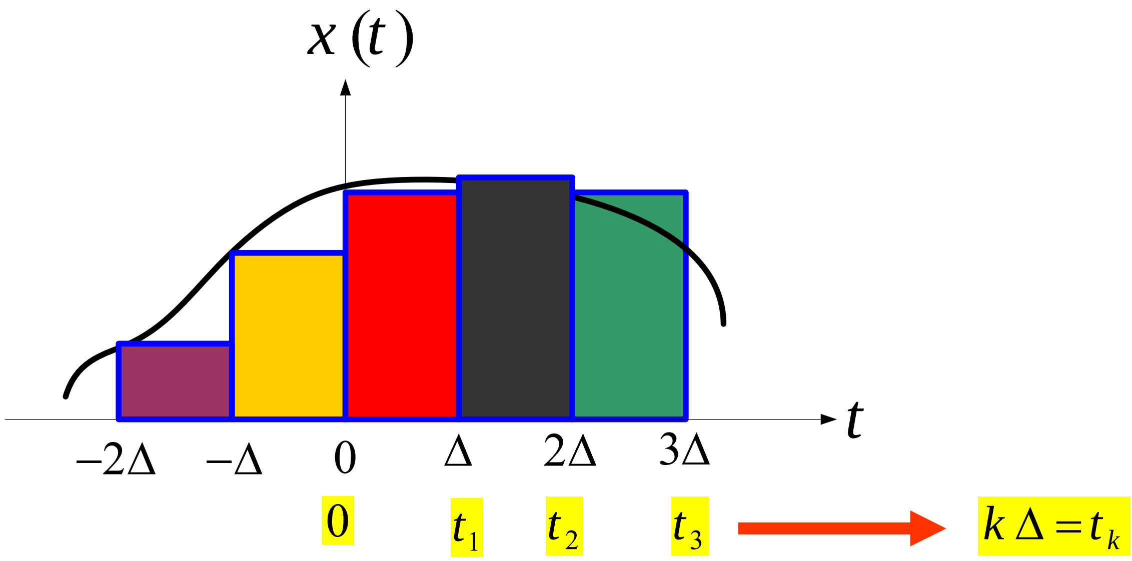






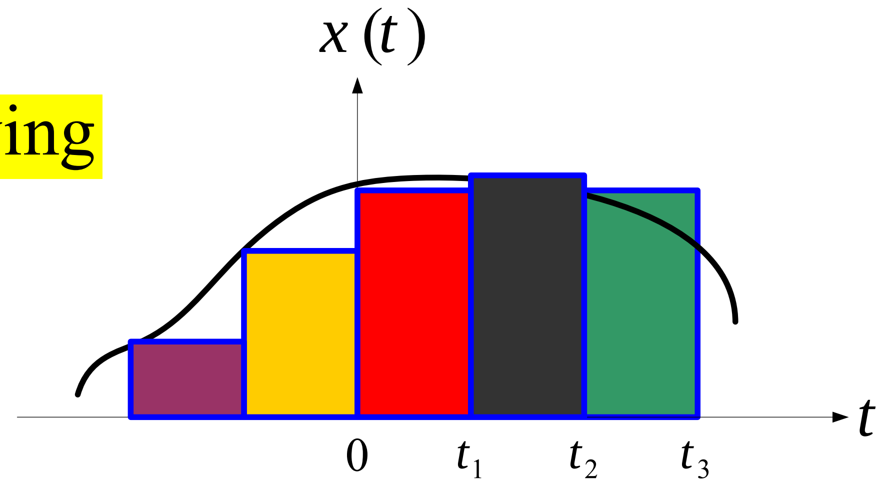
$$x(t) \approx \hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

$$x(t) \approx \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$



$$x(t) \approx \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta = \sum_{k=-\infty}^{\infty} x(t_k)\delta_{\Delta}(t-t_k)\Delta$$

$$x(t) \approx \hat{x}(t) = \sum_{k=-\infty}^{\infty} x(t_k)\delta_\Delta(t - t_k)\Delta$$



Now as $\Delta \rightarrow 0$ we have the following

$$\lim_{\Delta \rightarrow 0} \delta_\Delta(t) = \delta(t)$$

$$t_k \rightarrow \tau$$

$$\Delta \rightarrow d\tau$$

$$\sum_{k=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty}$$

This integral is called the convolution (التفاف الإلتواء) integral

$$x(t) = \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{\infty} x(t_k)\delta_\Delta(t - t_k)\Delta = \int_{-\infty}^{\infty} x(t)\delta(t - \tau)d\tau = x(t) * \delta(t)$$

Another proof for $$\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda = x(t)$$

$$x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda)$$ **Sifting properties**

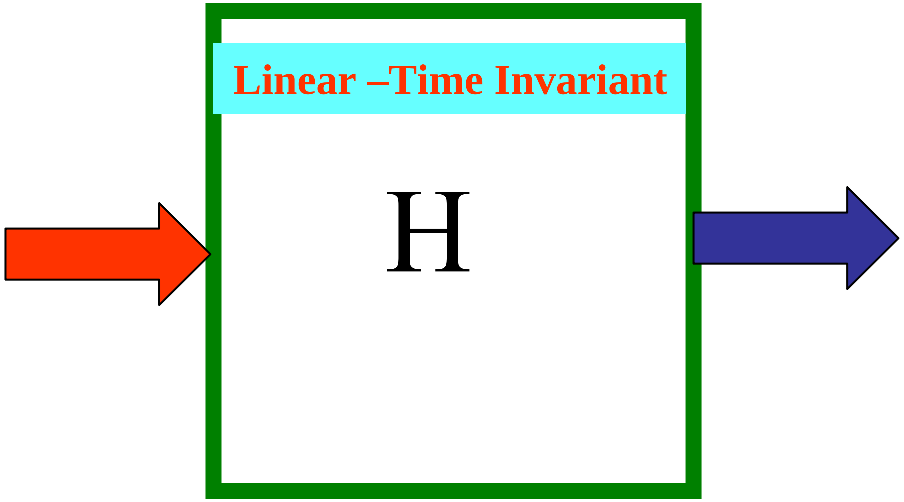
$$\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda = \int_{-\infty}^{\infty} x(t)\delta(t-\lambda)d\lambda$$

$$= x(t)\int_{-\infty}^{\infty} \delta(t-\lambda)d\lambda = x(t)$$

$x(t)$

$\delta(t)$

Impulse Input



$$H[x(t)] = y(t)$$

$$H[\delta(t)] = h(t)$$

Impulse response

$\delta(t-\lambda)$

Shifted Impulse Input

$$H[\delta(t-\lambda)] = h(t-\lambda)$$

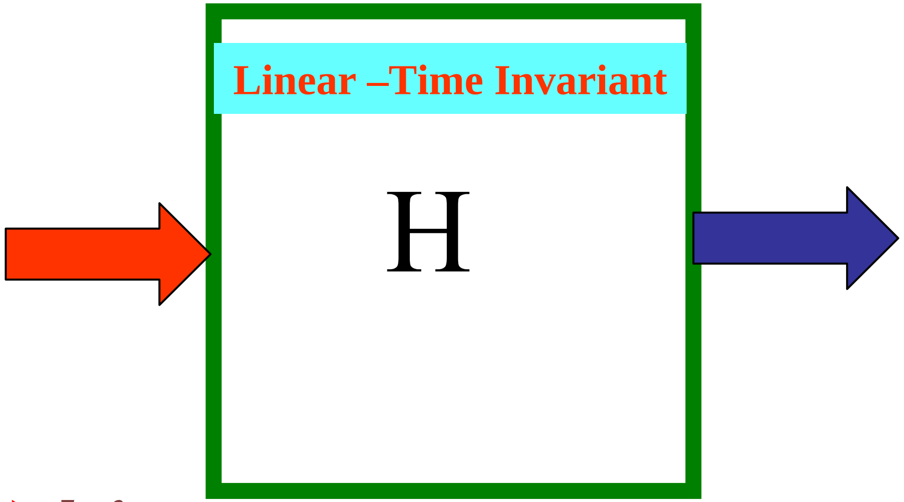
Shifted Impulse Response

$$\underbrace{x(\lambda)}_{\textbf{constant}}\,\delta(t-\lambda)$$

$$H[x(\lambda)\delta(t-\lambda)]$$
$$= x(\lambda)H[\delta(t-\lambda)]$$
$$= x(\lambda)h(t-\lambda)$$

$$\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda$$

$$\int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$



$$\underbrace{\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda}_{= x(t)}$$

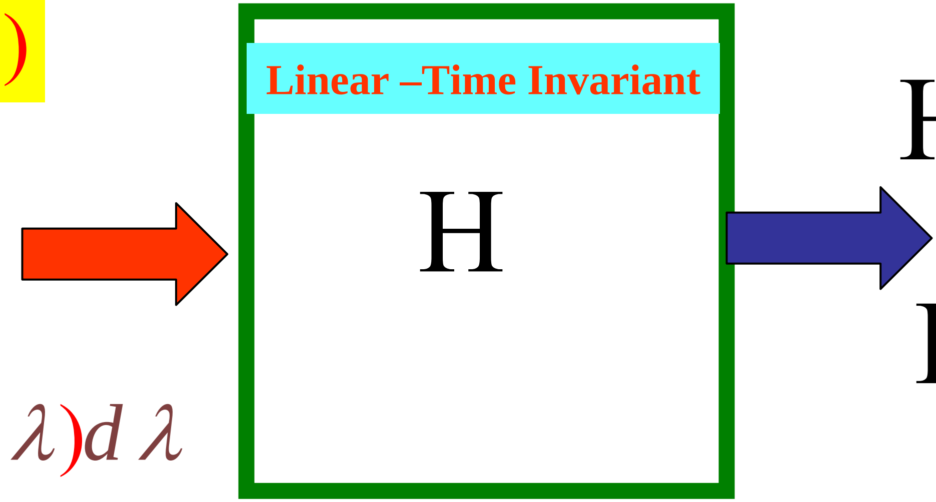
$$\int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda = y(t)$$

Convolution Integral

$x(t)$

$\delta(t)$

$\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda$



$H[x(t)] = y(t)$

$H[\delta(t)] = h(t)$

$$y(t) = H\left[\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda\right] = \int_{-\infty}^{\infty} H[x(\lambda)\delta(t-\lambda)]d\lambda$$

constant with respect to t

Operator with respect to t

Integration with respect to λ

$$= \int_{-\infty}^{\infty} x(\lambda)H[\delta(t-\lambda)]d\lambda = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

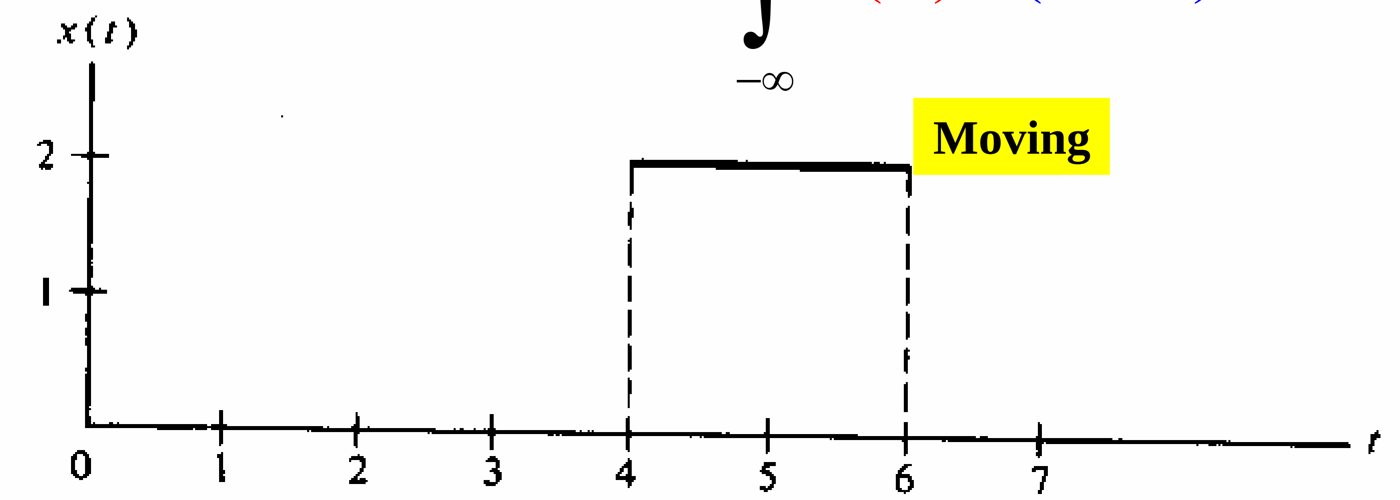
$$y(t) = x(t) * h(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

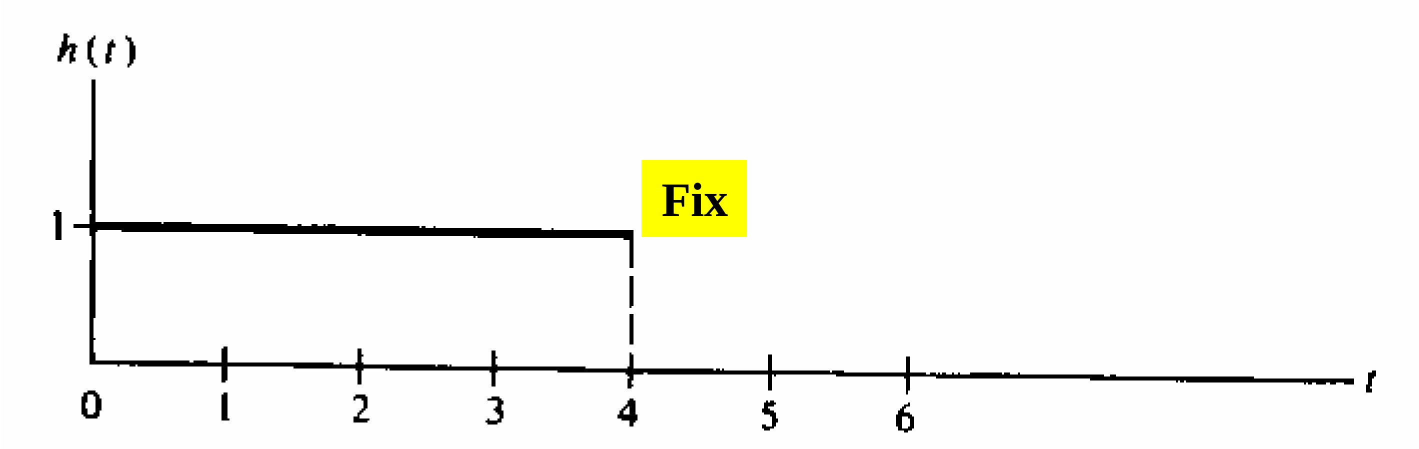
$$h(\tau) \xrightarrow{\text{Flip}} h(-\tau) \xrightarrow{\text{Slide}} h(t-\tau)$$

$$\xrightarrow{\text{Multiply}} x(\tau)h(t-\tau) \xrightarrow{\text{Integrate}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Example 2-7

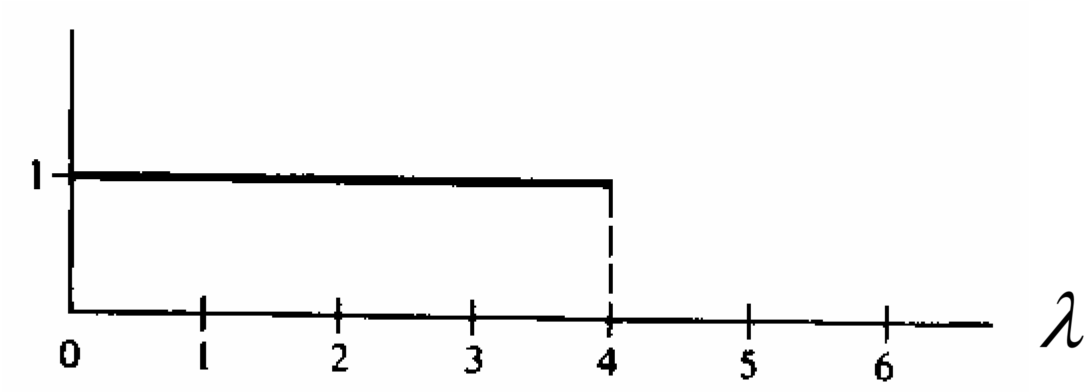
Evaluate $\int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$





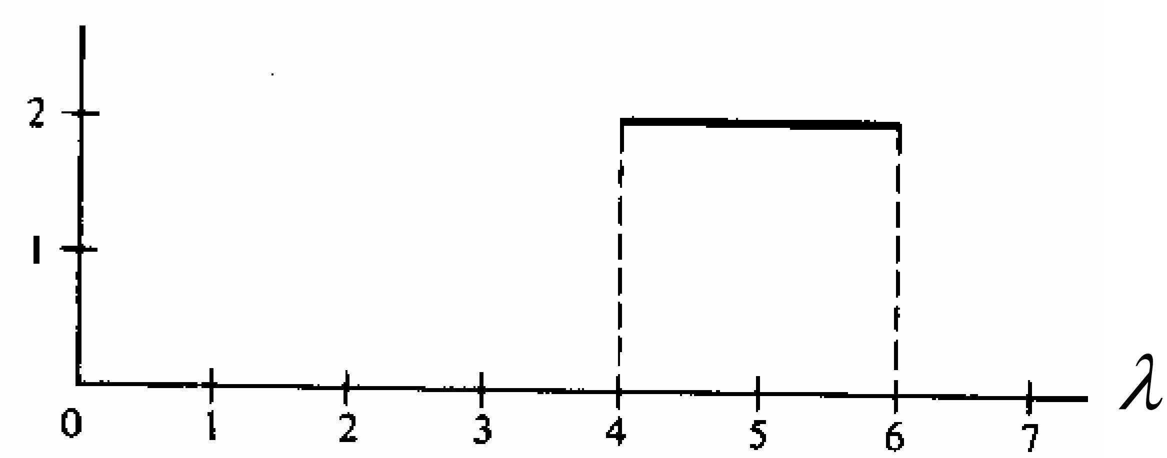
Sep 1 : make the functions or signals in terms of the variable λ

$h(\lambda)$

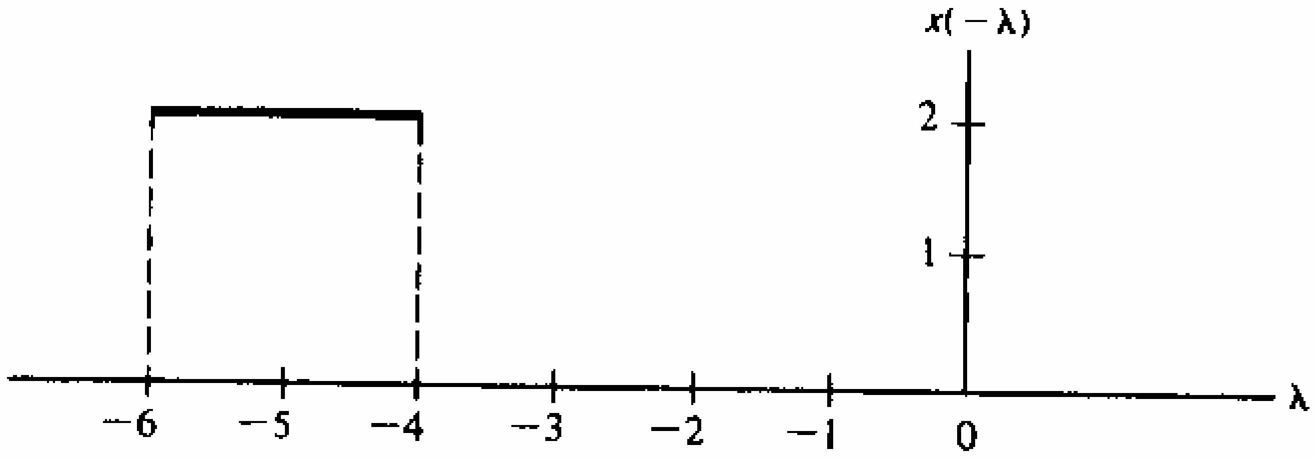


$$\int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

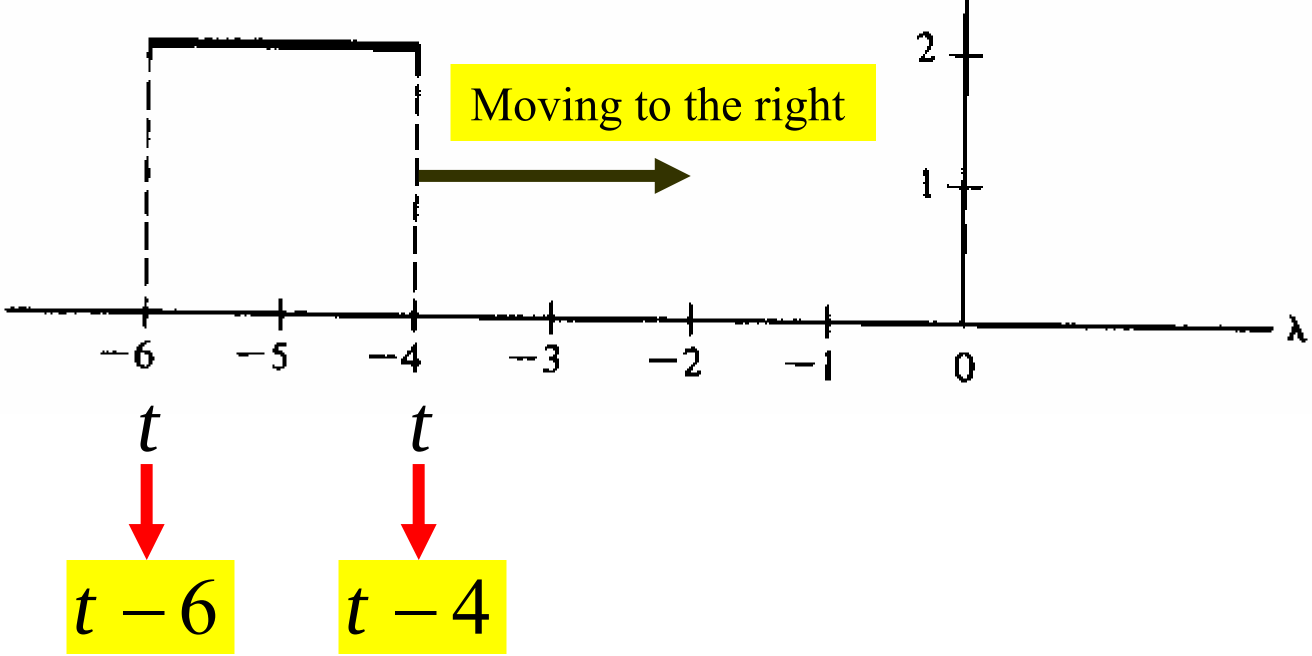
$x(\lambda)$

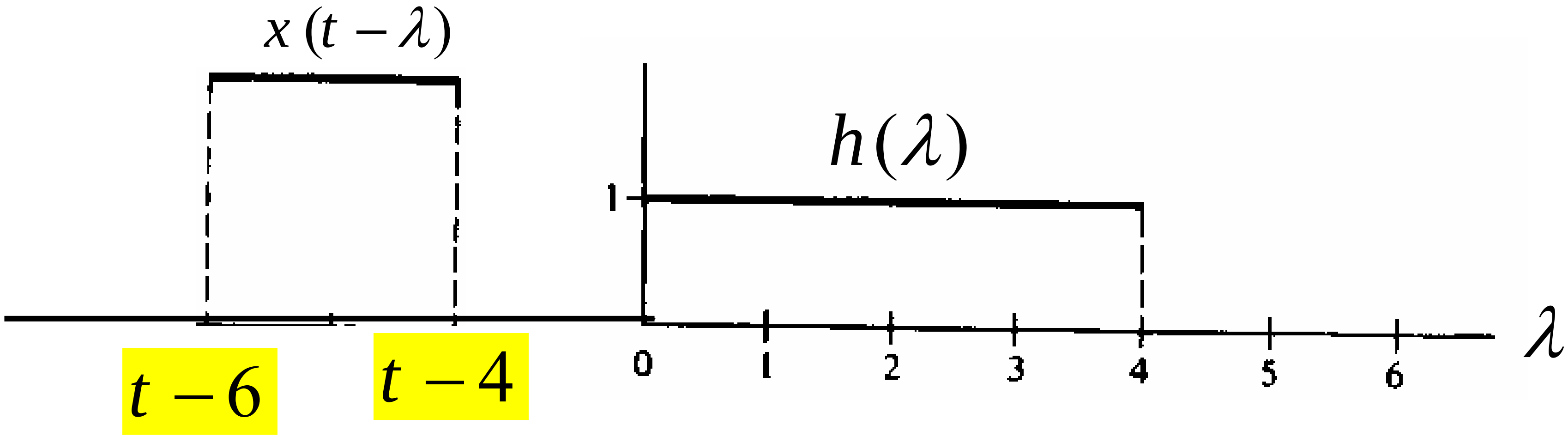


Sep 2 : make the moving function in terms of $-\lambda$



Sep 2 : add t to to form (t– λ)

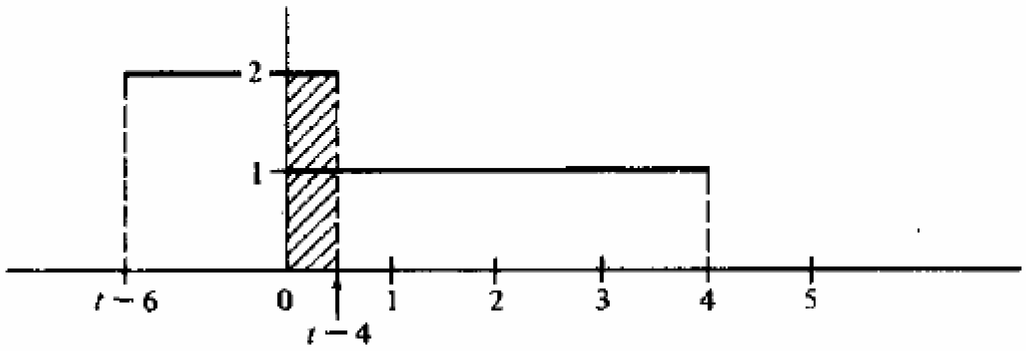




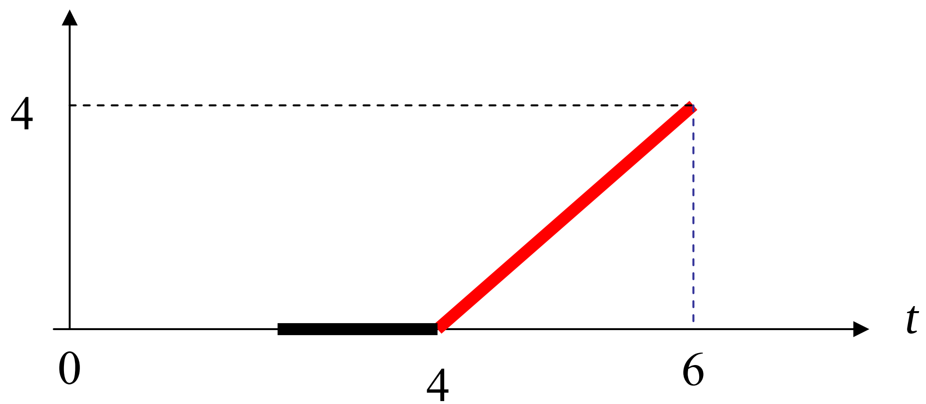
For $t \leq 4$ there is no overlapping between the functions

$$\int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda = 0$$

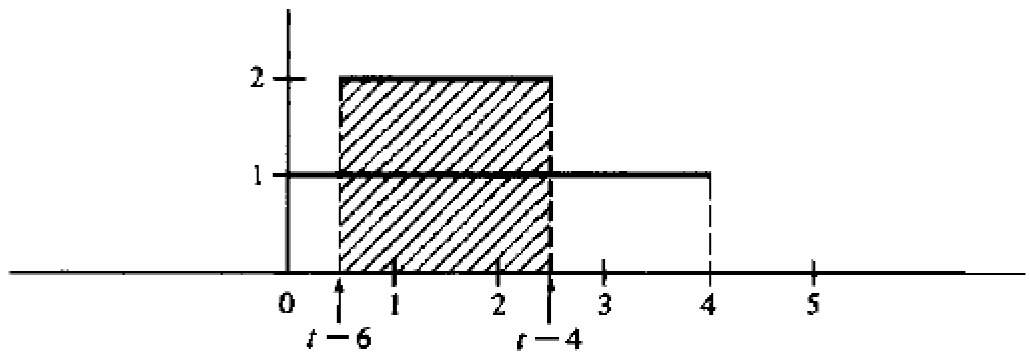
$4 < t \leqslant 6$



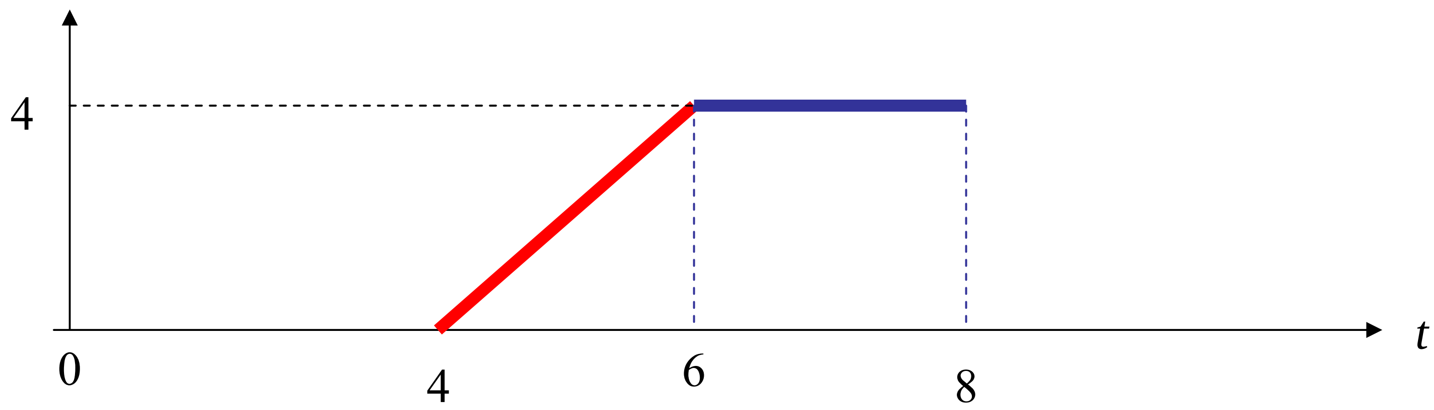
$$\int_{0}^{t-4} (1)(2)d\lambda = 2\lambda\Big|_{0}^{t-4} = 2((t-4)-0) = 2t-8$$



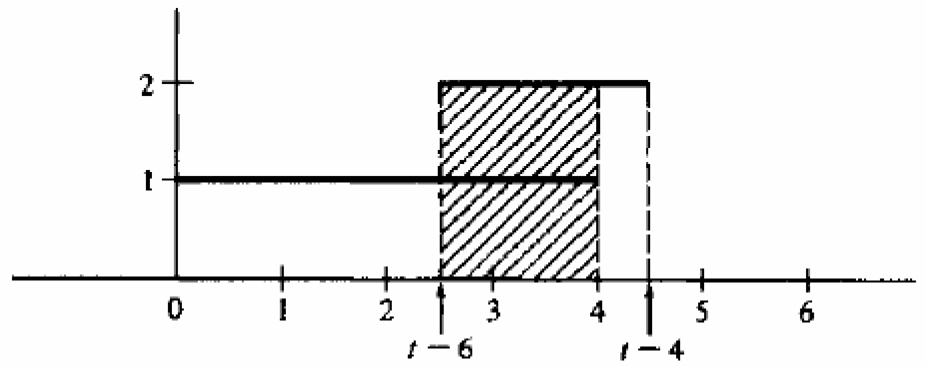
$6 < t \leq 8$



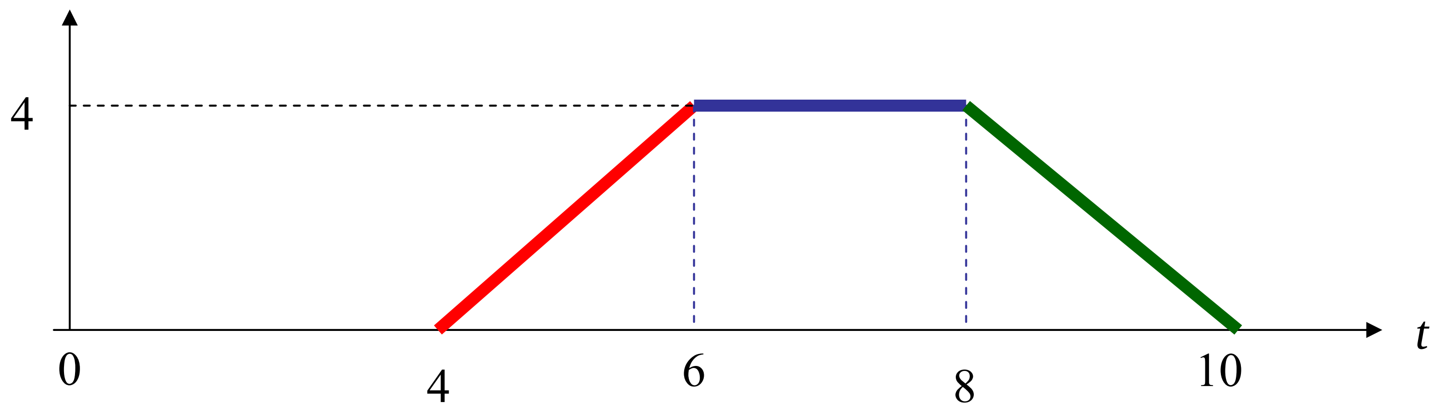
$$\int_{t-6}^{t-4} (1)(2)d\lambda = 2\lambda\Big|_{t-6}^{t-4} = 2((t-4)-(t-6)) = 4$$



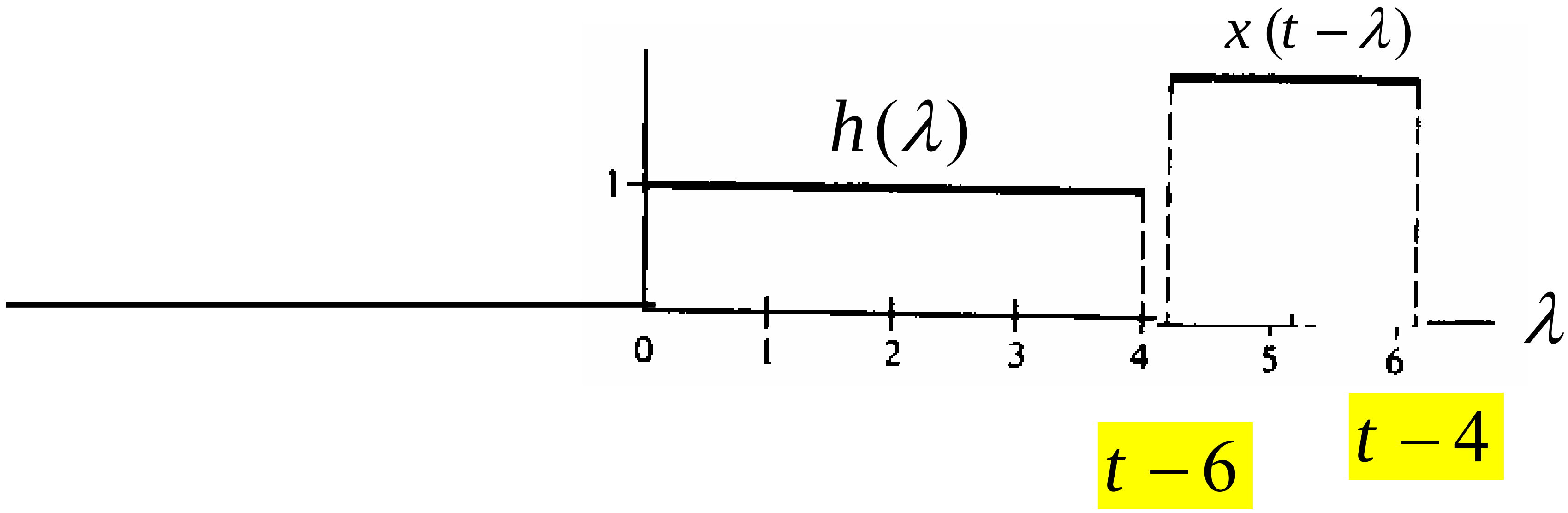
$8 < t \leqslant 10$



$$\int_{t-6}^{4} (1)(2)d\lambda = 2\lambda\Big|_{t-6}^{4} = 2((4)-(t-6)) = -2t+20$$



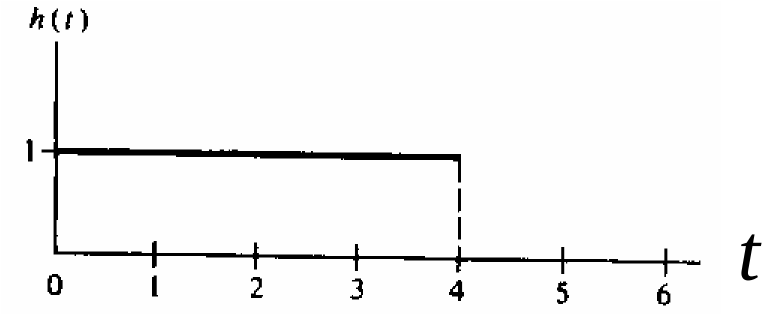
For $t \geq 10$

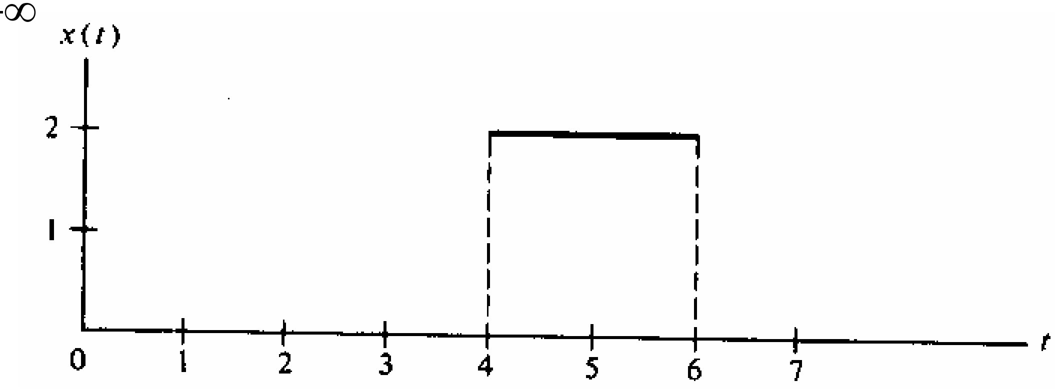


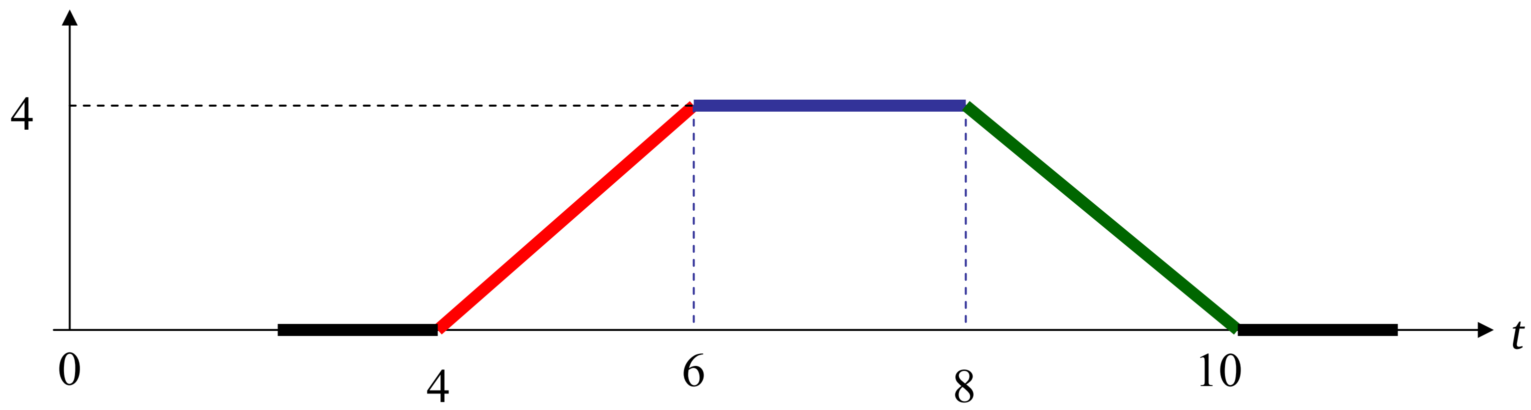
For $t \geq 10$ there is no overlapping between the functions

$$\Rightarrow \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda = 0$$

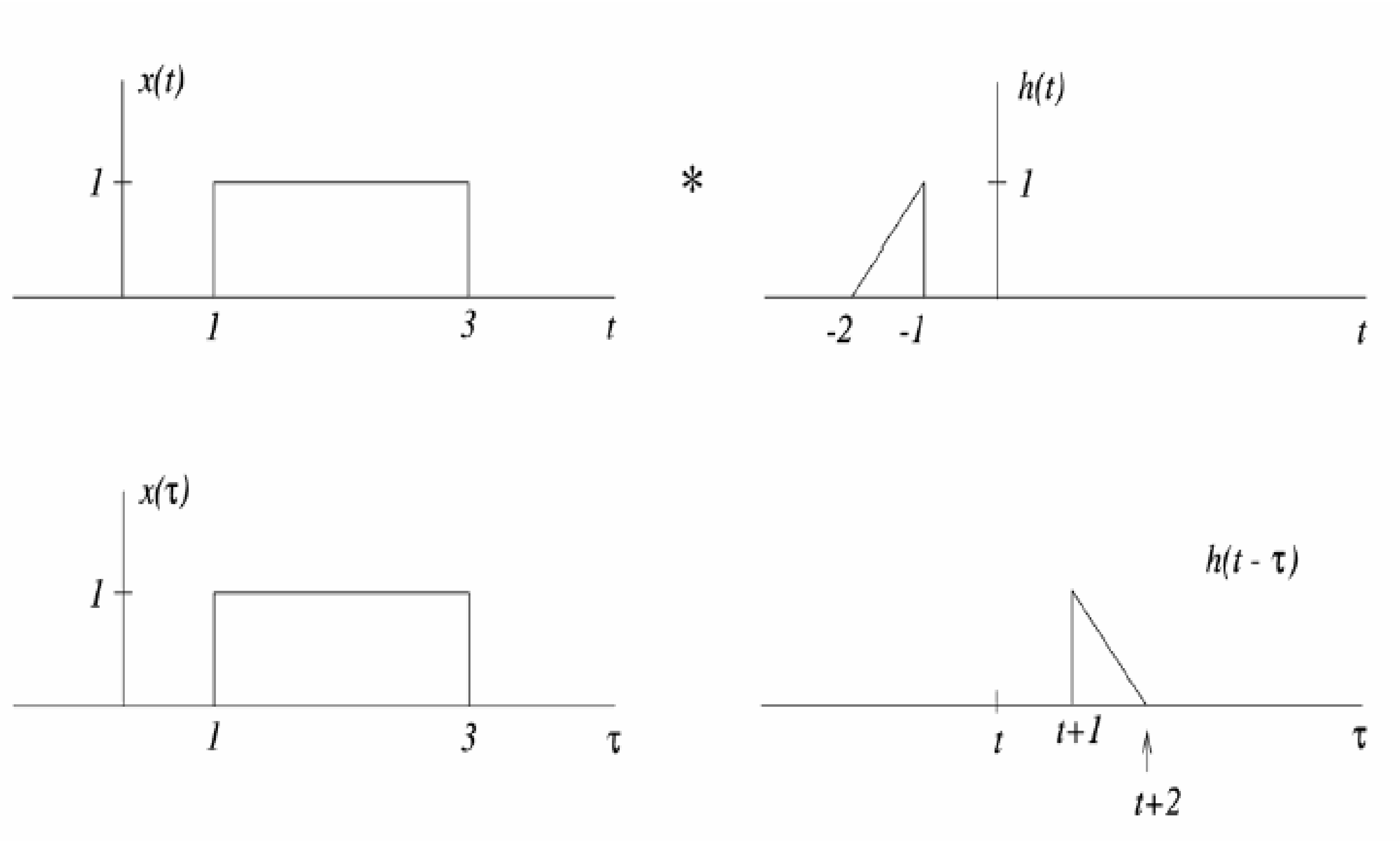
$$\int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

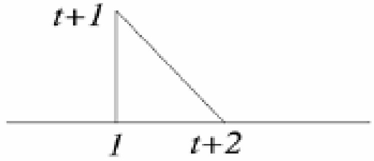
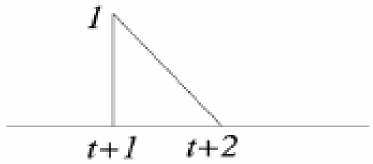
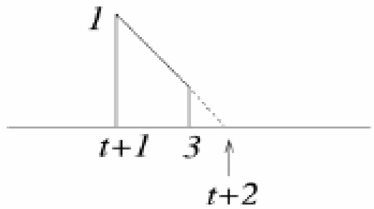


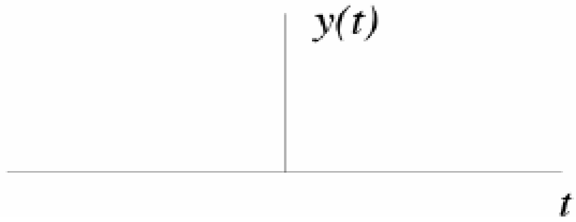


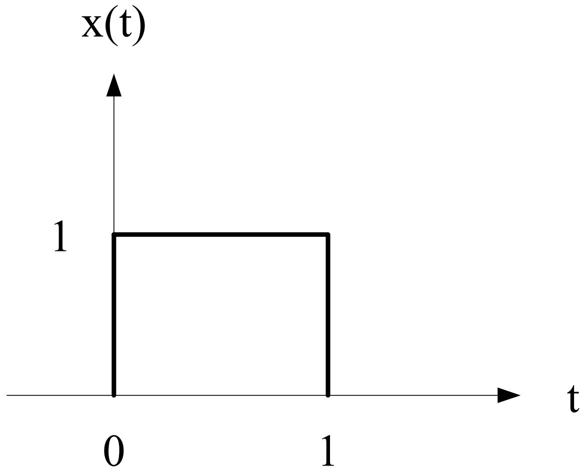


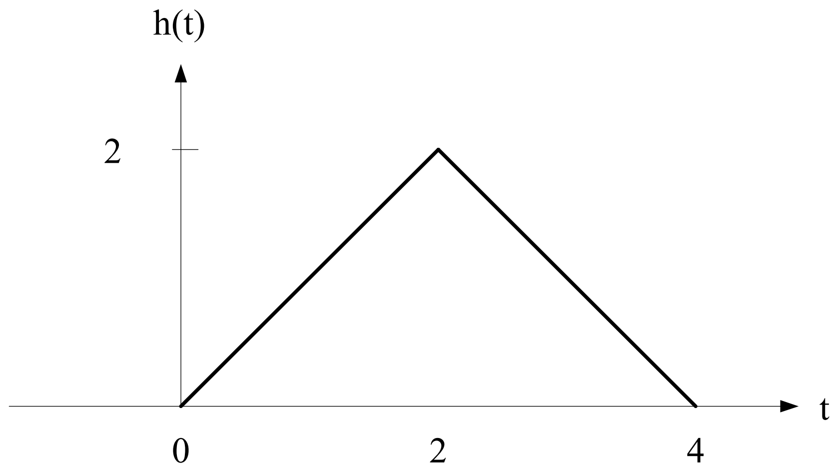
Example

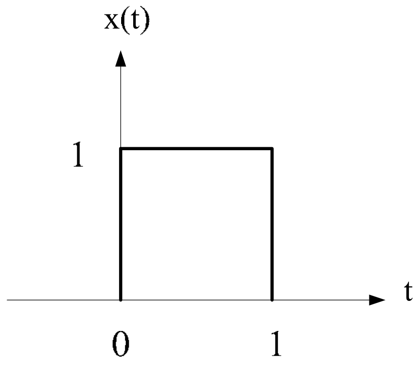


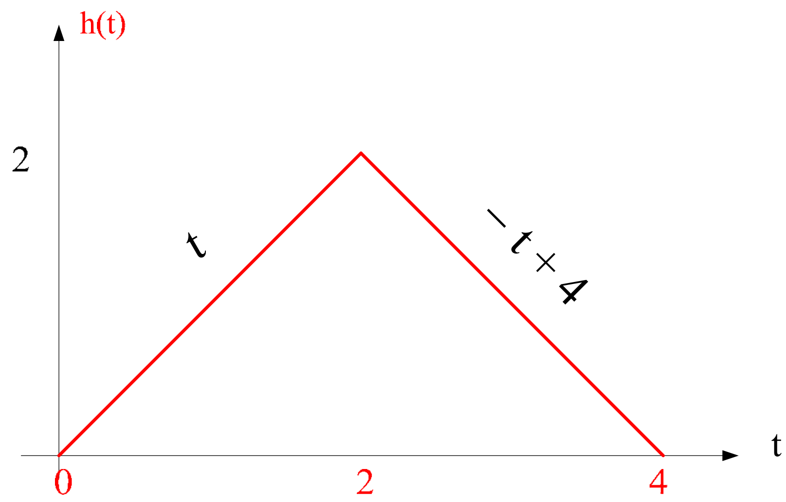
Time Interval	$x(\tau) \cdot h(t-\tau)$		Output
$t < -1$	0	$\Longrightarrow$	$y(t) = 0$
$-1 < t < 0$		$\Longrightarrow$	$y(t) = \frac{1}{2}(t+2)(t+2-1)$ $= \frac{1}{2}(t+1)^2$
$0 < t < 1$		$\Longrightarrow$	$y(t) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$
$1 < t < 2$		$\Longrightarrow$	$y(t) = \frac{1}{2} - \frac{1}{2}(t+2-3)(t-1$ $= \frac{1}{2} - \frac{1}{2}(t-1)^2$
$t > 2$	0	$\Longrightarrow$	$y(t) = 0$

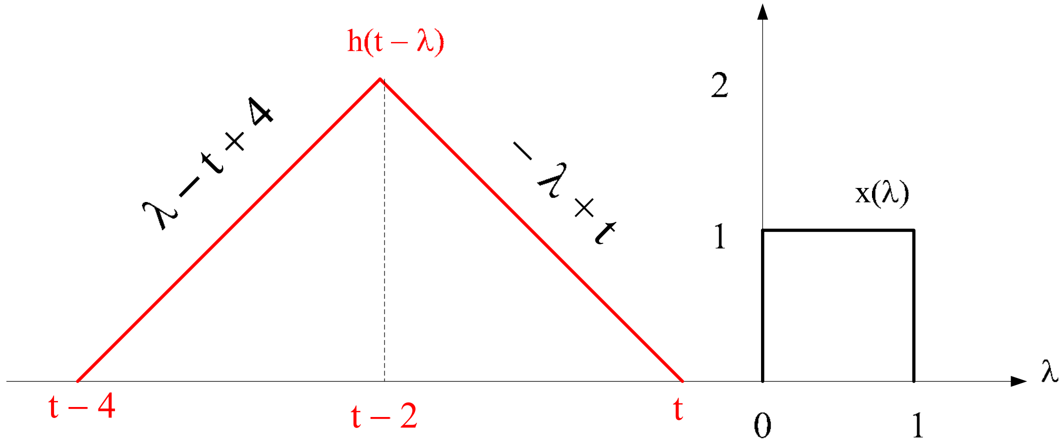


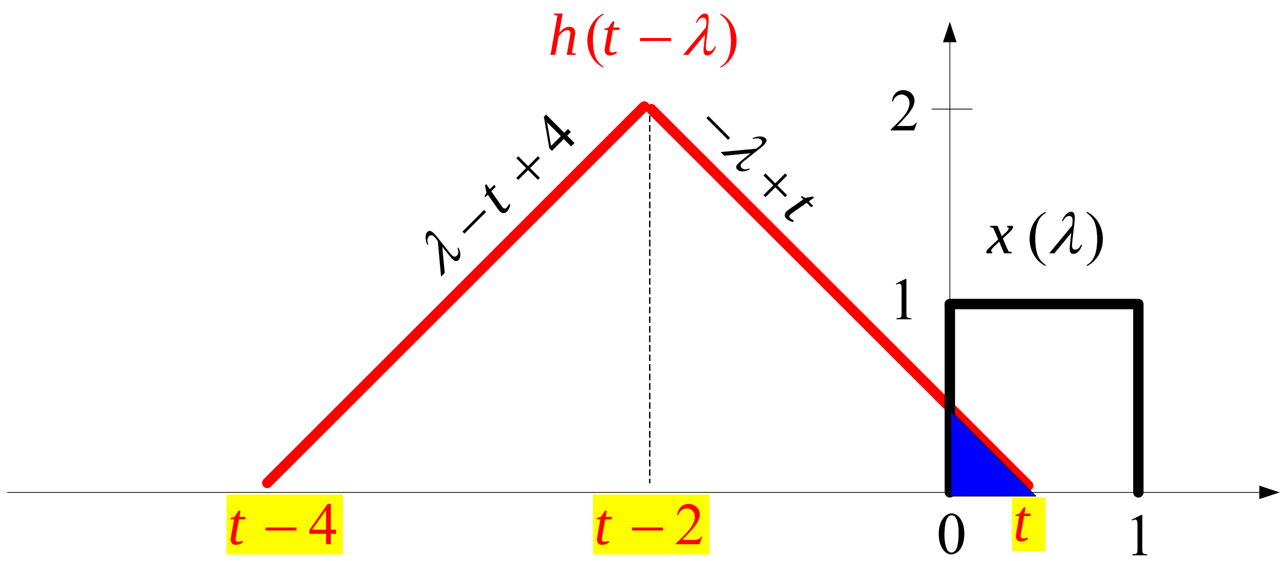

x(t)
1
t
0
1


h(t)
2
t
0
2
4

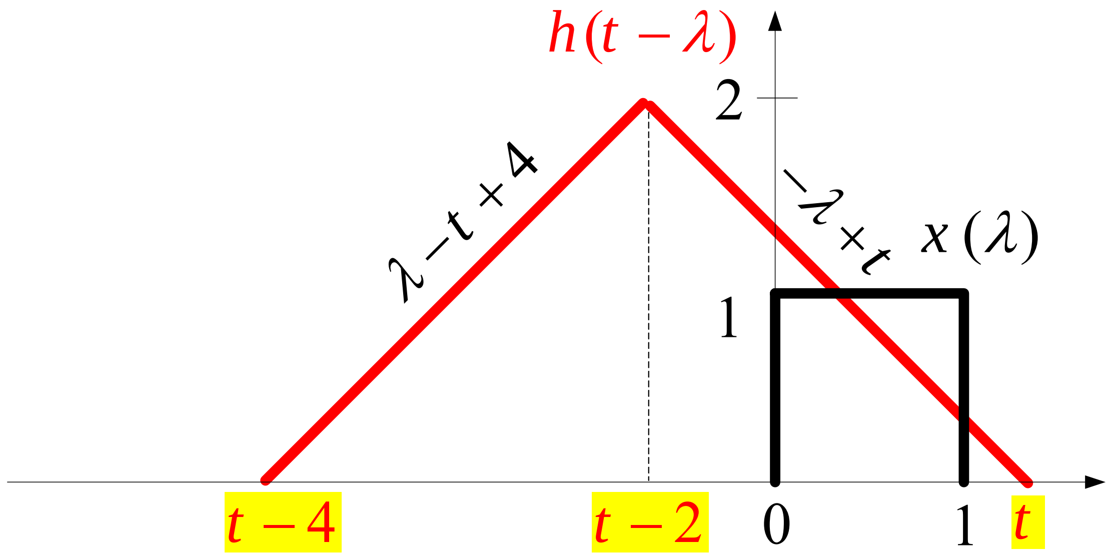

x(t)
1
t
0
1


h(t)
2
t
−t+4
t
0
2
4

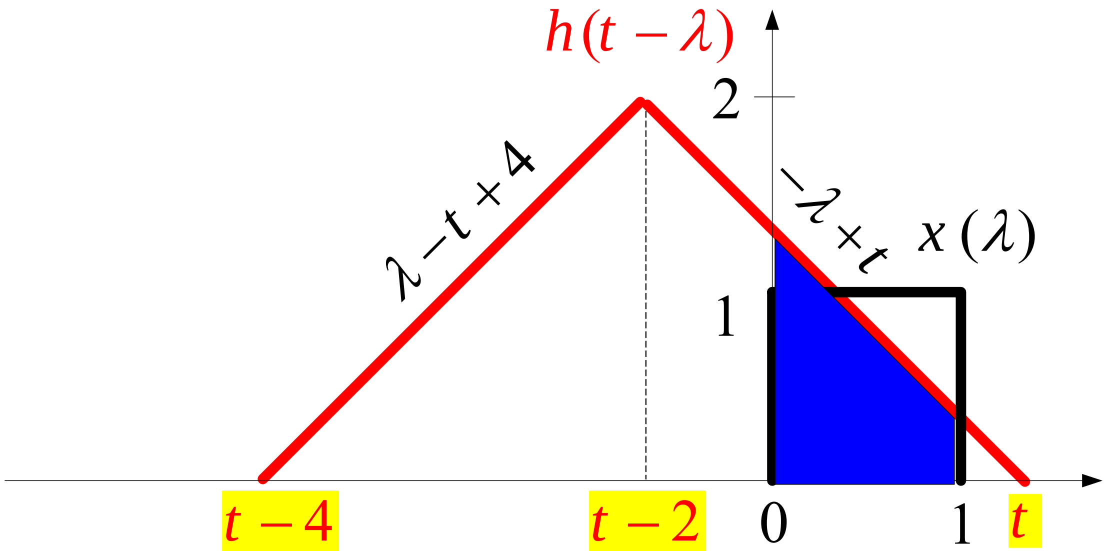




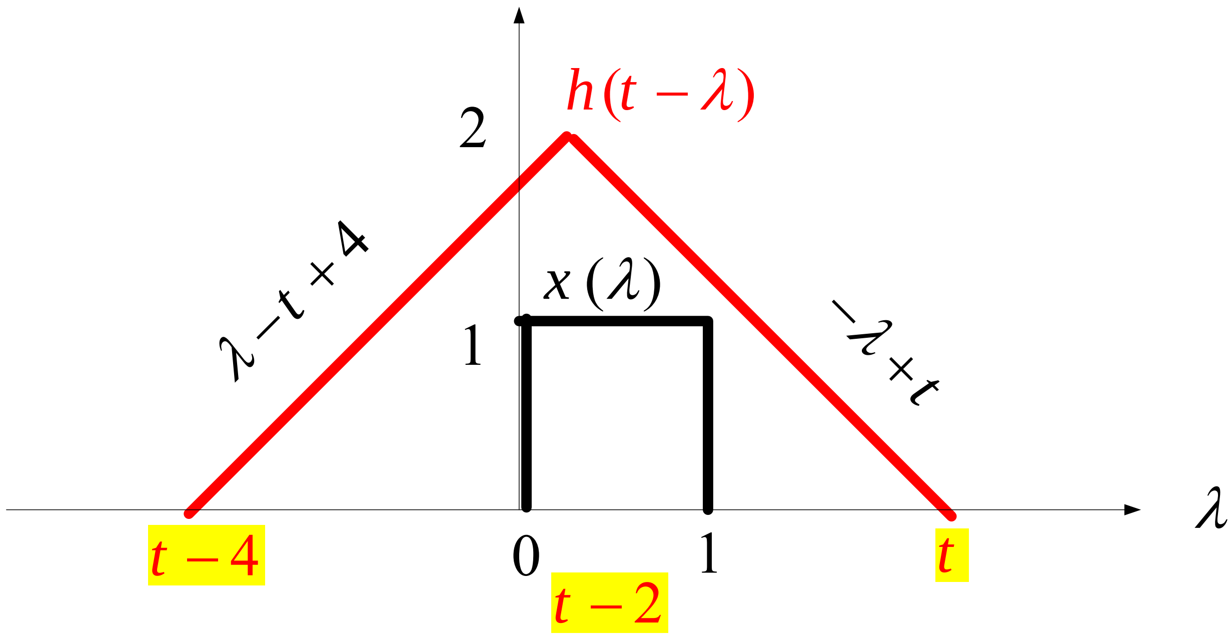
$$0 \le t \le 1 \quad \Rightarrow \quad x(t) * h(t) = \int_0^t \underbrace{(1)}_{x(\lambda)} \underbrace{(-\lambda + t)}_{h(t-\lambda)} d\lambda = \frac{t^2}{2}$$



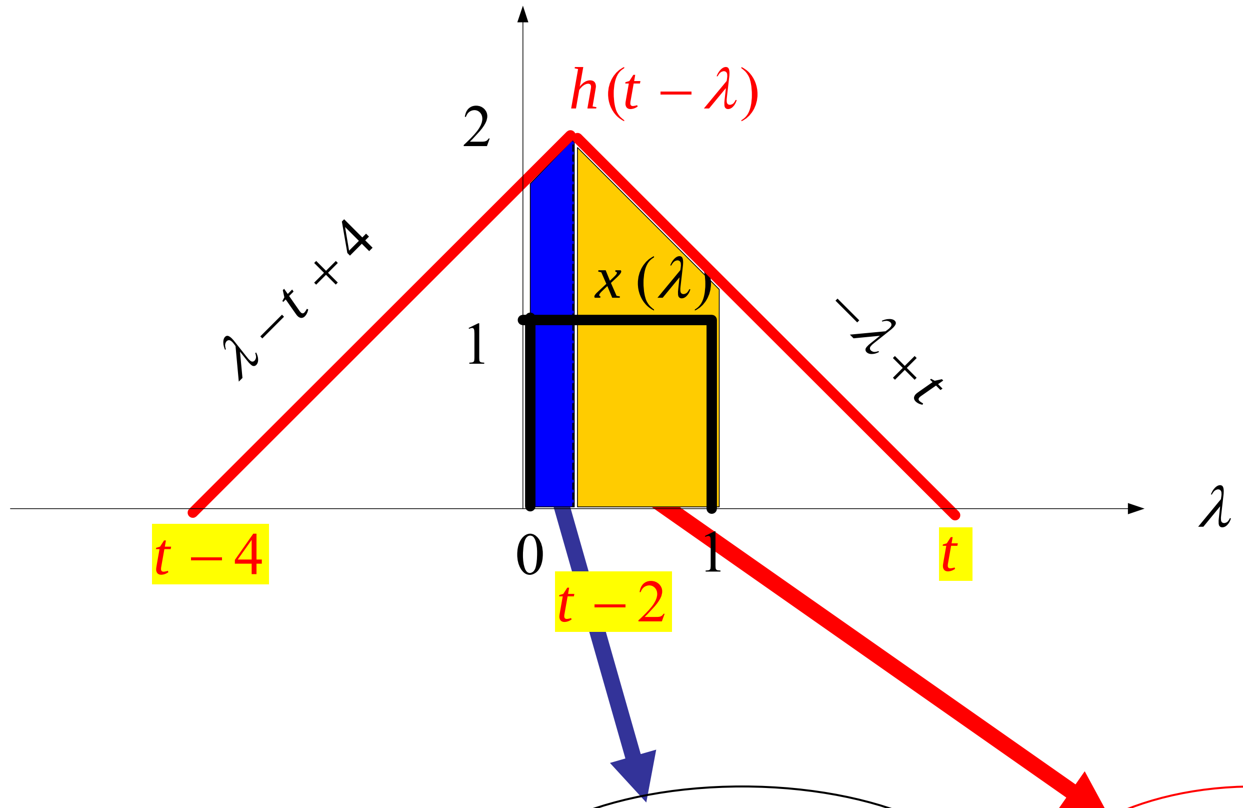
$h(t-\lambda)$
2
$\lambda - t + 4$
$-\lambda + t$
$x(\lambda)$
1
$t-4$
$t-2$
0
1
t



$$1 \le t \le 2 \quad \Rightarrow \quad x(t) * h(t) = \int_0^1 \underbrace{(1)}_{x(\lambda)} \underbrace{(-\lambda + t)}_{h(t-\lambda)} d\lambda = (t - \frac{1}{2})$$

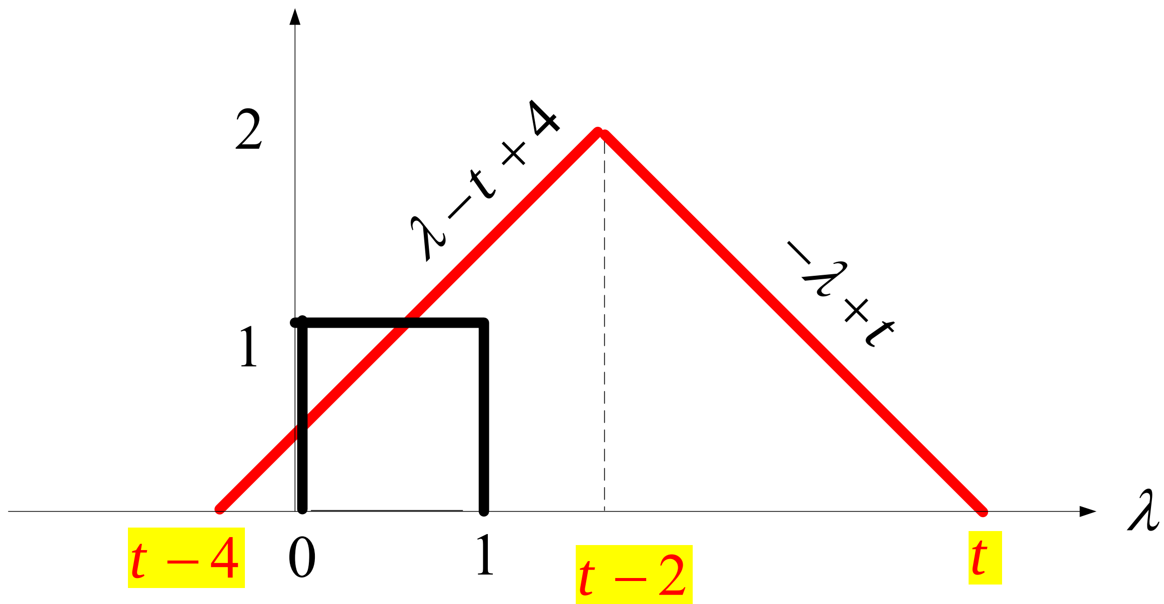


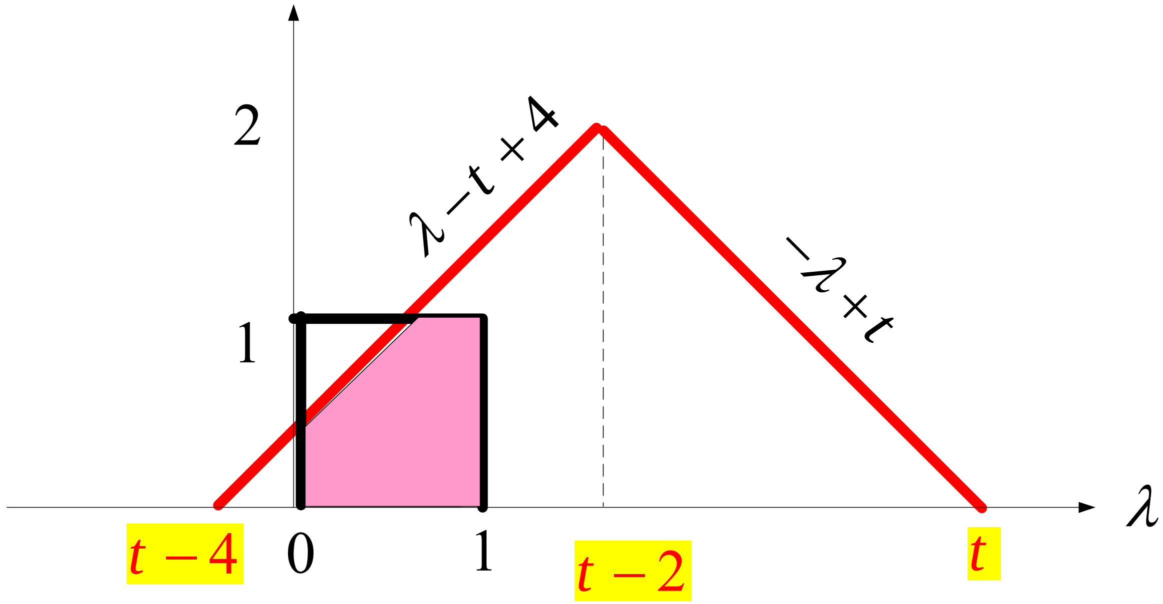
$h(t-\lambda)$
2
$\lambda - t + 4$
$x(\lambda)$
1
$-\lambda + t$
λ
$t-4$
0
1
t
$t-2$



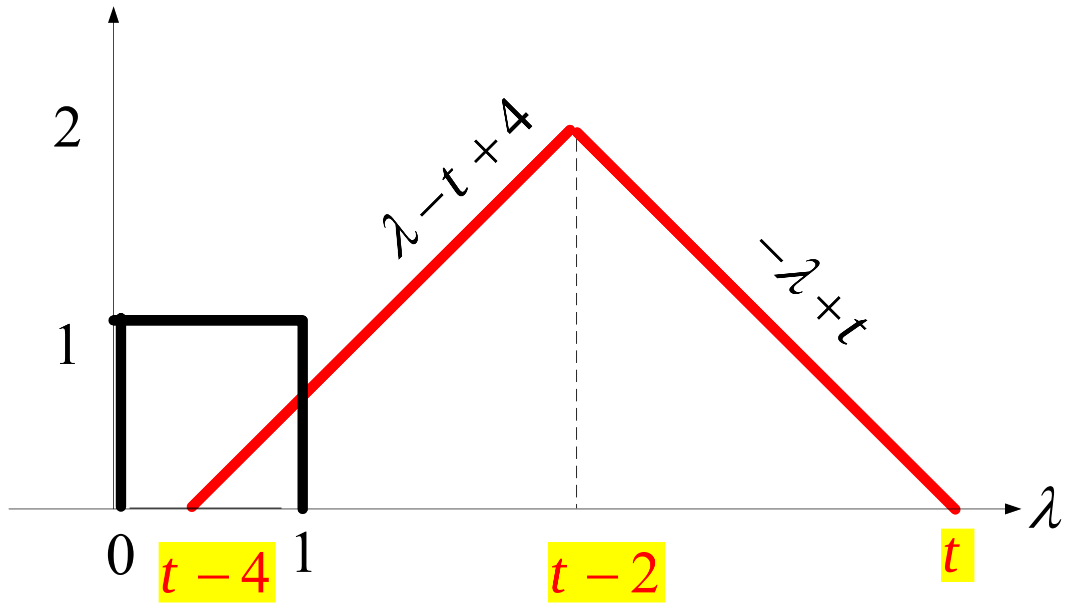
$$2 \le t \le 3 \quad \Rightarrow \quad x(t) * h(t) = \int_{0}^{t-2} (1)\,(\lambda - t + 4)\,d\lambda + \int_{t-2}^{1} (1)\,(-\lambda + t)\,d\lambda$$

$$= -t^2 + 5t - \frac{9}{2}$$

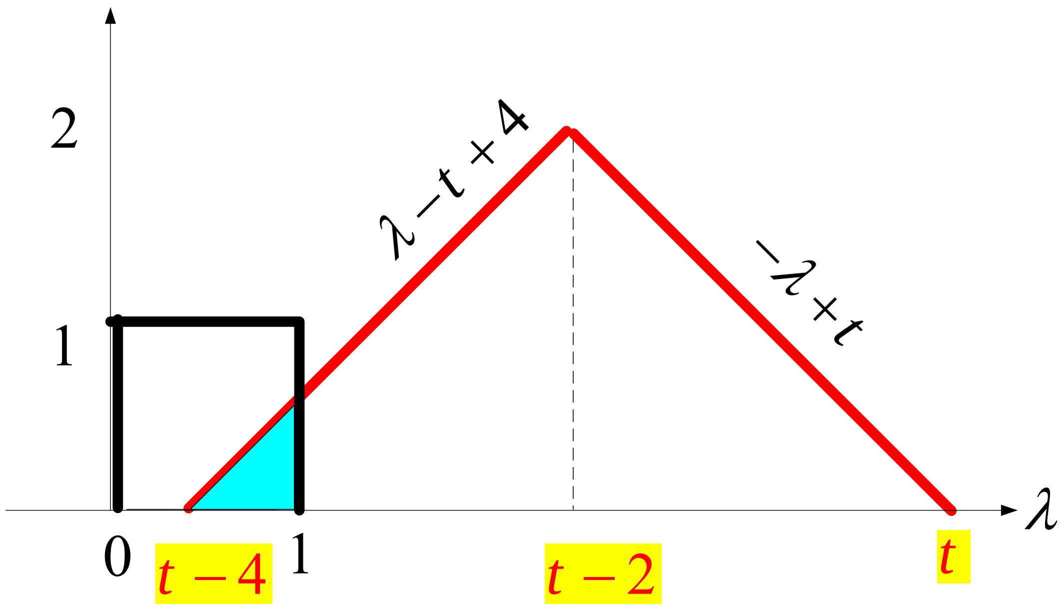

2
1
λ−t+4
−λ+t
λ
t−4
0
1
t−2
t



$$3 \le t \le 4 \quad \Rightarrow \quad x(t) * h(t) = \int_0^1 \underbrace{(1)}_{x(\lambda)} \underbrace{(\lambda - t + 4)}_{h(t-\lambda)} d\lambda = \frac{9}{2} - t$$



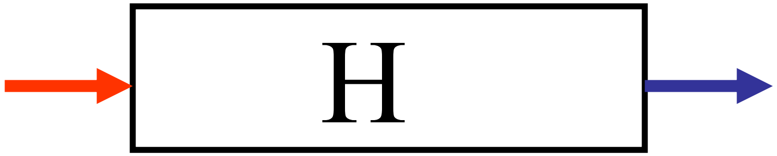
2
1
0
1
$\lambda - t + 4$
$-\lambda + t$
λ
$t-4$
$t-2$
t



$$4 \le t \le 5 \quad \Rightarrow \quad x(t) * h(t) = \int_{t-4}^{1} (1)\,(\lambda - t + 4)\,d\lambda$$

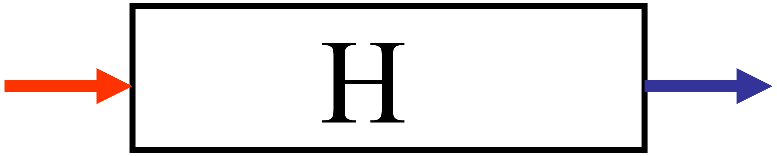
$$= \frac{t^2}{2} - 5t + \frac{25}{2}$$

2.6 Superposition Integral "convolution" in terms of step response

$\delta(t)$  $h(t) = \mathrm{H}[\delta(t)]$

Impulse response

Now if the input is a step function,

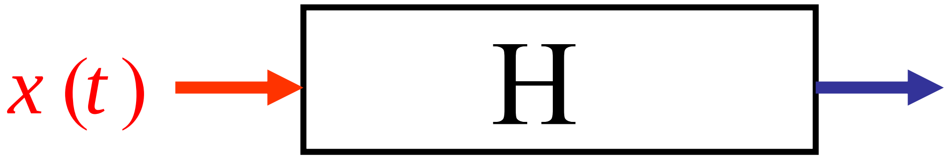
$u(t) = \int_{-\infty}^{t} \delta(t')dt'$  $a(t) = \mathrm{H}[\delta(t)]$

step response

$$a(t) = \mathrm{H}\left[\int_{-\infty}^{t} \delta(t')dt'\right] = \int_{-\infty}^{t} \mathrm{H}[\delta(t')]dt' = \int_{-\infty}^{t} h(t')dt'$$

step response

Now if the input is x(t) ,



$$y(t) = x(t) * h(t)$$

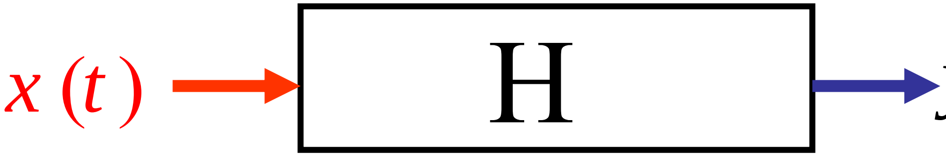
$$= \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda) d\lambda$$

$$= \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda) d\lambda$$

The output in terms of the impulse response $h(t)$

Objective is to write y(t) in terms of the step response $a(t)$

Now if the input is x(t) ,



$$y(t) = \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda) d\lambda$$

Integrating by parts , $\displaystyle\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

Let $u = x(t-\lambda)$ $\quad dv = h(\lambda) d\lambda$

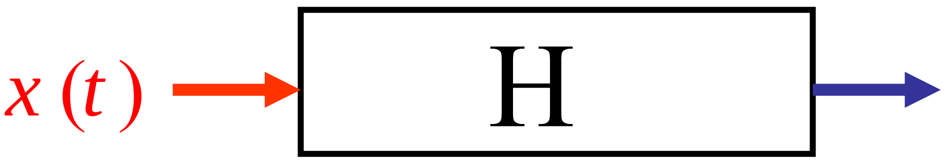
$$\rightarrow \quad v(\lambda) = \int_{-\infty}^{\lambda} dv = \int_{-\infty}^{\lambda} h(\lambda) d\lambda = a(\lambda)$$

step response

Over dot denotes differentiation

$$\frac{du(\lambda)}{d\lambda} = \frac{dx(t-\lambda)}{d(t-\lambda)} \frac{d(t-\lambda)}{d\lambda} = -\frac{dx(t-\lambda)}{d(t-\lambda)} = -\dot{x}(t-\lambda)$$

$$\rightarrow \quad du(\lambda) = -\dot{x}(t-\lambda) d\lambda$$



$$y(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

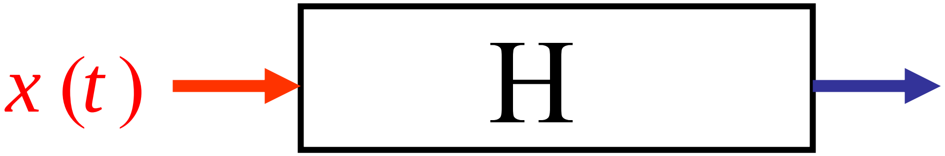
Integrating by parts , $\displaystyle\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,dv$

Let $u = x(t-\lambda)$ $\qquad dv = h(\lambda)d\lambda$ $\qquad v(\lambda) = a(\lambda)$

$$du(\lambda) = -\dot{x}(t-\lambda)d\lambda$$

Now we can write y(t) in terms of the step response ***a(t)***

$$y(t) = a(\lambda)x(t-\lambda)\Big|_{\lambda=-\infty}^{\infty} + \int_{-\infty}^{\infty} a(\lambda)\dot{x}(t-\lambda)d\lambda$$



$$y(t)=\int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

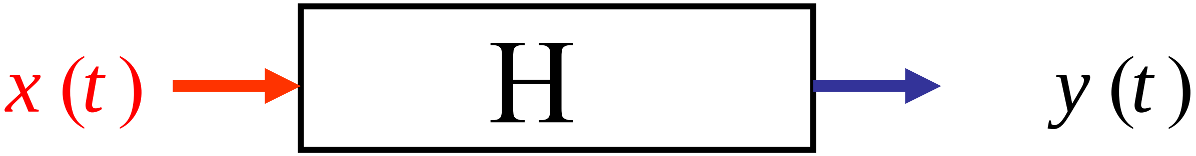
$$y(t)=a(\lambda)x(t-\lambda)\Big|_{\lambda=-\infty}^{\infty}+\int_{-\infty}^{\infty} a(\lambda)\dot{x}(t-\lambda)d\lambda$$

$$y(t)=\underbrace{a(\infty)x(t-\infty)}_{0}-\underbrace{a(-\infty)x(t+\infty)}_{0}+\int_{-\infty}^{\infty} a(\lambda)\dot{x}(t-\lambda)d\lambda$$

The system is initially unexcited → $a(-\infty)=0$ and $x(t-\infty)=0$

$$\rightarrow \quad y(t)=\int_{-\infty}^{\infty} a(\lambda)\dot{x}(t-\lambda)d\lambda=\int_{-\infty}^{\infty} \dot{x}(\lambda)a(t-\lambda)d\lambda$$

$$\rightarrow \quad y(t)=\dot{x}(t)*a(t)$$



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda) d\lambda$$

In term of impulse response

$$y(t) = \dot{x}(t) * a(t) = \int_{-\infty}^{\infty} a(\lambda) \dot{x}(t-\lambda) d\lambda$$

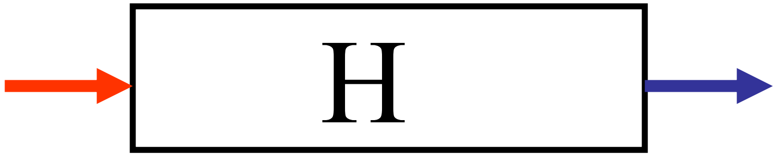
In term of step response

Note

$$\dot{x}(t) * a(t) = \frac{dx(t)}{dt} * a(t) = \frac{dx(t)}{dt} * \int_{-\infty}^{t} h(\lambda) d\lambda$$

$$= x(t) * \frac{d}{dt} \int_{-\infty}^{t} h(\lambda) d\lambda \quad = x(t) * h(t)$$

Impulse input $\delta(t)$



$h(t)$ **Impulse response**

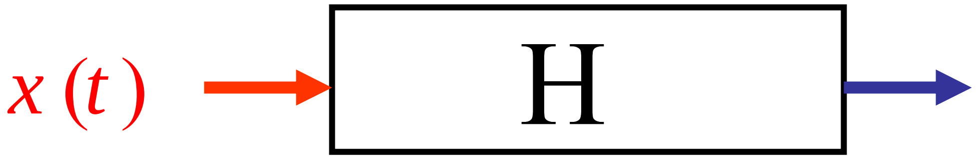
Step input $u(t) = \int_{-\infty}^{t} \delta(t')dt'$

$a(t)$ **step response**

Ramp input $r(t) = \int_{-\infty}^{t} u(t')dt'$

$b(t)$ **Ramp response**

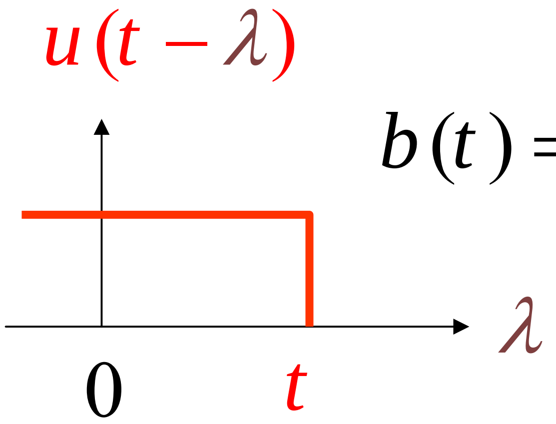
Objective is the ramp response $b(t)$



$$y(t) = \dot{x}(t) * a(t)$$

$$= \int_{-\infty}^{\infty} a(\lambda)\dot{x}(t-\lambda)d\lambda$$

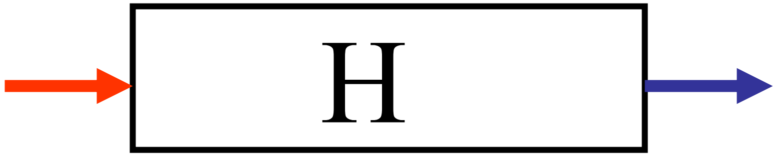
Now if $x(t)$ is the ramp $r(t)$ $\longrightarrow$ $\dot{x}(t-\lambda) = u(t-\lambda)$



$$b(t) = \int_{-\infty}^{\infty} a(\lambda)\dot{x}(t-\lambda)d\lambda = \int_{-\infty}^{\infty} a(\lambda)u(t-\lambda)d\lambda$$

$$= \int_{-\infty}^{t} a(\lambda)d\lambda$$

Impulse input $\delta(t)$



$h(t)$

Step input

$$u(t) = \int_{-\infty}^{t} \delta(t')dt' \qquad a(t) = \int_{-\infty}^{t} h(\lambda)d\lambda$$

Ramp input

$$r(t) = \int_{-\infty}^{t} u(t')dt' \qquad b(t) = \int_{-\infty}^{t} a(\lambda)d\lambda$$