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VIBRATION ANALYSIS OF SHROUDED BLADED-DISC ASSEMBLIES

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ABSTRACT

Engineering structures designed and assumed to be periodic rarely satisfy this condition in practice due to manufacturing errors or inherent minute differences in material properties. It is now known that such deviations from the ideal periodicity can give rise to unpredictable and undesirable dynamic behavior, which could even cause failure. In such cases, energy input into the nearly-periodic system can accumulate in only a some small region of the overall system, a phenomenon known as mode localization. One parameter for quantifying this phenomenon is the mode localization factor, which measures the exponential rate of energy decay from one station to the next in the nearly-periodic structure. In this paper, we investigate the dynamic behavior of linear and cyclic chains of cantilever beams, coupled by linear springs. This could, for instance, simulate the behavior of a shrouded bladed disk on a rotating shaft, where the shroud is modeled as massless linear springs. An exact solution is given for both the tuned (perfectly periodic) and mistuned (nearly periodic) cases, based on Green's functions.

Keywords: *Green's Functions, Cyclic, bladed-disc, mode localization, Monte Carlo.*

الملخص

إن المنشات الهندسية المصممة لكي تكون متماثلة لا تحقق هذه الخاصية من الناحية العملية بسبب أخطـاء التصـنيع أو بسبب اختلاف خواص المادة والتي يتعذر تحقق وجودها. ومن المعلوم الآن أن عدم التماثل الموجـود فـي المنشـآت الهندسية يسبب تصرفا ديناميكيا غير متوقع وقد يكون غير مرغوب فيه لأنه يقضى إلى الانهيار . وفى هـذه الحـالات فان طاقة تغذى بها المنشآت شبه المتماثلة يمكن أن تختزن في منطقة صغيرة فقط وهى خاصية تعـرف باسـم "نمـط المركزية" أو "نمط المحلية". ويمكن التعبير عن هذه الظاهرة كميا بواسطة "معامل نمط المركزية ", الذي يقيس معدل فقد الطاقة من منطقة إلى أخرى في هذه المنشآت شبه المتماثلة. وفى هذا البحث يتم فحص التصرف الدنيا لسلسـلة مـن العوارض الكابولية مدعمة بنوابض خطية . وهذه يمكن أن تمثل تصرف قرص دوار ذي ريش مغطاة من أطرافها حيث يمثل الغطاء بالنوابض الخطية و قد اعطى الحل الصحيح لكل حالتي المنشا المتماثل تمامـا و المنشـا شـبه المتماثـل بواسطة دوال جرين.

1. INTRODUCTION

Periodic structures are a common occurrence in a wide range of engineering and science fields. A perfectly periodic structure is modeled to have repeating sub-structures along one or more directions, all of which have identical properties. Typical examples are truss beams, rail road tracks, aircraft fuselages, ship hulls, and bladed-disk assemblies. The neglect of local variation in system properties from sub-structure to sub-structure is most critical for periodic structures. Departure from perfect periodicity always occurs because of manufacturing and material tolerances, operational wear, assembly errors etc. Depending on the relative magnitudes of the variations and internal coupling for the nearly periodic structure, these irregularities in properties may localize the free modes to small geometric regions and confine vibrational energy near the source of excitation. This phenomenon is referred to as normal mode localization and was first predicted in solid state physics by Anderson (1958). A periodic structure with dissimilar repeating units is said to be mistuned.

Even though researchers in structural dynamics, for instance [Meirovitch and Eagles, 1978], [Craig and Chung, 1985] and [Igusa, 1988] observed the high sensitivity of some periodic structures with mistuning, [Hodges, 1982] was the first to recognize that localization can occur in engineering structures and suggested that some of the knowledge acquired in physics could be applied to studies in structural dynamics. Using both wave and modal arguments, he discussed localization for chains of coupled pendulums and for beams on randomly spaced supports. Since then, numerous studies have been conducted in an attempt to understand the effects of mistuning on the dynamics of blade assemblies.

[Wei and Pierre, 1988, 1988] examined both free and forced localized responses of mistuned cyclic assemblies. They investigated how nonlinear dry friction damping affects localization, and [Wei and Pierre, 1989] also introduced stochastic techniques to calculate the forced response statistics. [Wei and Pierre, 1990], [Bendiksen, 1987], and [Cornwell and Bendiksen, 1989], examined mode localization for dish antennas numerically, while [Pierre and Cha, 1989] tackled localization effects analytically in assemblies of multi-mode component systems and showed that confinement increases rapidly with frequency. [Wei-Chau and Ariaratnam, 1996, 1996], using Lagrange's equations via component mode synthesis, studied localization in mistuned wrap-rib dish antennas. They calculated localization factors using transfer matrices and Furstenberg's theorem. They also calculated the localization factors using a Green function formulation. For a dish with a very large number of ribs, they stated that energy propagating in both directions meet at the rib of minimum vibration amplitude. The dish could then be cut open at the rib of minimum vibration amplitude to form a linear chain. Using this argument, they treated the cyclic dish as a linear chain for the purpose of calculating the localization factors.

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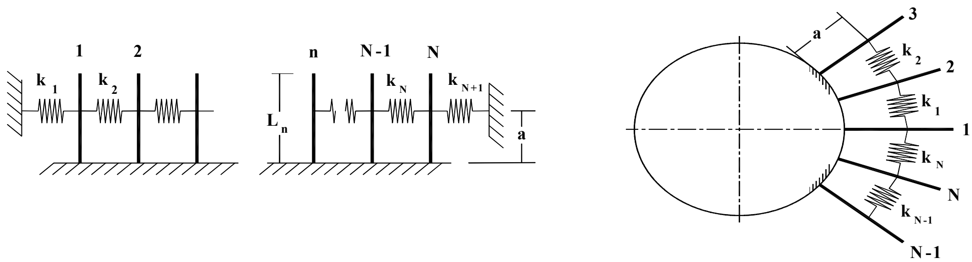
In this paper, we propose a mathematical formulation using Green's function of the unrestrained cantilever beam to study the dynamics of both tuned and mistuned, linear and cyclic bladed systems.

2. MATHEMATICAL MODEL OF THE MISTUNED LINEAR SYSTEM

In the derivation of the mathematical model for the linear chain, reference is made to Fig. 1a. N beams are linearly coupled at x = a by N+1 linear springs of constants $k_1, k_2, \ldots\ldots k_{N+1}$. The beams and springs are numbered from left to right, and are assumed to have properties uniformly statistically distributed with known means and assumed standard deviations, except the constraint point coordinate (x = a) which is assumed constant for all beams. For convenience, a dummy beam is added at each end, giving the system a total of N+2 beams with the condition that the displacements of the first and last beams are zero.

The transverse displacements of the n-th beam, $y_n(x, t)$, is obtained by the Euler-Bernoulli beam theory, thus:

$$EI_n \frac{\partial^4 y_n(x,t)}{\partial x^4} + m_n \frac{\partial^2 y_n(x,t)}{\partial t^2} = [k_{n+1} y_{n+1}(x,t) - (k_n + k_{n+1}) y_n(x,t) + k_n y_{n-1}(x,t)]\delta(x-a) \tag{1}$$



(a) Linear Chain (b) Cyclic Chain

Figure 1. Linear and cyclic chain configurations.

where EI_n and m_n are the flexural rigidity and mass per unit length respectively .

In (1), $\delta(x-a)$ is the Dirac delta function whose property relevant to this application is:

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a) \tag{2}$$

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As the vibrations are harmonic in time with frequency ω, one may assume a solution to (1) in the form

$$y_n(x,t) = Y_n(x)\exp(i\omega t) \tag{3}$$

and substituting (3) into (1) results in the fourth order shape function differential equation

$$\frac{d^4Y_n(x)}{dx^4} - \beta_n^4 Y_n(x) = \left[\frac{k_{n+1}}{EI_n}Y_{n+1} - \frac{k_n + k_{n+1}}{EI_n}Y_n + \frac{k_n}{EI_n}Y_{n-1}\right]\delta(x - a) \tag{4}$$

The abbreviation $Y_n \equiv Y_n(a)$ has been used, and $\beta_n^4 = m_n\omega^2 / EI_n$.

The Green's function for the n-th beam, $G_n(x,u:z)$ is now introduced, as the displacement of the beam at point x, due to the application of a transverse force of unit magnitude at the point x = u on the beam, and z is the frequency parameter defined later in equation (8).

This, from [Wylie, 1979], simply means that $G_n(x,u:z)$ is the solution of the differential equation:

$$\frac{d^4Y_n(x)}{dx^4} - \beta_n^4 Y_n(x) = \delta(x-u) \tag{5}$$

By superposition, we deduce that the solution to (4) is:

$$Y_n(x) = \int_{-\infty}^{+\infty} G_n(x,u:z)\left[\frac{k_{n+1}}{EI_n}Y_{n+1} - \frac{k_n + k_{n+1}}{EI_n}Y_n + \frac{k_n}{EI_n}Y_{n-1}\right]\delta(u - a)du \tag{6}$$

which integrates, by (2), to the nth beam mode shape functions of

$$Y_n(x) = G_n(x,a:z)\left[\frac{k_{n+1}}{EI_n}Y_{n+1} - \frac{k_n + k_{n+1}}{EI_n}Y_n + \frac{k_n}{EI_n}Y_{n-1}\right] \tag{7}$$

with the two boundary conditions that $Y_0 = 0$, $Y_{N+1} = 0$

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The n^{th} beam has a length, width and thickness of L_n, b_n and t_n respectively. For all N beams, the average length is L_a. Another quantity of importance is α_a, which is the average value of α_n over all the N beams, where $\alpha_n = (m_n / EI_n)^{1/4}$. The frequency parameter z is now conveniently defined in terms of average values and circular frequency as:

$$z = \alpha_a L_a \omega^{1/2} \tag{8}$$

If the mode shape equations (7) are evaluated at x = a for all the beams, the resulting equations can be cast in a compact matrix form as:

$$[D]\{Y\} = \{0\} \tag{9}$$

where

$\{Y\} = [Y_1 \quad Y_2 \quad . \quad . \quad . \quad . \quad Y_N]^T$ is the constraint point displacement vector, and [D] is a tri-diagonal matrix whose elements are as follows:

Letting

$$A_i = \frac{k_i L_a^3}{EI_i} \quad \text{and} \quad B_i = \frac{k_{i+1} L_a^3}{EI_i} \text{, then}$$

$$D_{i,i} = 2(z\alpha_i / \alpha_a)^3 \Delta_i(z) + (A_i + B_i) g_i(a, a : z) \tag{10a}$$

and

$$D_{i,j} = \begin{cases} -A_i g_i(a, a : z) .... i > j \\ -B_i g_i(a, a : z) .... i < j \end{cases} \tag{10b}$$

for the sub- and super-diagonals respectively, where $\Delta_i(z)$ and $g_i(a, a : z)$ are elements of the Greens Functions for beam i given in the Appendix.

In obtaining $\Delta_i(z)$ and $g_i(a, a : z)$ from the Appendix, the argument of the functions of the matrix $[e]$ in that Appendix is $(\alpha_i L_i / \alpha_a L_a)z$, while the argument of the trigonometric and hyperbolic functions of all other matrices is $(\alpha_i L_i / \alpha_a L_a)(a / L_i)z$. The matrix [D] itself expounds to:

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$$[D] = \begin{bmatrix} D_{1,1} & D_{1,2} & 0 & 0 & 0 & 0 & 0 \\ D_{2,1} & D_{2,2} & D_{2,3} & 0 & 0 & 0 & 0 \\ 0 & D_{3,2} & D_{3,3} & D_{3,4} & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & D_{N-1,N-2} & D_{N-1,N-1} & D_{N-1,N} \\ 0 & 0 & 0 & 0 & 0 & D_{N,N-1} & D_{N,N} \end{bmatrix} \tag{11}$$

The non-trivial solution of (9) requires that the determinant of [D], which is a function of z, be zero, from which any number of desired frequency parameters z can be determined. The frequencies in Hz are then calculated using (8), thus:

$$f_j = \frac{1}{2\pi}[z_j/(L_a\alpha_a)]^2 \tag{12}$$

where j represents the mode number.

3. THE CYCLIC MISTUNED SYSTEM

The derivation of the mode shape equations and characteristic determinant of the cyclic system, Figure 1 (b), is practically the same as that for the linear chain, except in the fact that the first and last beams are coupled. For a system of N beams, the numbering convention is to select an arbitrary beam as beam 1, and proceed to number the rest sequentially in the counter-clockwise direction. The N springs are numbered similarly with spring 1 immediately following beam 1 in the counter-clockwise chain. The mode shape functions for the 1^{st}, a general and the Nth beams are, respectively,

$$Y_1(x) = G_1(x,a:z)\left[\frac{k_N}{EI_1}Y_N - \frac{k_1 + k_N}{EI_1}Y_1 + \frac{k_1}{EI_1}Y_2\right] \tag{13a}$$

$$Y_n(x) = G_n(x,a:z)\left[\frac{k_{n-1}}{EI_n}Y_{n-1} - \frac{k_n + k_{n-1}}{EI_n}Y_n + \frac{k_n}{EI_n}Y_{n+1}\right] \tag{13b}$$

$$Y_N(x) = G_N(x,a:z)\left[\frac{k_N}{EI_N}Y_1 - \frac{k_N + k_{N-1}}{EI_N}Y_N + \frac{k_{N-1}}{EI_N}Y_{N-1}\right] \tag{13c}$$

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Evaluating equations (13a, b, c) at x = a for all the beams and gathering the resulting equations into matrix form yields an equation similar to (9), where the matrix [D] is now with elements defined as follows:

$$A_i = \frac{k_i L_a^3}{EI_i}\ ,\ \ B_i = \frac{k_{i-1} L_a^3}{EI_i} \quad \text{where } i > 1 \text{ in } B_i\ ,$$

$$D_{i,i} = 2(z\alpha_i / \alpha_a)^3 \Delta_i(z) + (A_i + B_i) g_i(a, a:z), \qquad i \neq 1, N \tag{14a}$$

$$D_{1,1} = 2(z\alpha_1 / \alpha_a)^3 \Delta_1(z) + \left(\frac{k_1 + k_N}{EI_1} L_a^3 \right) g_1(a, a:z) \tag{14b}$$

$$D_{N,N} = 2(z\alpha_N / \alpha_a)^3 \Delta_N(z) + \left(\frac{k_N + k_{N-1}}{EI_N} L_a^3 \right) g_N(a, a:z) \tag{14c}$$

$$D_{i,j} = \begin{cases} -B_i g_i(a, a:z) \ldots\ldots\ldots\ldots\ldots\ldots i > j, i \neq N \\ -A_i g_i(a, a:z) \ldots\ldots\ldots\ldots\ldots\ldots i < j, j \neq N \end{cases} \tag{14d}$$

$$D_{1,N} = -\left(k_N L_a^3 / EI_1\right) g_1(a, a:z) \tag{14e}$$

$$D_{N,1} = -\left(k_N L_a^3 / EI_N\right) g_N(a, a:z) \tag{14f}$$

All other elements of [D] are zero, and [D] itself takes the tri-cyclic form:

$$[D] = \begin{bmatrix} D_{1,1} & D_{1,2} & 0 & 0 & 0 & 0 & D_{1,N} \\ D_{2,1} & D_{2,2} & D_{2,3} & 0 & 0 & 0 & 0 \\ 0 & D_{3,2} & D_{3,3} & D_{3,4} & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & D_{N-1,N-2} & D_{N-1,N-1} & D_{N-1,N} \\ D_{N,1} & 0 & 0 & 0 & 0 & D_{N,N-1} & D_{N,N} \end{bmatrix} \tag{15}$$

The requirement that the determinant of [D] be zero yields the frequency equation.

4. FREE WAVE PROPAGATION CHARACTERISTICS

Wave progression through a periodic (or nearly periodic) structure is of paramount importance as it determines such important parameters as energy transfer and group velocities. For the perfectly periodic systems, exact solutions for the wave propagation parameters are possible, and the subject has received a great deal of attention in the literature, for example [Pierre and Cha, 1989]. In the current research, we shall describe wave propagation in terms of the components of the Green's Functions that have been used in our derivations. While the general solution procedure is the same, geometric interpretations of wave propagation and attenuation zones and their boundaries are given in terms of the Green's Function components, [Mohamad and Al-Jawi, 2000]. For the disordered systems, such simple and exact interpretations break down, leaving the resulting probabilistic systems to be tackled through perturbation or statistical simulation methods.

4.1. Propagation constants in periodic structures

In the tuned structure, all beams have identical inertia and geometric properties, so that all subscripts on variables pertaining to beam numbers are dropped. For the periodic structure, the general equation in (9) becomes:

$$Y_{n+1} - 2\left[1 + \frac{z^3 \Delta(z)}{Kg(a, a:z)}\right] Y_n + Y_{n-1} = 0 \tag{16}$$

where K is dimensionless spring stiffness given by $K = kL^3 / EI$. Δ(z) and g(a, a :z) remain as the elements of the Green's functions.

Equation (16) is a standard characteristic value problem discussed in detail in many sources, [Wylie, 1979], for example, with the solution depending on the quantity expressed in the square bracket.

The general solution of (16) can be assumed in the form

$$Y_n = Ae^{\gamma n} \tag{17}$$

where A is a constant, and γ is in general a complex constant of the form $\gamma = \gamma_r + i\gamma_i$, with γ_r and γ_i as the real and imaginary parts, and $i = \sqrt{-1}$. Equation (16) is recursively equivalent to:

$$Y_n = e^{\gamma} Y_{n-1} \tag{18}$$

γ is known in the literature as the wave propagation constant, or the Lyapunov exponent . The choice of the complex constant physically allows for wave decay (due to γ_r) from beam to beam, as well as a phase shift (due to γ_i) between adjacent beams [Wei-Chau and Ariaratnam, 1996, 1996]. From (18), γ_r represents the exponential rate of amplitude decay per bay, i.e. between adjacent beams.

When (17) is substituted into (16), the resulting equation simplifies to

$$\cosh(\gamma) = 1 + \frac{\overline{\Delta}}{Kg} \tag{19}$$

where

$\overline{\Delta} \equiv \overline{\Delta}(z) = z^3\Delta$ and $g \equiv g(a, a : z)$ Equation (19) may also be expanded as:

$$\cosh(\gamma_r)\cos(\gamma_i) + i\sinh(\gamma_r)\sin(\gamma_i) = 1 + \frac{\overline{\Delta}}{Kg} \tag{20}$$

The frequency spectrum across the first two pass bands of the complex propagation constant γ is calculated from (20) and given in Figure 2, for various values of K when a/L = 0.5. For the tuned problem, the real part, γ_r, is equal to the mode localization factor.

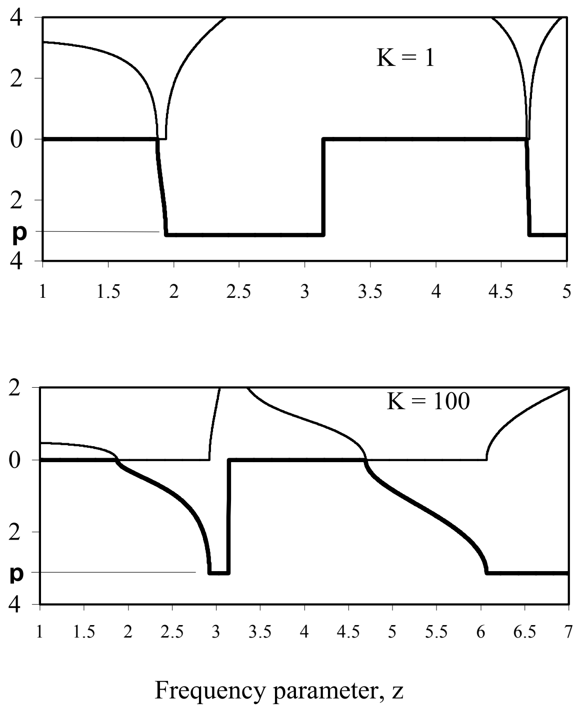

Figure 2. Propagation Constants. ________ γ_r, ________ γ_i

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5. MONTE CARLO SIMULATION FOR LOCALIZATION FACTORS

In order to calculate the mode localization constants for the linear mistuned chain, a sinusoidal force of magnitude F_o and frequency ω is applied to the constraint point of beam 1, and the constraint point response of beam N (last beam) observed. Under this forcing function, equation (9) is modified to read

$$[D(\omega)]\{Y\} = \{F\} \tag{21}$$

where the external force vector {F} is given by:

$$\{F\} = \begin{bmatrix} g_1\left(\dfrac{F_o L_1^3}{EI_1}\right) & 0 & 0 & . & . & . & 0 \end{bmatrix}^T \tag{22}$$

For a given frequency ω, the constraint point response vector $\{Y\}$ is calculated by inverting (21). The displacement $F_o L_1^3 / EI_1$ is set equal to unity in all cases, so that inverting (21) results in dimensionless displacements:

$$\{\overline{Y}\} = \{Y\}/(F_o L_1^3 / EI_1)\,. \tag{23}$$

It is assumed that the system properties of [D] are uniformly statistically distributed due to mistuning. As a result, the uniform probability distribution function is used to generate the random system variables. To generate a set of random variables p_i, i = 1, 2 …., uniformly distributed with mean p_{mean} and standard deviation σ given as a percentage of the mean, the properties of the uniformly distribution function gives

$$p_i = ((2r_i - 1)\sigma\sqrt{3} + 1)p_{mean}\,, \quad i = 1, 2\ldots\ldots \tag{24}$$

where $0 < r_i < 1$ are computer generated random numbers. A zero standard deviation corresponds to a tuned system with all parameters assuming mean values. Using this scheme, the system probabilistic input variables are generated. A large number of beams is used, and the response vector calculated from equation (21). The mode localization factor is calculated, in the case of a linear chain, from

$$\gamma = \lim_{N\to\infty}\left[-\frac{1}{N}\ln\left|\overline{Y}_N\right|\right] \tag{25}$$

As it is not computationally possible to let $N \to \infty$, several runs (realizations) of equation (25) are carried out for a specified number of beams, and the average value of γ calculated.

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In order to test the accuracy of the simulations, localization factors were calculated using 50 beams and 200 realizations, and a standard deviation of 0%, i.e., the tuned case. The results were compared with those obtained theoretically in Figure 2 where K = 1.0. The comparison is given in Figure 3, which indicates an excellent agreement. Localization factors for the linear chain of Figure 1a, in the first and second pass bands using equation (25) are given in Figure 4 for various degrees of mistuning. The properties of a system of 50 cantilevers were generated randomly, with a standard deviations of 0%, 5% and 10%. The expected values of properties used were, beam length of 350 mm, thickness of 4 mm, width of 30 mm and spring constant of 1600 N/m. In the average sense, this spring constant is equivalent to a dimensionless spring constant of K = 2.0. The simulations in each case were carried out over 200 realizations, and the average was assumed to be a good estimate of the localization factors.

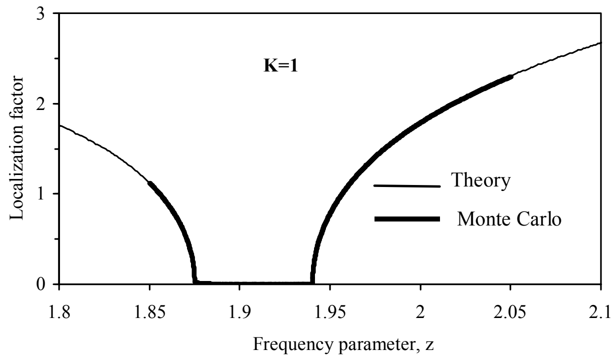
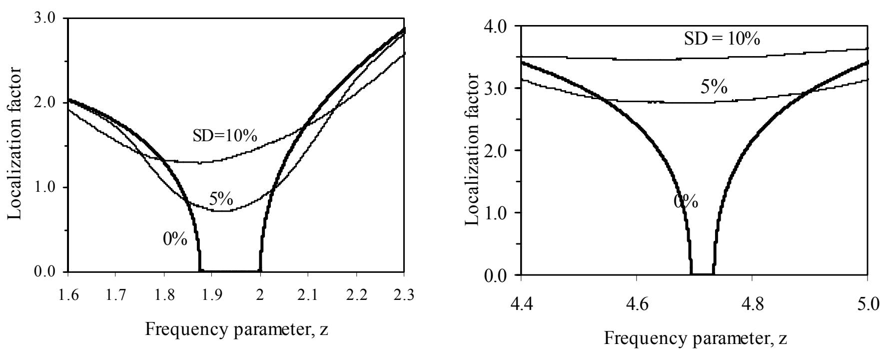


Figure 3. Theoretical and Monte Carlo simulations for the tuned case.



a) First pass band b) Second pass band

Figure 4. Mode localization factors for a linear chain

In the treatment of the cyclic chain, energy propagates in both clockwise and counterclockwise directions from the excitation beam, meeting at the beam of minimum amplitude, which is not known in advance. Figure 5 shows this statement. The Figure is established by first generating random properties of the system, and upon setting the determinant of [D(ω)] = 0, the frequencies of vibration are obtained. At each frequency, the theoretical constrained point displacement vector {Y} is then calculated. We use the scheme that at mode j,

$$\{Y_j\} = \alpha_j [C_{11} \quad C_{12} \quad C_{13} \quad \bullet \quad \bullet \quad \bullet \quad C_{1N}]_j^T$$

where C_{1k} are the cofactors of the first row of [D]. α_j is an arbitrary modal constant which may be set equal to one without any consequences. Thus evaluation of the first row co-factors yields the constraint point displacements. One observes from Figure 5 that the exponential rate of decay in the linear chain is the same as the exponential rate of decay in the cyclic chain in the counterclockwise direction, while the exponential rate of decay in the linear chain is the same as the exponential rate of amplitude growth in the clockwise direction of the cyclic chain, starting with the beam of minimum amplitude of the cyclic chain. This observation was made by [Wei-Chau and Ariaratnam, 1996,

1996]. The conclusion here is that the localization factors of a cyclic chain are approximately the same as those of the linear chain obtained by 'opening' the cyclic chain. To calculate the approximate localization factors for the cyclic chain by this method, the cyclic chain is opened by disconnecting first beam from the last beam, i.e. setting k_N equal to zero. The tri-cyclic matrix of equation (15) reduces to a tri-diagonal matrix. The application of equation (25) yields the approximate localization factors, which are shown in Figure 6a. Another way to estimate the localization factors for a cyclic chain without opening, is to say that for large N, based on the observation in Figure 5,

$$\gamma \approx -\frac{1}{N/2} \ln\left|\overline{Y}_{N/2}\right| \tag{26}$$

where N/2 should be rounded to the nearest integer. The results obtained by this equation are shown in Figure 6b. When drawn on the same frame, the localization factors for the cyclic chain are practically the same using the equivalent linear chain and equation (26). All system properties and simulation conditions are the same as those already stated for the linear chain.

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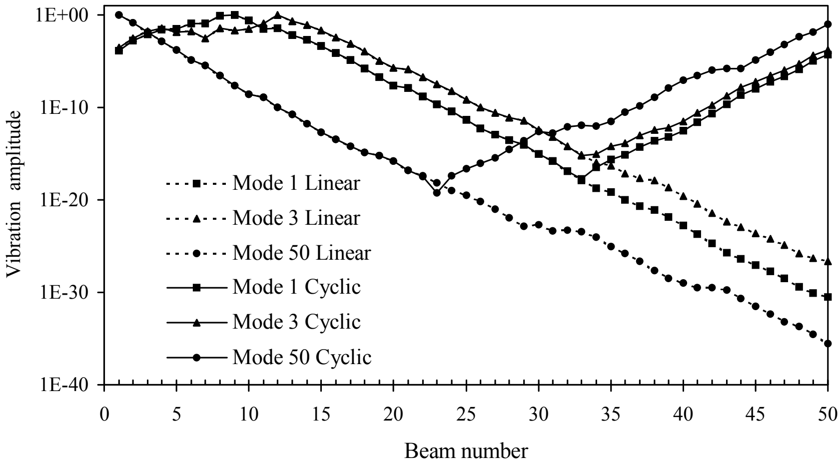

Figure 5. Constraint point model vibration amplitudes for a 50-beam mistuned linear chain

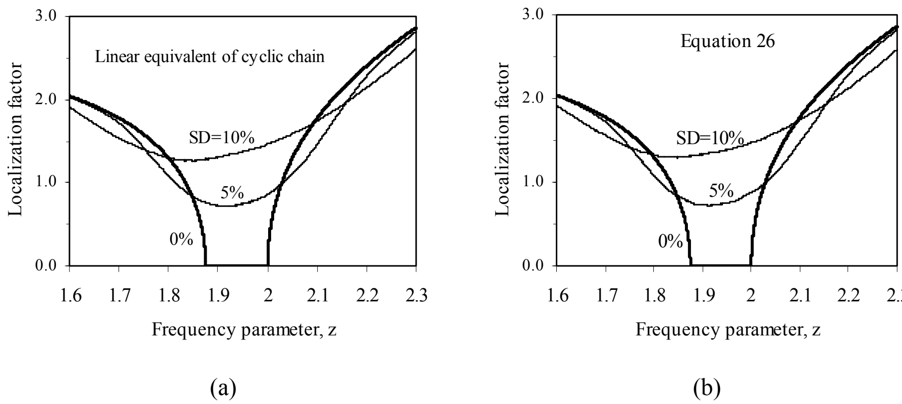

Figure 6. Mode localization factor for a cyclic chain

6. CONCLUSION

In this paper we have examined the dynamics of mono-coupled cantilever beams in linear and cyclic configurations using Green's functions. Forced response data is used to calculate the mode localization factors. For a system with N beams, matrices of size N x N must be inverted. If the same problem were treated using the component mode formulation with p participating component modes for example, the resulting matrices become N x N block tri-diagonal or cyclic block tri-diagonal, with each block having a size of p x p, or overall matrices of size pN x pN, for example [Wei-Chau and Ariaratnam, 1996]. As the labor effort in inverting a tri-diagonal matrix of size N is of order N, and storage requirement is also of order N, the computational superiority of using the Green's function is immediately apparent.

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APPENDIX : Green's Function for a Cantilever (Mohamad, 1994).

Let s = sin(z), c = cos(z), S = sinh(z) and C = cosh(z). Further , let $\tilde{s}$, $\tilde{c}$, $\tilde{S}$ and $\tilde{C}$ be the same functions but with arguments zx/L , while $\bar{s}$, $\bar{c}$, $\bar{S}$ and $\bar{C}$ are of arguments z(1-u/L). Then from [Mohamad, 1994] the Green's functions are given by

$$G(x, u : z) = \frac{L^3}{2z^3 \Delta(z)} \begin{cases} g(x, u : z), & 0 \le x \le u \\ g(u, x : z), & L \ge x \ge u \end{cases}$$

with

$$g(x, u : z) = \psi_{11}(u, z)\phi_{11}(x, z) - \psi_{21}(u, z)\phi_{21}(x, z)$$

where

$$\psi_{11}(u, z) = e_{22}(z)\varphi_{11}(u, z) - e_{12}(z)\varphi_{21}(u, z)$$
$$\psi_{21}(u, z) = e_{11}(z)\varphi_{21}(u, z) - e_{21}(z)\varphi_{11}(u, z)$$

and

$$[e] = \begin{bmatrix} -c - C & -s - S \\ s - S & -c - C \end{bmatrix}, \qquad [\phi] = \begin{bmatrix} \tilde{c} - \tilde{C} & -\tilde{s} - \tilde{S} \\ \tilde{s} - \tilde{S} & \tilde{c} - \tilde{C} \end{bmatrix},$$

$$[\varphi] = \begin{bmatrix} -\bar{s} - \bar{S} & \bar{c} + \bar{C} \\ -\bar{c} - \bar{C} & -\bar{s} + \bar{S} \end{bmatrix}$$

$\Delta(z)$ is the determinant of $[e]$.

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