Boolean Algebra and Logic Gates

COE 233

Digital Logic and Computer Organization

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Presentation Outline

- ❖ Boolean Algebra, truth tables, and DeMorgan's theorem
- ❖ Algebraic manipulation and expression simplification
- ❖ From truth table to logic expression: minterms and maxterms
- ❖ Logic gates, logic diagrams, and standard forms
- ❖ Additional gates: NAND, NOR, XOR, XNOR

Boolean Algebra

- Introduced by George Boole in 1854
- A set of two values: *B* = {0, 1}
- Three basic operations: **AND**, **OR**, and **NOT**
- The **AND** operator is denoted by a dot (•)
  - $x \cdot y$ or xy is read: x **AND** y
- The **OR** operator is denoted by a plus (**+**)
  - $x + y$ is read: x **OR** y
- The **NOT** operator is denoted by a quotation **'** or an overbar ¯
  - x' or $\overline{x}$ is the complement of x
- Today, Boolean algebra is being used to design digital circuits

Postulates of Boolean Algebra

1. Closure: the result of any Boolean operation is in $B = \{0, 1\}$
2. Identity element with respect to + is 0: $x + 0 = 0 + x = x$

   Identity element with respect to $\cdot$ is 1: $x \cdot 1 = 1 \cdot x = x$
3. Commutative with respect to +: $x + y = y + x$

   Commutative with respect to $\cdot$: $x \cdot y = y \cdot x$
4. $\cdot$ is distributive over +: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

   + is distributive over $\cdot$: $x + (y \cdot z) = (x + y) \cdot (x + z)$
5. For every x in B, there exists x' in B (called complement of x) such that: $x + x' = 1$ and $x \cdot x' = 0$

AND, OR, and NOT Operators

- The following tables define $x \cdot y$, $x + y$, and x'
- $x \cdot y$ is the **AND** operator
- $x + y$ is the **OR** operator
- x' is the **NOT** operator

x y	x·y
0 0	0
0 1	0
1 0	0
1 1	1

x y	x+y
0 0	0
0 1	1
1 0	1
1 1	1

x	x'
0	1
1	0

Boolean Functions

❖ Boolean functions are described by expressions that consist of:

- ✧ Boolean variables, such as: x, y, etc.
- ✧ Boolean constants: 0 and 1
- ✧ Boolean operators: AND (·), OR (+), NOT (')
- ✧ Parentheses, which can be nested

❖ Example: $f = x(y + w'z)$

- ✧ The dot operator is implicit and need not be written

❖ Operator precedence: to avoid ambiguity in expressions

- ✧ Expressions within parentheses should be evaluated first
- ✧ The NOT (') operator should be evaluated second
- ✧ The AND (·) operator should be evaluated third
- ✧ The OR (+) operator should be evaluated last

Truth Table

- A truth table can represent a Boolean function
- List all possible combinations of 0's and 1's assigned to variables
- If *n* variables then 2^n rows
- Example: Truth table for $f = xy' + x'z$

x	y	z	y'	xy'	x'	x'z	f = xy'+ x'z
0	0	0	1	0	1	0	0
0	0	1	1	0	1	1	1
0	1	0	0	0	1	0	0
0	1	1	0	0	1	1	1
1	0	0	1	1	0	0	1
1	0	1	1	1	0	0	1
1	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0

DeMorgan's Theorem

- $(x + y)' = x' \, y'$
- $(x \, y)' = x' + y'$

Can be verified
Using a Truth Table

x	y	x'	y'	x+y	(x+y)'	x'y'	x y	(x y)'	x'+ y'
0	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	0	0	1	1
1	0	0	1	1	0	0	0	1	1
1	1	0	0	1	0	0	1	0	0

Identical **Identical**

- Generalized DeMorgan's Theorem:
- $(x_1 + x_2 + \cdots + x_n)' = x_1' \cdot x_2' \cdot \ \cdots \ \cdot \ x_n'$
- $(x_1 \cdot x_2 \cdot \cdots \cdot x_n)' = x_1' + x_2' + \ \cdots + \ x_n'$

Complementing Boolean Functions

- What is the complement of $f = x'yz' + xy'z'$?
- Use DeMorgan's Theorem:
  - Complement each variable and constant
  - Interchange AND and OR operators
- So, what is the complement of $f = x'yz' + xy'z'$?

  Answer: $f' = (x + y' + z)(x' + y + z)$

- **Example 2:** Complement $g = (a' + bc)d' + e$
- **Answer:** $g' = (a(b' + c') + d)e'$

Algebraic Manipulation of Expressions

- The objective is to acquire skills in manipulating Boolean expressions, to transform them into simpler form.

- **Example 1:** prove $x + xy = x$ (absorption theorem)

- **Proof:** $x + xy = x \cdot 1 + xy$ $\quad x \cdot 1 = x$

  $= x \cdot (1 + y)$ $\quad$ Distributive $\cdot$ over +

  $= x \cdot 1 = x$ $\quad (1 + y) = 1$

- **Example 2:** prove $x + x'y = x + y$ (simplification theorem)

- **Proof:** $x + x'y = (x + x')(x + y)$ $\quad$ Distributive + over $\cdot$

  $= 1 \cdot (x + y)$ $\quad (x + x') = 1$

  $= x + y$

Expression Simplification

- Using Boolean algebra to simplify expressions
- Expression should contain the smallest number of literals
- A literal is a variable that may or may not be complemented
- **Example:** simplify $ab + a'cd + a'bd + a'cd' + abcd$
- **Solution:** $ab + a'cd + a'bd + a'cd' + abcd$ (15 literals)

$= ab + abcd + a'cd + a'cd' + a'bd$ (15 literals)

$= ab + ab(cd) + a'c(d + d') + a'bd$ (13 literals)

$= ab + a'c + a'bd$ (7 literals)

$= ba + ba'd + a'c$ (7 literals)

$= b(a + a'd) + a'c$ (6 literals)

$= b(a + d) + a'c$ (5 literals only)

Summary of Boolean Algebra

	Property	Dual Property
Identity	$x + 0 = x$	$x \cdot 1 = x$
Complement	$x + x' = 1$	$x \cdot x' = 0$
Null	$x + 1 = 1$	$x \cdot 0 = 0$
Idempotence	$x + x = x$	$x \cdot x = x$
Involution	$(x')' = x$	
Commutative	$x + y = y + x$	$x\, y = y\, x$
Associative	$(x + y) + z = x + (y + z)$	$(x\, y)\, z = x\, (y\, z)$
Distributive	$x\,(y + z) = xy + xz$	$x + yz = (x + y)(x + z)$
Absorption	$x + xy = x$	$x(x + y) = x$
Simplification	$x + x'y = x + y$	$x(x' + y) = xy$
De Morgan	$(x + y)' = x'\, y'$	$(x\, y)' = x' + y'$

Importance of Boolean Algebra

- Our objective is to learn how to design digital circuits
- These circuits use signals with two possible values
- Logic **0** is a **low** voltage signal (0 volts)
- Logic **1** is a **high** voltage signal (for example, 1.2 volts)
- The physical value of a signal is the actual voltage it carries, while its logic value is either 0 (low) or 1 (high)
- Having only two logic values (0 and 1) simplifies the implementation of the digital circuit

Next . . .

- Boolean Algebra, truth tables, and DeMorgan's theorem
- Algebraic manipulation and expression simplification
- From truth table to logic expression: minterms and maxterms
- Logic gates, logic diagrams, and standard forms
- Additional gates: NAND, NOR, XOR, XNOR

From a Truth Table to Logic Expression

❖ Given the truth table of a Boolean function f, how to obtain the logic expression ?

Truth Table

x y z	f
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

What is the logic expression of f?

To answer this question, we need to define Minterms and Maxterms

Minterms and Maxterms

- **Minterms** are AND terms with every variable present in either true or complement form
- **Maxterms** are OR terms with every variable present in either true or complement form

Minterms and Maxterms for 2 variables x and y

x	y	index	Minterm	Maxterm
0	0	0	$m_0 = x'y'$	$M_0 = x + y$
0	1	1	$m_1 = x'y$	$M_1 = x + y'$
1	0	2	$m_2 = xy'$	$M_2 = x' + y$
1	1	3	$m_3 = xy$	$M_3 = x' + y'$

- For *n* variables, there are 2^n Minterms and Maxterms

Minterms and Maxterms for 3 Variables

x	y	z	index	Minterm	Maxterm
0	0	0	0	$m_0 = x'y'z'$	$M_0 = x + y + z$
0	0	1	1	$m_1 = x'y'z$	$M_1 = x + y + z'$
0	1	0	2	$m_2 = x'yz'$	$M_2 = x + y' + z$
0	1	1	3	$m_3 = x'yz$	$M_3 = x + y' + z'$
1	0	0	4	$m_4 = xy'z'$	$M_4 = x' + y + z$
1	0	1	5	$m_5 = xy'z$	$M_5 = x' + y + z'$
1	1	0	6	$m_6 = xyz'$	$M_6 = x' + y' + z$
1	1	1	7	$m_7 = xyz$	$M_7 = x' + y' + z'$

Maxterm M_i is the **complement** of Minterm m_i

$M_i = m_i'$ and $m_i = M_i'$

Sum-Of-Minterms (SOM) Canonical Form

Truth Table

x y z	f	Minterm
0 0 0	0	
0 0 1	0	
0 1 0	1	$m_2 = x'yz'$
0 1 1	1	$m_3 = x'yz$
1 0 0	0	
1 0 1	1	$m_5 = xy'z$
1 1 0	0	
1 1 1	1	$m_7 = xyz$

Sum of Minterm entries that evaluate to '**1**'

Focus on the '**1**' entries

$$f = m_2 + m_3 + m_5 + m_7$$

$$f = \sum(2, 3, 5, 7)$$

$$f = x'yz' + x'yz + xy'z + xyz$$

Product-Of-Maxterms (POM) Canonical Form

Truth Table

x y z	f	Maxterm
0 0 0	0	$M_0 = x + y + z$
0 0 1	0	$M_1 = x + y + z'$
0 1 0	1	
0 1 1	1	
1 0 0	0	$M_4 = x' + y + z$
1 0 1	1	
1 1 0	0	$M_6 = x' + y' + z$
1 1 1	1	

Product of Maxterm entries that evaluate to '**0**'

Focus on the '**0**' entries

$$f = M_0 \cdot M_1 \cdot M_4 \cdot M_6$$

$$f = \prod(0, 1, 4, 6)$$

$$f = (x + y + z)(x + y + z')(x' + y + z)(x' + y' + z)$$

Purpose of the Index

- Minterms and Maxterms are designated with an index
- The **index** for the Minterm or Maxterm, expressed as a **binary number**, is used to determine whether the variable is shown in the true or complemented form
- For Minterms:
  - '**1**' means the variable is **Not Complemented**
  - '**0**' means the variable is **Complemented**
- For Maxterms:
  - '**0**' means the variable is **Not Complemented**
  - '**1**' means the variable is **Complemented**

Examples of Sum-Of-Minterms

- $f(a,b,c,d) = \sum(2,3,6,10,11)$
- $f(a,b,c,d) = m_2 + m_3 + m_6 + m_{10} + m_{11}$
- $f(a,b,c,d) = a'b'cd' + a'b'cd + a'bcd' + ab'cd' + ab'cd$
- $g(a,b,c,d) = \sum(0,1,12,15)$
- $g(a,b,c,d) = m_0 + m_1 + m_{12} + m_{15}$
- $g(a,b,c,d) = a'b'c'd' + a'b'c'd + abc'd' + abcd$

Examples of Product-Of-Maxterms

- $f(a,b,c,d) = \prod(1,3,11)$
- $f(a,b,c,d) = M_1 \cdot M_3 \cdot M_{11}$
- $f(a,b,c,d) = (a+b+c+d')(a+b+c'+d')\,(a'+b+c'+d')$
- $g(a,b,c,d) = \prod(0,5,13)$
- $g(a,b,c,d) = M_0 \cdot M_5 \cdot M_{13}$
- $g(a,b,c,d) = (a+b+c+d)(a+b'+c+d')\,(a'+b'+c+d')$

Conversions between Canonical Forms

- The same Boolean function f can be expressed in two ways:
  - Sum-of-Minterms $f = m_0 + m_2 + m_3 + m_5 + m_7 = \sum(0,2,3,5,7)$
  - Product-of-Maxterms $f = M_1 \cdot M_4 \cdot M_6 = \prod(1,4,6)$

Truth Table

x y z	f	Minterms	Maxterms
0 0 0	1	$m_0 = x'y'z'$	
0 0 1	0		$M_1 = x + y + z'$
0 1 0	1	$m_2 = x'yz'$	
0 1 1	1	$m_3 = x'yz$	
1 0 0	0		$M_4 = x' + y + z$
1 0 1	1	$m_5 = xy'z$	
1 1 0	0		$M_6 = x' + y' + z$
1 1 1	1	$m_7 = xyz$	

To convert from one canonical form to another, interchange the symbols $\sum$ and $\prod$ and list those numbers missing from the original form.

Function Complement

Truth Table

x y z	f	f'
0 0 0	1	0
0 0 1	0	1
0 1 0	1	0
0 1 1	1	0
1 0 0	0	1
1 0 1	1	0
1 1 0	0	1
1 1 1	1	0

Given a Boolean function f

$$f(x,y,z) = \sum(0,2,3,5,7) = \prod(1,4,6)$$

Then, the complement f' of function f

$$f'(x,y,z) = \prod(0,2,3,5,7) = \sum(1,4,6)$$

The complement of a function expressed by a Sum of Minterms is the Product of Maxterms with the same indices. Interchange the symbols Σ and Π, but keep the same list of indices.

Summary of Minterms and Maxterms

- There are 2^n Minterms and Maxterms for Boolean functions with n variables, indexed from 0 to $2^n - 1$
- Minterms correspond to the **1-entries** of the function
- Maxterms correspond to the **0-entries** of the function
- Any Boolean function can be expressed as a Sum-of-Minterms and as a Product-of-Maxterms
- For a Boolean function, given the list of Minterm indices one can determine the list of Maxterms indices (and vice versa)
- The complement of a Sum-of-Minterms is a Product-of-Maxterms with the same indices (and vice versa)

Next . . .

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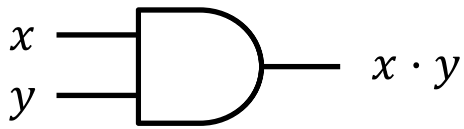
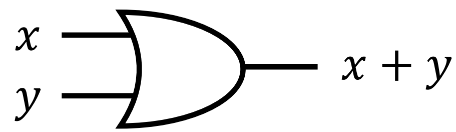
- ❖ Additional gates: NAND, NOR, XOR, XNOR

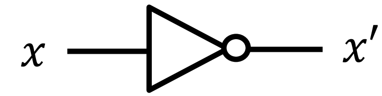
Logic Gates and Symbols



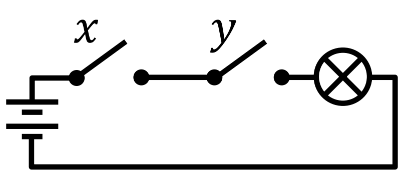
AND gate



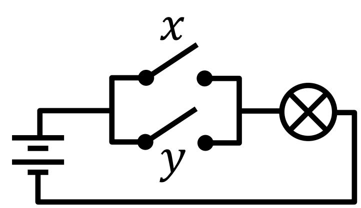
OR gate



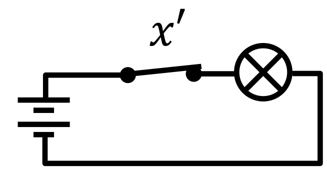
NOT gate (inverter)



AND: Switches in series
logic 0 is open switch



OR: Switches in parallel
logic 0 is open switch



NOT: Switch is normally
closed when x is 0

- In the earliest days of computers, relays were used as mechanical switches controlled by electricity (coils)

- Today, tiny transistors are used as electronic switches that implement the logic gates (CMOS technology)

Truth Table and Logic Diagram

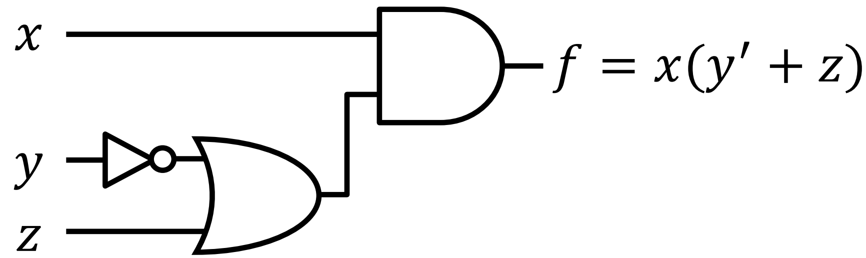
❖ Given the following logic function: $f = x(y' + z)$

❖ Draw the corresponding truth table and logic diagram

Truth Table

x y z	y'+z	f = x(y'+z)
0 0 0	1	0
0 0 1	1	0
0 1 0	0	0
0 1 1	1	0
1 0 0	1	1
1 0 1	1	1
1 1 0	0	0
1 1 1	1	1

Logic Diagram



Truth Table and Logic Diagram describe the same function f. Truth table is unique, but logic expression and logic diagram are not. This gives flexibility in implementing logic functions.

Standard Forms

- Boolean functions are usually expressed in **standard** form
- Two standard forms: **Sum-of-Products** and **Product-of -Sums**
- Sum of Products (**SOP**)
  - Boolean expression is the **ORing** (sum) of **AND terms** (products)
  - Examples: $f_1 = x'y + xz$ $\quad f_2 = y + w'xz$
- Products of Sums (**POS**)
  - Boolean expression is the **ANDing** (product) of **OR terms** (sums)
  - Examples: $f_3 = (x + z)(x' + y')$ $\quad f_4 = x(w' + y' + z)$

From Sum-of-Minterms to Sum-of-Products

- Simplify $f(x, y, z) = \sum(2, 3, 5, 7)$ into minimal sum-of-products
- First, write f as sum-of-minterms canonical form
- $f(x, y, z) = \sum(2, 3, 5, 7) = x'yz' + x'yz + xy'z + xyz$
- Then, simplify expression using properties of Boolean Algebra

$f = x'yz' + x'yz + xy'z + xyz$ (12 literals)

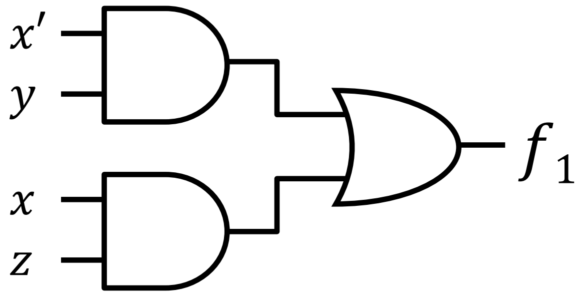
$f = x'y(z' + z) + xz(y' + y)$

$f = x'y\,(1) + x\,z\,(1)$

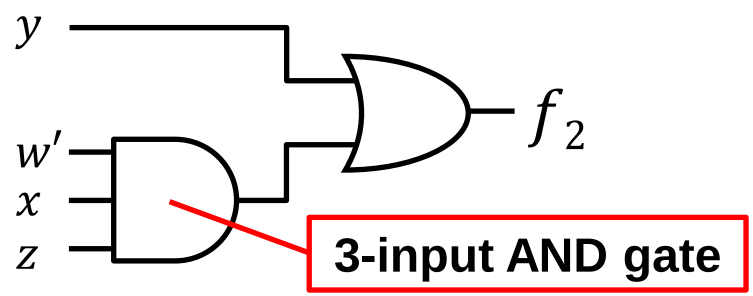
$f = x'y + xz$ (4 literals)

Two-Level Gate Implementation

$$f_1 = x'y + xz$$

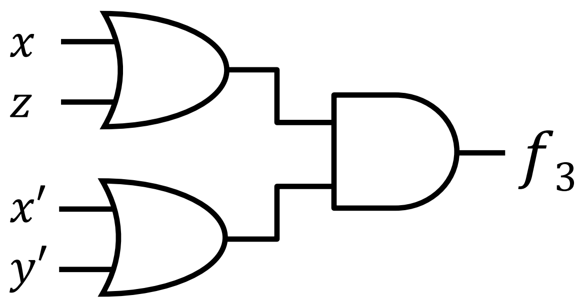


$$f_2 = y + w'xz$$

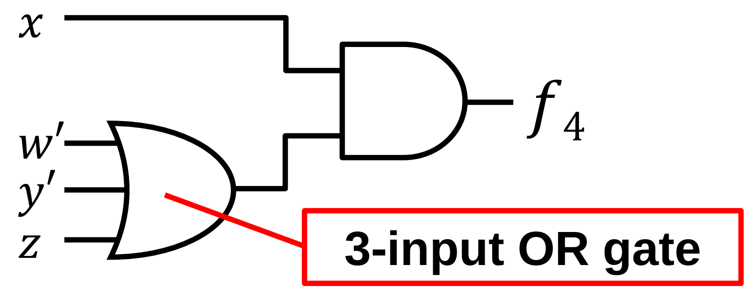


AND-OR implementations

$$f_3 = (x + z)(x' + y')$$



$$f_4 = x(w' + y' + z)$$



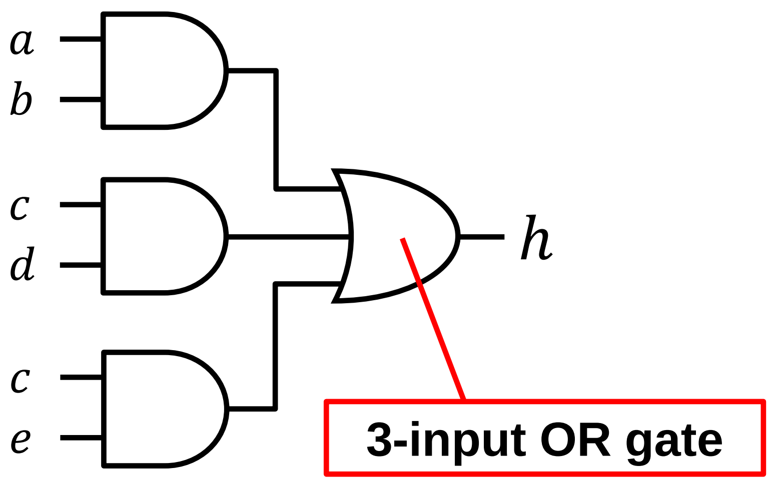
OR-AND implementations

Two-Level vs. Three-Level Implementation

- $h = ab + cd + ce$ (6 literals) is a sum-of-products
- h may also be written as: $h = ab + c(d + e)$ (5 literals)
- However, $h = ab + c(d + e)$ is a non-standard form
  - $h = ab + c(d + e)$ is not a sum-of-products nor a product-of-sums

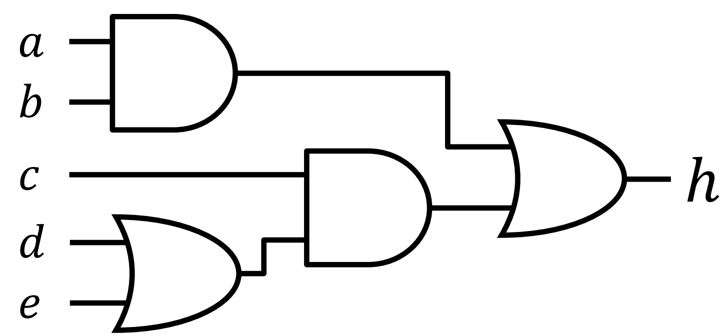
2-level implementation

$$h = ab + cd + ce$$



3-level implementation

$$h = ab + c(d + e)$$



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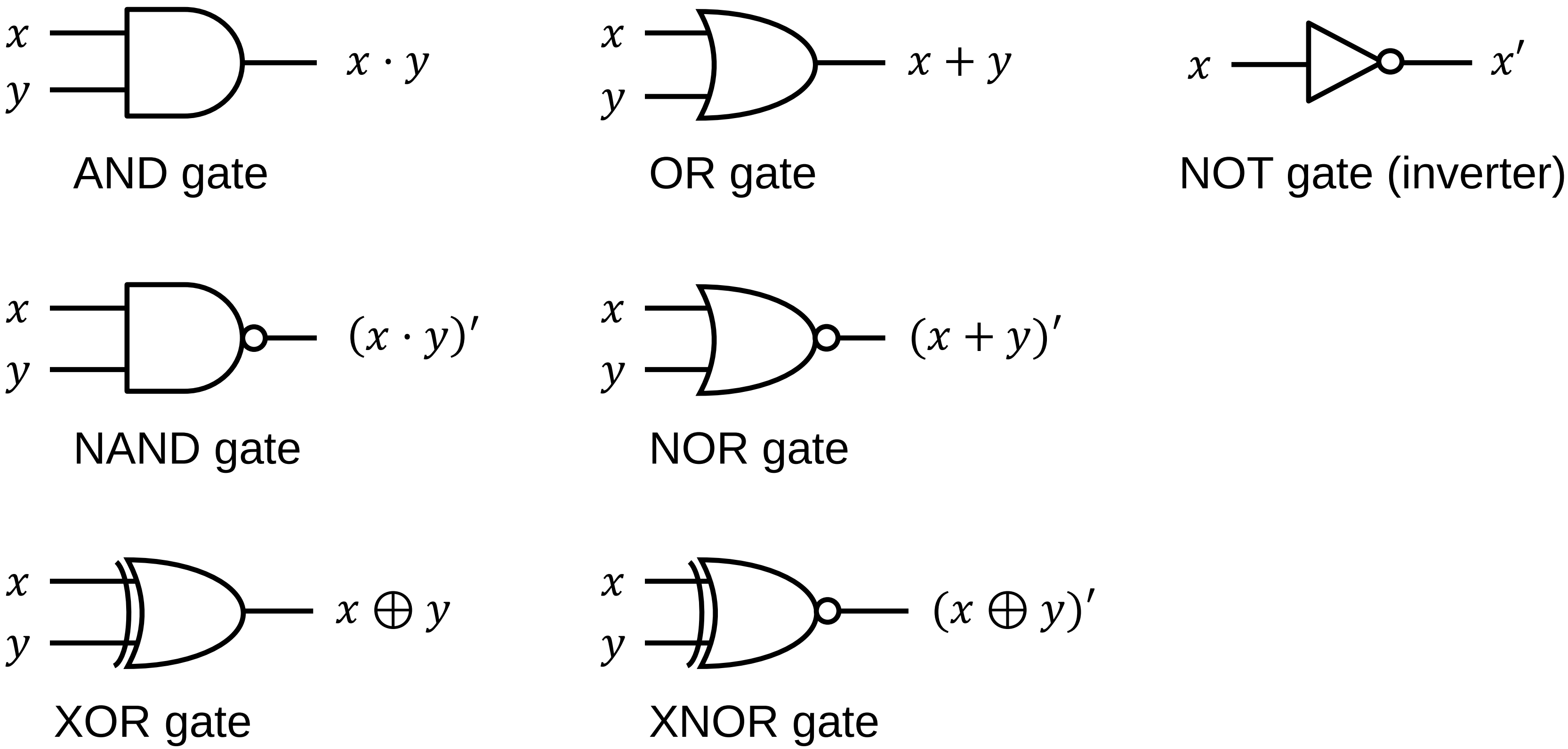
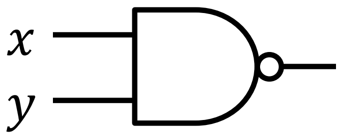
Additional Logic Gates and Symbols

- Why?
  - Low-cost implementation
  - Useful in implementing Boolean functions



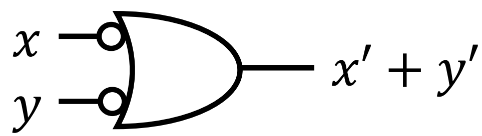
NAND Gate

- The NAND gate has the following symbol and truth table
- NAND represents **N**OT **AND**
- The small bubble circle represents the invert function



$(x \cdot y)' = x' + y'$

NAND gate



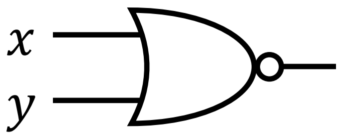
Another symbol for NAND

x	y	NAND
0	0	1
0	1	1
1	0	1
1	1	0

- NAND gate is implemented efficiently in CMOS technology
  - In terms of chip area and speed

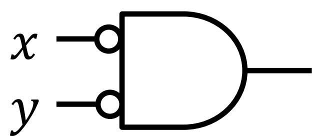
NOR Gate

- The NOR gate has the following symbol and truth table
- NOR represents **N**OT **OR**
- The small bubble circle represents the invert function



$(x + y)' = x' \cdot y'$

NOR gate



$x' \cdot y'$

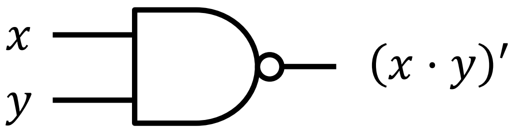
Another symbol for NOR

x	y	NOR
0	0	1
0	1	0
1	0	0
1	1	0

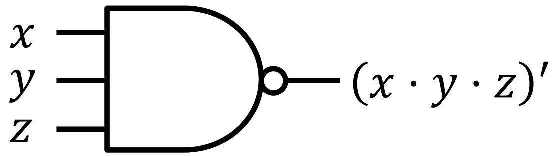
- NOR gate is implemented efficiently in CMOS technology
  - In terms of chip area and speed

Multiple-Input NAND / NOR Gates

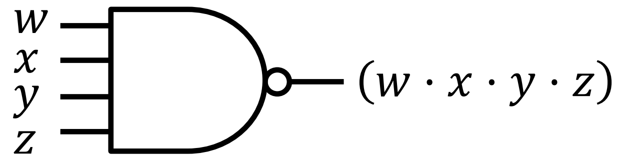
NAND/NOR gates can have multiple inputs, similar to AND/OR gates



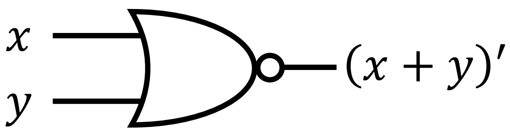
2-input NAND gate



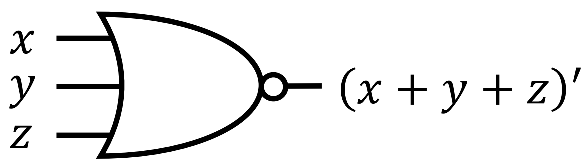
3-input NAND gate



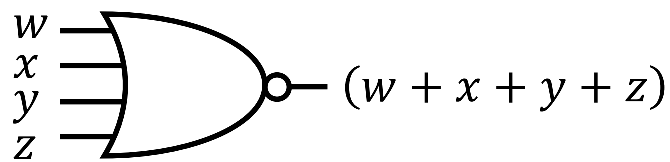
4-input NAND gate



2-input NOR gate



3-input NOR gate



4-input NOR gate

Note: a 3-input NAND is a single gate, NOT a combination of two 2-input gates.
The same can be said about other multiple-input NAND/NOR gates.

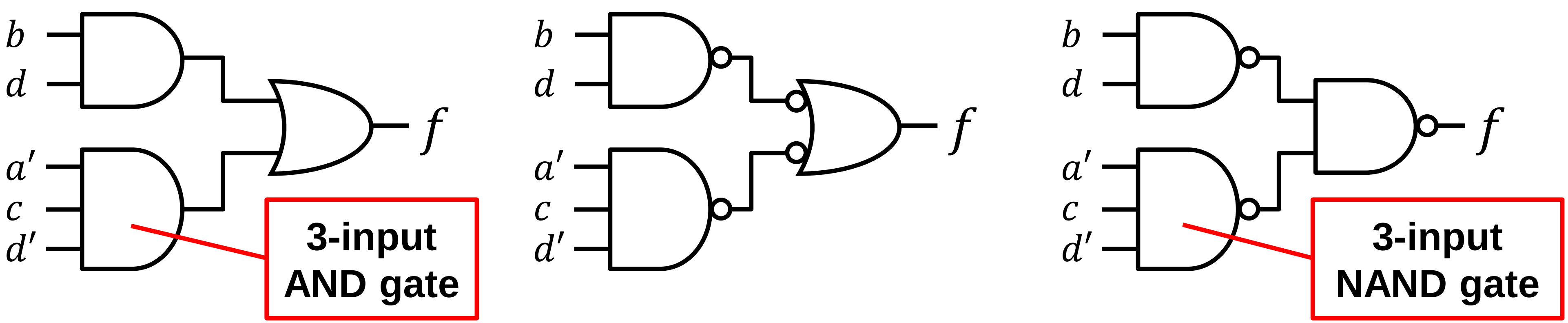
NAND - NAND Implementation

- Consider the following sum-of-products expression:

  $$f = bd + a'cd'$$

- A 2-level **AND-OR** circuit can be converted easily to a 2-level **NAND-NAND implementation**

2-Level AND-OR | **Inserting Bubbles** | **2-Level NAND-NAND**



Two successive bubbles on same line cancel each other

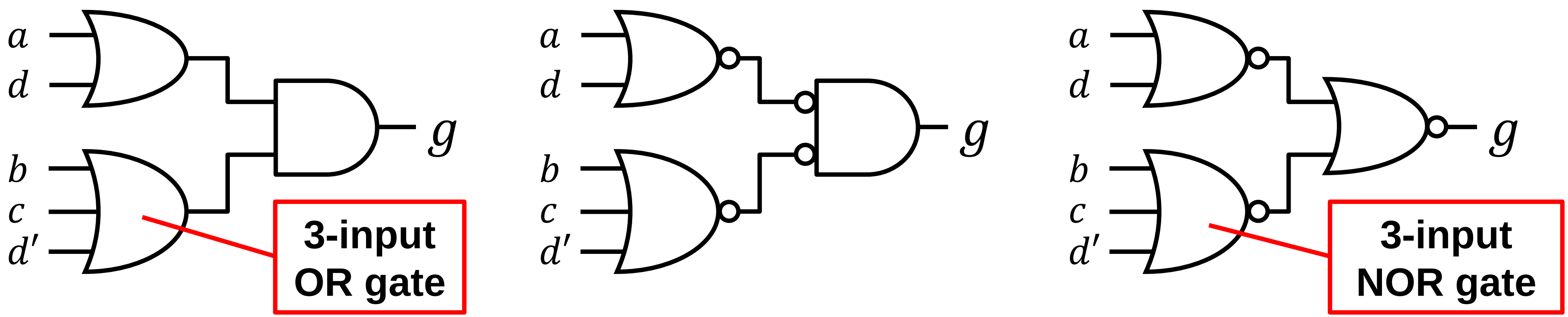
NOR - NOR Implementation

- Consider the following product-of-sums expression:

$$g = (a + d)(b + c + d')$$

- A 2-level **OR-AND** circuit can be converted easily to a 2-level **NOR-NOR implementation**

2-Level OR-AND | **Inserting Bubbles** | **2-Level NOR-NOR**



Two successive bubbles on same line cancel each other

Exclusive OR / Exclusive NOR

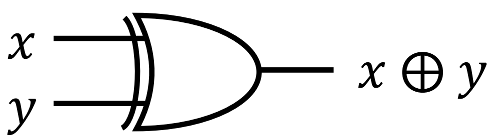
❖ Exclusive OR (XOR) is an important Boolean operation used extensively in logic circuits

❖ Exclusive NOR (XNOR) is the complement of XOR

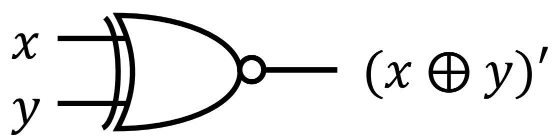
x	y	XOR
0	0	0
0	1	1
1	0	1
1	1	0

x	y	XNOR
0	0	1
0	1	0
1	0	0
1	1	1

XNOR is also known as **equivalence**



XOR gate



XNOR gate

XOR / XNOR Functions

- The XOR function is: $x \oplus y = xy' + x'y$
- The XNOR function is: $(x \oplus y)' = xy + x'y'$
- XOR and XNOR gates are complex
  - Can be implemented as a true gate, or by
  - Interconnecting other gate types
- XOR and XNOR gates do not exist for more than two inputs
  - For 3 inputs, use two XOR gates
  - The cost of a 3-input XOR gate is greater than the cost of two XOR gates
- Uses for XOR and XNOR gates include:
  - Adders, subtractors, multipliers, incrementers, and decrementers

XOR and XNOR Properties

- $x \oplus 0 = x$ $\qquad$ $x \oplus 1 = x'$
- $x \oplus x = 0$ $\qquad$ $x \oplus x' = 1$
- $x \oplus y = y \oplus x$
- $x' \oplus y' = x \oplus y$
- $(x \oplus y)' = x' \oplus y = x \oplus y'$

XOR is an **associative** operation

- $(x \oplus y) \oplus z = x \oplus (y \oplus z) = x \oplus y \oplus z$

Odd Function

- Output is 1 if the **number of 1's is odd in the inputs**
- Output is the XOR operation on all input variables

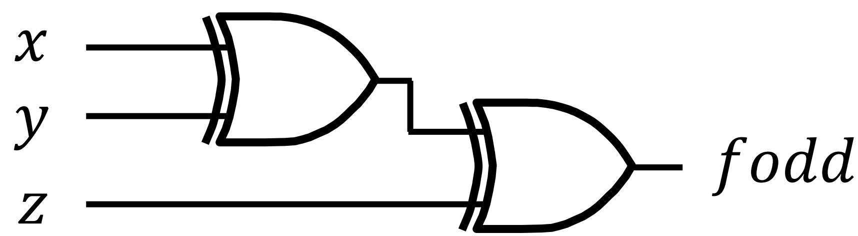
Odd Function with 3 inputs

x y z	fodd
0 0 0	0
0 0 1	1
0 1 0	1
0 1 1	0
1 0 0	1
1 0 1	0
1 1 0	0
1 1 1	1

$$fodd = \sum(1, 2, 4, 7)$$

$$fodd = x'y'z + x'yz' + xy'z' + xyz$$

$$fodd = x \oplus y \oplus z$$



Implementation using two XOR gates

Even Function

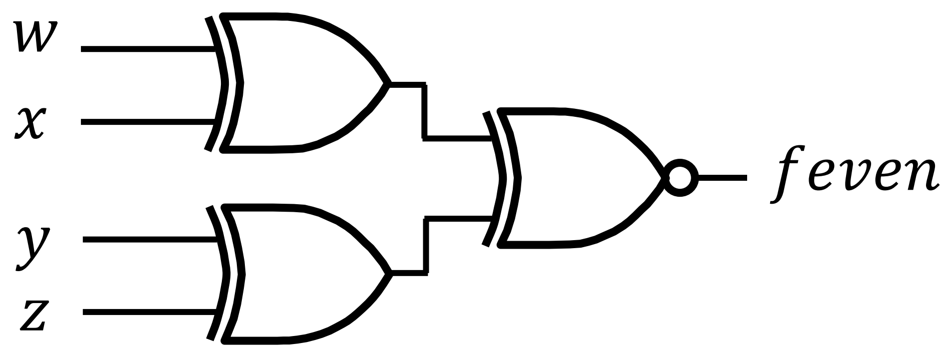
Even Function with 4 inputs

w x y z	feven
0 0 0 0	1
0 0 0 1	0
0 0 1 0	0
0 0 1 1	1
0 1 0 0	0
0 1 0 1	1
0 1 1 0	1
0 1 1 1	0
1 0 0 0	0
1 0 0 1	1
1 0 1 0	1
1 0 1 1	0
1 1 0 0	1
1 1 0 1	0
1 1 1 0	0
1 1 1 1	1

- Output is 1 if the **number of 1's is even** in the inputs (complement of odd function)
- Output is the XNOR operation on all inputs

$$feven = \sum(0, 3, 5, 6, 9, 10, 12, 15)$$

$$feven = (w \oplus x \oplus y \oplus z)'$$



Implementation using two XOR gates and one XNOR