Lecture 3

9/9/2020

Def 2.3.2

A set of vectors $\{V_0, V_1, \dots, V_{n-1}\}$ is linearly independent if

$$c_0 V_0 + c_1 V_1 + \dots + c_{n-1} V_{n-1} = 0$$

implies that $c_0 = c_1 = c_2 = \dots = c_{n-1} = 0$

Ex

a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly ind.

$$x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$x = y = z = 0$$

b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$

$x = -2, \ 2$

$z = 1, \ -1$

$$x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = 0 \quad y = -3, \ -3$$

Def 2.3.3

A set $\mathcal{B} = \{V_0, V_1, \ldots, V_{n-1}\}$ is called a Basis of a set vector space $\mathbb{V}$ if

1. every $V \in \mathbb{V}$ can be written combination of $\mathcal{B}$
2. $\mathcal{B}$ is linearly independent

Ex:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$ a basis of $\mathbb{C}^3$ ($\mathbb{R}^3$)

* A canonical space (standard basis)
  - in $\mathbb{R}^3$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$
  - in $\mathbb{C}^3$ (In general) $E_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $E_1 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ... $E_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$

Every vector $[c_0, c_1, \ldots, c_{n-1}]$ can be written

$$\sum_{j=0}^{n-1} (c_j \cdot E_j)$$

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Ex

In $\mathbb{R}^3$ has dimension of 3

In $\mathbb{C}^n$ has dimension of $\underline{n}$ (complex number)

* Moving from one basis to another

Ex: Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix} \right\}$ be the basis $\mathbb{R}^2$

$V = \begin{bmatrix} 7 \\ -17 \end{bmatrix}$ can be written as

$$\begin{bmatrix} 7 \\ -17 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -3 \end{bmatrix} - 2 \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$V_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

If e is Caninocal basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$V_e = \begin{bmatrix} 7 \\ -17 \end{bmatrix}$ $\quad V = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Consider the basis $\mathcal{D} = \left\{ \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \end{bmatrix} \right\}$

what is $V_{\mathcal{D}}$?

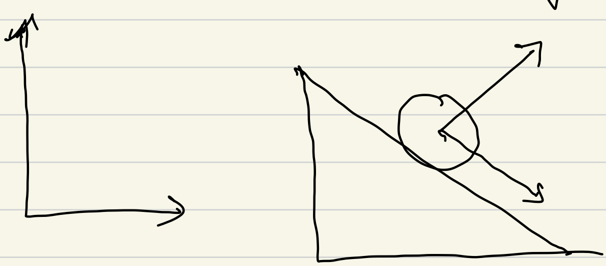
$$V_{\mathcal{D}} = M_{\underline{\mathcal{D}} \leftarrow \underline{\mathcal{B}}} V_{\mathcal{B}}$$

$$\hookrightarrow \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}$$

$$V_{\mathcal{D}} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ -14 \end{bmatrix}$$

$$9 \begin{bmatrix} -7 \\ 9 \end{bmatrix} - 14 \begin{bmatrix} -5 \\ 7 \end{bmatrix} = \begin{bmatrix} \overset{7}{-63 + 70} \\ 81 - 98 \\ -17 \end{bmatrix}$$

$$\neq V$$



* Hadamard Matrix

$$H = 1/\sqrt{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

It takes the canonical basis

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$$

Def 2.4.4

Inner product is a function

$$\langle \_, \_ \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$$

that satisfy

(i) non-degenerate :

$$\langle V, V \rangle \geq 0, \langle V, V \rangle = 0 \text{ if } V = 0$$

(ii) $\langle V_1 + V_2, V_3 \rangle = \langle V_1, V_3 \rangle + \langle V_2, V_3 \rangle \quad (\langle V_1, V_2 + V_3 \rangle)$

(iii) $\langle c.V_1, V_2 \rangle = c. \langle V_1, V_2 \rangle \quad (\langle V_1, c.V_2 \rangle = \bar{c} \langle V_1, V_2 \rangle$

(iv)

$$\langle V_1, V_2 \rangle = \overline{\langle V_2, V_1 \rangle}$$

$$\langle V_1, V_2 \rangle = V_1^T V_2 = \sum_{j=0}^{n-1} V_1^T[j] V_2[j]$$

Ex

$$\left\langle \begin{bmatrix} 5 \\ 3 \\ -7 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} \right\rangle = \begin{bmatrix} 5 & 3 & -7 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix}$$

$$= 36$$

Def 2.4.3

Norm of a vector V is defined as

$$|V| = \sqrt{\langle V, V \rangle}$$

Ex

$$V = \begin{bmatrix} -3 \\ -6 \\ 2 \end{bmatrix} \qquad |V| = \sqrt{\langle V, V \rangle}$$

$$= \sqrt{49} = 7$$

* In general, $V = [x \ y \ z]$

$$|V| = \sqrt{x^2 + y^2 + z^2}$$

* A norm has the following prop.

① $|V| > 0$ if $V \neq 0$ (if $|0| = 0$)

② $|V + W| \leq |V| + |W|$

③ $|c . V| = |c| \times |V|$

Def 2.4.4

Distance function between two vectors

$$d(\_,\_) : V \times V \to \mathbb{R}$$

$$d(V_1, V_2) = |V_1 - V_2| = \sqrt{\langle V_1 - V_2, V_1 - V_2 \rangle}$$

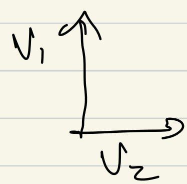
Ex

$$V_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \quad d(V_1, V_2)$$

$$V_1 - V_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \quad d(V_1, V_2) = \sqrt{11}$$

Def

Two vectors V_1 & V_2 are orthogonal if $\langle V_1, V_2 \rangle = 0$



Def 2.4.6

A basis $\mathcal{B} = \{V_0, V_1, \dots, V_{n-1}\}$ is orthogonal if vectors are pairwise orthogonal each other

$(\langle V_j, V_k \rangle = 0 \ \forall j \neq k)$.

(Cont)

An orthonormal basis if every vector is the basis is of norm 1

$$\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right)$$

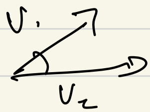
$$\langle v_j, v_k\rangle = \delta_{jk} = \begin{cases}1 & \text{if } j=k\\ 0 & \text{if } j\neq k\end{cases}$$

Kronecker's fun

delta

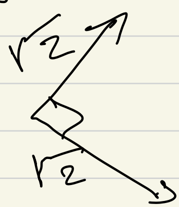
Ex

a) $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}$ $\quad \left\langle \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix} \right\rangle = 1$

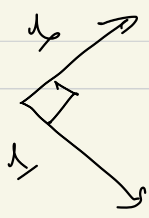


b) $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$ $\quad \left\langle \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \right\rangle = 0$

$\left| \begin{bmatrix}1\\1\end{bmatrix} \right| = \sqrt{2}$



c) $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$



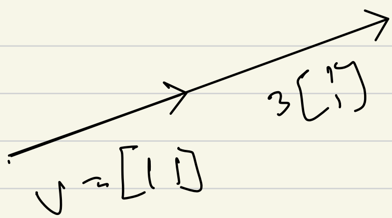
* Eigen Value & Vector

Ex:

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$$

a) $$\underbrace{\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{V} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \underbrace{3}_{c} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{V}$$

$$AV = c.V$$



b) $$\underbrace{\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{V} \underset{=}{=} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \underbrace{2}_{c} . \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{V}$$

* A matrix $A \in \mathbb{R}^{n \times n}$ is symmetric

$$A = A^T \quad \text{i.e. } (A[j,k] = A[k,j])$$

<u>Def 2.6.1</u>

An n-by-n matrix A is called Hermitian if $A^{\dagger} = A$ $\left(A[j,k] = \overline{A[k,j]}\right)$

<u>Def 2.6.2</u>

If A is Hermitian then the operator that it represents is called <u>self-adjoint</u>

<u>Ex:</u>

$$A = \begin{bmatrix} 5 & 4+5i & 6-16i \\ 4-5i & 13 & 7 \\ 6+16i & 7 & -2.1 \end{bmatrix}$$

Prop 2.6.2

If A is hermitian, then all eiganvalues are real

Prop 2.6.3

For a given hermition matrix, distinct eigan vector that have distinct eiganvalue are orthogonal

Def 2.6.3

A diagonal matrix is a square matrix whose only non-Zero elements are on the diagonal

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 1 \end{bmatrix}$$

(Prop 2.6.4) (Spectral theorem)

- Every self-adjoint matrix can be represented ~~as~~ by a diagonal matrix with entries are the eigenvalues of A and eigenvectors form an orthonormal basis of V.

- A matrix is invertible if there exist. a matrix A^{-1} s.t. $AA^{-1} = A^{-1}A = I_n$