Lecture 3

Review: Complex Vector Space

9/7/2020

* Conjugation of Complex number

if $c = a + bi$, then conjugate of c

$$\bar{c} = a - bi$$

* Conj. Properties

$$\bar{c}_1 + \bar{c}_2 = \overline{c_1 + c_2}$$

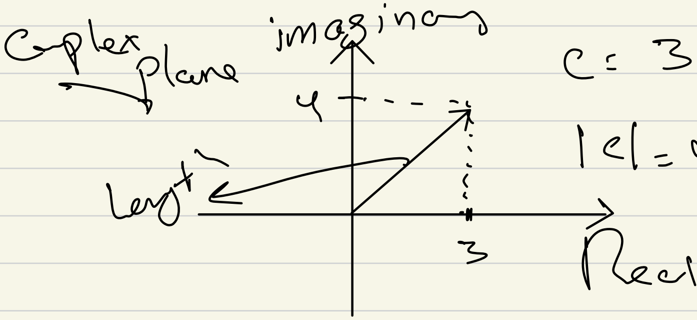
$$\bar{c}_1 \times \bar{c}_2 = \overline{c_1 \times c_2}$$

* $c \times \bar{c} = |c|^2$

<u>Ex 1</u>

$$(3+2i)(3-2i) = 9 + 4 = 13$$

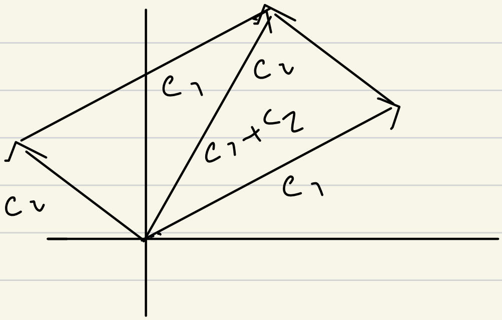
* Geometry of Complex numbers



$c = 3 + 4i$

$|c| = \sqrt{a^2 + b^2} = 5$

* Add. two complex numbers is geometr-
"parallelogram"



* $C=(a,b)$ is call "cartesian" repr..
* The "Polar" rep.

$$\underbrace{(a,b)}\longmapsto(\underbrace{\rho}_{\text{modulus}},\ \underbrace{\theta}_{\text{angle}})$$

$$\rho=\sqrt{a^2+b^2}\quad,\ \theta=\tan^{-1}\left(\frac{b}{a}\right)$$

$$a=\rho\cos\theta\quad,\ b=\rho\sin(\theta)$$

Ex 2

Let $c = 1 + i$, what is Polar rep.?

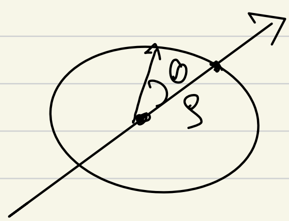
$$\rho = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \pi/4$$

$$c = (\sqrt{2}, \pi/4)$$

$$a = \sqrt{2}\cos(\pi/4) = \sqrt{2}\ 1/\sqrt{2} = 1$$
$$b = \sqrt{2}\sin(\pi/4) = 1$$

*



* $(\rho_1, \theta_1)(\rho_2, \theta_2) = (\rho_1\rho_2, \theta_1 + \theta_2)$

Complex Vector space $\mathbb{C}^n$

$\mathbb{C}^n$ is a vector space $\mathbb{C}^n = \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \dots \times \mathbb{C}$

complex number

Ex 3

$$\mathbb{C}^4 = \begin{bmatrix} 6-4i \\ 7+3i \\ 4.2+8.1i \\ -3i \end{bmatrix} = V$$

$V[i]$ ith element of V $i \in 0, 1, 2, 3$

$V[0] = 6-4i$, $V[3] = -3i$

* Addition

$$(V+W)[j] = V[j] + W[j]$$

Ex 4

$$V \quad , W = \begin{bmatrix} 16+2i \\ -7i \\ 6 \\ -4i \end{bmatrix} \quad V+W = \begin{bmatrix} 22-2i \\ 7-4i \\ 10.2+8.1i \\ -7i \end{bmatrix}$$

* Associativeness of addition

$$(V+W)+X = V+(W+X)$$

* $$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$0+V=V$$

* $\forall V \in \mathbb{C}^n$, there is a vector $-V \in \mathbb{C}^n$

$$V-V=0$$

$$V = \begin{bmatrix} 6+3i \\ 0 \\ 5+i \\ 4 \end{bmatrix} \qquad c = 3+2i$$

$$c.V = (3+2i)\begin{bmatrix} 6+3i \\ 0 \\ 5+i \\ 4 \end{bmatrix} = \begin{bmatrix} 12+21i \\ 0 \\ 13+13i \\ 12+8i \end{bmatrix}$$

$$(a_1,b_1)(a_2,b_2) = (a_1a_2-b_1b_2,\ a_1b_2+a_2b_1)$$

* Scalar mult. Prop.

1) $1 \cdot V = V$
2) $c_1 \cdot (c_2 \cdot V) = (c_1 \times c_2) \cdot V$
3) $c(V + W) = cV + cW$
4) $(c_1 + c_2)V = c_1 V + c_2 V$

* Abelian group with add., invers, and zero

* Abelian group + Scalar mult. is called <u>Complex Vector space</u>

Def 2.2.1 page 34

- A matrix $A \in \mathbb{C}^{m \times n}$ has complex entry $A[j,k]$ (a_{jk})

$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0\,n-1} \\ a_{10} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m-1\,0} & \cdots & \cdots & a_{m-1\,n-1} \end{bmatrix}$$

* Addition of matrices

$$(A+B)^{m \times n}[j,k] = A[j,k] + B[j,k]$$

* $(-A)[j,k] = -(A[j,k])$
* $(c.A)[j,k] = c \times (A[j,k])$

$\quad c = -1$

$\mathbb{C}^{m \times n}$: $m = 1$. $\mathbb{C}^{1 \times n} \to$ row vector (Bra)

$n = 1$ $\mathbb{C}^{m \times 1} \to$ col. vector (Ket)

* Transpose of A is A^T

$$A^T[j,k] = A[k,j]$$

* Conjugate of A

$$\overline{A}[j,k] = \overline{A[j,k]}$$

$A[j,k]$

a_{jk}

dagger

* Adjoint matrix $A^{\dagger} = (\overline{A})^T = \overline{(A)^T}$

$$A^{\dagger}[j,k] = \overline{A[k,j]}$$

Ex 5: let $A = \begin{bmatrix} 6-3i & 2+12i & -19i \\ 0 & 5+2i & 17 \\ 1 & 2+5i & 3-4.5i \end{bmatrix}$

$A^T = \begin{bmatrix} 6-3i & 0 & 1 \\ 2+12i & 5+2.1i & 2+5i \\ -19i & 17 & 3-4.5i \end{bmatrix}$, $\overline{A} = \begin{bmatrix} 6+3i & 2-12i & 19i \\ 0 & 5-2i & 17 \\ 1 & 2-5i & 3+4.5i \end{bmatrix}$

$A^{\dagger}$?

* Some Prop.

A) Idempotency

$$(A^T)^T = A \,,\ \overline{(\overline{A})} = A \,,\ (A^\dagger)^\dagger = A$$

B) Addition

$$(A+B)^T = A^T + B^T$$

$$\overline{(A+B)} = \overline{A} + \overline{B}$$

$$(A+B)^\dagger = A^\dagger + B^\dagger$$

c) Scalar mult

$$(c \cdot A)^T = c \cdot A^T$$

$$\overline{(c \cdot A)} = \overline{c}\,\overline{A}$$

$$(c \cdot A)^\dagger = \overline{c}\,A^\dagger$$

Matrix multiplication

$$\mathbb{C}^{m\times n} \times \mathbb{C}^{n\times p} \longrightarrow \mathbb{C}^{m\times p}$$

$$(A*B)[j,k] = \sum_{h=0}^{n-1} (A[j,h] \times B[h,k])$$

Def: Identity $I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \vdots & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & & 1 \end{bmatrix}$

Prop:

1) $I_n A = A = A I_n$
2) $(A*B)*C = A*(B*C)$
3) $A*(B+C) = A*B + A*C$
4) $(B+C)*A = B*A + C*A$
5) $(A*B)^T = B^T A^T$

   $(A*B)^+ = B^+ A^+$

$$\overline{(AB)} = \overline{A} * \overline{B}$$

* $A \in \mathbb{C}^{m\times n}$ can be though of as an operator that acts on vector $b \in \mathbb{C}^n$. This yields $c \in \mathbb{C}^n$ (i.e, $c = Ab$)

* $\underbrace{y}_{\text{new State}} = \underbrace{A}_{\text{operator}} x \rightarrow$ current State

<u>Def</u> $V, \hat{V} \in \mathbb{C}^n$. A linear map from V to $\hat{V}$ is a function $f: V \rightarrow \hat{V}$ s.t.

i) $f(V + \hat{V}) = f(V) + f(\hat{V})$

ii) $f(c.V) = c.f(V)$

Def 2.3.1

Let $\mathbb{V}$ be a complex vector space, $V \in \mathbb{V}$ is a linear combination of $V_0, V_1, \ldots, V_{n-1}$ if it can be written as

$$V = C_0 \cdot V_0 + C_1 V_1 + \ldots + C_{n-1} V_{n-1}$$

for some $C_0, C_1, \ldots, C_{n-1} \in \mathbb{C}$

Ex 6

$$\begin{array}{cccccccc} & 1 & & 2 & & 3 & & 4 \\ 3 & \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} & +5 & \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} & -4 & \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} & +2 & \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \end{array}$$

a linear combination $\begin{bmatrix} 45.3 \\ -2.9 \\ 31.1 \end{bmatrix} = V$

Def:

A set $\{V_0, V_1, \ldots, V_{n-1}\}$ is linearly independent if $C_0 V_0 + C_1 V_1 + \ldots + C_{n-1} V_{n-1} = 0$ implies $C_0 = C_1 = C_2 = \ldots = C_{n-1} = 0$