COE 202: Digital Logic Design
Number Systems
Part 1

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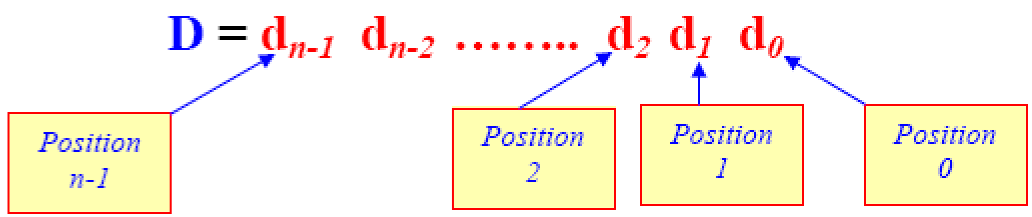
Objectives

1. Weighted (positional) number systems
2. Features of weighted number systems.
3. Commonly used number systems
4. Important properties

Introduction

- A ***number system*** is a set of ***numbers*** together with one or more ***operations*** (e.g. add, subtract).
- Before digital computers, the only known number system is the ***decimal number system*** (النظام العشري)
  - It has a total of ten digits: {0,1,2,….,9}
- From the previous lecture:
  - Digital systems deal with the binary system of numbering i.e. only 0’s and 1’s
  - Binary system has more reliability than decimal
- All these numbering systems are also referred to as ***weighted numbering systems***

Weighted Number System



- A number **D** consists of *n* digits and each digit has a *position.*
- Every digit *position* is associated with a *fixed weight*.
- If the weight associated with the *i*th. position is w_i, then the value of **D** is given by:

$$D = d_{n-1}\ w_{n-1} + d_{n-2}\ w_{n-2} + \ldots\ldots + d_1 w_1 + d_0 w_0$$

- Also called ***positional number system***

Example

First Position Index

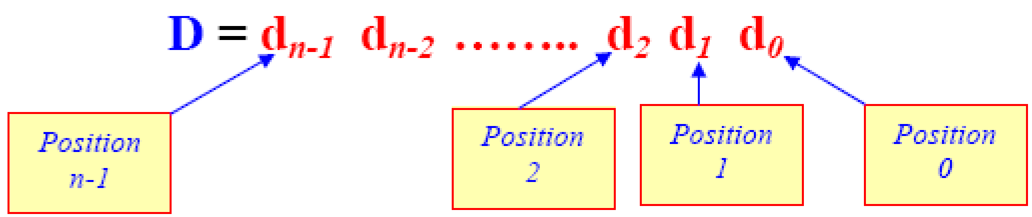
Position	3	2	1	0
Number	9	3	7	5
Weight	1000	100	10	1
Value	9 x 1000	3x100	7x10	5x1
Value	9000 +	300 +	70 +	5

First Position Index (0)

9375

- The Decimal number system is a weighted number system.
- For Integer decimal numbers, the weight of the rightmost digit (*at position 0*) is **1**, the weight of *position 1* digit is **10**, that of *position 2* digit is **100**, *position 3* is **1000**, etc.

The Radix (Base)



- A digit d_i, has a weight which is a power of some constant value called **radix (*r*)** or **base** such that $w_i = r^i$.
- A number system of radix r, has r allowed digits $\{0,1,\ldots (r-1)\}$
- The leftmost digit has the highest weight and called **Most Significant Digit (MSD)**
- The rightmost digit has the lowest weight and called **Least Significant Digit (LSD)**

Example

- Decimal Number System
- Radix (base) = 10
- $w_i = r^i$, so
  - $w_0 = 10^0 = 1$,
  - $w_1 = 10^1 = 10$
  - .
  - $w_n = r^n$
- Only 10 allowed digits {0,1,2,3,4,5,6,7,8,9}

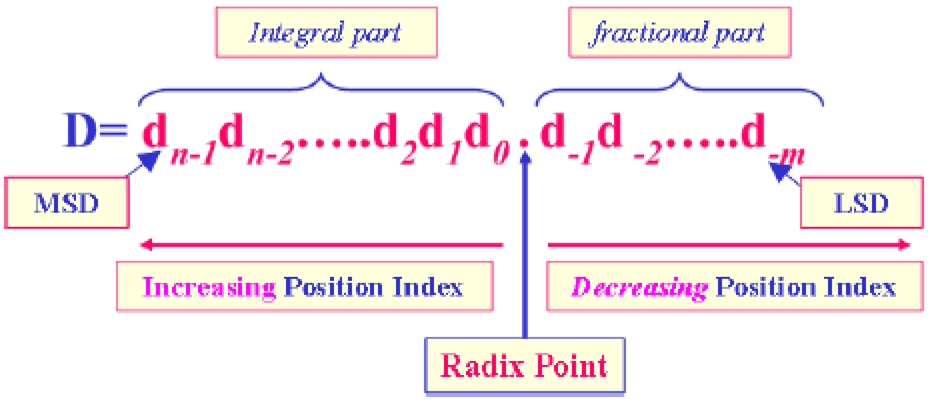
MSD LSD

$$9375 = 5x10^0 + 7x10^1 + 3x10^2 + 9x10^3$$

$$= 5x1 + 7x10 + 3x100 + 9x\,1000$$

Position	3	2	1	0
Weight	1000 $= 10^3$	100 $= 10^2$	10 $= 10^1$	1 $= 10^0$

Fractions (Radix point)



$$D = \sum_{i=-m}^{n-1} d_i r^i$$

- A number D has *n* ***integral*** digits and *m* ***fractional*** digits
- Digits to the left of the radix point (*integral digits*) have ***positive*** position indices, while digits to the right of the radix point (*fractional digits*) have ***negative*** position indices
- The ***weight*** for a digit position *i* is given by $\mathbf{w}_i = \mathbf{r}^i$

Example

- For D = 57.6528
  - n = 2
  - m = 4
  - r = 10 (decimal number)
- The weighted representation for D is:

  i = -4 $\quad d_i r^i = 8 \times 10^{-4}$

  i = -3 $\quad d_i r^i = 2 \times 10^{-3}$

  i = -2 $\quad d_i r^i = 5 \times 10^{-2}$

  i = -1 $\quad d_i r^i = 6 \times 10^{-1}$

  i = 0 $\quad d_i r^i = 7 \times 10^{0}$

  i = 1 $\quad d_i r^i = 5 \times 10^{1}$

D = 5 2 . 9 4 6

d_1 d_0 . d_{-1} d_{-2} d_{-3}

Number	5	2	.	9	4	6
Position	1	0	.	-1	-2	-3
Weight	10^1 = 10	10^0 = 1	.	10^{-1} = 0.1	10^{-2} = 0.01	10^{-3} = 0.001
Value	5 x 10	2 x 1	.	9 x 0.1	4 x 0.01	6 x 0.001
Value	50 + 2 + 0.9 + ~~0.02~~ + 0.006					

$$D = 5\times10^{1} + 2\times10^{0} + 9\times10^{-1} + 4\times10^{-2} + 6\times10^{-3}$$

0.04

Notation

A number D with base *r* can be denoted as $(D)_r$,

Decimal number 128 can be written as $(128)_{10}$

Similarly a binary number is written as $(10011)_2$

Question: Are these valid numbers?

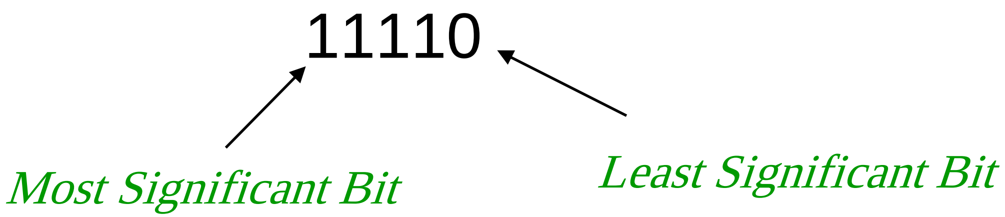
- $(9478)_{10}$
- $(1289)_2$
- $(111000)_2$
- $(55)_5$

Common Number Systems

- Decimal Number System (base-10)
- Binary Number System (base-2)
- Octal Number System (base-8)
- Hexadecimal Number System (base-16)

Binary Number System (base-2)

- r = 2
- Two allowed digits {0,1}
- A <u>B</u>inary Dig<u>it</u> is referred to as bit
- Examples: 1100111, 01, 0001, 11110
- The left most bit is called the *Most Significant Bit (MSB)*
- The rightmost bit is called the *Least Significant Bit (LSB)*



Binary Number System (base-2)

- The decimal equivalent of a binary number can be found by expanding the number into a power series:

Example

MSB LSB

- $(1\ 0\ 1)_2 = 1x2^0 + 0x2^1 + 1x2^2$
- $= 1x1 + 0x2 + 1x\ 4$
- $= (5)_{10}$

Question:

What is the decimal equivalent of $(110.11)_2$?

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- $= 1x1 + 0x2 + 1x\ 4$
- $= (5)_{10}$

Question:

What is the decimal equivalent of $(110.11)_2$?

Answer: **$(6.75)_{10}$**

Octal Number System (base-8)

- r = 8
- Eight allowed digits {0,1,2,3,4,5,6,7}
- Useful to represent binary numbers indirectly
  - Octal and binary are nicely related; i.e $8 = 2^3$
  - Each octal digit represent 3 binary digits (bits)
  - Example: $(101)_2 = (5)_8$
- Getting the decimal equivalent is as usual

MSD LSD

Example

$$(375)_8 = 5x8^0 + 7x8^1 + 3x8^2$$
$$= 5x1 + 7x8 + 3x64$$
$$= (253)_{10}$$

Hexadecimal Number System (base-16)

- r = 16
- 16 allowed digits {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}
- Useful to represent binary numbers indirectly
  - Hex and binary are nicely related; i.e $16 = 2^4$
  - Each hex digit represent 4 binary digits (bits)
  - Example: $(1010)_2 = (A)_{16}$
- Getting the decimal equivalent is as usual

Example

MSD LSD

$$(3B.C)_{16} = Cx16^{-1} + Bx16^{0} + 3x16^{1}$$
$$= 12x16^{-1} + 11x16^{0} + 3x16$$
$$= (59.75)_{10}$$

Question:
$(9E1)_{16} = (?)_{10}$

Hexadecimal Number System (base-16)

- r = 16
- 16 allowed digits {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}
- Useful to represent binary numbers indirectly
  - Hex and binary are nicely related; i.e $16 = 2^4$
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  - Example: $(1010)_2 = (A)_{16}$
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Example

MSD LSD

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$$= (59.75)_{10}$$

Question:
$(9E1)_{16} = (?)_{10}$

MSD LSD

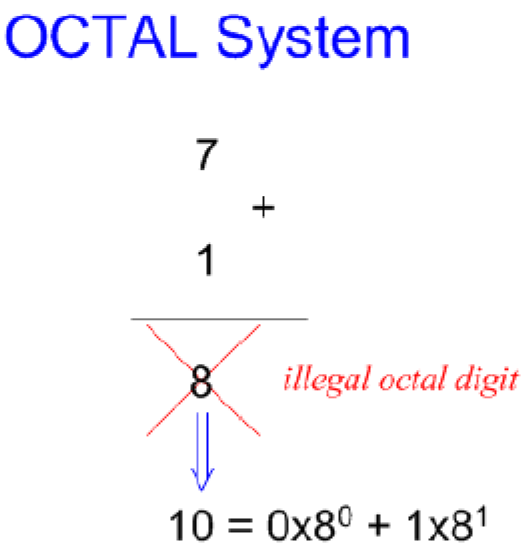
$$(9E1)_{16} = 1x16^{0} + Ex16^{1} + 9x16^{2}$$
$$= 1x1 + 14x16 + 9x256$$
$$= (2529)_{10}$$

Examples

Question: What is the result of adding 1 to the largest digit of some number system?

- $(9)_{10} + 1 = (10)_{10}$
- $(7)_8 + 1 = (10)_8$
- $(1)_2 + 1 = (10)_2$
- $(F)_{16} + 1 = (10)_{16}$

Conclusion: Adding 1 to the largest digit in any number system always has a result of (10) in that number system.



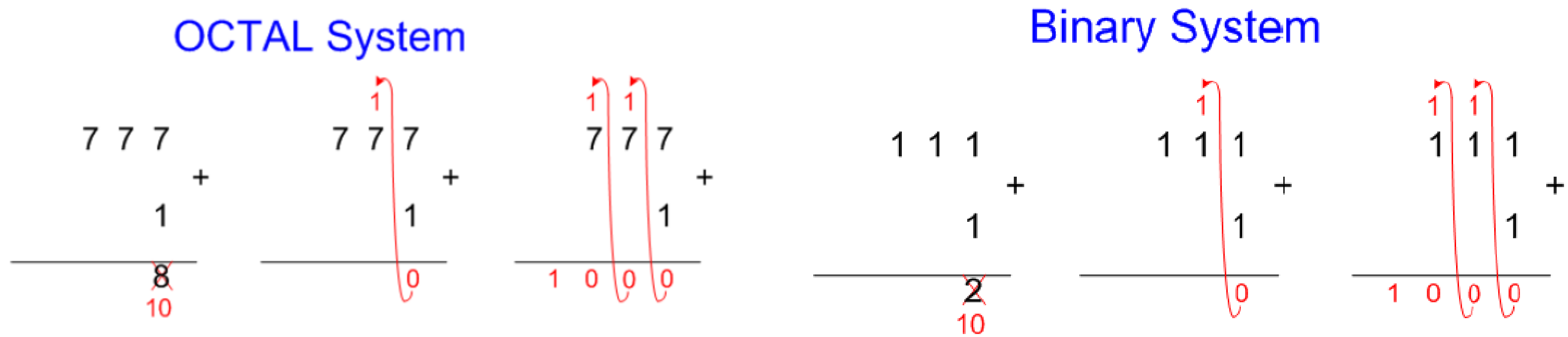
Examples

Question: What is the largest value representable using 3 integral digits?

Answer: The largest value results when all 3 positions are filled with the largest digit in the number system.

- **For** the decimal system, it is $(999)_{10}$
- **For** the octal system, it is $(777)_8$
- **For** the hex system, it is $(FFF)_{16}$
- **For** the binary system, it is $(111)_2$

Examples



Question: What is the result of adding 1 to the largest 3-digit number?

- For the decimal system, $(1)_{10} + (999)_{10} = (1000)_{10} = (10^3)_{10}$
- For the octal system, $(1)_8 + (777)_8 = (1000)_8 = (8^3)_{10}$

In general, for a number system of radix **r**, adding 1 to the largest *n*-digit number = $\mathbf{r}^n$

Accordingly, the value of largest *n*-digit number = $\mathbf{r}^n - 1$

Important Properties

- The number of possible digits in any number system with radix ***r*** equals ***r***.
- The smallest digit is ***0*** and the largest digit has a value **(*r* - *1*)**
  - Example: Octal system, ***r = 8***, smallest digit = ***0***, largest digit = ***8 – 1 = 7***
- The Largest value that can be expressed in **n** integral digits is $(r^n - 1)$
  - Example: ***n = 3, r = 10***, largest value = $10^3 - 1 = 999$

Important Properties

- The Largest value that can be expressed in m fractional digits is $(1 - r^{-m})$
  - Example: $n=3, r = 10$, largest value = $1\text{-}10^{-3} = 0.999$
- Largest value that can be expressed in n integral digits and m fractional digits is equal to $(r^{n} – r^{-m})$
- Total number of values (patterns) representable in n digits is r^{n}
  - Example: $r = 2, n = 5$ will generate ***32*** possible unique combinations of binary digits such as ***(00000 ->11111)***
  - Question: What about Intel 32-bit & 64-bit processors?

Conclusions

- A weighted (positional) number system has a radix (base) and each digit has a position and weight
- Commonly used number systems are decimal, binary, octal, hexadecimal
- A number D with base *r* can be denoted as $(D)_r$,
- To convert from base-r to decimal, use

$$(D)_r = \sum_{i=-m}^{n-1} d_i r^i$$

- Weighted (positional) number systems have several important properties