COE 202- Digital Logic

Number Systems I

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Objectives

- Introduction
- Weighted (positional) number systems
- Features of weighted number systems
- Commonly used number systems
- Important properties

Introduction

- Computers only deal with binary data (0s and 1s), hence all data manipulated by computers must be represented in binary format.
- Machine instructions manipulate many different forms of data:
  - Numbers:
    - Integers: 33, +128, -2827
    - Real numbers: 1.33, +9.55609, -6.76E12, +4.33E-03
  - Alphanumeric characters (letters, numbers, signs, control characters): examples: A, a, c, 1 ,3, ", +, Ctrl, Shift, etc.
  - Images (still or moving): Usually represented by numbers representing the Red, Green and Blue (RGB) colors of each pixel in an image,
  - Sounds: Numbers representing sound amplitudes sampled at a certain rate (usually 20kHz).
- So in general we have two major data types that need to be represented in computers; numbers and characters.

Numbering Systems

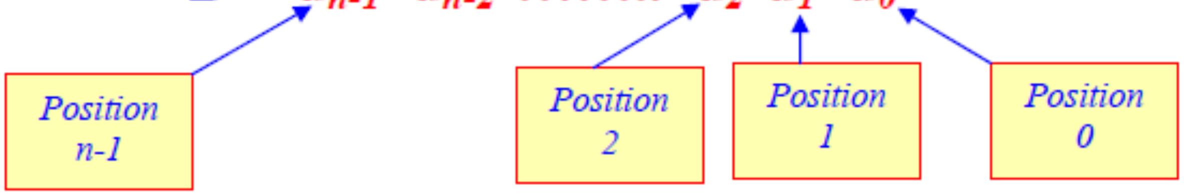
- Numbering systems are characterized by their base number.
- In general a numbering system with a base r will have r different digits (including the 0) in its number set. These digits will range from 0 to r-1.
- The most widely used numbering systems are listed in the table below:

Numbering System	Base	Digits Set
Binary	2	1 0
Octal	8	7 6 5 4 3 2 1 0
Decimal	10	9 8 7 6 5 4 3 2 1 0
Hexadecimal	16	F E D C B A 9 8 7 6 5 4 3 2 1 0

Weighted Number System

- A number D consists of n digits with each digit having a particular position.

$$D = d_{n-1}\ d_{n-2}\ \ldots\ldots\ d_2\ d_1\ d_0$$



- Every digit *position* is associated with a *fixed weight*.

- If the weight associated with the *ith* position is w_i, then the value of D is given by:

$$D = d_{n-1}\, w_{n-1} + d_{n-2}\, w_{n-2} + \ldots + d_2\, w_2 + d_1\, w_1 + d_0\, w_0$$

- Also called ***positional number system***

Example

First Position Index

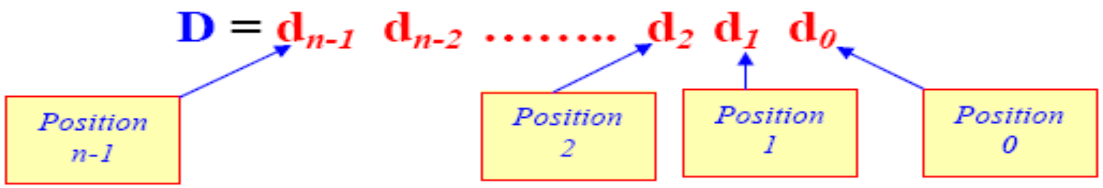
Position	3	2	1	0
Number	9	3	7	5
Weight	1000	100	10	1
Value	9 x 1000	3x100	7x10	5x1
Value	9000 + 300 + 70 + 5			

First Position Index (0)

9375

- The Decimal number system is a weighted number system.
- For Integer decimal numbers, the weight of the rightmost digit (at position 0) is 1, the weight of position 1 digit is 10, that of position 2 digit is 100, position 3 is 1000, etc.

The Radix (Base)



- A digit di, has a weight which is a power of some constant value
- called **radix (*r*)** or **base** such that $w_i = r^i$.
- A number system of radix *r*, has *r* allowed digits {0,1,... (r-1)}
- The leftmost digit has the highest weight and called **Most Significant Digit (MSD)**
- The rightmost digit has the lowest weight and called **Least Significant Digit (LSD)**

The Radix (Base)

- Example: Decimal Number System
- Radix (Base) = 10
- Since $w_i = r^i$, *then*
  - $w_0 = 10^0 = 1$,
  - $w_1 = 10^1 = 10$,
  - $w_2 = 10^2 = 100$,
  - $w_3 = 10^3 = 1000$, *etc.*
- Number of Allowed Digits is Ten:
  - {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

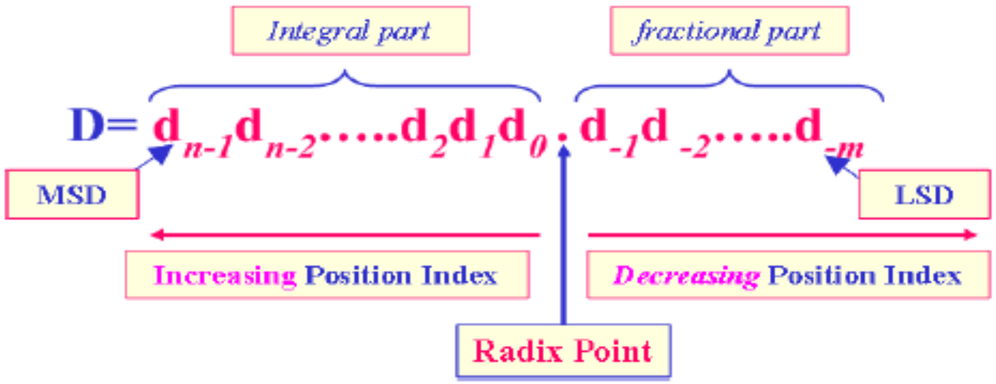
MSD LSD

$$9375 = 5x10^0 + 7x10^1 + 3x10^2 + 9x10^3$$
$$= 5x1 + 7x10 + 3x100 + 9x\ 1000$$

Position	3	2	1	0
Weight	1000 $= 10^3$	100 $= 10^2$	10 $= 10^1$	1 $= 10^0$

Fractions (Radix point)

- A number ***D*** of *n* integral digits and *m* fractional digits is represented as shown:



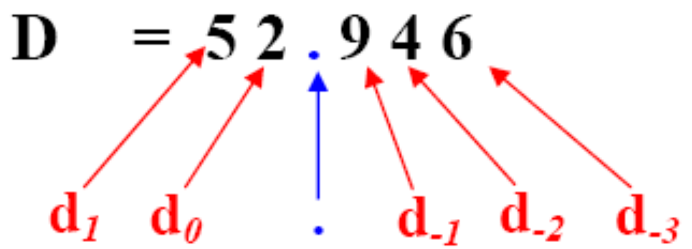
$$D = \sum_{i=-m}^{n-1} d_i \, r^i$$

- Digits to the left of the radix point (*integral digits*) have ***positive*** position indices, while digits to the right of the radix point (fractional digits) have ***negative*** position indices.

- The ***weight*** for a digit position *i* is given by $w_i = r^i$

Examples

- For D = 57.6528
  - $n = 2$
  - $m = 4$
  - $r = 10$ (decimal number)
- The weighted representation for D is:
  - i = -4 $d_i r^i = 8 \times 10^{-4}$
  - i = -3 $d_i r^i = 2 \times 10^{-3}$
  - i = -2 $d_i r^i = 5 \times 10^{-2}$
  - i = -1 $d_i r^i = 6 \times 10^{-1}$
  - i =0 $d_i r^i = 7 \times 10^{0}$
  - i =1 $d_i r^i = 5 \times 10^{1}$



Number	5	2	.	9	4	6
Position	1	0	.	-1	-2	-3
Weight	10^1 = 10	10^0 = 1	.	10^{-1} = 0.1	10^{-2} = 0.01	10^{-3} = 0.001
Value	5 x 10	2 x 1	.	9 x 0.1	4 x 0.01	6 x 0.001
Value	50 + 2 + 0.9 + ~~0.02~~ + 0.006					

0.04

$$D = 5\times10^{1} + 2\times10^{0} + 9\times10^{-1} + 4\times10^{-2} + 6\times10^{-3}$$

Notation

- A number D with base r can be denoted as $(D)_r$, Decimal number 128 can be written as $(128)_{10}$. Similarly a binary number is written as $(10011)_2$
- Question: Are these valid numbers?
  - $(9478)_{10}$
  - $(1289)_2$
  - $(111000)_2$
  - $(55)_5$

Common Number Systems

- Decimal Number System (base-10)
- Binary Number System (base-2)
- Octal Number System (base-8)
- Hexadecimal Number System (base-16)

Binary Number System (base-2)

- $r = 2$
- Two allowed digits {0,1}
- A Binary Digit is referred to as bit
- Examples: 1100111, 01, 0001, 11110
- The left most bit is called the Most Significant Bit (MSB)
- The rightmost bit is called the Least Significant Bit (LSB)

11110

Most Significant Bit *Least Significant Bit*

Binary Number System (base-2)

- The decimal equivalent of a binary number can be found by expanding the number into a power series:

- Examples: Find the decimal value of the two Binary numbers $(101)_2$ and $(1.101)_2$

MSB LSB

- $(1\ 0\ 1)_2 = 1x2^0 + 0x2^1 + 1x2^2$
- $= 1x1 + 0x2 + 1x\ 4$
- $= (5)_{10}$

- Question: What is the decimal equivalent of (110.11)2 ?

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- $= 1x1 + 0x2 + 1x\ 4$
- $= (5)_{10}$

- Question: What is the decimal equivalent of (110.11)2 ?
  - Answer: $(6.75)_{10}$

Octal Number System (base-8)

- *r* = 8
- Eight allowed digits {0,1,2,3,4,5,6,7}
- Useful to represent binary numbers indirectly
- Octal and binary are nicely related; i.e. 8 = 23
- Each octal digit represent 3 binary digits (bits)
- Example: $(101)_2 = (5)_8$
- Getting the decimal equivalent is as usual

Octal Number System (base-8)

- **Examples**

MSD LSD

$$(375)_8 = 5x8^0 + 7x8^1 + 3x8^2$$
$$= 5x1 + 7x8 + 3x64$$
$$= (253)_{10}$$

MSD LSD

$$(2.746)_8 = 2x8^0 + 7x8^{-1} + 4x8^{-2} + 6x8^{-3}$$
$$= (2.94921875)_{10}$$

Hexadecimal Number System (base-16)

- *r* = 16
- 16 allowed digits {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}
- Useful to represent binary numbers indirectly
- Hex and binary are nicely related; i.e. 16 = 24
- Each hex digit represent 4 binary digits (bits)
- Example: $(1010)_2 = (A)_{16}$
- Getting the decimal equivalent is as usual

Hexadecimal Number System (base-16)

Examples

MSD LSD

$$(9E1)_{16} = 1x16^0 + Ex16^1 + 9x16^2$$

$$= 1x1 + 14x16 + 9x256$$

$$= (2529)_{10}$$

MSD LSD

$$(3B.C)_{16} = Cx16^{-1} + Bx16^0 + 3x16^1$$

$$= 12x16^{-1} + 11x16^0 + 3x16$$

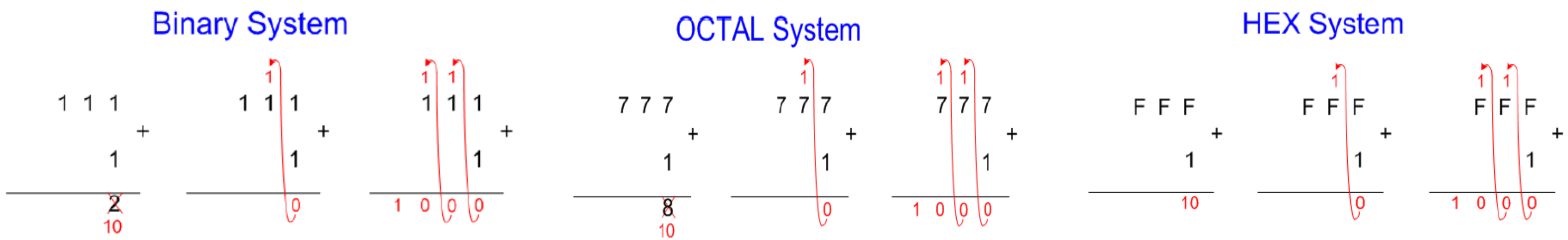
$$= (59.75)_{10}$$

Important Properties

- The number of possible digits in any number system with radix ***r*** equals ***r***.
- The smallest digit is ***0*** and the largest digit has a value ***(r - 1)***
  - Example: Octal system, r = 8, smallest digit = 0, largest digit = 8 – 1 = 7
- The Largest value that can be expressed in **n** integral digits is ***$(r^n - 1)$***
  - Example: n = 3, r = 10, largest value = $10^3 - 1 = (999)_{10}$
  - Example: n = 3, r = 8, largest value = $8^3 - 1 = (777)_8$
  - Example: n = 3, r = 16, largest value = $16^3 - 1 = (FFF)_{16}$
  - Example: n = 3, r = 2, largest value = $2^3 - 1 = (111)_2$

Important Properties

- **Question:** What is the result of adding 1 to the largest 3-digit number?
  - $(9)_{10} + 1 = (10)_{10}$
  - $(7)_8 + 1 = (10)_8$
  - $(1)_2 + 1 = (10)_2$
  - $(F)_{16} + 1 = (10)_{16}$



- In general, for a number system of radix r, adding 1 to the largest *n-digit* number = r^n.

Important Properties

- The Largest value that can be expressed in ***m*** fractional digits is ($1 - r^{-m}$)
  - Example: n=3, r = 10, largest value = $1\text{-}10^{-3}$ = 0.999
- Largest value that can be expressed in ***n*** integral digits and ***m*** fractional digits is equal to ($r^{n} - r^{-m}$)
- Total number of values (patterns) representable in ***n*** digits is r^n
  - Example: r = 2, n = 5 will generate 32 possible unique combinations of binary digits such as (00000 ->11111)
  - Question: What about Intel 32-bit & 64-bit processors?

Conclusions

- A weighted (positional) number system has a radix (base) and each digit has a position and weight
- Commonly used number systems are decimal, binary, octal, hexadecimal
- A number D with base r can be denoted as (D)r,
- To convert from base-r to decimal, use: $$D = \sum_{i=-m}^{n-1} d_i r^i$$
- Weighted (positional) number systems have several important properties