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CSE 642 - Computer Systems
Performance
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## Markov Chains: Applications to Multiplexing And Access

- Chapter 5 in Hayes's Textbook
- Plan:
- Time-Division Multiplexing
- Arrival Process
- Asynchronous Time-Division Multiplexing
- Synchronous Time-Division Multiplexing
- Random Access Techniques (ALOHA)
- In this chapter the Markov chain is used to model techniques for multiplexing and access in telecommunications networks


## Time-Division Multiplexing (TDM)

- Example - T1 Line
- Rate: $\mathbf{1 5 . 4 4} \mathbf{~ M b} / \mathrm{s}$
- Frame length = 125 micro second
- 24 channels with one framing bit
- Traffic flow is segmented into fixed-length slots (payloads)
- Markov chain is used to model the sequences formed by the information units (cells, bytes, packet, payloads, etc.) in the system at the slot boundaries
- Analysis of TDM and TimeDivision Multiple Access (TDMA) are identical.



## The Arrival Process

- The output of a traffic source is segmented into fixed-size units (cells, packets, etc.) Packetization
- E.g. ATM cell $=48 B$ data + 5B header
- In analysis, packetization is equivalent to transformation of random variables.


## The Arrival Process - cont'd

- Example: Message of i bits segmented into I-bit packets
- Let $\mathbf{B}(\mathbf{i})=$ Prob[ message $=\mathbf{i}$ bits]
- Then the probability distribution of number of packets in a message is given by

$$
M(k)=P(k \text { packets in a message })=\sum_{i=(k-1) I+1}^{k I} B(i)
$$

- Stuff bits are used to round messages into an integral number of packets
- For 0 overhead bit in each packet


## The Arrival Process - cont'd

- For 0 overhead bit in each packet, the total number of bits in a packetized message has the distribution

$$
P(k) \doteq B(k(I+O))
$$

## The Arrival Process - cont'd

- Example 5.1
- Suppose the number of bits in a message is the sum of a constant number of overhead bits $\mathbf{O}$ plus a geometrically distributed components with distribution

$$
B(i)=B^{i-1}(1-B) ; \quad i=1,2, \cdots
$$

Let the number of information bits in a packet be $I$ and assume $I>0$. What is the distribution of the number of packets in a message

## The Arrival Process - cont'd

- Example 5.1-Solution
- Let the distribution of number of packets, $k$, in a message be denoted by $\mathbf{P ( k )}$
- For $\mathbf{k}=1$

$$
\begin{aligned}
P(\text { one packet in a message }) & =P(M \leq I-O) \\
& =\sum_{j=1}^{I-O} B^{j-1}(1-B) \\
& =1-B^{I-O}
\end{aligned}
$$

## The Arrival Process - cont'd

- Example 5.1 - Solution - cont'd
- For k > 1

$$
\begin{aligned}
P(k \text { packets in a message } ; k>1) & =\sum_{i=I-O+1+(k-2) I}^{I-O+(k-1) I} B(i) \\
& =\sum_{j=0}^{I-1} B^{j+(k-1) I-O}(1-B) \\
& =B^{(k-1) I-O}\left(1-B^{I}\right)
\end{aligned}
$$

The Arrival Process - cont'd

- Example 5.1 - Solution - cont'd
- It can be shown that
$M(z)=\Sigma z^{\wedge} k$ prob(k packets) can be given by

$$
M(z)=z\left(1-B^{I-O}\right)+\frac{z B^{I-O}\left(1-B^{I}\right)}{1-z B^{I}}
$$

- For $\mathbf{0}=\mathbf{0}$ (i.e. no overhead header), M(z) reduces to

$$
M(z)=\frac{z\left(1-B^{I}\right)}{1-z B^{I}}
$$

- This mean the number of packets is geometrically distributed with mean $1 /\left(1-B^{1}\right)$


## Compound Arrivals

- Messages arrive to a slot containing a RANDOM number of packets
- The arrival pattern itself may be also RANDOM
- Example: Assume $\mathbf{n}$ sources are connected to a multiplexer
- Each source generates a message in a slot with probability P (i.e. Bernoulli arrivals)
- Total number of messages in a slot follows a Binomial distribution with parameters n and P.
- When $\mathbf{n}$ is large, the limiting distribution is Poisson with average equal to $\boldsymbol{\lambda}=\mathbf{n P}$.


## Compound Arrivals - Generalized

 Case- Suppose the number of message arrivals in a slot has an arbitrary distribution with probabilities a1, a2, ....
- The corresponding PGF for number of arrived MESSAGES, $A_{s}(z)$, is given by $\Sigma z^{k} a_{k}$.
- Conditioning on $j$ messages arrive in a time slot, the corresponding PGF for number of arrived PACKETS is denoted by $\mathrm{M}^{\mathrm{j}}(\mathrm{z})$
- Therefore, the PGF for the PACKET arrival process should be given by

$$
A(z)=\sum_{k=0}^{\infty} a_{k} M^{k}(z)=A_{s}(M(z))
$$

## Compound Arrivals - Generalized

## Case

- Let $\rho$ be average packet arrival rate.
- Using the PGF A(z) we have

$$
\rho=\bar{A}=\left.\frac{d A_{s}(M(z))}{d z}\right|_{z=1}=A_{s}^{\prime}(M(1)) M^{\prime}(1)=\bar{A}_{s} \bar{M}
$$

where $\bar{A}_{s}$ is the average number of messages arriving in a slot and $\bar{M}$ is the average number of packets per slot.

## Compound Arrivals - Example

- Example 5.2 - Suppose that the arrival process is Poisson with an average of $\lambda$ messages per time slot, and the messages have two lengths: $P$ (one packet) $=0.3$ and $P($ four packets $)=0.7$.

1. Compute the PGF for number of arriving packets per slot
2. What is the mean number of arriving packets.

## Compound Arrivals - Example

- Solution:
- The PGF is given by
$A(z)=\sum_{k=0}^{\infty} \frac{(\lambda T)^{k} e^{-\lambda T}}{k!}\left(0.3 z+0.7 z^{4}\right)^{k}=e^{-\lambda T\left(1-0.3 z+0.7 z^{4}\right)}$
- You can use a tool such as Matlab or Maple to evaluate the PMF and CDF. Figure 5.2 in textbook page 190 shows the CDF for different values of $\boldsymbol{\lambda}$.


## Asynchronous Time-Division Multiplexing (ATDM)

- For bursty sources
- No capacity is dedicated for the sources
- In contrast, for Synchronous Time-Division Multiplexing (STDM) capacity is reserved for every source
- Assume aggregate arrival process from all sources: $\lambda$ messages per second
- The transmission rate on the synchronous line out of the buffer is $1 / T$ slots per second - a slot carries exactly one packet
- Newly arriving packets are stored in the buffer

Sources


Figure 5.3 Asynchronous time-division multiplexing.

## Asynchronous Time-Division <br> Multiplexing (ATDM) - Analysis

- Embed a Markov chain at the slot boundaries
- Let
- $\mathbf{N}_{\mathrm{i}}$ be the number of packets in the buffer at the end of the ith slot
- $A_{i}\left(N_{i-1}\right)$ denote the number of packets that arrive during the ith slot, assuming that there are $\mathrm{N}_{\mathrm{i}-1}$ packets at the end of $(\mathbf{i}-1)^{\text {st }}$ slot in the system.
- Assume finite buffer
- Packets arriving during (in) a slot can not be transmitted until the beginning of the next slot


## Asynchronous Time-Division <br> Multiplexing (ATDM) - Analysis - cont'd

- Suppose that the ith departing message leaves behind a nonempty system (i.e. $\mathrm{N}_{\mathrm{i}}>0$ ).
- The state of the system is the end of next slot is given by

$$
\mathbf{N}_{\mathrm{i}+1}=\mathbf{N}_{\mathrm{i}}-\mathbf{1}+\mathrm{A}_{\mathrm{i}+1}\left(\mathbf{N}_{\mathrm{i}}\right) ; \quad \mathbf{N}_{\mathrm{i}}>=\mathbf{1}
$$

- If the system is empty at the beginning of a slot (i.e. $\mathbf{N i}=\mathbf{0}$ ), then $\mathrm{Ni}+1$ is given by

$$
N_{i+1}=A_{i+1}\left(N_{i}\right) ; \quad N_{i}=0 ;
$$

- The above two equations can be combined to describe the system dynamics as

$$
N_{i+1}=N_{i}-U\left(N_{i}\right)+A_{i+1}\left(N_{i}\right) ; N_{i}=0,1,2, \ldots
$$

where $U($.$) is 1$ when the argument is greater than zero and zero otherwise.

## Asynchronous Time-Division <br> Multiplexing (ATDM) - Finite Buffer

- Buffer can hold at most B packets
- State transition matrix has B+1 rows and B+1 columns
- Review the MUX problem in Section 2.7.2
- Let an be prob [n arrivals in a slot]
- Therefore, the state transition matrix can be written as
$R=\left[\begin{array}{cccccc}a_{0} & a_{1} & a_{2} & \ldots & a_{B-1} & \sum_{j=\beta}^{\infty} a_{j} \\ a_{0} & a_{1} & a_{2} & \ldots & a_{B-1} & \sum_{j=\beta}^{\infty} a_{j} \\ 0 & a_{0} & a_{1} & \ldots & a_{B-2} & \sum_{j=B-1}^{\infty} a_{j} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_{1} & \sum_{j=2}^{\infty} a_{j} \\ 0 & 0 & 0 & \ldots & a_{0} & \sum_{j=1}^{\infty} a_{j}\end{array}\right]$

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Asynchronous Time-Division
Multiplexing (ATDM) - Finite Buffer -
cont'd
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- Let Pi be the steady state probability that there are i packets in buffer at the beginning of a slot
- Pis are encapsulated in the raw vector $\mathrm{P}=$ (P0, P1, ..., PB).
- From material in chapter 2 , the vector $P$ should satisfy

$$
\mathbf{P}=\mathbf{P} \mathbf{R}
$$

- The above system of equations can be solved, together with the constraint that $\Sigma \mathrm{Pi}=1$, using the Matlab
Get_Steady_State_Distribution routine provided by instructor for MUX problems.


## Asynchronous Iime-Division Multiplexing (ATDM) - Finite Buffer cont'd

- The previous system of equations can be solved also as follows
- The linear system of equations can be written as

$$
P_{i}=\sum_{j=0}^{B} P_{j} r_{i j}=\left\{\begin{array}{lr}
P_{0} a_{i}+\sum_{j=0}^{i} P_{j+1} a_{i-j} & 0 \leq i<B \\
P_{0} \sum_{j=B}^{\infty} a_{j}+\sum_{j=0}^{B-1} P_{j+1} \sum_{k=B-j}^{\infty} a_{k} & i=B
\end{array}\right.
$$

- We can solve for $\mathbf{P}_{\mathbf{j}+\mathbf{1}}$ as

$$
P_{j+1}=\frac{P_{i}-\sum_{k=1}^{i} P_{k} a_{i-k+1}-P_{0} a_{i}}{a_{0}} ; 0 \leq i<B
$$

where $a_{0}>0$. If $a_{0}<=0$, the system would be full all the time

- Assume $\mathrm{P}_{\mathbf{0}}=\mathbf{1}$, use the above equation to compute $\mathrm{P}^{\prime}{ }_{1}, \mathrm{P}^{\prime}{ }_{2 \prime}$ $\ldots, \mathbf{P}_{\mathrm{B}}^{\prime}$, then normalize $\mathbf{P}_{i}^{\prime}$ 's to obtain $\mathrm{P}_{0}, \mathrm{P}_{1}, \ldots, \mathbf{P}_{\mathrm{B}}$ as

$$
P_{i}=P_{i}^{\prime} / \sum_{j=0}^{B} P_{i}^{\prime} ; \quad i=0,1, \cdots, B
$$

## Asynchronous Time-Division Multiplexing (ATDM) - Finite Buffer Performance Figures

- Given the Pi's one can compute several quantities of interest
- The expected number of packets in the buffer is given by

$$
\bar{N}_{p}=\sum_{i=0}^{B} i P_{i}
$$

- The rate on the output line is (1-P0)/T packets per second. The rate of packet arrivals is given by $\Sigma \mathrm{ia}_{\mathrm{i}} / \mathrm{T}$ packets/sec. Therefore, the loss rate should be given by

$$
\bar{L}=1-\left(1-P_{0}\right) / \sum_{i=1}^{\infty} i a_{i}
$$

- Using Little's formula, the average delay of $s$ packet in seconds is given by

$$
\bar{D}_{P}=T \bar{N}_{P} /\left(1-P_{0}\right)=T \sum_{i=0}^{B} i P_{i} /\left(1-P_{0}\right)
$$

# Asynchronous Time-Division Multiplexing (ATDM) - Finite Buffer Example 

- Example 5.3 (textbook page 193):
- For Quiz5 - Generate Figures 5.4a and 5.4b. Due Monday Jan $11^{\text {th }}-$ class time.
- In addition plot the average number of packets in the buffer as a function of offered load.


## Asynchronous Time-Division Multiplexing (ATDM) - Infinite Buffer Analysis

- For the finite buffer case, the system dynamics were governed by

$$
N_{i+1}=N_{i}-U\left(N_{i}\right)+A_{i+1}\left(N_{i}\right) ; N_{i}=0,1,2, \ldots
$$

- The infinite buffer case can be used to approximate the case when the buffer is so large and the overflow probability is negligible
- If the steady state distribution exists, then

$$
\begin{array}{r}
\lim _{i \rightarrow \infty} E\left[N_{i+1}\right]=\lim _{i \rightarrow \infty} E\left[N_{i}\right]=\bar{N} \\
\lim _{i \rightarrow \infty} E\left[A_{i+1}\left(N_{i}\right)\right]=\lim _{i \rightarrow \infty} E\left[A_{i+1}\right]=\bar{A}
\end{array}
$$

where $\bar{A}=\rho$ is the average number of arrivals per time slot. The dependence on the number of packets in the buffer is dropped since the buffer is of infinite size and not packets are lost.

- The term $\mathbf{U}(\mathbf{N i})$ is the indicator function of the event that the number of packets in the system is greater than 0 . Therefore
and

$$
\begin{gathered}
E\left[U\left(N_{i}\right)\right]=P\left(N_{i}>0\right)=1-P_{0} \\
P_{0}=1-\rho
\end{gathered}
$$

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Asynchronous Time-Division
Multiplexing (ATDM) - Infinite Buffer -
Analysis - cont'd
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- The goal is to obtain the PFG for number of packets in the buffer defined as

$$
P_{i}(z)=E\left(z^{N_{i}}\right)=\sum_{i=0}^{\infty} P_{i} z^{i}
$$

- Using the equation: $\mathbf{N}_{\mathrm{i}+1}=\mathbf{N}_{\mathrm{i}}-\mathbf{U}\left(\mathbf{N}_{\mathrm{i}}\right)+\mathbf{A}_{\mathrm{i}+1}\left(\mathbf{N}_{\mathrm{i}}\right) ; \mathbf{N}_{\mathrm{i}}=$ $0,1,2, \ldots$ we can write

$$
P_{i+1}(z) \doteq E\left(z^{N_{n+1}}\right)=E\left(z^{N_{1}^{N-V}\left(N_{N}\right)+A_{1+1}}\right)=E\left(z^{N_{1}-U\left(N_{1}\right)}\right) E\left(z^{A_{n+1}}\right)
$$

since the arrivals are independent of the content of the buffer

- Now we can also write

$$
\begin{aligned}
& E\left(z^{x+v(N)}\right)=\sum_{k=1}^{n} z^{k=v(k)} P\left(N_{i}=k\right)=P_{0}+\sum_{i=1}^{n} z^{k-1} P\left(N_{i}=k\right) \\
& =P_{0}^{=+z^{-1}}\left[\sum_{i=1}^{x} z^{\star} P\left(N_{i}=k\right)-P_{0}\right]=P_{0}+z^{-1}\left[P(z)-P_{0}\right]
\end{aligned}
$$

- Combining the above two results we get

$$
P_{i+1}(z)=\left\{P_{0}+z^{-1}\left[P_{i}(z)-P_{0}\right]\right\} A_{i+1}(z)
$$

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Asynchronous Time-Division
Multiplexing (ATDM) - Infinite Buffer -
Analysis - cont'd
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- If the Ni process is stationary, then

$$
\lim _{i \rightarrow \infty} P_{i+1}(z)=\lim _{i \rightarrow \infty} P_{i}(z)=P(z)
$$

- Therefore, the steady state distribution is given by

$$
P(z)=\frac{P_{0}(1-z) A(z, T)}{A(z, T)-z}
$$

- This PGF can used to find the moments of the distribution.
- The PFG can also be used to compute Pi's through successive differentiations and setting $z=1$.
- Be careful in regard to the notation -
- Pi is the ss probability that the buffer has i packets
- $\mathbf{P}(\mathbf{i})$ is the PGF evaluated at $\mathbf{z}=\mathbf{i}$;


## Asynchronous Time-Division Multiplexing (ATDM) - Infinite Buffer Performance Figures

- From the PGF we can write $P(z)[A(z, T)-z]=P_{0}(1-z) A(z, T)$
- Differentiating yields $P^{\prime}(z)[A(z, T)-z]+P(z)\left[A^{\prime}(z, T)-1\right]=P_{0}(-1) A(z, T)+P_{0}(1-z) A^{\prime}(z, T)$
- Now if we set $\mathbf{z}$ to $\mathbf{1}$, we have $A(1, T)=P(1)=1, A^{\prime}(1, T)=\lambda T \bar{M}=\rho$, and $P_{0}=1-\lambda T \bar{M}=1-\rho$
- The system is stable as long as $\rho<1$
- Differentiating again results in $P^{\prime \prime}(z)[A(z, T)-z]+2 P^{\prime}(z)\left[A^{\prime}(z, T)-1\right]+P(z) A^{\prime \prime}(z, T)$ $=2 P_{0}(-1) A^{\prime}(z, T)+P_{0}(1-z) A^{\prime \prime}(z, T)$
- Now setting z to $\mathbf{1}$, we have

$$
\bar{N}_{P}=P^{\prime}(1)=\rho+A^{\prime \prime}(1, T) /[2(1-\rho)]
$$

Main result (1)

- Since the buffer is infinite and there is no packet loss $\rightarrow$ average arrival rate is $\bar{A} / T=A^{\prime}(1, T) / T$
- Therefore, the average packet delay is given by

$$
\bar{D}_{P}=T+T A^{\prime \prime}(1, T) /[2 \rho(1-\rho)]
$$

## Asynchronous Time-Division Multiplexing (ATDM) - Infinite Buffer Example

- Example 5.4: Assume that traffic is generated by $\mathbf{N}$ sources. In a slot, each source, acting independently, generates a message with probability $P$. The messages have two possible lengths: one packet and four packets. The probability of one packet message is designated as $\mathbf{Q}$.

1. Write the PGF for the number of packet arrivals to the system in a slot
2. Write the PGF for the number of packets in the buffer
3. For $\mathbf{P}=\mathbf{0 . 0 5}, \mathbf{Q}=\mathbf{0 . 8}$, and $\mathbf{N}=10$, Compute the load, the mean number of packets in the buffer, and the mean delay

## Asynchronous Time-Division Multiplexing (ATDM) - Infinite Buffer Example - cont'd

- Solution:

H From the given info, $M(z)=Q z+(1-Q) z^{4} . A_{s}(z)=P z+(1-P)$, therefore $A(z)=A_{s}(M(z))$ should be given by $A(z)=[P(Q z+(1-$ Q) $z^{4}$ ) + 1-P] .
from this expression $\rho=A^{\prime}(z=1)=N P(3 Q+4)$, and $P_{0}=1-\rho$. Also $A^{\prime \prime}(z=1)=N(N-1)\left[P(3 Q+4]^{\wedge} 2+N P(12(1-Q))\right.$
\& The PGF $P(z)$ is equal to $P_{0}(1-z) A(z) /(A(z)-z)$ or (1-NP(3Q+4))(1$z) /\left(1-z /\left[P\left(Q z+(1-Q) z^{4}\right)+1-P\right]^{N}\right)$
\& The mean number packets in buffer is given by $\rho=A^{\prime \prime}(1) /\left(2^{*}(1-\right.$ $\rho)$ ) - Substitute in the above expressions ...
\& The mean packet delay in the system is given by $\mathbf{T}+$ TA'" $^{\prime \prime}(1) /\left(2^{*}\right.$ $\rho(1-\rho))$ - Assume $T=1$, and substitute in the above expression

## Synchronous Time-Division Multiplexing (STDM)

- For STDM each source is allocated a specific capacity justified if the source is mostly active (high rate).
- Flow on the line is blocked into fixed-length frames
- Since capacities allocated to sources are independent of one another, queueing for each source may be analyzed independently!!



## Synchronous Time-Division <br> Multiplexing (STDM) - Analysis

## - Assumptions

- Fixed length frame, $T_{F}$ - includes all slots, guard time, etc.
- A source may transmit UP TO b >= 1 slots during a frame
- Gated service - as opposed to exhaustive. Packets arriving during the frame must wait until the next frame
- The equation for the embedded Markov chain for the number of packets in the buffer is given by

$$
N_{i+1}=\max \left(0, N_{i}-b\right)+A_{i+1}
$$

where $\mathbf{N}_{\mathbf{i}}$ is the number of packets in the system at the beginning of the ith frame, $A_{i}$ is the number of packets arriving during the ith frame

## Synchronous Time-Division <br> Multiplexing (STDM) - Analysis - cont'd

- The objective is to compute the PGF for the number packets in the system

$$
P_{i+1}(z)=E\left(z^{N_{i+1}}\right)=E\left(z^{\max \left(0, N_{i}-b\right)+A_{i+1}}\right)=E\left(z^{\max \left(0, N_{i}-b\right)}\right) E\left(z^{A_{i+1}}\right)
$$

- After simple manipulation

$$
P_{i+1}(z)=\left[\sum_{j=0}^{b-1} P_{j}^{i}+\sum_{j=b}^{\infty} P_{j}^{i} z^{j-b}\right] E\left[z^{A_{i+1}}\right]
$$

- Assume the steady state solution exists and that Lim Pji $=\mathbf{P j}$, then

$$
\begin{aligned}
P(z) & =\left[\sum_{j=0}^{b-1} P_{j}+z^{-b}\left(\sum_{j=b}^{\infty} P_{j} z^{j}-\sum_{j=0}^{b-1} P_{j} z^{j}\right)\right] A\left(z, T_{F}\right) \\
& =A\left(z, T_{F}\right) \sum_{j=0}^{b-1} P_{j}\left(1-z^{-b+j}\right)+z^{-b} A\left(z, T_{F}\right) P(z)
\end{aligned}
$$

where $A(z, T F)$ is the PGF for the number of arriving packets during a frame

- Therefore, the required PGF is given by

$$
P(z)=\frac{A\left(z, T_{F}\right) \sum_{j=0}^{b-1} P_{j}\left(z^{b}-z^{j}\right)}{z^{b}-A\left(z, T_{F}\right)}
$$



- There are b unknowns: P0, P1, ..., Pb-1
- Note that if $\mathbf{b}=\mathbf{1}$, the above PGF reduces the one on slide 26!!


## Synchronous Time-Division Multiplexing (STDM) - Analysis

- How to find the unknowns PO, P1, ..., Pb-1?
- If it was only for PO (i.e. $\mathbf{b}=1$ ), then the condition that $\mathbf{P ( 1 )}$ (i.e. PGF evaluated at $\mathrm{z}=1$ ) must be $\mathbf{1}$ is sufficient to find the unknown PO.
- For the case of $\mathbf{b}>1$, we need extra b-1 conditions or equations to find P1, P2, ..., Pb-1.
- These equations can be obtained by applying Rouche's theorem.
- The provided solution restricts the arrival process to Poisson
- But the concept is applicable to any arrival process.


## Synchronous Iime-Division <br> Multiplexing (STDM) - Rouche's Theorem

- Consider the properties of the PGF P(z), $z^{b}$, and $A\left(z, T_{F}\right)$.
- $P(z)$ is analytic within the unit disk $|z| \leq 1$

$$
|P(z)|=\left|\sum_{j=0}^{\infty} z^{j} P_{j}\right| \leq \sum_{j=0}^{\infty}\left|z^{j}\right| P_{j} \leq \sum_{j=0}^{\infty} P_{j}=1
$$

- The arrival process is assumed to Poisson; then

$$
A\left(z, T_{F}\right)=\exp \left(-\lambda T_{F}(1-M(z))\right)
$$

- For stability, we require that average packet arrivals not exceed b.

$$
b>\lambda T_{F} M^{\prime}(1)=\lambda T_{F} \bar{M}
$$

- Therefore, where $\bar{M}$ is the mean number of packets in a message.
- Now consider the zeros, $z_{i}^{\prime}$ 's where $\left|z_{i}\right|<=1$ for the denominator of P(z)

$$
z_{i}^{b}=A\left(z_{i}, T_{F}\right)=e^{-\lambda T_{F}\left(1-M\left(z_{i}\right)\right)}
$$

- The textbook shows that for all these $z_{i}$ 's the root MUST be simple


## Synchronous IIme-Division

Multiplexing (STDM) - Rouche's
Theorem - cont'd

- Let $\mathbf{z}^{\mathrm{b}}=\mathrm{f}(\mathrm{z}),-\mathrm{e}^{-\lambda \mathrm{TF}(1-\mathrm{M}(\mathrm{z}))}=\mathbf{g}(\mathrm{z})$, region $\mathrm{R}|\mathrm{z}| \leq \mathbf{1}+\delta$, where $\delta>0$.
- The objective is to show that for a closed contour $C$ in $R$, if $f(z)$ is not zero and $|f(z)|>|g(z)| \rightarrow$ then by the theorem $f(z)$ and $f(z)+g(z)$ have the same NUMBER of roots with in C.
- For small enough $\delta, f(z)$ and $g(z)$ are analytic in $R$
- Define the contour $\mathbf{C}$ to be $|\mathrm{z}|=\mathbf{1 +} \delta^{\prime}$ where $\mathbf{0}<\delta^{\prime}<\delta$. Then using Tailor series expansions we can write
- And

$$
\left|z^{b}\right|=\left|1+\delta^{\prime}\right|^{b} \cong 1+b \delta^{\prime}
$$

$$
\left|-e^{-\lambda T_{F}(1-M(z))}\right| \cong e^{-\lambda T_{F}(1-M(1))}\left|1+\lambda T_{F} M^{\prime}(1) \delta^{\prime}\right|=1+\lambda T_{F} \bar{M} \delta^{\prime}
$$

- Then $|f(z)|>|g(z)|$ on $C$


## Synchronous Iime-Division <br> Multiplexing (STDM) - Rouche's Theorem - cont'd

- This means that the conditions for Rouche's theorem are satisfied. Since $f(z)=z^{b}$ has $b$ (repeated) roots at $z=0$, then $f(z)+g(z)=z^{b}$ - $e^{-\lambda \operatorname{TF}(1-M(z))}$ must have $b$ roots.
- The textbook shows that these b roots MUST be distinct (refer to page 199).
- Denote these roots by z0, z1, ..., zb-1.
- We know that $\mathbf{z 0}$ is equal to 1.
- These $b$ roots provide the needed set of equations to compute P0, P1, ..., Pb-1.

> Synchronous Time-Division
> Multiplexing (STDM) - Rouche's
> Theorem - cont'd

- The b equations are:

$$
\begin{aligned}
& \sum_{j=0}^{b-1} P_{j}\left(z_{i}^{b}-z_{i}^{j}\right)=0 ; \quad i=1,2, \cdots, b-1 \quad \text { and } \\
& \sum_{j=0}^{b-1} P_{j}(b-j)=b-A\left(1, T_{F}\right)
\end{aligned}
$$

- These are sufficient to find P0, P1, ..., Pb-1
- The textbook proves that a solution must exist
- The determinant of $\Delta$ can not be zero, and therefore the coefficient matrix is not singular
- The textbook proposes a simple method for finding the roots of $z^{b}-A\left(z, T_{F}\right)$ when the arrival process is Poisson (refer to equation 5.38)

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Synchronous Time-Division
    Multiplexing (STDM) - Mean Number of
    Packets in the System
```

- Refer to equation 5.39


## Synchronous Time-Division <br> Multiplexing (STDM) - Example

- Example 5.5
- The author uses Matlab's function "root" to find the needed four roots $\mathrm{z0}, \mathrm{z1}, \mathrm{z2}$, and $\mathrm{z3}$.
- The rest is direct substitution in the formulas
- Result: P0 = 0.1037, P1 = 0.1778, P2 = 0.1926 , and $P 3=0.1666$. This completes the specification of $\mathbf{P ( z )}$.

