King Fahd University of Petroleum & Minerals
Computer Engineering Dept

CSE 642 – Computer Systems Performance

Term 091

Dr. Ashraf S. Hasan Mahmoud

Rm 22-148-3

Ext. 1724

Email: ashraf@kfupm.edu.sa

Time Reversed Processes

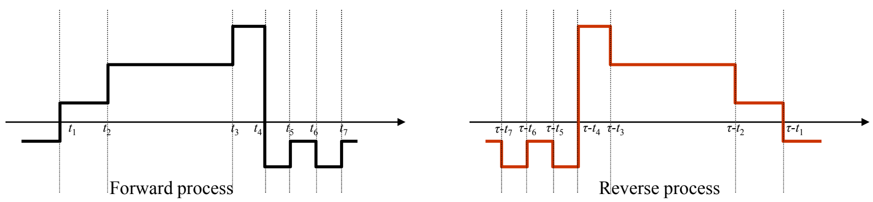
- **Refer to Leon Garcia's textbook section 8.5 for discussion of Time-Reversed Markov Chains**
- **Consider a continuous-time process X(t)**
- **Define the following process $X^r(t) = X(T-t)$, for an arbitrary T. For simplicity we set T to 0**
- **$X^r(t)$ is the reverse process for X(t)**

Time Reversed Processes – cont'd

- **Time reversibility – a process is time reversible if the following is true**

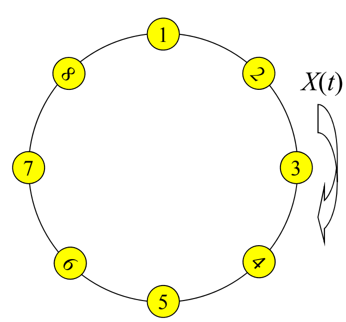
$$P(X(t_1)=i_1, X(t_2)=i_2, \cdots, X(t_m)=i_m)$$
$$= P(X(\tau-t_m)=i_m, X(\tau-t_{m-1})=i_{m-1}, \cdots, X(\tau-t_1)=i_1)$$

- **We say the process reversed in time has the same probabilistic properties as the forward process**



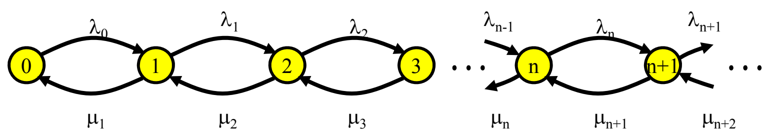
Time Reversed Processes – Observations

- **For a process to be reversible, it is not enough for the marginal probabilities for the forward and reverse processes to be equal.**
- **Example: consider the process X(t) depicted in figure. X(t) goes clockwise through states i→(i+1) mod 8. The probability of X(t) being in any of the state is 1/8. The same is true for the reverse process which travels counter clockwise. However, the process is clearly not reversible since state 2 can not follow state 1 in the reverse process, for example.**



Reversibility and Birth and Death Processes

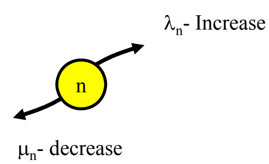
- **Review:**



- **Local balance equation: $P_n \lambda_n = P_{n+1} \mu_n$ – where P_n is the probability of being in state n**

Reversibility and Birth and Death Processes – cont'd

- **State = population equal to n**
  - **Time spends in state is exponentially distributed with mean $1/(\lambda_n+\mu_n)$**
  - **Probability of decrease (i.e. jumping to state n-1) = Probability of an departure occurring before an arrival**
  - **Probability of increase (i.e. jumping to state n+1) = Probability of an arrival occurring before an departure**
- **Show that:**
  - **Prob[Increase] = $\lambda_n/(\lambda_n+\mu_n)$**
  - **Prob[Decrease] = $\mu_n/(\lambda_n+\mu_n)$**



Reversibility and Birth and Death Processes – cont'd

- **A birth-death process is reversible if and only if the local balance equations hold**
- **Proof: Refer to textbook – section 4.2.2**

Reversibility and Birth and Death Processes – Proof Part 1

- **Reversible birth-death process → Local balance equations hold**
- **Proof: Assume X(t) is a reversible BD process →**

$$P(X(t)=j, X(t+\delta) = k) = P(X(t)=k, X(t+\delta) = j)$$

Let $P_j = P(X(t)=j)$ and $P_k = P(X(t)=k)$, for $\delta \to 0$, we can assume $k = j + 1$ (i.e. one transition is possible), we can write:

$$P_j\, P(X(t+\delta) = j+1 \,/\, X(t) = j) = P_{j+1}\, P(X(t+\delta) = j \,/\, X(t) = j+1)$$

recognizing that,

$$\lim_{\delta \to 0} \frac{P(X(t+\delta) = j+1/X(t)=j)}{\delta} = \lambda_j$$

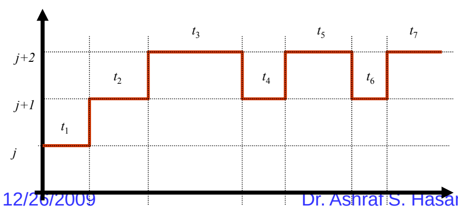
$$\lim_{\delta \to 0} \frac{P(X(t+\delta) = j/X(t)=j+1)}{\delta} = \mu_{j+1}$$

Therefore, $P_j\, \lambda_j = P_{j+1}\, \mu_{j+1}$

Reversibility and Birth and Death Processes – Proof Part 2

- **Local balance equations hold → Reversible birth-death process**
- **Proof:**
  Consider the sample path depicted of a birth-death process, the probability of the process taking this "exact path" is given by the following expression:

$$
\begin{aligned}
&P_j \times \frac{\lambda_j}{\lambda_j+\mu_j} \times \left(\lambda_j+\mu_j\right) \times e^{-(\lambda_j+\mu_j)t_1} dt_1 \times \frac{\lambda_{j+1}}{\lambda_{j+1}+\mu_{j+1}} \times \left(\lambda_{j+1}+\mu_{j+1}\right) \times e^{-(\lambda_{j+1}+\mu_{j+1})t_2} dt_2 \\
&\times \frac{\mu_{j+2}}{\lambda_{j+2}+\mu_{j+2}} \times \left(\lambda_{j+2}+\mu_{j+2}\right) \times e^{-(\lambda_{j+2}+\mu_{j+2})t_3} dt_3 \times \frac{\lambda_{j+1}}{\lambda_{j+1}+\mu_{j+1}} \times \left(\lambda_{j+1}+\mu_{j+1}\right) \times e^{-(\lambda_{j+1}+\mu_{j+1})t_4} dt_4 \\
&\times \frac{\mu_{j+2}}{\lambda_{j+2}+\mu_{j+2}} \times \left(\lambda_{j+2}+\mu_{j+2}\right) \times e^{-(\lambda_{j+2}+\mu_{j+2})t_5} dt_5 \times \frac{\lambda_{j+1}}{\lambda_{j+1}+\mu_{j+1}} \times \left(\lambda_{j+1}+\mu_{j+1}\right) \times e^{-(\lambda_{j+1}+\mu_{j+1})t_6} dt_6 \times e^{-(\lambda_{j+2}+\mu_{j+2})t_7}
\end{aligned}
$$



Note:
- P_j is the probability of starting from state j
- The term $(\lambda_j+\mu_j)\exp[-(\lambda_j+\mu_j)t_1]dt_1$ is probability density of the first interval
- $\exp[-(\lambda_{j+2}+\mu_{j+2})t_7]$ is the probability that the process remains in state j+2 for at least t_7

Reversibility and Birth and Death Processes – Proof Part 2 cont'd

- **The previous expression is simplified to be**

$$
\begin{aligned}
&P_j \lambda_j e^{-(\lambda_j+\mu_j)t_1} dt_1 \times \lambda_{j+1} e^{-(\lambda_{j+1}+\mu_{j+1})t_2} dt_2 \\
&\times \mu_{j+2} e^{-(\lambda_{j+2}+\mu_{j+2})t_3} dt_3 \times \lambda_{j+1} \times e^{-(\lambda_{j+1}+\mu_{j+1})t_4} dt_4 \\
&\times \mu_{j+2} e^{-(\lambda_{j+2}+\mu_{j+2})t_5} dt_5 \times \lambda_{j+1} e^{-(\lambda_{j+1}+\mu_{j+1})t_6} dt_6 \times e^{-(\lambda_{j+2}+\mu_{j+2})t_7} dt_7
\end{aligned}
$$

- **Using the local balance equation ($P_j \lambda_j = P_{j+1} \mu_j$, we can write:**

$$
P_j \lambda_j \lambda_{j+1} \mu_{j+2} \lambda_{j+1} \mu_{j+2} \lambda_{j+1} = P_{j+2} \mu_{j+2} \lambda_{j+1} \mu_{j+2} \lambda_{j+1} \mu_{j+2} \mu_{j+1}
$$

Reversibility and Birth and Death Processes – Proof Part 2 cont'd

- **Substitute the above equation in the former path evolution probability formula, we obtain**

$$
\begin{aligned}
&P_{j+2}\mu_{j+2}e^{-(\lambda_{j+2}+\mu_{j+2})t_1}dt_7 \times \lambda_{j+1}e^{-(\lambda_{j+1}+\mu_{j+1})t_6}dt_6 \\
&\times \mu_{j+2}e^{-(\lambda_{j+2}+\mu_{j+2})t_5}dt_5 \times \lambda_{j+1} \times e^{-(\lambda_{j+1}+\mu_{j+1})t_4}dt_4 \\
&\times \mu_{j+2}e^{-(\lambda_{j+2}+\mu_{j+2})t_3}dt_3 \times \mu_{j+1}e^{-(\lambda_{j+1}+\mu_{j+1})t_2}dt_2 \times e^{-(\lambda_j+\mu_j)t_1}
\end{aligned}
$$

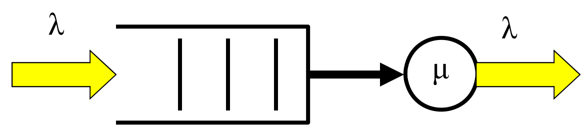
- **Which is the probability of the process starting with state j+2 and taking the same sample path in reverse**

- **➔ Process is reversible**

Burke's Theorem

- **Consider an M/M/1 or M/M/S or M/M/∞ queueing system at steady state with arrival rate λ, then**
  - **The departure process is Poisson with rate λ**
  - **At each time t, the number of customers in the system N(t) is *independent* of the sequence of departure times prior to t.**

M/M/1 example

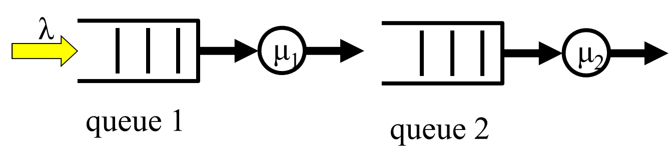


Burke's Theorem – cont'd

- **Theorem:** The departure process from an M/M/S queue is Poisson and is independent of the content of the queue
- **Proof:** Refer to textbook pages 118 and 119.
  - Part 1: The departure process from an M/M/S queue is a Poison
  - Part 2: The number of messages in a system at time t is independent of the sequence of departures prior to t.

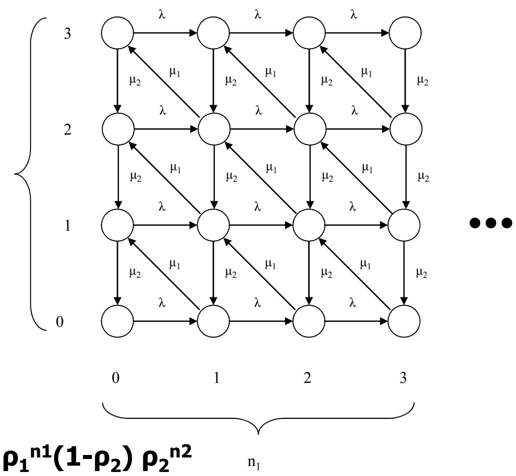
FeedForward Networks

- Consider the system depicted in figure where two are in tandem.
- How would the following system be analyzed?
- Assume infinite storage and a single server with exponential service time for each queue



Two Queues in Tandem :

- Approach 1 – using global balance equations
- We can use the state diagram to solve this problem:
  - State = (n_1, n_2) where n_i is the number of customers in i^{th} queue
- Refer to Garcia's textbook section 9.8 for solution of this two dimensional state diagram



- We can show

$P(N_1 = n_1, N_2 = n_2) = (1-\rho_1)\, \rho_1^{n1}(1-\rho_2)\, \rho_2^{n2}$

Where $\rho_1 = \lambda_1/\mu_1$ and $\rho_2 = \lambda_2/\mu_2$

Two Queues in Tandem – cont'd

- Since for a single queue,

$$P(N_1 = n_1) = (1-\rho_1)\, \rho_1^{n1}$$

Then it is clear that

$$P(N_1 = n_1, N_2 = n_2) = P(N_1 = n_1)P(N_2 = n_2)$$

- **Therefore, the number of customers at queue 1 and the number of customers at queue 2 are independent random variables!!**

Two Queues in Tandem – cont'd

- <u>Approach 2:</u> using Burke's Theorem:
- Since the second queue does not affect the first queue, the

$$P(N_1 = n_1) = (1-\rho_1)\,\rho_1^{n1}$$

  where $\rho_1 = \lambda/\mu_1$.
- For the second queue – apply Burke's theorem: the departure process of the first queue is a Poisson process. Therefore, by

$$P(N_2 = n_2) = (1-\rho_2)\,\rho_2^{n2}$$

  where $\rho_2 = \lambda/\mu_2$.
- The joint pmf is given by $P(N_1 = n_1, N_2 = n_2)$ can be computed using the 2nd part of Burke's theorem, therefore,

$$P(N_1 = n_1, N_2 = n_2) = P(N_1 = n_1)\,P(N_2 = n_2) = (1-\rho_1)\,\rho_1^{n1}\,(1-\rho_2)\,\rho_2^{n2}$$

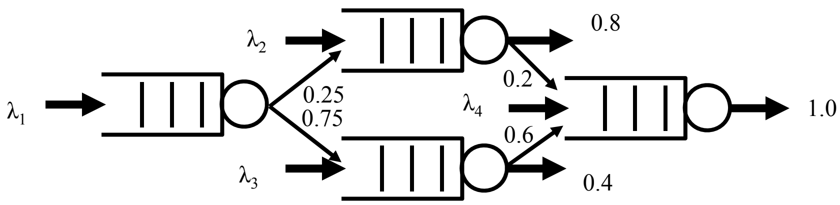
Feedforward Networks – Example 1

- An application of Burk's theorem
- Example 1: M queues in tandem
- Direct extension to the two queues in tandem case

$$P(Q_1(t)=k_1, Q_2(t)=k_2, \cdots, Q_M(t)=k_M) = \prod_{i=1}^{M}(1-\rho_i)\rho_i^{k_i}$$

Feedforward Networks – Example 2

- **Example2: feedforward acyclic networks (i.e. no feedback paths)**



- **Since joining and splitting of Poisson streams results in Poisson streams – Burks' theorem still applicable**
- **Solution key: deal with the individual queues after determining the total flow to the queue**

Traffic Equation and Routing Matrix

- **Consider a network of N queues, each having an independent exponential server and an infinite buffer**
- **External arrivals at each node – Poisson with rate λ_i**
- **Messages are routed probabilistically:**
  - **q_{ji}: i,j = 1, 2, 3, …, N is the probability of a message being routed from node j to node i**
  - **q_{jN+1}: is the probability of a message being routed outside the network**
  - **Note:** $\sum_{i=1}^{N+1} q_{ji} = 1$

Traffic Equation and Routing Matrix – cont'd

- **Let Λ_i: total flow into the i^{th} node**
- **Clearly, one can write**

$$\Lambda_i = \lambda_i + \sum_{j=1}^{N} q_{ji}\Lambda_j$$

- **The matrix version is**

$$[\Lambda_1 \quad \Lambda_2 \quad \cdots \quad \Lambda_N] = [\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_N] + [\Lambda_1 \quad \Lambda_2 \quad \cdots \quad \Lambda_N]\begin{bmatrix} 0 & q_{12} & \cdots & q_{1N} \\ q_{21} & 0 & \cdots & q_{2N} \\ \vdots & & 0 & \\ q_{N1} & q_{N2} & \cdots & 0 \end{bmatrix}$$

or

$$\Lambda = \lambda + \Lambda Q$$

Note:
$q_{jj} = 0$ – no routing to same source node

Traffic Equation and Routing Matrix – cont'd

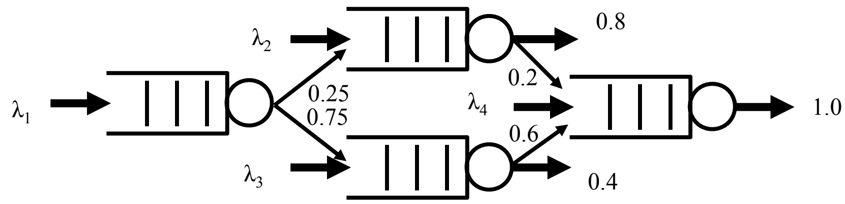
- **Normally the inputs and the routing matrix are known, the total flow to each node can be found using**

$$\Lambda = \lambda[I - Q]^{-1}$$

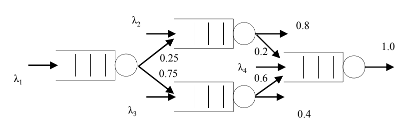
where *I* is the NxN identity matrix

Example:

- **<u>Problem</u>: Consider the network of queue depicted in figure. If the arrival rates are given by $\boldsymbol{\lambda}$ = [2.0, 1.0, 0.5, 3.0], and the service rates are $\boldsymbol{\mu}$ = [4.0, 6.0, 11.0, 9.9],**
  - **a) compute the total flow into each node**
  - **b) Find the joint pmf for number of customers in queues**



Example:



- **<u>Solution</u>:**

a) The routing matrix for the network is given by

$$Q = \begin{bmatrix} 0 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, total flows are given by

$\boldsymbol{\Lambda}$ = [2.0, 1.5, 2.0, 4.5]

b) The loads for the queues are given by

$\boldsymbol{R} = \boldsymbol{\Lambda}./\boldsymbol{\mu}$ (./ is the element-by-element division – Matlab notation)

= [1/2 , 1/4, 2/11, 5/11]

The joint pmf for the number of customers is given by

$$P(Q_1 = k_1, Q_2 = k_2, Q_3 = k_3, Q_4 = k_4) = \prod_{i=1}^{4} (1 - R_i) R_i^{k_i}$$

$$= (81/484)(1/2)^{k_1} (1/4)^{k_2} (2/11)^{k_3} (5/11)^{k_4}$$

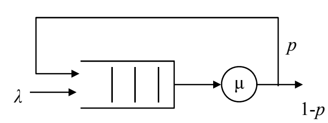
Open Network – Flows within Feedback Paths

- **Open networks – at least one external source of arrivals ➔ there must be a flow to outside network (exit path)**
  - **i.e. sum q_{ki} < 1 for at least one k = 1, 2, …, N**

Example: M/M/1 queue with Feedback

- **<u>Problem</u>: Consider the following system – Find the pmf for number of customers in the system.**
- **<u>Solution</u>:**

$\Lambda = \lambda + p\Lambda$ ➔ $\Lambda = \lambda/(1-p)$
Therefore, traffic load, R is given by
$R = \Lambda/\mu = \lambda/[\mu(1-p)]$
$Prob(N = k) = (1-R)R^k$ k=0, 1, 2, …



Note R < 1 ➔ $\lambda/[\mu(1-p)] < 1$ or $\lambda < \mu(1-p)$ – this imposes a limit on the maximum arrival rate

What is the average number of customer visits to the queue?

$E[N] = R/(1-R)$
$E[T] = E[N] / \lambda$ - direct application of Little's formula

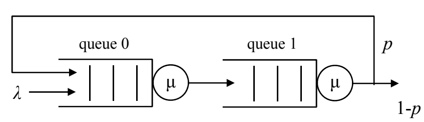
For a general solution of an M/G/1 with Bernoulli feedback check: L. Takács, "A Single-Server Queue with Feedback," Bell Technical Journal, March 1963, pp. 505-519.

Exercise – Using OPNET

- **Use example 9.23 of Leon Garcia's textbook to do the following:**
  - **(a) Using the theoretical analysis supplied in the solved example in the textbook:**
    - **Plot the average total number of customers in network versus the external arrival rate**
    - **Plot the average end-to-end delay of a customer versus the external arrival rate**
  - **(b) Develop an opnet simulation model and produce simulation results and compare them against those obtained in part (a)**
    - **Produce comparative curves similar to those found on slide 39.**
  - **For part (a) and (b) use p = 0.9 and p = 0.6 (note p is the probability of exiting the network from queue 1)**

Feedback Violates the Poisson Departure process

- **Let us examine the following system***



- **<u>Solution:</u> The analytic solution for the depicted system is as follows:**

$\Lambda = \lambda + p\Lambda \Rightarrow \Lambda = \lambda/(1-p)$
Therefore, traffic load, R is given by
$R_0 = R_1 = R = \Lambda/\mu = \lambda/[\mu(1-p)]$
$\text{Prob}(N_0 = k) = \text{Prob}(N_1 = k) = (1-R)R^k \quad k=0, 1, 2, \ldots$

Note $R < 1 \Rightarrow \lambda/[\mu(1-p)] < 1$ or $\lambda < \mu(1-p)$
$E[N_0] = E[N_1] = R/(1-R) = \lambda/[\mu(1-p) - \lambda]$
$E[N] = E[N_0] + E[N_1] = 2R/(1-R) = 2\lambda/[\mu(1-p) - \lambda]$
End-to-end delay for a customer is computed as
$E[T] = E[N] / \lambda = 2/[\mu(1-p) - \lambda]$

*I can show the same behavior using the single-server queue system considered in the last example – but I am using the system proposed the textbook to give another example on opnet modeling

Feedback Violates the Poisson Departure process – cont'd

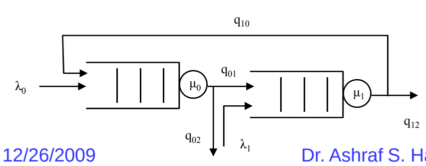
- **To depict the violation of the Poisson arrival/departure process**
  - **Assume a very low external arrival rate λ – say 1 packet every 2 hours – i.e. mean interarrival time of 7200 seconds**
  - **Assume a very small mean service time 1/μ – say 10^{-9} second**
  - **Let p = 0.999**
- **This setting translates the following:**
  - **One customer arrives – the next arrival is 1000s of seconds away on average**
  - **The customer is TRAPPED in the system circulating (since p ≈ 1**
  - **So if we monitor the customer departures of queue 0 or 1 (prior to the feedback branching) – we expect to see departure bursts**

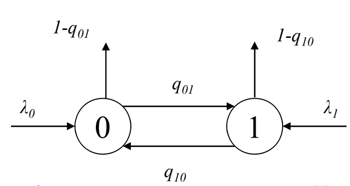
Proof Of the Product Form: Two-Node Network

- **Consider the two queues network depicted in figure**
- **The total flow equations are given by**

$$[\Lambda_0 \quad \Lambda_1] = [\lambda_0 \quad \lambda_1] + [\Lambda_0 \quad \Lambda_1] \begin{bmatrix} 0 & q_{01} \\ q_{10} & 0 \end{bmatrix}$$

- **System state: (k_0, k_1) where k_0 is number of customers in queue 0 while k_1 is number of customers in queue 1**
- **Define $P(Q_0(t)=k_0, Q_1(t)=k_1) = P(k_0, k_1; t)$ – as the probability of k_i customers in the respective queue at time t.**





Proof Of the Product Form: Two-Node Network – cont'd

- **The Kolmogorov differential equation for $P(k_0, k_1; t)$:**

$$\begin{aligned} P(k_0, k_1; t+\delta) = {} & P(k_0-1, k_1; t)\lambda_0\delta + P(k_0, k_1-1; t)\lambda_1\delta \\ & + P(k_0+1, k_1; t)\mu_0(1-q_{01})\delta \\ & + P(k_0, k_1+1; t)\mu_1(1-q_{10})\delta \\ & + P(k_0+1, k_1-1; t)\mu_0 q_{01}\delta \\ & + P(k_0-1, k_1+1; t)\mu_1 q_{10}\delta \\ & + P(k_0, k_1; t)(1-(\lambda_0+\lambda_1+\mu_0+\mu_1))\delta \end{aligned}$$

- **The terms on the RHS:**
  - First two terms – arrivals to either queues
  - Second pair – departures from system
  - Third pair – transfers between queues
  - Final term – no arrivals, departures, or transfers
- **The Kolmogorov D.E is given by**

$$\begin{aligned} dP(k_0, k_1; t)/dt = {} & P(k_0-1, k_1; t)\lambda_0 + P(k_0, k_1-1; t)\lambda_1 \\ & + P(k_0+1, k_1; t)\mu_0(1-q_{01}) \\ & + P(k_0, k_1+1; t)\mu_1(1-q_{10}) \\ & + P(k_0+1, k_1-1; t)\mu_0 q_{01} \\ & + P(k_0-1, k_1+1; t)\mu_1 q_{10} \\ & - P(k_0, k_1; t)(\lambda_0+\lambda_1+\mu_0+\mu_1) \end{aligned}$$

- **At steady state $dP(k_0, k_1; t)/dt = 0$**

Proof Of the Product Form: Two-Node Network – cont'd

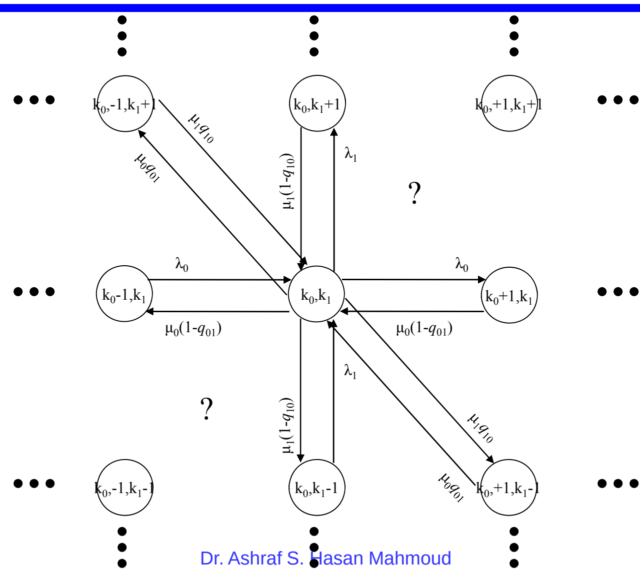
- **The steady state probabilities are then given by:**

$$\begin{aligned} P(k_0, k_1)(\lambda_0+\lambda_1+\mu_0+\mu_1) = {} & P(k_0-1, k_1)\lambda_0 + P(k_0, k_1-1)\lambda_1 \\ & + P(k_0+1, k_1)\mu_0(1-q_{01}) \\ & + P(k_0, k_1+1)\mu_1(1-q_{10}) \\ & + P(k_0+1, k_1-1)\mu_0 q_{01} \\ & + P(k_0-1, k_1+1)\mu_1 q_{10} \quad \forall\, k_0,k_1 \geq 0 \end{aligned}$$

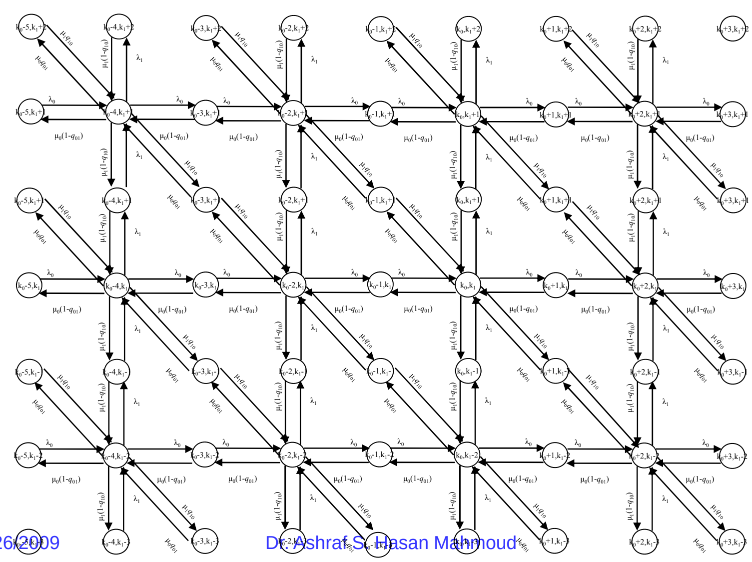
- **Note that the set of equilibrium equations stated above together with the normalizing condition $\sum_{\forall k_0,k_1} P(k_0,k_1)=1$ can be solved to obtain the complete pmf – however, as we will show, the closed form solution turns out to be simple**

- **The state transition flow diagram is shown on the next slide**

Proof Of the Product Form: Two-Node Network – State Transition Flow Diagram



Proof Of the Product Form: Two-Node Network – State Transition Flow Diagram – more detailed



Proof Of the Product Form: Two-Node Network – Rewriting the equations In Terms of Total Flow

- **Rewriting the pervious equations in terms of the total flows Λ_0 and Λ_1 results in:**

$$\begin{aligned}
&P(k_0, k_1)(\Lambda_0+\Lambda_1+\mu_0+\mu_1) \\
&+ P(k_0-1, k_1)q_{10}\Lambda_1 + P(k_0, k_1-1)\, q_{01}\Lambda_0 \\
&+ P(k_0+1, k_1)\mu_0 q_{01} + P(k_0, k_1+1)\mu_1 q_{10} \\
&\qquad = P(k_0+1, k_1)\mu_0 + P(k_0, k_1+1)\mu_1 \\
&\qquad + P(k_0-1, k_1)\, \Lambda_0 + P(k_0, k_1-1)\, \Lambda_1 \\
&\qquad + P(k_0+1, k_1-1)\mu_0 q_{01} + P(k_0-1, k_1+1)\mu_1 q_{10} \\
&\qquad + P(k_0, k_1)(q_{01}\Lambda_0+q_{10}\Lambda_1) \quad \forall\, k_0,k_1 \geq 0
\end{aligned}$$

- **Solving the above equations, yields**

$$P(k_0, k_1)\Lambda_0 = P(k_0+1, k_1)\mu_0$$
$$P(k_0, k_1+1)\mu_1 = P(k_0, k_1)\Lambda_1$$

- **These can be written as**

$$P(k_0+1, k_1) = \rho_0\, P(k_0, k_1)$$
$$P(k_0, k_1+1) = \rho_1\, P(k_0, k_1)$$

where ρ_i is given by Λ_i/μ_i

Proof Of the Product Form: Two-Node Network – Final Solution

- **Therefore, solving iteratively and using the normalizing condition, one can write**

$$P(k_0,k_1) = (1-\rho_0)(1-\rho_1)\, \rho_0^{k0}\, \rho_1^{k1} \qquad k_0,k_1 = 0, 1, \ldots$$

i.e. the product form applies.

Example: Two-Node Network

- **<u>Problem</u>: Let λ = [2.0, 1.0], and the service rates are μ = [15.625, 3.75] – Let the routing parameters q01 = 0.4 and q10 = 0.5.**
  - **A) compute the total flow into each queue**
  - **B) Compute the traffic utilization of each queue – write an expression for the joint pmf of number of customers in the network**
  - **C) Compute the average number of messages in node 0 and the average number of messages in node 1**
  - **D) Calculate the average end-to-end delay for a customer**

Example: Two-Node Network – cont'd

- **<u>Solution</u>:**

A) The total flow is found by solving the following set of equations:

$$[\Lambda_0\ \Lambda_1] = [\lambda_0\ \lambda_1] + [\Lambda_0\ \Lambda_1] \begin{bmatrix} 0 & q01 \\ q10 & 0 \end{bmatrix}$$

Therefore, $[\Lambda_0\ \Lambda_1]$ = [3.125 2.25]

B) Traffic Utilization: R = $\Lambda./\mu$

= [0.2 0.6]

$P(k_0, k_1) = 0.32(0.2)^{k0}(0.6)^{k1}$ for k0, k1 = 0, 1, …

C) $E[N_0] = R_0/(1 - R_0) = 0.25$

$E[N_1] = R_1/(1 - R_1) = 1.5$

Note that $E[N] = E[N_0] + E[N_1]$

= 1.75

D) $E[T] = E[N]/(\lambda_0+\lambda_1)$ = 0.583 seconds

N-Node Open Jackson Networks – Problem Specification

- **Consider an N-Node open network that is characterized by**
  - **Probabilistic routing matrix Q = $\{q_{ij}\}$**
  - **Set of external flows λ_i; i=1, 2, …, N - Poisson processes**
  - **Infinite storage at each node**
- **Assume S_i servers at each node i – each having exponentially distributed service time**
  - **Departure rate from state k_i (i.e. k_i customers in node i) is equal to**

$$\mu_i d(k_i) = \begin{cases} \mu_i k_i & k_i \le S_i \\ \mu_i S_i & k_i > S_i \end{cases}$$

N-Node Open Jackson Networks – Problem Specification

- **The total flow to the ith node is computed using:**

$$\Lambda_i = \lambda_i + \sum_{j=1}^{N} q_{ji} \Lambda_j$$

- **The queue at node i is stable if $\Lambda_i < \mu_i S_i$; i=1,2, …, N**

N-Node Open Jackson Networks – Global Balance Equations

- **Along the same lines we followed for the two-node system, the global balance equations are given by**

$$P(k_1,k_2,\ldots,k_N)\left[\sum_{i=1}^{N}(\lambda_i+\mu_i d(k_i))\right] = \begin{cases} \sum_{i=1}^{N} P(k_1,k_2,\ldots,k_i-1,\ldots,k_N)\lambda_i \\ +\sum_{i=1}^{N} P(k_1,k_2,\ldots,k_i+1,\ldots,k_N)\mu_i d(k_i+1)q_{i,N+1} \\ +\sum_{i=1}^{N}\sum_{j=1}^{N} P(k_1,k_2,\ldots,k_i+1,\ldots,k_j-1,\ldots,k_N)\mu_i d(k_i+1)q_{i,j} \\ \forall k_1,k_2,\ldots,k_N \geq 0 \end{cases}$$

Flow out of state (k_1,k_2, …, k_N)

Flow into of state (k_1,k_2, …, k_N)

Due to an arrival

Due to a departure to outside the network

Due to a departure from node i to node j

N-Node Open Jackson Networks – Global Balance Equations – cont'd

- **We use the following substitutions in the previous global balance equation:**

$$\lambda_i = \Lambda_i - \sum_{j=1}^{N}\Lambda_i q_{ji} \qquad i=1,2,\ldots,N$$

$$q_{i,N+1} = 1-\sum_{j=1}^{N} q_{ij} \qquad i=1,2,\ldots,N$$

N-Node Open Jackson Networks – Global Balance Equations – cont'd

- **Rewriting the global balance equation:**

$$
\begin{aligned}
& P(k_1,k_2,\ldots,k_N)\Lambda_i + \sum_{i=1}^{N} P(k_1,k_2,\ldots,k_i-1,\ldots,k_N)\mu_i d(k_i) \\
& \quad + \sum_{i=1}^{N}\sum_{j=1}^{N} P(k_1,k_2,\ldots,k_j-1,\ldots,k_N)\Lambda_i q_{ij} \\
& \quad + \sum_{i=1}^{N}\sum_{j=1}^{N} P(k_1,k_2,\ldots,k_i+1,\ldots,k_N)\mu_i d(k_i+1) q_{ij} \\
& = \sum_{i=1}^{N} P(k_1,k_2,\ldots,k_i+1,\ldots,k_N)\mu_i d(k_i+1) \\
& \quad + \sum_{i=1}^{N} P(k_1,k_2,\ldots,k_i-1,\ldots,k_N)\Lambda_i \\
& \quad + \sum_{i=1}^{N}\sum_{j=1}^{N} P(k_1,k_2,\ldots,k_i+1,\ldots,k_j-1,\ldots,k_N)\mu_i d(k_i+1) q_{ij} \\
& \quad + \sum_{i=1}^{N}\sum_{j=1}^{N} P(k_1,k_2,\ldots,k_N)\Lambda_i q_{ij}
\end{aligned}
$$

N-Node Open Jackson Networks – Joint Probability Mass Function

- **The global balance equation is satisfied if**

$$\mu_i d(k_i+1) P(k_1,k_2,\ldots,k_i+1,\ldots,k_N) = \Lambda_i P(k_1,k_2,\ldots,k_i,\ldots,k_N)$$

or

$$P(k_1,k_2,\ldots,k_i+1,\ldots,k_N) = \frac{R_i}{d(k_i+1)} P(k_1,k_2,\ldots,k_i,\ldots,k_N)$$

where $R_i = \Lambda_i / \mu_i$

- **The solution to the above equation is given by**

$$P(k_1,k_2,\ldots,k_N) = G^{-1}\prod_{i=1}^{N}\frac{R_i^{k_i}}{\prod_{j=1}^{k_i} d(j)}; \quad \forall k_1,k_2,\ldots,k_N \ge 0$$

where G^{-1} is the normalization constant

N-Node Open Jackson Networks – Joint Probability Mass Function – Single Server Nodes

- **Assume single server nodes – i.e. $S_i = 1$ $\forall$ i = 1, 2, …, N**
- **Therefore,** $d(j) = 1; \quad j = 1,2,\ldots,N$
- **The joint PMF is given by**

$$P(k_1, k_2, \ldots, k_N) = G^{-1} \prod_{i=1}^{N} R_i^{k_i}; \quad \forall k_1, k_2, \ldots, k_N \geq 0$$

- **Hence, the normalization constant is given by**

$$G^{-1} = \prod_{i=1}^{N} (1 - R_i)$$

- **Rewriting the PMF results in**

$$P(k_1, k_2, \ldots, k_N) = \prod_{i=1}^{N} (1 - R_i) R_i^{k_i}; \quad \forall k_1, k_2, \ldots, k_N \geq 0$$

N-Node Open Jackson Networks – Joint Probability Mass Function – Infinite Server Nodes

- **Assume infinite server nodes – i.e. $S_i = \infty$ $\forall$ i = 1, 2, …, N**
- **Therefore,** $d(j) = j; \quad j = 1,2,\ldots,N$

  and $\prod_{j=1}^{k_i} d(j) = k_i!$
- **The joint PMF is given by**

$$P(k_1, k_2, \ldots, k_N) = G^{-1} \prod_{i=1}^{N} \frac{R_i^{k_i}}{k_i!}; \quad \forall k_1, k_2, \ldots, k_N \geq 0$$

- **Hence, the normalization constant is given by**

$$G^{-1} = \prod_{i=1}^{N} e^{-R_i}$$

- **Rewriting the joint PMF results in**

$$P(k_1, k_2, \ldots, k_N) = \prod_{i=1}^{N} \frac{e^{-R_i}}{k_i!} R_i^{k_i}; \quad \forall k_1, k_2, \ldots, k_N \geq 0$$

N-Node Open Jackson Networks – Performance Calculations

- **The marginal probability of node i having k_i messages is given by**

$$P(Q_i = k_i) = \left(1 - R_i^{k_i}\right) R_i^{k_i}\,; \quad i = 1, 2, \ldots, N$$

- **If the nodes in the system have limited buffer size M, then probability of buffer overflow may be approximated by**

$$P(Q_i \geq M) = \sum_{k_i = M}^{\infty} \left(1 - R_i^{k_i}\right) R_i^{k_i} = R_i^{M}\,; \quad i = 1, 2, \ldots, N$$

- **The mean and variance of total number of customers in N nodes is given by**

- Prove these two equations?
- For large number of nodes, one may invoke the central-limit theorem to approximate the distribution of the total number of customers in the system by a Gaussian distribution

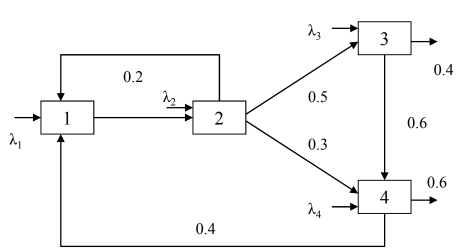
$$E[k_1 + k_2 + \ldots + k_N] = \sum_{i=1}^{N} \frac{R_i}{1 - R_i}$$

$$Var[k_1 + k_2 + \ldots + k_N] = \sum_{i=1}^{N} \frac{R_i}{(1 - R_i)^2}$$

Example: N-Node Open Jackson Networks

- **<u>Problem</u>: Consider the network of queues depicted in figure. Assume λ = [2.0, 1.0, 0.5, 0.3] and μ = [0.1, 0.07, 0.03, 0.075]**

a) **Find the routing matrix**
b) **Calculate the total traffic flow vector, and the resulting loads at each queue node**
c) **Approximate the probability mass function of the total number of customers in the system using the Gaussian distribution**
d) **Find the exact probability mass function of the total number of customers and compare it to the one obtained in part (c).**



Example: N-Node Open Jackson Networks – cont'd

- **Solution**:

a) **The routing matrix, Q, is given by**

$$Q = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ 0.2 & 0 & 0.5 & 0.3 \\ 0 & 0 & 0 & 0.6 \\ 0.4 & 0 & 0 & 0 \end{bmatrix}$$

b) **Therefore the total traffic flow is given by and the loads are**

$$\Lambda = \lambda[I - Q]^{-1} = [4.7857 \quad 5.7857 \quad 3.3929 \quad 4.0714]$$

$$R = [0.4786 \quad 0.4050 \quad 0.1018 \quad 0.3054]$$

c) **The mean total messages in system is given by while the variance is given**

$$E[k_1 + k_2 + k_3 + k_4] = m = \sum_{i=1}^{4} \frac{R_i}{1 - R_i} = 2.1514$$

$$Var[k_1 + k_2 + k_3 + k_4] = \sum_{i=1}^{4} \frac{R_i}{(1 - R_i)^2} = 3.6632$$

or the standard deviation is $\sigma = \sqrt{\sum_{i=1}^{4} \frac{R_i}{(1 - R_i)^2}} = 1.9139$

Example: N-Node Open Jackson Networks – cont'd

- **Solution-cont'd**:

c) Assuming the total number of customers can be approximated by a Gaussian distribution, then $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}$

where m and σ quantities are computed earlier.

Then CDF of total number of customers can be computed using $F(x) = 0.5 + 0.5 erf\left(\frac{x - m}{\sigma\sqrt{2}}\right)$

Refer to the Gaussian distribution material

Example: N-Node Open Jackson Networks – cont'd

- **Solution-cont'd:**

d) To calculate the exact distribution we need to evaluation the following

$$\Pr ob(total = j) = \sum_{\forall (k_1+k_2+k_3+k_4=j)} \prod_{i=1}^{4} (1-R_i) R_i^{k_i} \qquad j = 0,1,2,\ldots$$

to obtain the PMF function.

Subsequently, the CDF is given by

$$F(k) = \sum_{j=0}^{k} \Pr ob(total = j) \qquad k = 0,1,2,\ldots$$

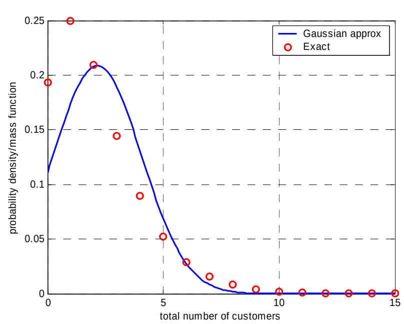
A Matlab code is used to find all the (k_1, k_2, k_3, k_4) that add to a particular j and then the k_is are substituted in the above expression to calculate the PMF.

Example: N-Node Open Jackson Networks – cont'd

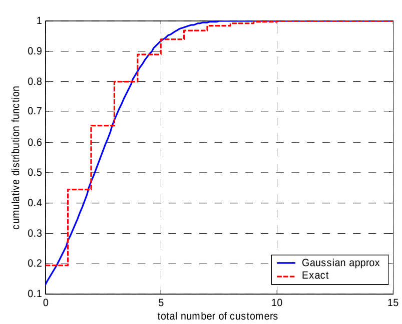
- **Solution-cont'd:**

d) The probability density/mass functions for the total number of customers in system are shown in figure (I)

The cumulative probability functions for the total number of customers in system are shown in figure (II)



(I)



(II)

Example: N-Node Open Jackson Networks – cont'd (Matlab Code)

- **Main Code: Example4_3_1.m**

```
%
% example 4.3 of Hayes
clear all
LineWidth = 2;
Lambda = [2.0, 1.0,  0.5,  0.3];
M      = 1./[0.1, 0.07, 0.03, 0.075];
Q = [0    1.0  0    0  ; ...
     0.2  0    0.5  0.3; ...
     0    0    0    0.6; ...
     0.4  0    0    0];

Omega = Lambda * inv(eye(4) - Q);
R     = Omega ./ M;

MeanTotal = 0;
VarTotal  = 0;
for i=1:4
    MeanTotal = MeanTotal + R(i)/(1-R(i));
    VarTotal  = VarTotal  + R(i)/(1-R(i))^2;
end
Nmax = 15; % max range for probability functions
%
% compute the Gaussian
nG  = 0:0.1:Nmax;
f1  = 1/sqrt(2*pi*VarTotal)*exp(-(nG-
    MeanTotal).^2/(2*VarTotal));
F1  = 0.5 + 0.5*erf((nG-MeanTotal)/(sqrt(2*VarTotal)));
%
% Compute the exact
f2  = GetExactDistribution(R, Nmax);
F2 = cumsum(f2);
[nE, FF2] = stairs(0:15, F2);
%
% plot results
figure(1);
h = plot(nG, f1, '-', 0:Nmax, f2,'or');
set(h, 'LineWidth', LineWidth);
ylabel('probability density/mass function');
xlabel('total number of customers');
legend('Gaussian approx', 'Exact');
grid

figure(2);
h = plot(nG, F1, '-', nE, FF2,'--r');
set(h, 'LineWidth', LineWidth);
ylabel('cumulative distribution function');
xlabel('total number of customers');
legend('Gaussian approx', 'Exact');
grid
```

- **GetExactDistribution function**

```
function P = GetExactDistribution(R, N);
%
% example 4.3 of Hayes
% computation of exact pmf for total
% number of customers, N
% R is a vector of loads

P = zeros(1,N+1);
for i=0:N;
    [AllKs, m] = GetAllKs( i );
    for j=1:m
        P(i+1) = P(i+1) +
  ComputeFromJoint(AllKs(j,:), R);
    end
end
```

- GetAllKs function

```
function [AllKs, m] = GetAllKs( N );
%
% function returns all possible 4 numbers that add
  to N
m     = 0;
AllKs = [];
for i1=0:N
    for i2=0:N
        for i3=0:N
            for i4=0:N
                if (N == (i1+i2+i3+i4)) % found one
                    % check if it is not aready
  included
                    %[i1 i2 i3 i4]
                    if (NotIncludedYet([i1 i2 i3
  i4], AllKs, m))
                        %
                        % include it
                        m = m + 1;
                        AllKs(m,:) = [i1 i2 i3 i4];
                    end
                end
            end
        end
    end
end
```


Example: N-Node Open Jackson Networks – cont'd (Matlab Code)

- ComputeFromJoint fuction

```
function P = ComputeFromJoint(Vector, R);
%
% use the product form to evaluate
P = 1;
for i=1:length(R)
    P = P*(1-R(i))*(R(i)^Vector(i));
end
```

- NotIncludedYet function

```
function Flag = NotIncludedYet(Vector, AllKs, m)
%
% check if Vector is already included in AllKs
Flag = 1;
if (m)
    for i=1:m
        if (sum(Vector == AllKs(i,:)) == 4)
            % i.e. vector found
            Flag = 0;
            break;
        end
    end
end
```

Average Message Delay

- **In an open network of N nodes, we have shown that the mean number of customers in network is given by** $E[k_1 + k_2 + \ldots + k_N] = \bar{\rho} = \sum_{i=1}^{N} \frac{R_i}{1 - R_i}$
- **Therefore, using Little's formula we have** $\bar{\rho} = E[T] \sum_{i=1}^{N} \lambda_i$ **where E[T] is the mean delay for a customer**
- **We can also apply Little's formula on the individual nodes as in** $\bar{\rho} = \sum_{i=1}^{N} \lambda_i E[T_i]$

  where $E[T_i]$ is the mean delay of a customer originating at node i
- **However, obtaining $E[T_i]$ is not straight forward!!**

Average Message Delay – cont'd

- **Assume Λ_i is the total flow into node i and D_i is the customer delay at node i – then another application of Little's formula results in** $\bar{\rho} = \sum_{i=1}^{N} \Lambda_i E[D_i]$
- **Combining the previous results, we obtain**

$$E[T] = \frac{\sum_{i=1}^{N} \Lambda_i E[D_i]}{\sum_{i=1}^{N} \lambda_i}$$

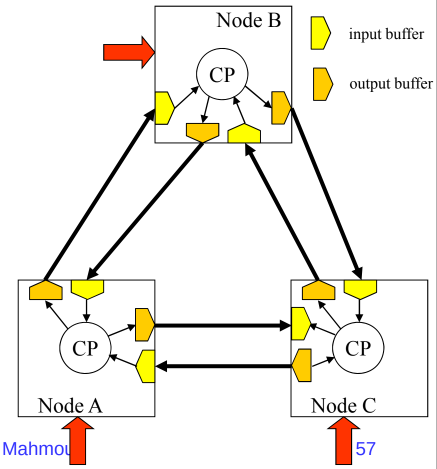
Remember
$\Lambda_i E[D_i] = R_i/(1-R_i)$

- **Therefore, the final result is given by** $E[T] = \frac{\sum_{i=1}^{N} R_i/(1 - R_i)}{\sum_{i=1}^{N} \lambda_i}$

  which is the results used in previous examples to compute the mean end-to-end delay

Store-and-Forward Message Switched Networks

- Consider the 3-node store-and-forward network depicted in figure
- The focus is on the queueing delay in the output buffers
- Arrival of messages to output buffers is rapid and can be modeled by Poisson arrivals
  - Prob of more than one arrival in an infinitesimal time period ≈ 0
  - Accumulation of traffic from multiple lines
  - Independence!!
- Is message service time independent of the arrival process?
- Independent Assumption: the service time of a message is chosen independently at each node
  - Many sources feed into one queue – valid approximation



Store-and-Forward Message Switched Networks – Delay Optimization

- **Objective**: Determine link capacities (bit/sec) so that mean end-to-delay for network is minimum
- **Assumptions and notations:**
  - Packet size = B bits
  - ith link/node/server capacity = C_i bit/sec → mean service time is equal to $E[M_i] = E[B]/C_i$
- **Previously, we have derived the mean delay to be**

$$E[T] = \frac{1}{\alpha}\sum_{i=1}^{N}\frac{\Lambda_i E[B]}{C_i - \Lambda_i E[B]} = \frac{1}{\alpha}\sum_{i=1}^{N}\frac{I_i}{C_i - I_i}$$

Note that $R_i = \Lambda_i/\mu_i = \Lambda_i E[M_i] = \Lambda_i E[B]/C_i$ – $\alpha = \Sigma\lambda_i$ (i.e. sum of external arrivals) – $I_i = \Lambda_i E[B_i]$

Store-and-Forward Message Switched Networks – Delay Optimization – cont'd

- **One can refine the mean delay formula by including the link propagation time, P_i – The resulting formula is**

$$E[T] = \frac{1}{\alpha}\sum_{i=1}^{N}\left(\frac{I_i}{C_i - I_i} + \Lambda_i P_i\right)$$

Store-and-Forward Message Switched Networks – Delay Optimization – cont'd

- **Let us define the following performance figure**

$$E[T^k] = \frac{1}{\alpha}\left[\sum_{i=1}^{N}\left(\frac{I_i}{C_i - I_i}\right)^k\right]^{1/k}$$

- **Special cases:**
  - **For k=1 ➔ mean delay**
  - **For k=2 ➔ standard deviation of delay (assuming the delays are independent)**

Store-and-Forward Message Switched Networks – Delay Optimization – cont'd

- **Special cases (cont'd):**
  - **For k=∞ (refer to reference below)**

$$E\left[T^{\infty}\right]=\lim_{k\to\infty}\frac{1}{\alpha}\left[\sum_{i=1}^{N}\left(\frac{I_i}{C_i-I_i}\right)^k\right]^{1/k}=\frac{1}{\alpha}\max_i\left(\frac{I_i}{C_i-I_i}\right)=\frac{I_{k^*}}{\alpha\left(C_{k^*}-I_{k^*}\right)}$$

where k* is the value of i for which $I_i/(C_i-I_i)$ is maximum over i=1,2,…,N

- **Two applications:**
  - **Given I_i; i=1,2, …, N determine C_is such that mean delay is minimum – This will be tackled later**
  - **Given C_i; i=1,2, …, N determine the routing scheme (i.e. I_is) such that mean delay is minimum**

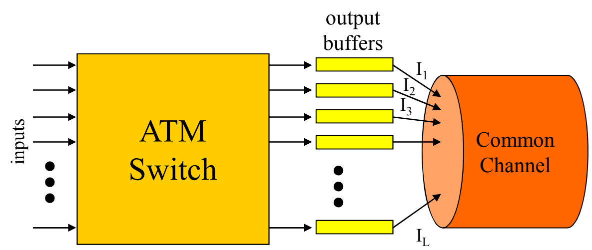
Meister, B.; Muller, H.; Rudin, H., Jr.; "New Optimization Criteria for Message-Switching Networks," IEEE Transactions on Communications, Volume: 19 , Issue: 3 , Jun 1971, Pages:256 - 26

Example: Store-and-Forward Message Switched Networks

- **<u>Problem</u>: Example 4.5 in textbook is missing information (average packet size for example – refer to Errata sheet)**
- **Network topology is a bit challenging**
- **Given table is packet routing table and NOT our Q (probabilistic routing matrix)!!**
- **Mere numerical substitution once the missing information is given**

Capacity Allocation

- Consider the following optimization problem that occurs in store-and-forward networks (switches for example)
- The switch hardware/software allocate capacities for the individual output links such that the sum does exceed the total available capacity
- How would the capacities be allocated?
  - How about minimizing the average delay?



ATM Multiplexing

Capacity Allocation – cont'd

- We have shown the average delay to be

$$E[T] = \frac{1}{\alpha}\sum_{i=1}^{L}\frac{I_i}{C_i - I_i}$$

assuming we have L links. We require the sum of the individual capacities be less than some upper bound C, i.e.

$$C \geq \sum_{i=1}^{L} C_i$$

- You can show that the individual C_ls should be give by

$$C_l = I_l + \frac{\left(C - \sum_{i=1}^{N} I_i\right)\sqrt{I_l}}{\sum_{i=1}^{N}\sqrt{I_i}}; \quad l = 1,2,\ldots,L$$

Capacity Allocation – cont'd

- **Note the allocated capacity is the minimum required (I_l) plus a fraction of the remaining capacity ($C-\Sigma I_l$)**
- **The minimum average is equal to**

$$E[T]_{\min} = \frac{1}{\alpha}\frac{\left(\sum_{i=1}^{N}\sqrt{I_i}\right)^2}{C-\sum_{i=1}^{N} I_i}$$

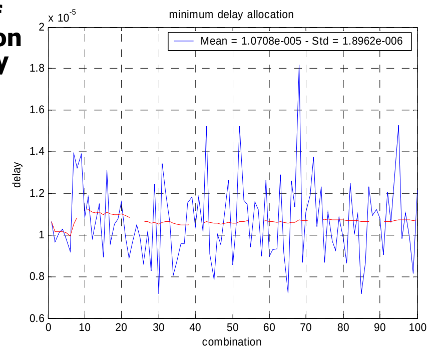
Example: Capacity Allocation

- **<u>Problem</u>: Assume 10 OC-1 (51.84 Mb/s) inputs are multiplexed on an output link who is total capacity is OC-12 (622.08 Mb/s) – If the volume of the input lines is chosen at random, determine the optimal allocation for each case and the average and standard deviation of the overall delay.**

For a nice table of the OC hierarchy please refer to http://www.linktionary.com/o/oc.html

Example: Capacity Allocation – cont'd

- **<u>Solution</u>: It can be seen that if the optimum capacity allocation is always used, the mean delay is 1.07e-5 while the standard deviation is 1.9e-6**




```
clear all
OC1  = 51.84e6;
OC12 = 622.08e6;
L    = 10; % L inputs
N    = 100;
for i=1:N
    Is   = OC1 * rand(1,L);
    RemC = OC12 - sum(Is);
    Cs   = Is + RemC * sqrt(Is)./sum(sqrt(Is));
    Alpha = sum(Is)/(53*8);
    T(i) = sum(sqrt(Is))^2/RemC/Alpha;
end
for i=1:N % compute mean
    TM(i) = mean(T(1:i));
end

figure(1);
h = plot(1:N, T, '-', 1:N, TM,'--r');
title('minimum delay allocation');
xlabel('combination');
ylabel('delay');
legend(['Mean = ' num2str(mean(T)) ' - Std = '
  num2str(std(T))]);
grid
```

What are the units of the delay in the above curve?

Closed Jackson Networks

- **Closed: fixed number of messages circulate within the network with neither arrivals to nor departures from the network**
- **Classic application – computer system**
  - **Over a short period it can be assumed that tasks/processes/customers neither enter nor leave the system**

Closed Jackson Networks – Traffic Equation

- **Since there are no external arrivals, the traffic equation reduces to**

$$\Lambda_i = \sum_{j=1}^{N} q_{ji}\Lambda_j$$

$$[\Lambda_1 \quad \Lambda_2 \quad \cdots \quad \Lambda_N] = [\Lambda_1 \quad \Lambda_2 \quad \cdots \quad \Lambda_N]\begin{bmatrix} 0 & q_{12} & \cdots & q_{1N} \\ q_{21} & 0 & \cdots & q_{2N} \\ \vdots & & 0 & \\ q_{N1} & q_{N2} & \cdots & 0 \end{bmatrix}$$

$$\Lambda = \Lambda Q$$

$$\Lambda^T = Q^T \Lambda^T$$

Note that Λ is the transpose of the eigenvector for the matrix Q^T corresponding to the eigenvalue 1!!

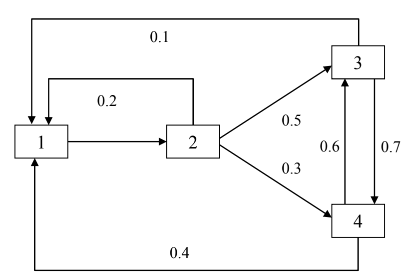
For a nice and very breif introduction to eigenvalues and eigenvectors please refer to http://www.sosmath.com/matrix/eigen0/eigen0.html or http://maths.ucd.ie/courses/math1200/algebra/algebranotes7-1.pdf
In Matlab do "help eig" to get help regarding eigenvalues/vectors calculations

Example: Traffic Equation for Closed Networks

- **Problem: For the closed network shown in figure,**

a) Find the routing matrix, Q?

b) Compute the total flows into each node?

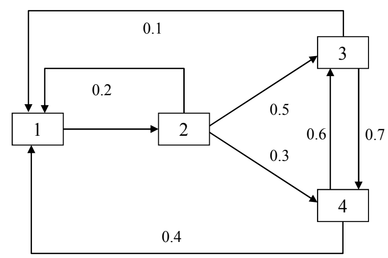


Example: Traffic Equation for Closed Networks

- **Solution:**

A) The routing matrix, Q:

$$Q = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ 0.2 & 0 & 0.5 & 0.3 \\ 0.1 & 0.2 & 0 & 0.7 \\ 0.4 & 0 & 0.6 & 0 \end{bmatrix}$$



Example: Traffic Equation for Closed Networks

- **Solution:**

B) The eigenvectos/values are calculated as shown

Hence, the *relative* flows are

Λ = [1.0 1.31 1.53 1.46]

```
>> [Vectors, Values] = eig(Q');
>> Vectors

Vectors =

   0.3728             0.0490 - 0.2601i   0.0490 + 0.2601i  -0.2709
   0.4870            -0.7672            -0.7672             0.5362
   0.5710             0.2160 + 0.1956i   0.2160 - 0.1956i  -0.6822
   0.5458             0.5022 + 0.0645i   0.5022 - 0.0645i   0.4169

>> Values

Values =

   1.0000                  0                  0                  0
        0            -0.1202 + 0.2880i        0                  0
        0                  0            -0.1202 - 0.2880i        0
        0                  0                  0            -0.7596

>> Vectors(:,1)./Vectors(1,1)

ans =

    1.0000
    1.3063
    1.5315
    1.4640
```

Closed Jackson Networks – Global Balance Equations

- **Same assumptions as before**
  - **Exponential and independent service time**
  - **Si servers at node i**
- **K – total number of customers**
- **An easy extension to the equations derived for open networks**

$$P(k_1,k_2,\ldots,k_N)\sum_{i=1}^{N}\mu_i d(k_i)=\sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i d(k_i+1)q_{ij}P(k_1,k_2,\ldots,k_i-1,\ldots,k_j+1,\ldots,k_N)$$

$$\forall k_1,k_2,\ldots,k_N\geq 0$$

Closed Jackson Networks – Global Balance Equations – cont'd

- **It can be shown (following the same derivation process as that for open networks), the joint pmf is given by**

$$P(k_1,k_2,\ldots,k_N)=G(K,N)^{-1}\prod_{i=1}^{N}\left[R_i^{k_i}\Big/\prod_{j=1}^{k_i}d(j)\right]$$

$$=\begin{cases} G(K,N)^{-1}\prod_{i=1}^{N}R_i^{k_i} & \text{single server nodes} \\ G(K,N)^{-1}\dfrac{\prod_{i=1}^{N}R_i^{k_i}}{k_i!} & \text{infinite server nodes} \end{cases}$$

where Λ_1, Λ_2, …, Λ_N, is the solution to the traffic equation. $R_i = \Lambda_i/\mu_i$ and G(K,N) is a the normalization constant

Convolution Algorithm

- **How to calculate the normalization constant?**
- **Exhaustive method: find all (k_1, k_2, …, k_N) such that Σk_i = K – substitute in joint pmf and compute the constant G(K,N) such that the sum is equal to 1.**
  - **There are (N+K-1)!/(K!(N-1)!) ways - e.g. N = 4, K = 7 → 120 combinations!!**
  - **Prohibitive!!**
- **Use convolution algorithms**

Convolution Algorithm – Buzen – Simplified Version

- **Single server nodes → service rate is always μ – does not depend on number of customers at node**
- **Define S(k,n) = {k_1, k_2, …, k_n / Σk_i = k; 0≤k≤K; 1≤n≤N}**
- **Define G(k,n) by summing over the set S(k,n)**

$$G(k,n) = \sum_{S(k,n)} \prod_{i=1}^{n} R_i^{k_i}$$

- **G(k, n) is the sum over all possible ways of dispersing k messages among n nodes.**

Convolution Algorithm – Buzen – Simplified Version – cont'd

- **How to compute G(k,n)? – consider splitting the summation into**
  - **kn = 0**
  - **kn > 0**
- **Therefore we can write G(k,n) as**

$$G(k,n)=\sum_{\substack{S(k,n)\\k_n=0}}\prod_{i=1}^{n}R_i^{k_i}+\sum_{\substack{S(k,n)\\k_n>0}}\prod_{i=1}^{n}R_i^{k_i}$$

- **But the first summation is just the sum over the first n-1 nodes since the nth node is empty. i.e. G(k,n-1)**
- **For the second summation – there is at least one message in node n – i.e. there are at most k-1 *other* messages in the total network. i.e. G(k-1,n)**
- **Hence, the G(k,n) can be written as**

$$G(k,n)=G(k,n-1)+R_nG(k-1,n)$$

Convolution Algorithm – Buzen – Simplified Version – cont'd

- **We can show that**

$$G(k,n)=G(k,n-1)+R_nG(k-1,n)$$

- **The initiating values:**

$$G(k,1)=R_1^k\,;\quad k=1,2,\ldots,K$$

$$G(0,n)=1;\quad n\geq 1$$

- **What is G(1,n) equal to for n >0?**

Example: Convolution Algorithm

- **<u>Problem</u>: Assume a four node network with the routing matrix Q**

  Assume μ = [2.5 2.5 2.5 2.5] and finite population of K = 7.

  A) Find the relative total flows

  B) Compute the joint distribution $P(k_1, k_2, k_3, k_4)$

$$Q = \begin{bmatrix} 0 & 0.75 & 0.25 & 0 \\ 0.05 & 0 & 0.15 & 0.8 \\ 0.25 & 0.25 & 0 & 0.5 \\ 0.4 & 0.35 & 0.25 & 0 \end{bmatrix}$$

Example: Convolution Algorithm

- **<u>Solution</u>: Closed Network K = 7, N = 4**

  A) Relative flows (found in the same manner as previous example

  Λ = [1.0000 1.5844 0.9195 1.7273]

  B) To compute the joint distribution $P(k_1, k_2, k_3, k_4)$, we need to compute:

  $R = \Lambda./\mu$

  = [0.4000 0.6338 0.3678 0.6909]

  Note R is the RELATIVE loading

  We also need to compute G(K,N) using Buzen's convolution algorithm

Example: Convolution Algorithm – cont’d

- **Solution: cont’d**

  Using the recursive algorithm (Refer to Matlab code) – G(7,4) = 1.7036

Therefore the joint pmf is equal to

$$P(k_1, k_2, \ldots, k_N) = \prod_{i=1}^{N} R_i^{k_i} \Big/ G(K, N)$$
$$= (0.4)^{k_1} (0.6338)^{k_2} (0.0.3678)^{k_3} (0.0.6910)^{k_4} / 1.7036$$

Example: Convolution Algorithm – cont’d

- **Solution: cont’d**

The following code implements the recursive alogorithm:

```
%
% Example 4.8

K = 7;
N = 4;
M = 2.5*ones(1,N);

Q = [0    0.75  0.25  0; ...
     0.05 0     0.15  0.8; ...
     0.25 0.25  0     0.5; ...
     0.4  0.35  0.25  0];

[Vectors, Values] = eig(Q');
RFlows = Vectors(:,1)'./Vectors(1,1); % relative flows
RR     = RFlows./M; % compute relative loads

ks = 0:K;
ns = 1:N;
G_K_N = zeros(K+1,N);
%
% fill initial values
G_K_N(1,:) = ones(1,N);
G_K_N(:,1) = RR(1).^ks';
%
% fill the remaining of the matrix
for n=2:N
    for k=1:K
        G_K_N(k+1, n) = G_K_N(k+1, n-1) + RR(n)*G_K_N(k, n);
    end
end
```

Program output:

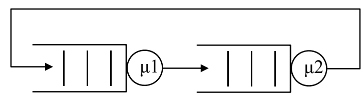
```
>> RFlows

RFlows =

    1.0000    1.5844    0.9195    1.7273

>> RR

RR =

    0.4000    0.6338    0.3678    0.6909

>> G_K_N

G_K_N =

    1.0000    1.0000    1.0000    1.0000
    0.4000    1.0338    1.4016    2.0925
    0.1600    0.8152    1.3306    2.7764
    0.0640    0.5806    1.0700    2.9882
    0.0256    0.3936    0.7871    2.8517
    0.0102    0.2597    0.5492    2.5195
    0.0041    0.1687    0.3707    2.1114
    0.0016    0.1085    0.2449    1.7036

>>
```

Example: Two-Node Network

- **<u>Problem</u>: Assume a network as shown in Figure with K total number of customers. The service rate for nodes 1 and 2 are μ1 and μ2, respectively. Using the theory of closed networks**

A) Derive the joint probability mass function

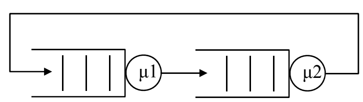
B) Derive the marginal distributions for each of the nodes



K circulating customers

Example: Two-Node Network – cont'd

- **<u>Solution</u>:**

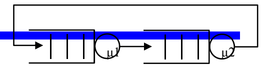


K circulating customers

This problem was solved in Assignment #2 as an example of a birth-and-death process.

Apply the theory of closed networks and make sure get matching answers

Example: Two-Node Network – cont'd



K circulating customers

- **<u>Solution</u>:**

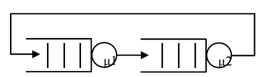
The routing matrix is given by $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Q^T has two eigen values: 1 and -1. The vector corresponding to the eigen value 1 is √2(1,1)^T.

Therefore, the relative total flow is given by Λ = [1 1] and the relative loading is given by R = [1/μ1 1/μ2]

Example: Two-Node Network – cont'd

- **<u>Solution</u>:**



K circulating customers

From the closed network theory, the joint pmf is given by

$$P(k_1,k_2) = \frac{R_1^{k_1} R_2^{k_2}}{G(K,2)}$$

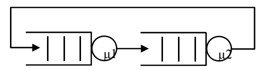
To find G(K,2) we either use the exhaustive method or follow the convolution algorithm explained in class

<u>The exhaustive method</u>: all (k_1, k_2) states such that k_1+k_2 = K can be written as (k_1, K-k_1) for k_1 = 0, 1, …, K

Therefore $G(K,2) = \sum_{i=0}^{K} R_1^{k_1} R_2^{K-k1}$

Example: Two-Node Network – cont'd

- **<u>Solution</u>:**



K circulating customers

<u>The convolution method:</u> is shown in table on the side – Again, G(K,2) is given by

$$G(K,2)=\sum_{i=0}^{K} R_1^{\,i} R_2^{\,K-i}$$

	1	2
0	1	1
1	R_1	R_1+R_2
2	R_1^2	$R_1^2+R_1R_2+R_2^2$
3	R_1^3	$R_1^3+R_1^2R_2+R_1R_2^2+R_2^3$
.		.
.		.
K	R_1^K	$\sum_{i=0}^{K} R_1^{K-i} R_2^{i}$

Therefore the final joint PMF is given by

$$P(k_1,k_2)=\frac{R_1^{\,k_1} R_2^{\,k_2}}{\sum_{i=0}^{K} R_1^{\,i} R_2^{\,K-i}}$$

Example: Two-Node Network – cont'd

- **<u>Solution</u>:**

The marginal distribution for node 1 is given by

$$P(k_1)=P(k_1,K-k_1)=\frac{R_1^{\,k_1} R_2^{\,K-k_1}}{\sum_{i=0}^{K} R_1^{\,i} R_2^{\,K-i}}$$

$$=\frac{R_2^{\,K}(R_1/R_2)^{k_1}}{R_2^{\,K}\sum_{i=0}^{K}(R_1/R_2)^{i}}$$

The same result obtained before

$$=\frac{(\mu_2/\mu_1)^{k_1}}{\sum_{i=0}^{K}(\mu_2/\mu_1)^{i}}$$

$$=\frac{1-\mu_2/\mu_1}{1-(\mu_2/\mu_1)^{K+1}}\left(\frac{\mu_2}{\mu_1}\right)^{k_1}$$

Mean Number of Message in Each Queue

- **Consider a closed network with K messages and N nodes**
- **Probability of node i having k customers or more (i.e. K-k or less are dispersed in the rest of N-1 nodes) is given by**

$$\Pr ob(Q_i \geq k) = \frac{\sum_{\substack{S(K,N) \\ k_i \geq k}} \prod_{i=1}^{N} R_i^{k_i}}{G(K,N)} = \frac{R_i^k \sum_{S(K-k,N)} \prod_{i=1}^{N} R_i^{k_i}}{G(K,N)}$$

$$= \frac{R_i^k G(K-k,N)}{G(K,N)}; \quad k \geq 1$$

Mean Number of Message in Each Queue – cont'd

- **Therefore, Probability of node i having exactly k customers is given by**

$$\Pr ob(Q_i = k) = \Pr ob(Q_i \geq k) - \Pr ob(Q_i \geq k+1)$$

$$= \frac{R_i^k G(K-k,N) - R_i^{k+1} G(K-k-1,N)}{G(K,N)}$$

- **Therefore, the mean number of customers in node i is given by**

$$E[K_i] = \sum_{k=0}^{K} k \Pr ob(Q_i = k)$$

$$= \frac{1}{G(K,N)} \sum_{k=0}^{K} k \left[R_i^k G(K-k,N) - R_i^{k+1} G(K-k-1,N) \right]$$

Mean Number of Message in Each Queue – cont'd

- **The previous formula can be simplified to be**

$$E[K_i] = \frac{1}{G(K,N)} \sum_{k=1}^{K} R_i^k G(K-k,N); \quad i = 1,2,\ldots,N$$

- **Can you do the above simplification?**
- **What is $\sum_{i=1}^{N} E[K_i]$ equal to? Prove?**

Absolute Flows

- **The derived quantities Λi and Ri were all relative**
- **Let us derive ρ_i or the i^{th} node utilization**
- **From definition of ρ_i = Prob(Qi ≥ 1), therefore**

$$\Pr ob[Q_i \geq 1] = \rho_i = \frac{R_i G(K-1,N)}{G(K,N)}$$

- **The absolute flow is equal to ρ_i divided by the average service time, i.e.**

$$\Omega_i = \frac{\Lambda_i G(K-1,N)}{G(K,N)}; \quad i = 1,2,\ldots,N$$

Remember $R_i = \Lambda_i / \mu_i = \Lambda_i M_i$, where M_i is the average service time

Message Delays

- **The average delay of a message through the ith node can be derived from the mean number of message in node through the application of Little's formula**

$$E[D_i] = \frac{E[K_i]}{\Omega_i}$$

Example: Four-node Networks

- **<u>Problem</u>: Using the four node network specified on slide 77**

  a) Compute the mean number of customers in each of the four nodes

  b) Compute the absolute flow into each node

  c) Compute the mean delay through each of the four nodes

Example: Four-node Networks

- **<u>Solution</u>: Refer to Matlab code on next slide for implementation of previous formula**

a) The mean number of customers per node is given by [0.9071 2.3589 0.7858 2.9483]
b) Absolute flows: [1.2393 1.9636 1.1395 2.1407]
c) Mean delays: [0.7319 1.2013 0.6896 1.3773]

Note that sum of mean number of customers should be equal to K = 7!!

- **Regarding the Matlab code implementation: note that G_K_N matrix is of size K+1 by N – where 1st row corresponds to k=0, 2nd row to k=1, …, and K+1st row to k=K.**
  - **Therefore, G(K-1,N) in the previous formulas corresponds to G_K_N(K,N) in the Matlab code. Similary, G(K-2,N) in formula corresponds to G_K_N(K-1,N) in the Matlab code, and so on**

Example: Four-node Networks

<u>Solution</u>: Matlab code for example

```
%
% Example 4.8
K = 7;
N = 4;
M = 2.5*ones(1,N);

Q = [0    0.75  0.25  0; ...
     0.05 0     0.15  0.8; ...
     0.25 0.25  0     0.5; ...
     0.4  0.35  0.25  0];

[Vectors, Values] = eig(Q');
RFlows = Vectors(:,1)'./Vectors(1,1); % relative flows
RR     = RFlows./M; % compute relative loads

ks = 0:K;
ns = 1:N;
G_K_N = zeros(K+1,N);
%
% fill initial values
G_K_N(1,:) = ones(1,N);
G_K_N(:,1) = RR(1).^ks';
%
% fill the remaining of the matrix
for n=2:N
    for k=1:K
        G_K_N(k+1, n) = G_K_N(k+1, n-1) + RR(n)*G_K_N(k, n);
    end
end
%
% Mean numbers
for i=1:N
    Kmean(i) = sum(RR(i).^(1:K)' .* G_K_N(K:-1:1,N))/...
               G_K_N(K+1,N);
end
Omega = RFlows*G_K_N(K,N)/G_K_N(K+1,N);
Dmean = Kmean./Omega;
```

```
>> Example_4_8
>> RFlows

RFlows =

    1.0000    1.5844    0.9195    1.7273

>> RR

RR =

    0.4000    0.6338    0.3678    0.6909

>> Kmean

Kmean =

    0.9071    2.3589    0.7858    2.9483

>> Omega

Omega =

    1.2393    1.9636    1.1395    2.1407

>> Dmean

Dmean =

    0.7319    1.2013    0.6896    1.3773

>>
```

Infinite Server Case

- **How to compute the normalizing constant**

$$G(k,n) = \sum_{S(k,n)} \prod_{i=1}^{n} \frac{R_i^{k_i}}{k_i!}$$

for an infinite server case

- **The above is the multinomial expansion, i.e.**

$$G(k,n) = \frac{\left[\sum_{i=1}^{n} R_i\right]^k}{k!}$$

The binomial expansion is given by

$$(p+q)^k = \sum_{i=0}^{k} \binom{k}{i} p^i q^{k-i}$$

This is generalized by the multinomial expansion:

$$(x_1 + x_2 + \ldots + x_n)^k = \sum_{k_1 + k_2 + \ldots + k_n = k} \frac{k!}{k_1! k_2! \ldots k_n!} x_1^{k_1} x_2^{k_2} \ldots x_{k_n}^{k_n}$$

Infinite Server Case – Joint and Marginal Distributions

- **Therefore, the joint pmf for the closed network case with infinite servers case is given by**

$$P(k_1, k_2, \ldots, k_N) = \frac{K! \prod_{i=1}^{N} \left[R_i^{k_i} / k_i!\right]}{\left[\sum_{i=1}^{N} R_i\right]^K}$$

- **The marginal distribution of the Nth node can also be found as an application of the multinomial expansion:**

$$P(K_N = m) = \sum_{S(K-m,N-1)} \frac{K! \prod_{i=1}^{N} \left[R_i^{k_i} / k_i!\right]}{\left[\sum_{i=1}^{N} R_i\right]^K} = \binom{K}{m} R_N^m \frac{\left[\sum_{i=1}^{N-1} R_i\right]^{K-m}}{\left[\sum_{i=1}^{N} R_i\right]^K}$$

Finite Server Case – Joint and Marginal Distributions – cont'd

- **The previous expression can be applied to obtain the marginal distribution for any node**

Finite Server Case – Mean Number of Customers & Absolute Flows

- **Show that the mean number of customers in the i^{th} node is equal to**

$$E[K_i] = \frac{KR_i}{\sum_{i=1}^{N} R_i}; \quad i = 1,2,\ldots,N$$

Note that the sum of the mean is equal to K!!

- **The i^{th} absolute flow is given by**

$$\Omega_i = \frac{K\Lambda_i}{\sum_{i=1}^{N} R_i}; \quad i = 1,2,\ldots,N$$

Example: Infinite Server Case

- **Problem: Consider the previous four node problem where the single servers are replaced with infinite server models.**

a) Calculate the average number of customers in each of the 4 nodes?

b) Calculate the absolute flows to each of the 4 nodes

Example: Infinite Server Case

- **Solution:**

a) K = 7, N = 4

In the previous example, relative flows and loadings were found to be:

Λ = [1.0000 1.5844 0.9195 1.7273]

$R = \Lambda./\mu$

= [0.4000 0.6338 0.3678 0.6909]

$E[K_i] = KR_i / \Sigma R_i$

$E[K_i]$ = [1.3381 2.1202 1.2304 2.3113]

Check $\Sigma K_i = K = 7$.

b) The absolute flows are given by:

Ω = [3.3453 5.3004 3.0760 5.7783]

Example: Infinite Server Case

- **Solution:**

```
%
% Example 4.8
K = 7;
N = 4;
M = 2.5*ones(1,N);

Q = [0    0.75  0.25  0; ...
     0.05 0     0.15  0.8; ...
     0.25 0.25  0     0.5; ...
     0.4  0.35  0.25  0];

[Vectors, Values] = eig(Q');
RFlows = Vectors(:,1)'./Vectors(1,1); % relative flows
RR     = RFlows./M; % compute relative loads

KMean = K*RR./sum(RR);
Omega = K*RFlows./sum(RR);
```

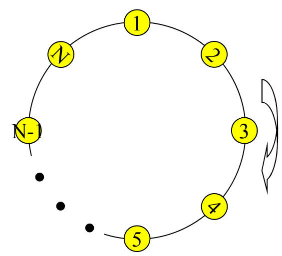
```
>> Example_4_9
>> KMean

KMean =

    1.3381    2.1202    1.2304    2.3113

>> sum(KMean)

ans =

     7

>> Omega

Omega =

    3.3453    5.3004    3.0760    5.7783

>>
```


Mean Value Analysis

- **Numerical problems arise when attempting to compute the normalization constant using the convolutional method**
- **Alternative – Mean value analysis**
  - **It yields averages rather than distributions**
  - **Usually sufficient for most applications.**
- **Based on the arrival theorem:**
  - **Within a closed chain containing k messages, the distribution of the number of messages of it own class seen by a message arriving at a node is the steady-state distribution for the case of one less message in the chain, k-1.**
  - **In contrast – for Poisson arrivals in an open network, the steady-state distribution and the distribution seen by an arriving message are identical**
- **For a simplified proof of the arrival theorem, refer to chapter 9 of Leon Garcia's textbook.**

Mean Value Analysis – Closed Chains

- **Consider a closed chain as shown in figure**
- **Assume:**
  - **no of circulating messages is k**
  - **Service time in node i is M_i ; i=1,2, …, N**



Mean Value Analysis – Closed Chains – cont'd

- **Delay at node i, is $d_i(k) = M_i[1+n_i(k-1)]$; k=1,2, …, K – where $n_i(k-1)$ is the average number of customers found in queue (or when there are k-1 customers circulating) by the k^{th} customer**
- **Throughput $\lambda(k) = k/ \Sigma d_i(k)$; k=1,2, …, K – the sum is carried over all N nodes**
  - **Note $\Sigma d_i(k)$ is the total delay around the chain**
- **Applying Little's formula again $n_i(k) = \lambda(k)\, d_i(k)$; i=1,2, …, N; and k=1,2, …, K.**
- **The above procedure is used iteratively to find $d_i(K)$ and $n_i(K)$**
  - **Initially $n_i(0) = 0$; for i=1,2, …, N**

Example: Mean Value Analysis – Closed Chains

- **Problem: Consider the example of a closed chain with K = 14, and N = 6. Assume M = [2.5, 0.75, 0.03, 0.2, 0.5, 1.2].**

  a) Compute the mean number of message at each node.

  b) Use the detailed method outlined on slide 89 to calculate the mean number of message at each node

Example: Mean Value Analysis – Closed Chains

- **Solution:**

  Direct application of the iterative algorithm reveals that

$n_i(K)$ = [12.2999 0.4285 0.0121 0.0870 0.2500 0.9225]

$d_i(K)$ = [30.7514 1.0713 0.0304 0.2174 0.6250 2.3063], and

$\lambda(k) = 0.400$

The following slide shows the intermediate solutions

Example: Mean Value Analysis – Closed Chains

- **Solution:**

```
%
% Example 4.10a
K = 14;
N = 6;
M = [2.5 0.75 0.03 0.2 0.5 1.2];

n_k_i = zeros(K+1,N);
d_k_i = zeros(K,N);
for k=1:K
    d_k_i(k,:) = M .* ( 1 + n_k_i(k,:) );
    Lambda(k)  = k./sum(d_k_i(k,:));
    n_k_i(k+1,:) = Lambda(k) .* d_k_i(k,:);
end
```


Example: Mean Value Analysis – Closed Chains

- **Solution:**

Output is as shown:

```
>> Example_4_10a
>> n_k_i
n_k_i =
```

	i=1	i=2	i=3	i=4	i=5	i=6
k=0	0	0	0	0	0	0
k=1	0.4826	0.1448	0.0058	0.0386	0.0965	0.2317
k=2	1.0855	0.2514	0.0088	0.0608	0.1606	0.4328
	1.7990	0.3239	0.0104	0.0732	0.2002	0.5933
	2.6043	0.3695	0.0113	0.0799	0.2234	0.7116
.	3.4791	0.3966	0.0117	0.0834	0.2362	0.7930
.	4.4022	0.4118	0.0119	0.0852	0.2430	0.8459
	5.3568	0.4200	0.0120	0.0861	0.2465	0.8786
.	6.3308	0.4242	0.0121	0.0865	0.2483	0.8980
	7.3163	0.4264	0.0121	0.0868	0.2492	0.9092
	8.3083	0.4275	0.0121	0.0869	0.2496	0.9156
	9.3040	0.4281	0.0121	0.0869	0.2498	0.9190
	10.3018	0.4283	0.0121	0.0869	0.2499	0.9209
k=13	11.3006	0.4285	0.0121	0.0869	0.2500	0.9220
$n_i(K)$ ⇒ k=14	12.2999	0.4285	0.0121	0.0870	0.2500	0.9225

Example: Mean Value Analysis – Closed Chains

- **Solution:**

Output is as shown:

i=1 i=2 i=3 i=4 i=5 i=6

```
>> d_k_i

d_k_i =

k=0     2.5000    0.7500    0.0300    0.2000    0.5000    1.2000
k=1     3.7066    0.8586    0.0302    0.2077    0.5483    1.4780
k=2     5.2137    0.9386    0.0303    0.2122    0.5803    1.7194
        6.9975    0.9929    0.0303    0.2146    0.6001    1.9119
        9.0109    1.0272    0.0303    0.2160    0.6117    2.0539
 .     11.1978    1.0474    0.0304    0.2167    0.6181    2.1516
 .     13.5056    1.0588    0.0304    0.2170    0.6215    2.2151
       15.8920    1.0650    0.0304    0.2172    0.6233    2.2543
 .     18.3270    1.0682    0.0304    0.2173    0.6241    2.2776
       20.7907    1.0698    0.0304    0.2174    0.6246    2.2911
       23.2708    1.0706    0.0304    0.2174    0.6248    2.2987
       25.7601    1.0710    0.0304    0.2174    0.6249    2.3029
k=13   28.2544    1.0712    0.0304    0.2174    0.6250    2.3051
k=14   30.7514    1.0713    0.0304    0.2174    0.6250    2.3063
```

$d_i(K)$

$\lambda(K)$

```
>> Lambda

Lambda =
  Columns 1 through 7
    0.1931    0.2929    0.3450    0.3722    0.3861    0.3931    0.39[illegible]
  Columns 8 through 14
    0.3984    0.3992    0.3996    0.3998    0.3999    0.4000    0.4000
```


Example: Mean Value Analysis – Closed Chains

- **Solution:**

  b) Using the detailed method: The code on slide 96 is modified to solve for this particular network.

  The routing matrix Q is changed to reflect the new routing policy for this chain.

  Furthermore, the eigen vector corresponding to eigen value 1 is the 6th column.

Example: Mean Value Analysis – Closed Chains

- **Solution:**

```
%
% Example 4.10a
clear all
K = 14;
N = 6;
M = 1./[2.5 0.75 0.03 0.2 0.5 1.2];

Q = [0  1  0  0  0  0; ...
     0  0  1  0  0  0; ...
     0  0  0  1  0  0; ...
     0  0  0  0  1  0; ...
     0  0  0  0  0  1; ...
     1  0  0  0  0  0];

[Vectors, Values] = eig(Q');
RFlows = Vectors(:,6)'./Vectors(1,1); % relative flows
RR     = RFlows./M; % compute relative loads

ks = 0:K;
ns = 1:N;
G_K_N = zeros(K+1,N);
%
% fill initial values
G_K_N(1,:) = ones(1,N);
G_K_N(:,1) = RR(1).^ks';
%
% fill the remaining of the matrix
for n=2:N
    for k=1:K
        G_K_N(k+1, n) = G_K_N(k+1, n-1) + RR(n)*G_K_N(k,
    n);
    end
end
%
% Mean numbers
for i=1:N
    Kmean(i) = sum(RR(i).^(1:K)' .* G_K_N(K:-1:1,N))/...
               G_K_N(K+1,N);
end
Omega = RFlows*G_K_N(K,N)/G_K_N(K+1,N);
Dmean = Kmean./Omega;
```

```
>> G_K_N
G_K_N =

  1.0e+006 *

    0.0000    0.0000    0.0000    0.0000    0.0000    0.0000
    0.0000    0.0000    0.0000    0.0000    0.0000    0.0000
    0.0000    0.0000    0.0000    0.0000    0.0000    0.0000
    0.0000    0.0000    0.0000    0.0000    0.0000    0.0001
    0.0000    0.0001    0.0001    0.0001    0.0001    0.0001
    0.0001    0.0001    0.0001    0.0002    0.0002    0.0004
    0.0002    0.0003    0.0004    0.0004    0.0005    0.0009
    0.0006    0.0009    0.0009    0.0010    0.0012    0.0023
    0.0015    0.0022    0.0022    0.0024    0.0030    0.0057
    0.0038    0.0054    0.0055    0.0060    0.0075    0.0144
    0.0095    0.0136    0.0138    0.0150    0.0187    0.0360
    0.0238    0.0341    0.0345    0.0375    0.0468    0.0900
    0.0596    0.0851    0.0862    0.0937    0.1171    0.2251
    0.1490    0.2129    0.2155    0.2342    0.2927    0.5629
    0.3725    0.5322    0.5386    0.5855    0.7319    1.4074

>> Kmean
Kmean =

   12.2999    0.4285    0.0121    0.0870    0.2500    0.9225

>> Dmean
Dmean =

   30.7514    1.0713    0.0304    0.2174    0.6250    2.3063

>> Omega
Omega =

    0.4000    0.4000    0.4000    0.4000    0.4000    0.4000
```


Mean Value Analysis - Generalization

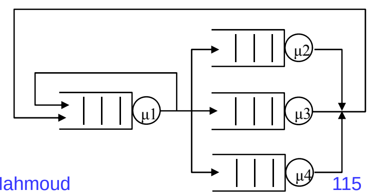
- **The previous iterative algorithm is a closed chain only**
- **The MVA algorithm is modified to accommodate general N-node closed networks:**
  - $d_i(k) = M_i[1+n_i(k-1)]$
  - $\lambda(k) = k/ \Sigma[\ \Lambda_i\ d_i(k)\ ];$ $G(k) = G(k-1)/\ \lambda(k)$
  - $n_i(k) = \Lambda_i\ \lambda(k)\ d_i(k)$; $i=1,2, \ldots, N$
  - **Initially** $n_i(0) = 0$; $i=1,2, \ldots, N$; $G(0) = 1$;
  - **The iterations are carried over** $k = 0, 1, \ldots, K$
- **The above algorithm also computes the normalization constant required for the joint pmf distribution!**
- **The above is valid for one class of users – but can be generalized for C classes of users (refer to Hayes's textbook)**

Example: Mean Value Analysis - Generalization

- **<u>Problem</u>: Consider the network shown in Figure for K = 6. The mean service times are given by M =[0.02 0.2 0.4 0.6]. Furthermore, the relative flows are given by Λ =[1 0.4 0.2 0.1].**

  a) Find the mean number of customers and mead delay for each node.

  b) Use the detailed method to verify your answer



Example: Mean Value Analysis - Generalization

- **<u>Solution</u>:**

  a) Applying the algorithm outlined for the MVA for closed networks, we find

  $n_i(K)$ = [0.2436 2.2610 2.2610 1.2343]
  $d_i(K)$ = [0.0246 0.5698 1.1397 1.2443], and
  λ(k) = 9.9198

  Furthermore, the normalization constant G(K,N) is equal to 5.7562e-006

Example: Mean Value Analysis – Generalization – cont'd

- **Solution:**

```
%
% Example MVA for closed network
K = 6;
N = 4;
M = [0.02 0.2 0.4 0.6];
L = [1 0.4 0.2 0.1];
n_k_i = zeros(K+1,N);
d_k_i = zeros(K,N);
G     = ones(K+1,1);
for k=1:K
    d_k_i(k,:) = M .* ( 1 + n_k_i(k,:) );
    Lambda(k)  = k./sum(L.*d_k_i(k,:));
    G(k+1)     = G(k)/Lambda(k);
    n_k_i(k+1,:) = L .* Lambda(k) .* d_k_i(k,:);
end
```


Example: Mean Value Analysis – Generalization – cont'd

- **Solution:**

```
> n_k_i

n k i =

         0         0         0         0
    0.0833    0.3333    0.3333    0.2500
    0.1398    0.6882    0.6882    0.4839
    0.1791    1.0608    1.0608    0.6993
    0.2072    1.4485    1.4485    0.8958
    0.2279    1.8491    1.8491    1.0738
    0.2436    2.2610    2.2610    1.2343

>> d_k_i

d_k_i =

    0.0200    0.2000    0.4000    0.6000
    0.0217    0.2667    0.5333    0.7500
    0.0228    0.3376    0.6753    0.8903
    0.0236    0.4122    0.8243    1.0196
    0.0241    0.4897    0.9794    1.1375
    0.0246    0.5698    1.1397    1.2443
```

```
>> G

G =

    1.0000
    0.2400
    0.0372
    0.0047
    0.0005
    0.0001
    0.0000      5.7562e-006

>> Lambda

Lambda =

    4.1667    6.4516    7.8547
     8.7860    9.4401    9.9198
```

Example: Mean Value Analysis – Generalization – cont'd

- **Solution:**

b) using the detailed method

```
%
% Example MVA for closed networks - b
clear all
K = 6;
N = 4;
M = 1./[0.02 0.2 0.4 0.6];
L = [1 0.4 0.2 0.1];
RFlows = L; % relative flows
RR     = RFlows./M; % compute relative loads

ks = 0:K;
ns = 1:N;
G_K_N = zeros(K+1,N);
%
% fill initial values
G_K_N(1,:) = ones(1,N);
G_K_N(:,1) = RR(1).^ks';
%
% fill the remaining of the matrix
for n=2:N
    for k=1:K
        G_K_N(k+1, n) = G_K_N(k+1, n-1) + RR(n)*G_K_N(k, n);
    end
end
%
% Mean numbers
for i=1:N
    Kmean(i) = sum(RR(i).^(1:K)' .* G_K_N(K:-1:1,N))/...
                  G_K_N(K+1,N);
end
Omega = RFlows*G_K_N(K,N)/G_K_N(K+1,N);
Dmean = Kmean./Omega;
```

Same as $G(k)$ computed by the MVA algorithm

```
>> Example_MVA_CNb
>> G_K_N

G_K_N =

    1.0000    1.0000    1.0000    1.0000
    0.0200    0.1000    0.1800    0.2400
    0.0004    0.0084    0.0228    0.0372
    0.0000    0.0007    0.0025    0.0047
    0.0000    0.0001    0.0003    0.0005
    0.0000    0.0000    0.0000    0.0001
    0.0000    0.0000    0.0000    0.0000

>> Kmean

Kmean =

    0.2436    2.2610    2.2610    1.2343

>> Dmean

Dmean =

    0.0246    0.5698    1.1397    1.2443
```


BCMP Networks

- **BCMP = Baskette, Chandy, Muntz, Palacios = 1975 paper**
- **Generalization of the product forms obtained for Jackson networks for FCFS**
- **The product form holds for**
  1. **FCFS with exponential service times – studies in previous sections (and chapter 3)**
  2. **Infinite server model: a message immediately assigned a server as soon as it enters the system – all messages are simultaneously in service**
  3. **Processor sharing: each message in the queue receives equal simultaneous service – all messages are simultaneously in service**
  4. **Preemptive resume last-come first-served: newly arrived messages are served immediately – displaced messages are re-queued and resume server only when the server is available again**

BCMP Networks – FEATURES

1. **More than one class of messages is allowed**
2. **For 2, 3, and 4: the product form holds also for arbitrary service times too!!**
   - **An arbitrary service time distribution may be approximated by a rational Laplace tranform, then the model tranforms to Cox network (refer to chapter 3)**
   - **Insensitivity property follows for state occupancy probabilities**
3. **For 2, 3, and 4: different classes of messages may have different service time distributions**
4. **For 2, 3, and 4: message are allowed to change class probabilistically**
5. **Networks may be mixed with respect to class: closed for one class, but open for another**
6. **Arrivals may be dependent on the state of the network, under certain conditions – e.g. limited storage problem**

Probabilistic and Markov Routing

- **Messages are allowed to switch classes probabilistically as they are routed between nodes**
- **Def: customer of class k leaving node i is switched to class j and routed to node j with probability** q_{ij}^{kl}
- **Traffic equation for N nodes and C customer classes is given by**

$$\Lambda_i^k = \lambda_i^k + \sum_{j=1}^{N} \sum_{l=1}^{C} \Lambda_j^l q_{ji}^{lk}$$

BCMP Networks – cont'd

- The following slides are going to show the results (product forms) for probability distribution of message in BCMP networks for SINGLE NODE case (for each of the 4 disciplines)
- Then the results will be extended to the case of a network of N nodes, again for each of the 4 disciplines
- The slides shown only the end results – the derivations are found in the textbook

BCMP Networks – Single Node with Exponential Server

- Poisson arrivals, exponential Service time, FCFS discipline
- For case of 1 class of users

$$P(Q=n)=(1-\rho)\rho^n; \quad n=0,\,1,\,2,\,\ldots$$

- C classes of users
- The joint pmf can be shown to be

$$P\left(N_1=n_1, N_2=n_2, \ldots, N_C=n_C, \sum_{i=1}^{C} n_i = m\right)=(1-\rho)m!\prod_{i=1}^{C}\frac{\rho_i^{n_i}}{n_i!}$$

- Note the product form holds for multiple classes in a single node network!!
- To verify above expression sum over all n_i

BCMP Networks – Single Node with Infinite Server

- **This problem has been handled when service time is exponential (textbook eq 3.55)**
  - **For one class** $P(Q=m)=\dfrac{e^{-\rho}\rho^{m}}{m!}; \quad m=0,1,2,\ldots$ **where ρ = λ/μ < 1**

- **Here, the result is extended to ARBITRARY service distribution**

BCMP Networks – Single Node with Infinite Server – cont'd

- **For an arbitrary service distribution – assuming ONE class of user**
  - **Employ Cox network with K stages with the typical parameters:**
    - **Initial and final routing probabilities** $q_0=1, q_K=0$
    - **External arrival rate** λ
    - **Average service rate for** i^{th} **stage is** ν_i
    - **Then arrivals to the ith stage are given by**
      $\omega_1=\lambda, \omega_{i+1}=\omega_i q_i; \quad i=1,2,\cdots,K-1$

BCMP Networks – Single Node with Infinite Server – cont'd

- **The textbook shows that the joint PMF for the customers in the K stages is given by**

$$P(k_1, k_2, \cdots, k_K) = G^{-1} \prod_{i=1}^{K} \frac{\gamma_i^{k_i}}{k_i!}$$

where $\gamma_i = \omega_i / \nu_i$ **and** G^{-1} **is some constant**

- **However, the interest is in the total number of customers** $m = \sum_{i=1}^{K} k_i$**, it is shown that**

$$P(Q = m) = P\left(\sum_{i=1}^{K} k_i = m\right) = \frac{e^{-\rho} \rho^m}{m!}; \quad m \geq 0$$

where ρ = λ/μ.

BCMP Networks – Single Node with Infinite Server – cont'd

- **The previous result may be extended to C classes of messages**
- **The joint PMF for messages of each class is given by**

$$P(n_1, n_2, \ldots, n_C) = e^{-\rho} \prod_{i=1}^{C} \frac{\rho_i^{n_i}}{n_i!}; \quad \forall n_i \geq 0$$

- **The PMF for the total number of messages is given by**

$$P\left(\sum_{i=1}^{C} n_i = m\right) = \frac{e^{-\rho} \rho^m}{m!}; \quad m \geq 0$$

BCMP Networks – Single Node with Processor Sharing

- C classes of message: i = 1, 2, …, C.
- Each has its mean service time M_i
  - i^{th} service time is modeled by Cox network with K_i stages
  - With q_{ij} transition probability and average transition rate ν_{ij} of message from i^{th} class and j^{th} stage
- Each has it own arrival rate λ_i
- Processor sharing – departure rate from a stage is $\nu_{ij} k_{ij} / m$ where $m = \sum_{i=1}^{C} \sum_{j=1}^{K} k_{ij}$

BCMP Networks – Single Node with Processor Sharing – cont'd

- The joint PMF is given by

$$P(n_1, n_2, \ldots, n_C) = (1-\rho) m! \prod_{i=1}^{C} \frac{\rho_i^{n_i}}{n_i!}; \quad m \geq 0$$

where ρ = Σ ρi, ρi= λ_i M_i and Σn_i = m

- The PMF for total number of messages, m, is given by

$$P\left(\sum_{i=1}^{C} n_i = m\right) = \frac{e^{-\rho} \rho^m}{m!}; \quad m \geq 0$$

BCMP Networks – Single Node with Pre-emptive Resume Last-Come First-Served Discipline

- **The textbook derives the PMF for m for one class of messages**
- **Then, the results is extended to C classes of messages**
  - **The joint PMF is given by**

$$P(N_1 = n_1, N_2 = n_2, \ldots, N_C = n_C) = (1-\rho)m! \prod_{i=1}^{C} \frac{\rho_i^{n_i}}{n_i!}$$

  - **The PMF for number of messages m is given by**

$$P\left(\sum_{i=1}^{C} n_i = m\right) = \frac{e^{-\rho}\rho^m}{m!}; \quad m \geq 0$$

where $\rho = \Sigma\lambda_i/\mu = \Sigma\, \rho_i$ – and $\Sigma n_i = m$

BCMP Networks – Single Node - Summary

- **We can summarize the results for the single node and C classes of customers as**

$$P(N_1 = n_1, N_2 = n_2, \ldots, N_C = n_C) = \begin{cases} (1-\rho)m!\prod_{i=1}^{C} \rho_i^{n_i}/n_i! & \text{(1) FCFS – exponential service} \\ (1-\rho)m!\prod_{i=1}^{C} \rho_i^{n_i}/n_i! & \text{(2) PS – arbitrary service} \\ e^{-\rho}\prod_{i=1}^{C} \rho_i^{n_i}/n_i! & \text{(3) Infinite servers – arbitrary service} \\ (1-\rho)m!\prod_{i=1}^{C} \rho_i^{n_i}/n_i! & \text{(4) Preemptive LCFS – arbitrary service} \end{cases}$$

The above schemes are referred to by the numbers (1), (2), (3) and (4) in the following slides.

$\rho = \Sigma\lambda_i M_i = \Sigma\, \rho_i$ – and $\Sigma n_i = m$

Network of BCMP Queues

- Nodes belong to one of the previous four types
- Assumption
  - Routing is probabilistic
  - Infinite storage at each queue
  - For open networks – Poisson arrivals
- The arrival rate of messages ith node is Λi as computed by the traffic equations
- Joint PMF for number of messages at each node is the PRODUCT of expressions given in previous slide.

Network of BCMP Queues - Example

- Example:
  - Assume N node network
  - Nodes 1 to i belong to disciplines (1), (2), or (4) Nodes i+1 to N belong to discipline (3)
- The joint pmf is given by

$$P(n_1, n_2, \ldots, n_N) = \prod_{j=1}^{i} (1 - R_j) R_j^{n_j} \prod_{k=i+1}^{N} \frac{e^{-R_k} R_k^{n_k}}{n_k!}$$

where $R_j = \Lambda_j \bar{M}_j$

- The average number of messages in each node is given by
  - Rj; j = i+1, i+2, …, N (i.e. for the infinite servers nodes)
  - Rj/(1-Rj); j = 1, 2, …, N

Network of BCMP Queues – Example – cont'd

- **If the network is closed – G(K, N) must be computed**
- **If all nodes have (1), (2), or (4) with single servers, the joint PMF is given by**

$$P(n_1, n_2, \ldots, n_N) = G(K,N)^{-1} \prod_{i=1}^{N} R_i^{n_i}$$

where G(K,N) can be found as before.

- **If all nodes have K servers or more – network will behave as if it has infinite servers. In this case the joint PMF is given by**

$$P(n_1, n_2, \ldots, n_N) = G(K,N)^{-1} \prod_{i=1}^{N} \frac{R_i^{n_i}}{n_i!}$$

where G(K,N) can be found using the convolution algorithm for infinite servers case (utilizing the multinomial expansion)

Store-and-Forward Message-Switched Nodes

- **Assumptions:**
  - **Poisson arrivals (multiple input lines)**
  - **Infinite storage space in the node**
- **Role of central processor**
  - **ACK/NACK is sent for correct/erroneous packets**
  - **ACK and NACK are piggybacked on information packets**
- **O output lines – based on destination - qi**

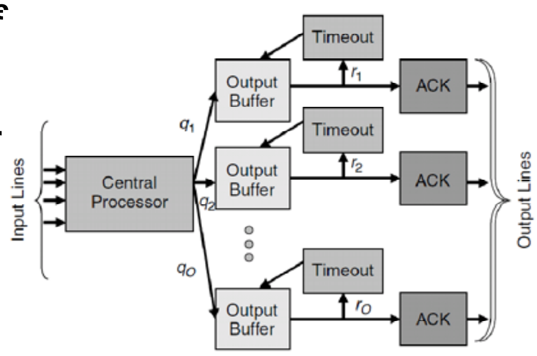


Figure 4.21 Store-and-forward packet-switching node.

Store-and-Forward Message-Switched Nodes – Modeling -cont'd

- **Central processor: processor sharing with constant service time**
- **Output buffers – service time is message transmit time (~ exponential)**
- **ARQ: timeout and ACK boxes**
  - **Prob rj; j=1, 2, …, O the attempted transmission over channel j fails (due to channel error or not enough storage)**
  - **Event enters the timeout box – stays for random time and then return to output buffer for retransmission**
- **For successful transmission (prob 1-rj), the even enters the ACK box for a random time and then it leaves the system**
- **Residency times in the timeout or ACK boxes represent the interval after transmission until NACK or ACK is received**
  - **Round trip propagation + processing time**
- **Timeout and ACK boxes are modeled as infinite servers nodes – we need to keep messages as much as needed**

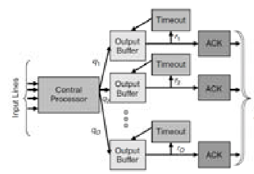


Figure 4.21 Store-and-forward packet-switching node.

Store-and-Forward Message-Switched Nodes – Traffic Equation

- **I input lines**
- **Let γi be arrival rate on the ith input line; i = 1, 2, …, I**
- **The total input (arrivals) ➔** $\Gamma = \sum_{i=1}^{I} \gamma_i$
- **Total flow into output buffer i ➔** $\Lambda_i^O = q_i \Gamma + \Lambda_i^T$

  where Λ_i^T **is the total flow into timeout box i.**
- **The flows into the timeout box and the ACK box are** $\Lambda_i^T = r_i \Lambda_i^O$ **and** $\Lambda_i^A = (1 - r_i)\Lambda_i^O$ **for i=1,2, …, O**
- **This mean, the total flow into ith output buffer is given by** $\Lambda_i^O = q_i \Gamma / (1 - r_i)$

Store-and-Forward Message-Switched Nodes – Store-and-Forward Node Model

- **Considering channel output buffer**
- **Simplified – ACK box not considered**
- **There is one Cox network (of M stages) for each of the infinite number of servers in the timeout box**
- **Arrival rate from central processor is $\lambda j = qj\Gamma$**
- **Output channel buffer service rate is μ_j.**
- **After residing in output buffer, message is routed to timeout box with prob rj**
- **Traffic equations for the branch are:**

$$\Lambda_i^O = \sum_{i=1}^{M-1} \omega_i (1 - P_i) + \omega_M + \lambda_j$$

$$\omega_1 = r_j \Lambda_j^O, \quad \omega_{i+1} = \omega_i P_i; \quad i = 1, 2, \cdots, M-1$$

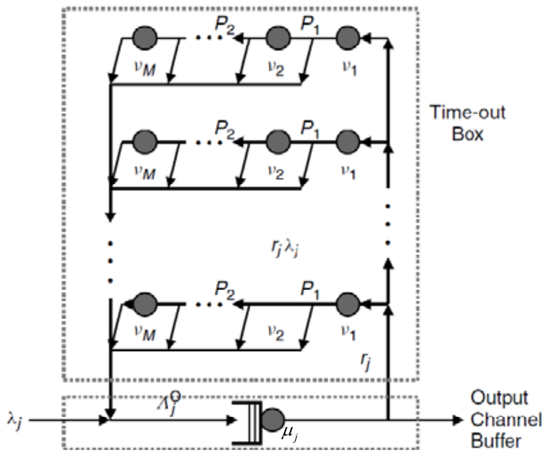


Figure 4.22 Portion of store-and-forward node.

Store-and-Forward Message-Switched Nodes – Store-and-Forward Node Model – cont'd

- **State of network: (M+1)-dimensional vector (n, k1, k2, …, kM)**
  - **n – # of messages in output channel buffer**
  - **ki - # of messages in the ith stage of timeout box**
- **The textbook shows that the joint PMF ois given by**

$$P(n, k_1, k_2, \ldots, k_M) = G^{-1} \left(\frac{\Lambda_j^O}{\mu_j} \right) \prod_{i=1}^{M} \frac{\rho_i^{k_i}}{k_i!}$$

where G^{-1} is a constant; $\rho_i = \omega_i / \nu_i$.

- **The interest is in the total number of messages in the timeout box, k = Σki. Therefore, the joint PMF for n and k is given by**

$$P\left(n, \sum_{i=1}^{M} k_i = k\right) = G^{-1} \left(\frac{\Lambda_j^O}{\mu_j} \right) \left(\sum_{i=1}^{M} \rho_i \right)^k \Big/ k!$$

- **Note that $\rho_T = \sum_{i=1}^{M} \rho i = \omega_1 \bar{M}_T$ where $\bar{M}_T$ is the mean processing time of message in timeout box.**

Store-and-Forward Message-Switched Nodes – Store-and-Forward Node Model – cont'd

- **Considering the rest of the node and expressing the arrival rates in terms of input rate**

$$\Lambda_i^O = \lambda_j/(1-r_i)$$
$$\omega_1 = r_j\Lambda_j^O = r_j\lambda_j/(1-r_i)$$
$$\Lambda_i^A = (1-r_i)\Lambda_i^O = \lambda_j$$

- **Again, first considering a single branch with its Timeout and ACK boxes**
- **Let l_j be the number of messages in the timeout and ACK boxes of the j^{th} branch, then the joint PMF for n and l_j is given by**

$$P(n_j, l_j) = G^{-1}\left(\frac{\Lambda_j^O}{\mu_j}\right)\left(r_j\Lambda_j^O\bar{M}_T + \lambda_j\bar{M}_A\right)^{l_j}/l!$$

where $\bar{M}_A$ is the average time spent in the ACK box. Note that λ_j is the rate of messages into the ACK box.

Store-and-Forward Message-Switched Nodes – Store-and-Forward Node Model – cont'd

- **Writing formulas in terms of link speeds (output buffer transmission speed)**
  - **Let Cj be transmission speed for jth output buffer; j = 1, 2, …, O.**
  - **Assume average message size in bit is $\bar{B}$**
- **Summing previous joint PMF over all nj and all lj and equating to one yields**

$$P(n_j, l_j) = \underbrace{\left(1-\rho_{O_j}\right)\rho_{O_j}^{n_i}}_{n_j \text{ messages in channel output buffer}}\;\underbrace{e^{-R_j}R_j^{l_j}/l!}_{l_j \text{ messages in timeout and ACK boxes}};\quad j = 1, 2, \ldots, O$$

where $R_j = r_j\Lambda_j^O\bar{M}_T + \lambda_j\bar{M}_A$ **and** $\rho_{O_j} = \Lambda_j^O\bar{B}/C_j$

Store-and-Forward Message-Switched Nodes – Store-and-Forward Node Model – cont'd

- Now consider all the O channel output buffers each with its own timeout and ACK boxes together with the central processor (i.e. entire store-and-forward node)
- The joint PMF is given by

Product of marginal distributions!!

$$P(n_0, n_1, n_2, \cdots, n_O, l_1, l_2, \cdots, l_O) = (1-\rho_P)\rho_P^{n_0}$$
$$\times \left[\prod_{j=1}^{O} (1-\rho_{O_j}) \rho_{O_j}^{n_j} \times e^{-R_j} R_j / l_j ! \right]$$

where n_0 is the number of messages at the central processor, and

$$\rho_P = \Gamma \overline{P}$$

- Remember that Γ is the total arrivals of messages per second to the central processor. $\overline{P}$ is the average processing time for a message.

Store-and-Forward Message-Switched Nodes – Total No of Messages

- The model assume infinite storage
- In practice this is not possible → Prob of overflow
- If Prob of overflow is small is can be approximated from the PMF for number of messages in node with infinite storage
  - Good approximation for small probability of flow.
- Towards this we need to
  - Compute the mean and variance for the total number of messages in the node (central processor, output buffers, and timeout/ACK boxes)
  - Assume a Gaussian distribution for the sum
  - Use the Gaussian PDF to estimate the overflow probability

Store-and-Forward Message-Switched Nodes – Total No of Messages – cont'd

- **The timeout/ACK boxes were modeled as infinite servers node → number of customers in each is Poisson**
- **The sum is also Poisson**
- **Let l be the total number of messages in all the timeout and ACK boxes**
  - **Mean of l is given by** $E[l] = R = \sum_{j=1}^{O} R_j = \sum_{j=1}^{O} \left(r_j \Lambda_j^O \bar{M}_T + \lambda_j \bar{M}_A \right)$
- **Therefore, the joint PMF can be rewritten as** $P(n_0, n_1, \cdots, n_O, l) = (1-\rho_P)\rho_P^{n_0} \times \left[\prod_{j=1}^{O} (1-\rho_{O_j})\rho_{O_j}^{n_j} \right] \times \frac{e^{-R} R^l}{l!}$

Store-and-Forward Message-Switched Nodes – Total No of Messages – cont'd

- **Let N^P – be number of messages in central processor** N_i^O, N_i^T and N_i^A **- be the number of message in the ith output buffer, timeout, and ACK boxes**
- **The means and variances are given by**

$$E\left[N^P\right] = \frac{\rho_P}{1-\rho_P} \qquad \mathrm{Var}\left[N^P\right] = \frac{\rho_P}{(1-\rho_P)^2}$$

$$E\left[N_i^O\right] = \frac{\rho_{O_i}}{1-\rho_{O_i}} \qquad \mathrm{Var}\left[N_i^O\right] = \frac{\rho_{O_i}}{(1-\rho_{O_i})^2}$$

$$E\left[N_i^T + N_i^A\right] = R_i = r_i \Lambda_i \bar{M}_T + \lambda_i \bar{M}_A \qquad \mathrm{Var}\left[N_i^T + N_i^A\right] = E\left[N_i^T + N_i^A\right]$$

for i = 1, 2, …, O.

- **Define N as the sum of N^P,** N_i^O, N_i^T and N_i^A**, the mean and variance of the Gaussian distribution are given by**

$$E[N] = E\left[N^P\right] + E\left[N_i^O\right] + E\left[N_i^T + N_i^A\right]$$

$$\mathrm{Var}[N] = \mathrm{Var}\left[N^P\right] + \mathrm{Var}\left[N_i^O\right] + \mathrm{Var}\left[N_i^T + N_i^A\right]$$

Store-and-Forward Message-Switched Nodes – Total No of Messages – cont'd

- **It would be interesting to compare the approximate (Gaussian) PDF to the actual PMF obtained numerically through successive convolutions of the previous JOINT PMF on slide 145.**

- **Bonus – 3 points in the final exam.**
  - **For the assignment problem (Assign #3 – Q 4), evaluate the approximate PDF and the actual PMF numerically (as on slide 52).**
  - **Submit a softcopy and a hardcopy of the bonus question containing the two curves (PDF and CDF as in slide 52) and the Matlab code used to obtain the result.**

Store-and-Forward Message-Switched Nodes – Average Message Delay – cont'd

- **Delay for a message going through a network is given by**
- **Refer to textbook eq 4.36** $E[T]=\frac{1}{\alpha}\sum_{\forall i}\Lambda_i\overline{D}_i$
- **Delay in central processor:** $\overline{D}_P=\frac{\overline{P}}{1-\rho_P}$
- **Delay in the output buffer:** $\overline{D}_j=\frac{\overline{B}/C_j}{1-\rho_{O_j}}=\frac{\overline{B}}{C_j-I_j};\quad I_j=\Lambda_j^O\overline{B}$
- **Delay through the timeout and ACK boxes:** $\overline{M}_A$, and $\overline{M}_T$
- **Therefore, overall delay**

$$\overline{D}_P=\frac{\overline{P}}{1-\rho_P}+\frac{1}{\Gamma}\sum_{j=1}^{O}\left[\frac{I_j}{C_j-I_j}+\lambda_j\overline{M}_A^j+\frac{\lambda_j}{1-r_j}\overline{M}_T^j\right]$$

Window Flow Control – A Closed Network Model

- **Assumptions:**
  - **N nodes**
  - **Independent and exponential service time ~ 1/μi**
  - **Forward path and return path modeled as chain**

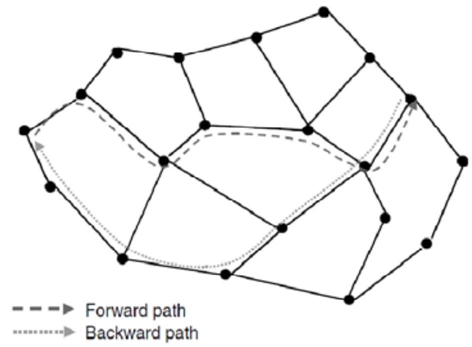


Figure 4.23 Forward and feedback paths through a network.

Window Flow Control – A Closed Network Model – cont'd

- **Packets are typically longer than ACKs – different service times**
- **Difficult to model the process of holding the message until there is room in the chain**
- **Assume messages arrive at the source node (node 1) at an average rate of λ0**
- **Messages arriving to a full chain are lost**
- **N original nodes + 1 phantom node (node 0)**
  - **Phantom node service rate = λ0**

- **Messages from source-destination pair that we are interested in, circulate in closed chain.**
- **If W internal messages are outstanding, the phantom node is empty – no new arrivals to node 1**
- **When there are less than W messages in nodes 1 to N, there is at least one message in the phantom node – arrival rate of messages from node 0 to node 1 is λ0.**
- **Other (external) traffic passing through the nodes 1 to N is modeled through arrivals of rate λi and exist rate λi.**

- **Refer to model details in textbook**

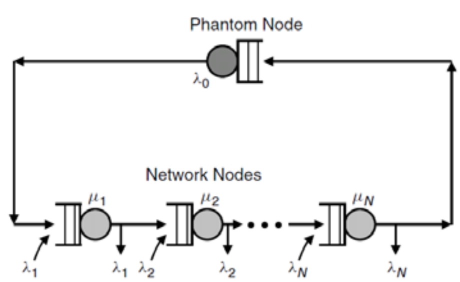


Figure 4.24 Logical network chain.

Window Flow Control – A Closed Network Model – Results

- Define state as (k_0, k_1, …, k_N, l_1, l_2, …, l_N) where ki is number of internal messages; li is number of external messages in node i.
- The join PMF is given by

$$P(k_0, k_1, \ldots, k_N; l_1, l_2, \ldots, l_N) = G^{-1} \prod_{i=1}^{N} \left(\frac{\lambda_0}{\mu_i}\right)^{k_i} \frac{\rho_i^{l_i}(k_i + l_i)!}{k_i! l_i!}$$

where $\rho i = \lambda i/\mu i$

- Summing over all li from 0 to infinity (since there is no storage limit on external message) – we get

$$P(k_0, k_1, \cdots, k_N) = G^{-1}(W, N) \prod_{i=0}^{N} R_i^{k_i}$$

where R0 = 1 (by definition), Ri = $\lambda 0/(\mu i - \lambda i)$

- Refer to the detailed derivation in textbook

Window Flow Control – A Closed Network Model – Results – cont'd

- New internal messages are blocked when the phantom node is empty – Blocking probability
- The join PMF is given by (eq 4.58)

$$P(k_0 = 0) = 1 - G^{-1}(W-1, N) / G^{-1}(W, N)$$

- Delay for external traffic increases due to internal traffic – What is the amount of increase?
- Let Lm and Km denote the number of external and interal message, respectively, at node m = 1, 2, …, N.
- You can show that:

$$E[L_m] = \frac{\rho_m}{1-\rho_m}\left(E[K_m] + 1\right)$$

- For E[Km], the mean is given by eq 4.58 derived earlier

$$E[K_m] = \frac{1}{G(W,N)} \sum_{k=1}^{W} R_m^k G(W-k, N); \quad m = 1, 2, \ldots, N$$

Window Flow Control – A Closed Network Model – Results – cont'd

- The normalized different in average delay due to internal traffic at node m is

$$\frac{\Delta D_m}{D_{m_0}} = \frac{\frac{\rho_m}{\lambda_m(1-\rho_m)} E[K_m]}{\frac{\rho_m}{\lambda_m(1-\rho_m)}} = E[K_m]$$

- Averaging over the message arrival rate for all nodes, we get

$$\frac{\Delta D}{D_0} = \frac{\sum_{m=1}^{N} \lambda_m E[K_m]}{\sum_{i=1}^{N} \lambda_i}$$

where Rm = λ0/(μm- λm). Note that λ0 is an arbitrary constant that is eliminated in the normalization process.

Window Flow Control – A Closed Network Model – Example 4.12

- **Problem:** Consider the case of a chain consisting of 5 nodes with a window of 10 messages. Assume λ0 is equal to 0.6. Also μ = [3 4 2 6 3], and λ = [2 0.5 0.8 4 1]
- Compute the joint PMF for number of internal messages in system
- Verify that sum of mean number of internal messages for the 5 nodes is equal to 10.
- Calculate the blocking probability
- What is the change of delay of external message due to internal traffic

Window Flow Control – A Closed Network Model – Example 4.12

- **Solution:**
- **Make sure you can get final answers obtained in textbook page 175**

Cellular Radio – Model

- **Service provided by base station**
- **Frequencies are assigned to the cells to minimize interference**
- **Mobile users move between cells – users switch frequency bands**
- **14 cells are considered**
- **Each is modeled as having an infinite number of servers**
- **There are L channels in each cell**

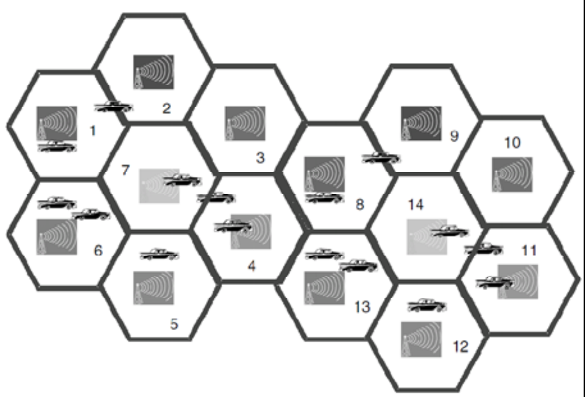


Figure 4.25 Mobile radio system.

Cellular Radio – Model/Results

- **The PMF for number of users in each cell is given by**

$$P(k_1, k_2, \cdots, k_N) = \prod_{i=1}^{N} \frac{e^{-\rho_i} \rho_i^{k_i}}{k_i!}$$

- **The marginal distribution is given by**

$$P(k) = \frac{e^{-\rho} \rho^k}{k!}; \quad k = 1, 2, \cdots$$

- **Each of the users contends for L channels ➔ finite-source model; A user is blocked if all channels are occupied.**
- **Probability of blocking, Q_L, is given by**

$$Q_L = \binom{K}{L} \gamma^L \Big/ \left[\sum_{i=0}^{K} \binom{K}{i} \gamma^i \right]$$

where $\gamma = \sigma/\mu$. σ is the probability that a single source goes from off t on in an incremental interval and μ is the probability of change in the opposite direction.

Cellular Radio – Results – cont'd

- **The overall blocking probability of**

$$\begin{aligned} P_B &= \sum_{k=L+1}^{\infty} \overbrace{\frac{e^{-\rho} \rho^k}{k!}}^{\text{number of users}} \times \overbrace{\binom{K}{L} \gamma^L \Big/ \left[\sum_{i=0}^{K} \binom{K}{i} \gamma^i \right]}^{\text{blocking probability}} \\ &= \frac{e^{-\rho} \gamma^L}{L!} \sum_{k=L+1}^{\infty} \frac{\rho^k}{k!(k-L)! \sum_{i=0}^{L} \gamma^i \big/ \left(i!(k-i)!\right)} \end{aligned}$$

Cellular Radio – Example 4.13

- **Problem: Assume 14-node system – the queue in a cell is modeled as an infinite number of serves. Let the residency time be exponentially distributed.**
- **Assume users are equally likely to move through EACH of the six cell boundaries**
  - **E.g. for cell 1, half of its traffic leaves the subsystem while for cell 7, all of its traffic remains in the subsystem**

Cellular Radio – Example 4.13 – cont'd

- **<u>Solution:</u>**
- **The routing matrix is as shown**
- **The arrival rate of users from adjacent cells is proportional to the number of sides of a cell, which interface the larger system ➔**

λ = [3 3 2 1 3 3 0 1 3 3 3 3 2 0]

- **The total flows into each cell is given by**

Λ = [6 6 6 6 6 6 6 6 6 6 6 6 6 6]

$$Q = \begin{bmatrix}
0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0
\end{bmatrix}$$

Cellular Radio – Example 4.13 – cont'd

- **<u>Solution:</u>**
- **If the mean residency time = 3 min ➔ the load per cell is 3 x 6 = 18.**
- **For L = 5 (available lines) in a cell and γ = 0.2 (source activity) ➔ blocking probability = 0.1142**
- **Student must perform the required calculations to arrive at the numerical answers.**