King Fahd University of Petroleum & Minerals Computer Engineering Dept

CSE 642 – Computer Systems Performance

Term 091

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Random/Stochastic Processes

- **Consider a random experiment specified by the outcomes ζ from some sample space S, by the events defined on S, and by the probabilities on these events. Suppose that every outcome $\zeta \in S$, we assign a function of time according to some rule:**

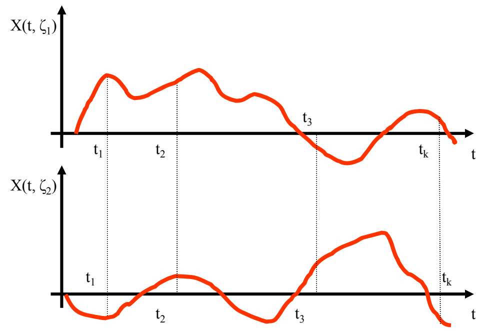
$$X(t, \zeta) \quad t \in I$$

The graph of $X(t, \zeta)$ versus t, for ζ fixed, is called a *REALIZATION* or *sample path* of the random process

A stochastic process is said to be discrete-time if the index set I is a countable set

Random/Stochastic Processes – cont'd

- **For each fixed t_k from the index set I, X(t_k, ζ) is a random variable**
- **{X(t, ζ) , t ∈ I} forms an indexed family of random variables ➔ a *random or stochastic process***



- **Figure shows two realizations of a random process**

Random/Stochastic Processes – cont'd

- **A stochastic process is said to be discrete-time if the index set I is a countable set**
- **Continuous-time stochastic process is one in which I is continuous**

Example 1: Discrete Random Process

- **Let ζ be a number selected at random from the interval S =[0,1], and let b1, b2, … be the binary expansion of ζ:**

$$\zeta = \sum_{i=1}^{\infty} b_i 2^{-i}$$

Define the discrete-time random process X(n, ζ) by

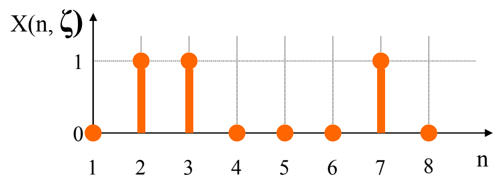
$$X(n, \zeta) = b_n \qquad \text{for } n = 1, 2, \ldots$$

Example 1: cont'd

- **Realizations of the random process**

$$X(n, \zeta) = b_n \qquad \text{for } n = 1, 2, \ldots$$

For $\zeta = 2^{-2}+2^{-3}+2^{-7}$
$= 0.3828125$



For any ζ, you can produce a realization of X(n, ζ)

Example 2: Discrete Random Process

- **<u>Problem</u>: For the random process defined in the previous example, compute**
  - $P(X(1,\zeta) = 0)$ and
  - $P(X(1,\zeta) = 0$ and $X(2,\zeta) = 1)$

- **<u>Solution</u>:**

  $P(X(1,\zeta) = 0) = P(0 \le \zeta \le 1/2) = 1/2$

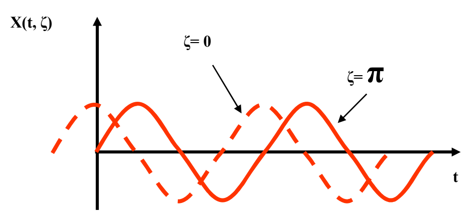
  $P(X(1,\zeta) = 0 \text{ AND } X(2,\zeta) = 1) = P(1/4 \le \zeta \le 1/2) = 1/4$

Example 3: Continuous Random Process

- **Let ζ be a number selected at random from the interval $S = [-\pi, \pi]$. Define the continuous-time random process $X(t, \zeta)$ by**

  $$X(t, \zeta) = \cos(2\pi t + \zeta) \quad -\infty < t < \infty$$

  The realizations of $X(t, , \zeta)$ are time-shifted versions of $\cos(2\pi t)$



Example 4: Continuous Random Process

- **Problem**: find the pdf of $X_0 = X(t_0, \zeta)$ in the previous example.

- **Solution**: For all values of t_0,

  $X_0 = \cos(\theta)$

  where θ is a uniform r.v. $\in [-\pi+\pi t_0, \pi+\pi t_0]$

  Therefore, using the techniques we learned (functions of random variables), it can be shown that the pdf of X0 is given

$$f_{X(t_0)}(x) = \frac{1}{\pi\sqrt{1-x^2}} \qquad |x| < 1$$

In general $f_{X(t0)}(x)$ is a function of t_0

Specifying A Random Process – Joint Distribution of Time Samples

- **Let $X_1, X_2, \ldots, X_k$ be the k random variables obtained by sampling the random process $X(t, \zeta)$ at the times $t_1, t_2, \ldots, t_k$:**

  $X_1=X(t_1, \zeta), X_2=X(t_2, \zeta), \ldots, X_k=X(t_k, \zeta)$

→ $(X_1, X_2, \ldots, X_k)$ vector of random variables

- **A stochastic process is specified by the collection of k^{th} order joint cumulative distribution function:**

$$F_{X_1,\ldots,X_k}(x_1, x_2, \ldots, x_k) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_k \le x_k)$$

for any k and

any choice of sampling instants $t_1, t_2, \ldots, t_k$.

Specifying A Random Process – Joint Distribution of Time Samples - cont'd

- **If the stochastic process is discrete-valued. The a collection of probability mass functions is used to specify the stochastic process:**

$$p_{X1,\ldots,Xk}(x_1, x_2, \ldots, x_k) = P(X_1=x_1, X_2=x_2, \ldots, X_k=x_k)$$

Example 5:

- **Let Xn be a sequence of iid Bernoulli random variables with p = ½. The joint pmf for any k time samples is then**

$$P(X_1=x_1, X_2=x_2, \ldots, X_k=x_k) = 2^{-k} \quad x_i \in \{0,1\} \ \forall \ I$$

This process is equivalent to that described in Example 1.

Moments of Random Process Time Samples

- **Partially specify the random process**
- **Mean, $m_X(t)$**

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)dx$$

- **Autocorrelation (joint moment of $X(t_1)$ and $X(t_2)$):**

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t_2)X(t_2)}(x,y)dxdy$$

Moments of Random Process Time Samples – cont'd

- **Autocovariance of $X(t_1)$ and $X(t_2)$**

$$C_X(t_1,t_2) = E[(X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))]$$

- **Correlation coefficient of X(t):**

$$\rho_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{\sqrt{C_X(t_1,t_1)}\sqrt{C_X(t_2,t_2)}}$$

it can be shown that:

$$C_X(t_1,t_2) = R_X(t_1,t_2) - m_X(t_1)m_X(t_2)$$

$$Var[X(t)] = C_X(t,t)$$

Example 6:

- **Let X(t) = cos(ωt+θ) – where θ is uniformly distributed in the interval (-π,π). Find $m_X(t)$ and $C_X(t_1,t_2)$.**

Example 6: cont'd

- **Solution:**

$$\begin{aligned} m_X(t) &= E[\cos(\omega t+\theta)] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega t+x)dx \\ &= 0 \end{aligned}$$

$$\begin{aligned} C_X(t_1,t_2) &= E[\cos(\omega t_1+\theta)\cos(\omega t_2+\theta)] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega t_1+x)\cos(\omega t_2+x)\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\{\cos(\omega(t_1-t_2))+\cos(\omega(t_1+t_2)+2x)\}dx \\ &= \frac{1}{2}\cos(\omega(t_1-t_2)) \end{aligned}$$

note that $C_X(t_1,t_2)$ does NOT depend on t_1 and t_2 but on $|t_1 - t_2|$

Examples of Discrete-Time Random Processes – iid Random Processes

- Let X_n be a discrete-time random process consisting of sequence of iid random variables with common CDF $F_X(x)$, mean m, and variance σ^2. The joint CDF for any time instants $n_1, n_2, \ldots, n_k$ is given by

$$F_{X1,\ldots,Xk}(x_1, x_2, \ldots, x_k) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_k \le x_k)$$
$$= F_X(x_1)\, F_X(x_2) \ldots F_X(x_k)$$

Note we used X_i to denote X_{n_i} for simplicity.

Furthermore,

$$m_X(n) = E[X_n] = m \quad \forall\, n$$

You can show that

$$C_X(n_1, n_2) = \sigma^2\, \delta(n_1-n_2)$$

where $\delta(v) = 1$ if $v = 0$
$= 0 \quad v \neq 0$

In the same fashion, $R_X(n_1, n_2) = \sigma^2\, \delta(n_1-n_2) + m^2$

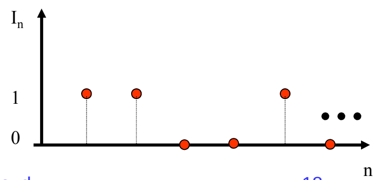
Example 7: Bernoulli Random Process

- Let I_n be sequence of iid Bernoulli random variables. I_n is then an iid random process taking on values from the set {0,1}

$$m_I(n) = p, \quad Var[I_n] = p(1-p)$$

- e.g:

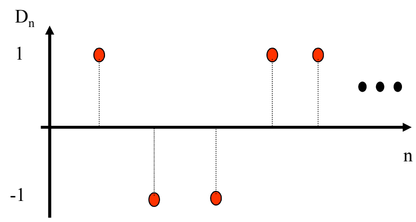
P(1001 sequence)
$= P(I_1=1, I_2=0, I_3=0, I_4=1)$
$= P(I_1=1)P(I_2=0)P(I_3=0)P(I_4=1)$
$= p^2(1-p)^2$



Example 8:

- **Let D_n be define as $2I_n - 1$ where I_n is the Bernoulli random process. Clearly**

$$D_n = \begin{cases} 1 & \text{if } I_n = 1 \\ -1 & \text{if } I_n = 0 \end{cases}$$



one sample function

$m_D(n) = E[D_n] = E[2I_n\text{-}1] = 2E[I_n]\text{-}1 = 2p\text{-}1$

$Var[D_n] = Var[2I_n\text{-}1] = 2^2 Var[I_n] = 4p(1\text{-}p)$

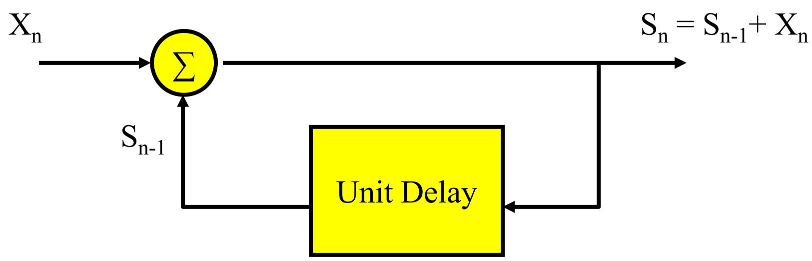
Sum Processes

- **Let $S_n = X_1 + X_2 + \ldots + X_n$, $n = 1, 2, \ldots$**
  $= S_{n-1} + X_n$

  where
  $S_0 = 0$
  $X_1, X_2, \ldots, X_n$ are iid random variables
- **S_n is a sum process**
- **PDF or pmf for S_n is found using the convolution or characteristic equation methods**
- **Note: S_n depends in the *past*, $S_1, S_2, \ldots, S_{n-1}$, ONLY through S_{n-1}. i.e. S_n is *independent of the past* when S_{n-1} is known!!**

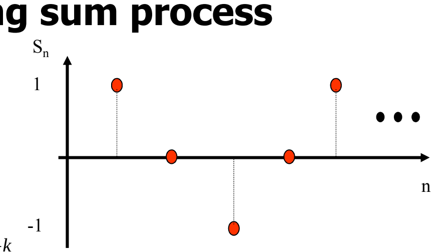
Sum Processes Generation

- **The sum process can be generated as follows:**



Example 9: One-Dimensional Random Walk

- **Let D_n be the iid process of ±1 random variables defined in previous example. Let S_n be the corresponding sum process**



- **The pmf of S_n is given by**

$$P(S_n = 2k - n) = \binom{n}{k} p^k (1-p)^{n-k}$$

for k ∈ {0,1, …, n}

Characteristics of Sum Processes of iid Random Variables

- **Independent Increments**

S_{n1}- S_{n2} and S_{n2}-S_{n3} are independent random variables

- **Stationary increments**

$$P(S_{n1} - S_{n2} = y) = P(S_{n1-n2} = y)$$

Examples Of Continuous-Time Random Processes – Poisson Process

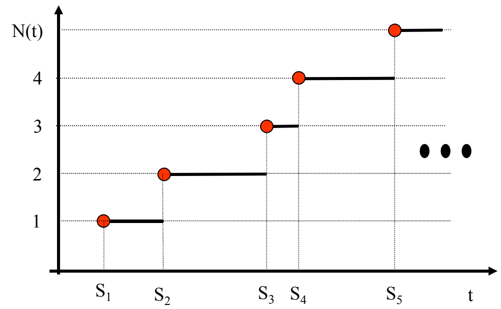
- **Assume events (e.g. arrivals) occur at rate of λ events per second. Let N(t) be the number of occurrences in the interval [0,t]**

➔ N(t) is non-decreasing integer-valued continuous-time random process

- **pmf for N(t) is given by**

$$P(N(t)=k)=\frac{(\lambda t)^k}{k!}e^{-\lambda t}$$

for k=0,1, …



Examples Of Continuous-Time Random Processes – Poisson Process – cont'd

- **Independent increments**

- **Stationary increments – the distribution for the number of event occurrences in *ANY interval of length t* is given by the previous pmf.**

Example 10: Poisson Process

- **Problem: Find $P(N(t_1)=i, N(t_2)=j]$ where $N(t)$ is a Poisson process.**

- **Solution:**

$$P(N(t_1)=i, N(t_2)=j]$$
$$= P(N(t_1)=i)P(N(t_2)-N(t_1)=j-i]$$
$$= P(N(t_1)=i)P(N(t_2-t_1)=j-i]$$

$$= \frac{(\lambda t_1)^i e^{-\lambda t_1}}{i!} \frac{(\lambda(t_2-t_1))^{j-i} e^{-\lambda t_2 - t_1}}{(j-i)!}$$

Example 11: Poisson Process

- **Problem: N(t) is a Poisson process – Show that T, the time between event occurrences is exponentially distributed**

- **Solution:**
  pmf of N(t) is given by

$$P(N(t)=k)=\frac{(\lambda t)^k}{k!}e^{-\lambda t} \qquad k=0,1,\ldots$$

P(T > t) = P(no events in t seconds)

= $e^{-\lambda t}$

Therefore, P(T ≤ t) = $F_T(t)$ = 1-$e^{-\lambda t}$ – i.e. T is exponentially distributed with mean 1/λ

Stationary Random Processes

- **Nature of randomness observed in the process does not change with time**

- **A discrete-time or continuous-time random process X(t) is stationary if the joint distribution of any set of sample does not depend on the placement of the time origin:**

- **Joints CDF of $X(t_1)$, $X(t_2)$, …, $X(t_k)$ is the same as joint CDF of $X(t_1+\tau)$, $X(t_2+\tau)$, …, $X(t_k+\tau)$, i.e.**

$$F_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1,x_1,\ldots,x_k)=F_{X(t_1+\tau),X(t_2+\tau),\ldots,X(t_k+\tau)}(x_1,x_1,\ldots,x_k)$$

for all τ and all k

Stationary Random Processes – cont'd

- **First order CDF of a stationary random process must be independent of time since**

$$F_{X(t_1)}(x) = F_{X(t_1+\tau)}(x) = F_X(x) \qquad \forall t, \tau$$

➔ mean and variance are independent of time, i.e.

$$m_X(t) = E[X(t)] = m \qquad \forall t$$

$$Var[X(t)] = E\left[(X(t) - m)^2\right] = \sigma^2 \qquad \forall t$$

Stationary Random Processes – cont'd

- **Second order CDF of a stationary random process depends only on the time difference between the samples**

$$F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(0),X(t_2-t_1)}(x_1, x_2) \qquad \forall t$$

➔ $R_X(t_1,t_2)$ and $C_X(t_1,t_2)$ are depend only on t_2-t_1, i.e.

$$R_X(t_1, t_2) = R_X(t_2 - t_1) \qquad \forall t_1, t_2$$

$$C_X(t_1, t_2) = C_X(t_2 - t_1) \qquad \forall t_1, t_2$$

Example: Stationary Random Processes

Problem: Is the sum process a discrete-time stationary process?

Solution: The sum process is defined by

$$S_n = X_1 + X_2 + \ldots + X_n, \quad n = 1, 2, \ldots$$

The process mean and variance are given by

$$m_S(n) = nm \quad \text{and} \quad Var[S_n] = n\,\sigma^2$$

where m and σ^2 are the mean and variance of X_n.

It can be noticed that the mean and variance of the process are not constant, but rather increase linearly with the time index n.

➔ The sum process is NOT stationary

Wide-Sense Stationary Random Processes

- **In many situations we can not determine whether a random process is stationary, but we can determine whether the mean is a constant:**

$$m_X(t) = m \quad \text{for all } t$$

And whether the autocovariance (or autocorrelation) is a function of t1-t2 only:

$$C_X(t_1,t_2) = C_X(t_1-t_2)$$

➔ X(t) is a wide-sense stationary (WSS) process

Example: Wide-Sense Stationary Random Processes

- **Problem**: Let X_n consist of two interleaved sequences of independent random variables. For n even, X_n assumes the values of ±1 with probability ½; for n odd, X_n assumes the values 1/3 and -3 with probabilities 9/10 and 1/10, respectively.

  Is X_n stationary?

Example: Wide-Sense Stationary Random Processes

- **Solution:**

  Xn is not stationary since pmf depends on the time index n

  $E[X_n] = +1/2 - 1/2 = 0$ for n even

  $= 1/3 \times 9/10 - 3 \times 1/10 = 0$ for n odd

  therefore

  $E[X_n] = 0$ for all n

  One can also show that

  $E[X_n^2] = 1$ for all n

  Therefore, $C_X(i,j) = E[X_i]E[X_j] = 0$ for $i \neq j$

  $E[X_i^2] = 1$ for $i = j$

  clearly X_n is a wide-sense stationary process

Time Averages Versus Ensample Average

- **Ensample average:**

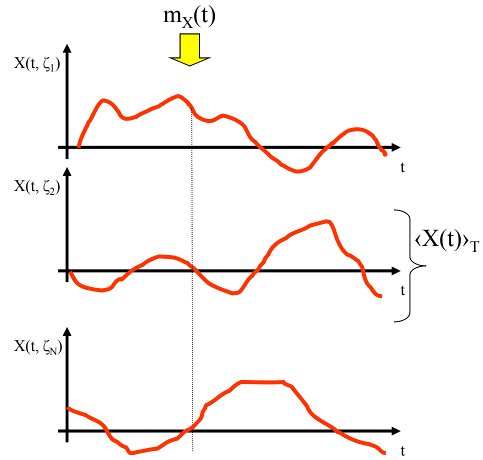
$$\hat{m}_X(t) = \frac{1}{N}\sum_{i=1}^{N} X(t,\zeta_i)$$

note the experiment is run N times (N realizations), and the expectations is approximated by the arithmetic average given above

- **Time average:**

$$\langle X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t,s)dt$$

Using one realization



Ergodic Processes

- **For Ergodic Processes time average converges to the expected mean of the random process**

Markov Process

- A random process X(t) is a Markov Process if the future of the process given the present is independent of the past.

- For arbitrary times: $t_1 < t_2 < \ldots < t_k < t_{k+1}$

$$\text{Prob}[X(t_{k+1}) = x_{k+1} / X(t_k)=x_k, \ldots, X(t_1)=x_1] = \text{Prob}[X(t_{k+1}) = x_{k+1} / X(t_k)=x_k]$$

Or (for discrete-valued)

$$\text{Prob}[a < X(t_{k+1}) \le b / X(t_k)=x_k, \ldots, X(t_1)=x_1] = \text{Prob}[a < X(t_{k+1}) \le b / X(t_k)=x_k]$$

Markov Property

Markov ≡ Memoryless

Markov Chain

- An integer-valued Markov random process is called a Markov Chain

- The joint pmf for k+1 arbitrary time instances is given by:

$$\text{Prob}[X(t_{k+1}) = x_{k+1}, X(t_k)=x_k, \ldots, X(t_1)=x_1]$$

$$= \underbrace{\text{Prob}[X(t_{k+1}) = x_{k+1}/X(t_k)=x_k] \times \text{Prob}[X(t_k) = x_k/X(t_{k-1})=x_{k-1}] \times \ldots \times \text{Prob}[X(t_2) = x_2/X(t_1)=x_1]}_{\text{transition probabilities}} \times \text{Prob}[X(t_1)=x_1]$$

← pmf of the initial time

Discrete-Time Markov Chains

- **Let X_n be a discrete-time integer values Markov Chain that starts at n = 0 with pmf**

$$p_j(0) = Prob[X_0 = j] \quad j=0,1,2, \ldots$$

$$Prob[X_n=i_n, X_{n-1}=i_{n-1},\ldots,X_0=i_0]$$
$$= Prob[X_n=i_n / X_{n-1}=i_{n-1}] \times Prob[X_{n-1}=i_{n-1} / X_{n-2}=i_{n-2}] \times \ldots . \; Prob[X_1=i_1 / X_0=i_0] \times Prob[X_0=i_0]$$

Same as the previous slide but for discrete-time

Discrete-Time Markov Chains – cont'd (2)

- **Assume the one-step state transition probabilities are fixed and do not change with time:**

$$Prob[X_{n+1}=j/X_n=i] = p_{ij} \quad \text{for all } n$$

➔ **X_n is said to be homogeneous in time**

- **The joint pmf for X_n, X_{n-1}, …, X_1,X_0 is then given by**

$$P[X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0]$$
$$= p_{i_{n-1},i_n} \times p_{i_{n-2},i_{n-1}} \times \ldots \times p_{i_0,i_1} \times p_{i_0}(0)$$

Discrete-Time Markov Chains – cont'd (3)

- **Thus X_n is completely specified by the initial pmf $p_i(0)$ and the matrix of one-step transition probabilities P:**

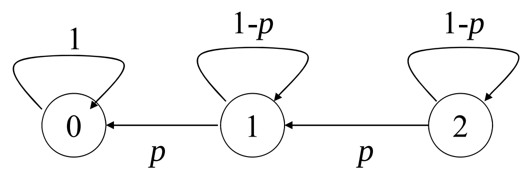
i.e. rows of P add to UNITY

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ . & . & . & . \\ p_{i0} & p_{i1} & p_{i2} & \cdots \\ . & . & . & . \end{bmatrix}$$

$$1 = \sum_j P[X_{n+1} = j / X_n = i] = \sum_j p_{ij}$$

Example 6: two-state Markov Chain

- **On day 0 a house has two new light bulbs in reserve. The probability that the house will need a single new light bulb during day n is p and the probability that it will not need any is q = 1-p. Let Yn be the number of new light bulbs left in house at the end of day n.**
- **Yn is a Markov chain with state transition probability as shown**



Example 6: two-state Markov Chain – cont'd

- **The state transition matrix P is given by**

$$
\begin{array}{cccc}
 & Yn = 0 & 1 & 2 \\
 & \downarrow & \downarrow & \downarrow \\
P = & \multicolumn{3}{c}{\begin{bmatrix} 1 & 0 & 0 \\ p & q & 0 \\ 0 & p & q \end{bmatrix}}
\end{array}
$$

The n-step Transition Probabilities

- **Let $P(n) = \{p_{ij}(n)\}$ be the matrix of n-step transition probabilities, where**

$p_{ij}(n) = Prob[X_{n+k} = j \,/\, X_k = i] \quad n \geq 0; i,j \geq 0$

Note:

$Prob[X_{n+k} = j \,/\, X_k = i] = Prob[X_n = j \,/\, X_0 = i]$ for all n – why?

Transition probabilities do not depend on time (homogeneous)

It can be shown that:

$P(n) = \{p_{ij}(n)\} = P^n$ – where P is the 1-step transition probability matrix

The State Probabilities

- **It can be shown that the state pmf at time n is obtained by multiplying the initial state pmf, p(0), by the n-step transition matrix, P(n), in other words**

$$p(n) = p(0)\,P(n)$$
$$= p(0)\,P^n$$

Make a distinction between small p and capital P!

Example 7:

- **Consider the problem given in Example 6 – find the n-step transition matrix and compute the state pmf p(n)**

Example 7: cont'd

Answer: The n-step transition matrix can be found by multiplying P (the 1-step transition matrix) by itself n times or alternatively we can use:

$p_{22}(n)$ = Prob[no new light bulbs needed in n days] = q^n
$p_{21}(n)$ = Prob[1 light bulb needed in n days] = n p q^{n-1}
$p_{20}(n)$ = Prob[2 light bulbs needed in n days]
= 1 – $p_{22}(n)$ – $p_{21}(n)$
$p_{10}(n)$ = Prob[the one light bulb is not needed in n days] = 1 - q^n
$p_{11}(n)$ = Prob[the one light bulb is not needed in n days] = q^n
$p_{12}(n)$ = 0

$p_{00}(n)$ = 1
$p_{01}(n)$ = 0
$p_{02}(n)$ = 0

Example 7: cont'd

- **Therefore, the n-step transition matrix is given by**

$$P^n = \begin{bmatrix} 1 & 0 & 0 \\ 1-q^n & q^n & 0 \\ 1-q^n-npq^n & npq^n & q^n \end{bmatrix}$$

Example 7: cont'd

- **Notes:**
  - **For all transition matrices, sum of any row SHOULD equal to ONE**
  - **For q = 1-p < 1 ➔ as n→∞, then P^n limit is**

$$\lim_{n\to\infty} P^n \to \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Example 7: cont'd

- **Therefore, if we start with 2 light bulbs, then the state pmf p(n) approaches**

p(n) = p(0) P^n

$$p(n) \to \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Meaning – if n approaches ∞, then it is almost certain we will end up in the 0 (no light bulbs) state

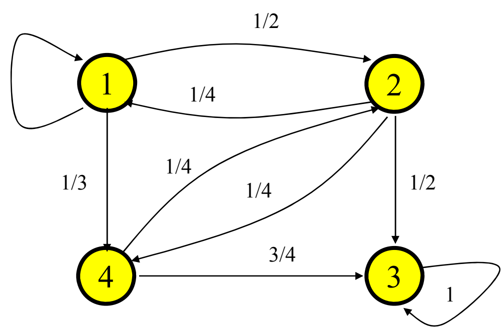
Example 7: Absorbing State

Problem: Consider the Markov chain depicted by the shown state diagram. Write the transition matrix. If the system is initially in state 1, find the state probability at the Nth step.

Solution:

The single step transition matrix P is given by

$$P = \begin{bmatrix} 1/6 & 1/2 & 0 & 1/3 \\ 1/4 & 0 & 1/2 & 1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 1/4 & 3/4 & 0 \end{bmatrix}$$



Example 7: Absorbing State – cont'd

Solution:

The state probabilities at the Nth step, p(n), are given by p(n-1)P

Matlab code:

```
clear all
P = [1/6 1/2  0  1/3; ...
     1/4  0  1/2 1/4; ...
      0   0   1   0; ...
      0  1/4 3/4  0];

P0 = [1 0 0 0];
N = 9;
State_Prob = zeros(N, 4);
State_Prob(1,:) = P0;
for i=2:N
    State_Prob(i,:) = State_Prob(i-1,:) * P;
end
State_Prob
```

Nth state probabilities:

Step	State 1	State 2	State 3	State 4
1	0.166667	0.500000	0.000000	0.333333
2	0.152778	0.166667	0.500000	0.180556
3	0.067130	0.121528	0.718750	0.092593
4	0.041570	0.056713	0.848958	0.052758
5	0.021107	0.033975	0.916884	0.028035
6	0.012011	0.017562	0.954897	0.015529
7	0.006392	0.009888	0.975325	0.008394
8	0.003537	0.005295	0.986565	0.004603

- **It can be shown that P(∞) = [0 0 1 0]**
  - **State 3 is known as an absorbing state**

Concepts - Terms

- **Absorbing state**
- **Closed set**
- **Irreducible Markov Chain**
- **Recurrent state**
- **Aperiodic state**
- **Persistent state**
- **Transient state**

See the textbook for formal definitions

Steady State Probabilities

- **Some Markov chains settle into stationary behavior. As $n \rightarrow \infty$, the n-step transition matrix approaches a matrix in which all rows are equal to the same pmf, that is**

$$p_{ij}(n) \rightarrow \pi_j$$

Therefore,

$$p_j(n) \rightarrow \sum_i \pi_j \, p_i(0) = \pi_j$$

$$\rightarrow \quad \pi_j = \sum_i p_{ij} \, \pi_j$$

Or in matrix form

$$\Pi = \Pi \mathbf{P} \quad \text{- where } \Pi = \{\pi_j\}$$

In general the above formation has n-1 linearly independent equations – the additional equation required is provided by

$$\sum_i \pi_i = 1 \quad \text{or}$$

Steady State Probabilities – cont'd 2

- **In other words: At steady state (n is very large) – the nth state pmf is the same as the n+1st state pmf**
- **Meaning the nth (n very large) state pmf is time invariant (steady state)**

$$\Pi = \Pi \mathbf{P}$$

Π ➔ **is the steady state pmf**

P ➔ is the 1-step transition matrix

Steady State Probabilities – cont'd 3

- **Checking the dimensions:**

Π ➔ **is the steady state pmf of dimensions = 1Xk - assuming k states**

= $[\pi_1\ \pi_2\ \pi_3 \ldots \pi_k]$ **where** π_i $1 \le i \le k$ **is the steady state probability for being in state i**

P ➔ is the 1-step transition matrix of dimensions k X k

= $\{p_{ij}\}$ **is the Probability of transitioning from state i to j**

Recall that all rows of P sum to 1

Example: 8

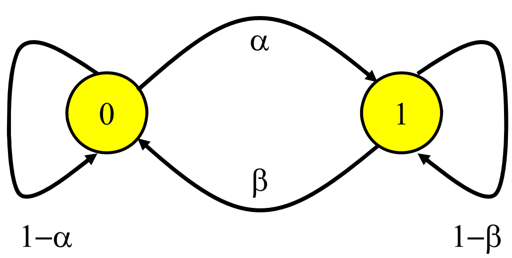
Problem: **A Markov model for packet speech assumes that if the nth packet contains silence then the probability of silence in the next packet is 1-α and the probability of speech activity is α. Similarly if the nth packet contains speech activity, then the probability of speech activity in next packet is 1-β and the probability of silence is β. Find the stationary state pmf.**

Example: 8 – cont'd

Answer: **The state diagram is as shown:**

The 1-step transition probability, P, is given by:

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$



State 0: silence
State 1: speech

Example: 8 – cont'd 2

Answer: The steady state pmf $\Pi = [\pi_0\ \pi_1]$ can be solved for using

$$\Pi = \Pi\,\mathbf{P}$$

Or

$$\begin{bmatrix}\pi_0 & \pi_1\end{bmatrix} = \begin{bmatrix}\pi_0 & \pi_1\end{bmatrix} \times \begin{bmatrix}1-\alpha & \alpha \\ \beta & 1-\beta\end{bmatrix}$$

Or

$$\pi_0 = (1-\alpha)\,\pi_0 + \beta\,\pi_1$$
$$\pi_1 = \alpha\,\pi_0 + (1-\beta)\,\pi_1$$

In addition to the constraint that $\pi_0 + \pi_1 = 1$

Example: 8 – cont'd 3

Answer: Therefore steady state pmf $\Pi = [\pi_0\ \pi_1]$ is given by:

$$\pi_0 = \beta / (\alpha + \beta)$$
$$\pi_1 = \alpha / (\alpha + \beta)$$

Note that sum of all π_i's should equal to 1!!

For $\alpha = 1/10$, $\beta = 1/5$ ➔ $\Pi = [2/3\ \ 1/3]$

Example: 8 – cont'd 4

Answer: Alternatively, one can find a general form for P^n and take the limit as $n \to \infty$.

P^n can be shown to be:

$$P^n = \frac{1}{\alpha+\beta}\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} + \frac{(1-\alpha-\beta)^n}{\alpha+\beta}\begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix}$$

Which clearly approaches:

$$\lim_{n\to\infty} P^n = \frac{1}{\alpha+\beta}\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix}$$

Example: 8 – cont'd 5

Answer: If the initial state pmf is $p_0(0)$ and $p_1(0) = 1-p_0(0)$

Then the n^{th} state pmf ($n \to \infty$) is given by:

p(n) as $n \to \infty$ = $[p_0(0) \;\; 1- p_0(0)]\, P^n$

$= [\beta/(\alpha+\beta) \quad \alpha/(\alpha+\beta)]$

Same as the solution obtained using the 1-step transition matrix!!

Example 9:

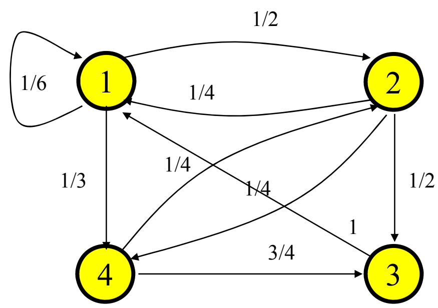
Problem: Assume the state diagram of Example 7 was modified as shown to eliminate the absorbing state. Find the steady state distribution of the chain.

Solution:
Using Matlab, p(n) = p(n-1), P remains ~ constant for n > 40. Therefore,

p(∞) = [0.359 0.223 0.243 0.175]

Nth state probability:

Step	State 1	State 2	State 3	State 4
33	0.358561	0.223111	0.243027	0.175301
34	0.358565	0.223106	0.243031	0.175298
35	0.358568	0.223107	0.243027	0.175298
36	0.358565	0.223109	0.243027	0.175300
37	0.358565	0.223107	0.243029	0.175299
38	0.358567	0.223107	0.243028	0.175298
39	0.358566	0.223108	0.243027	0.175299
40	0.358565	0.223108	0.243028	0.175299
41	0.358566	0.223107	0.243028	0.175299
42	0.358566	0.223108	0.243028	0.175299



Example 9: cont'd

Solution – cont'd:
Alternatively, one can solve the linear system

$$\Pi = \Pi \mathbf{P}$$

To find that
p(∞) = π
= [0.359 0.223 0.243 0.175]

A third alternative would be to compute P^∞, which happens to be as shown. Then,

$$p(\infty) = p(0)P^\infty$$

$$P^\infty = \begin{bmatrix} 0.359 & 0.223 & 0.243 & 0.175 \\ 0.359 & 0.223 & 0.243 & 0.175 \\ 0.359 & 0.223 & 0.243 & 0.175 \\ 0.359 & 0.223 & 0.243 & 0.175 \end{bmatrix}$$

which leads to

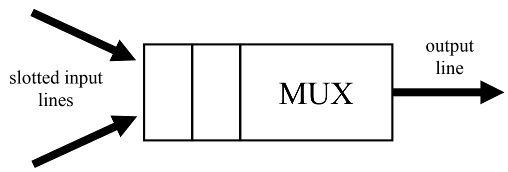
p(∞) = π
= [0.359 0.223 0.243 0.175]

Example 10: Multiplexer

Problem: Data in the form of fixed-length packets arrive in slots on both of the input lines of a multiplexer. A slot contains a packet with probability p, independent of the arrivals during other slots or on the other line. The multiplexer transmits one packet per time slot and has the capacity to store two messages only. If no room for a packet is found, the packet is dropped.

a) Draw the state diagram and define the matrix P

b) Compute the throughput of the multiplexer for p = 0.3



Example 10: Multiplexer – cont'd

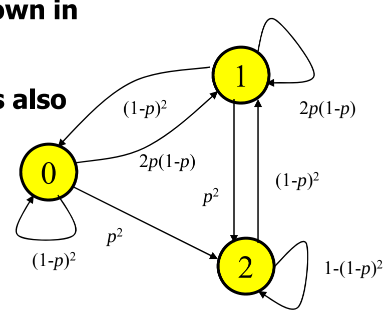
Solution: In any slot time, the arrivals pmf is given by

P(j cells arrive) = $(1-p)^2$ j=0
$2p(1-p)$ j=1
p^2 j=2

Let the state be the number of packets in the buffer, then the state diagram is shown in figure.

The corresponding transition matrix is also given below

$$P = \begin{bmatrix} (1-p)^2 & 2p(1-p) & p^2 \\ (1-p)^2 & 2p(1-p) & p^2 \\ 0 & (1-p)^2 & 1-(1-p)^2 \end{bmatrix}$$



Example 10: Multiplexer – cont'd

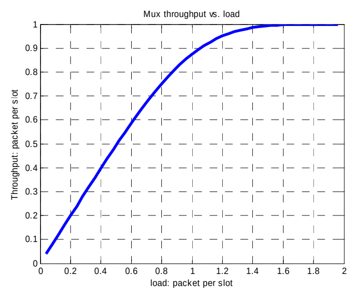
<u>Solution-cont'd</u>:

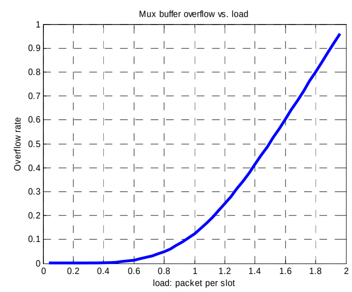
Load: average arrivals = 2*p* packets/slot

Throughput: π_1 + π_2 (the MUX outputs one packet per slot as long as it exists in states 1 and 2)

Buffer overflow = Prob(two packet arrivals while in state 2)
= Prob(two arrivals) X π_2
= p^2 π_2

The graphs below show the relation of load versus –throughput and buffer overflow for the MUX





Example 10: Multiplexer – cont'd

<u>Solution-cont'd</u>:

The matlab code used for plotted previous results is shown below.

Make sure you understand the matrix formulation and the solution for the steady state probability vector π

```
clear all
Step        = 0.02;
ArrivalProb = [Step:Step:1-Step];
A    = zeros(4,3);
E    = zeros(4,1);
E(4) = 1;
for i=1:length(ArrivalProb)
    p = ArrivalProb(i);
    P = [(1-p)^2  2*p*(1-p)  p^2; ...
         (1-p)^2  2*p*(1-p)  p^2; ...
          0        (1-p)^2    1-(1-p)^2];
   A(1:3,:) = (P - eye(3))';
   A(4,:)   = ones(1,3);
   E(4)     = 1;
   SteadyStateP = A\E;
   % Prob(packet is lost) = Prob(2 arrivals) X
   %                        Prob(being in state 2);
   DropProb(i) = p^2*SteadyStateP(3);
   Throughput(i) = sum(SteadyStateP(2:3));
end
```

```
% matlab code continued
figure(1),
h = plot(2*ArrivalProb, Throughput);
set(h, 'LineWidth', 3);
title('Mux throughput vs. load');
ylabel('throughput: packet per slot');
xlabel('load: packet per slot');
grid
figure(2),
h = plot(2*ArrivalProb, DropProb);
set(h, 'LineWidth', 3);
title('Mux buffer overflow vs. load');
ylabel('overflow rate');
xlabel('load: packet per slot');
grid
```

References

- **Alberto Leon-Garcia, Probability and Random Processes for Electrical Engineering, Addison Wesley, 1989**
- **J. Hayes, T. Babu, Modeling and Analysis of Telecommunications Networks, Wiley 2004**
- **L. Kleinrock. *Queueing Systems – Volume I: Theory*. Wiley, New York, 1975**