King Fahd University of Petroleum & Minerals Computer Engineering Dept

COE 540 – Computer Networks

Term 072

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Primer on Probability Theory

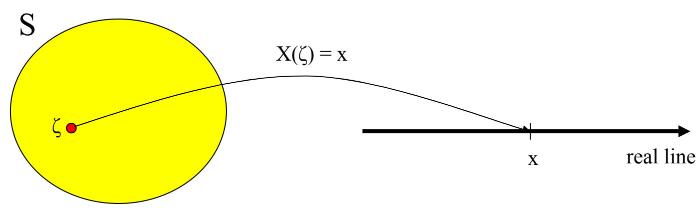
- **Source: Chapter 2 and 3 of:**

Alberto Leon-Garcia, Probability and Random Processes for Electrical Engineering, Addison Wisely

What is a Random Variable?

- **Random Experiment**
- **Sample Space**

- **Def: A random variable X is a function that assigns a number of X(ζ) to each outcome ζ in the sample space of S of the random experiment**



Set Functions

- Define **Ω** as the set of all possible outcomes
- Define **A** as set of events
- Define A as an event – subset of the set of all experiments outcomes
- Set operations:
  - Complementation A^c: is the event that event A does not occur
  - Intersection A ∩ B : is the event that event A and B occur
  - Union A ∪ B: is the event that event A or B occur
  - Inclusion A ⊆ B: An event A occurring implying events B occurs

Set Functions

- Note:
  - Set of events **A** is closed under set operations
  - Φ – empty set
  - $A \cap B = \Phi$ ➔ are mutually exclusive or disjoint

Axioms of Probability

- Let P(A) denote probability of event A:
  1. For any event A belongs **A**, P(A) ≥ 0;
  2. For set of all possible outcomes **Ω**, P(**Ω**) = 1;
  3. If A and B are disjoint events, P(A un B) = P(A) + P(B)
  4. For countably infinite sets, A_1, A_2, … such that A_i ins A_j = Φ for i≠j

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Additional Properties

- For any event, $P(A) \leq 1$
- $P(A^C) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A) \leq P(B)$ for $A \subseteq B$

Conditional Probability

- Conditional probability is defined as

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

- P(A/B) probability of event A conditioned on the occurrence of event B
- Note:
  - A and B are *independent* if $P(A \cap B) = P(A)P(B) \rightarrow P(A/B) = P(A)$
  - Independent IS NOT EQUAL TO mutually exclusive

The Law of Total Probability

- A set of events A_i, i = 1, 2, …, n partitions the set of experimental outcomes if

$$\bigcup_{i=1}^{n} A_i = \Omega$$

and

$$A_i \cap A_j = \Phi$$

Then we can write any event B in terms of A_i, i = 1, 2, …, n as

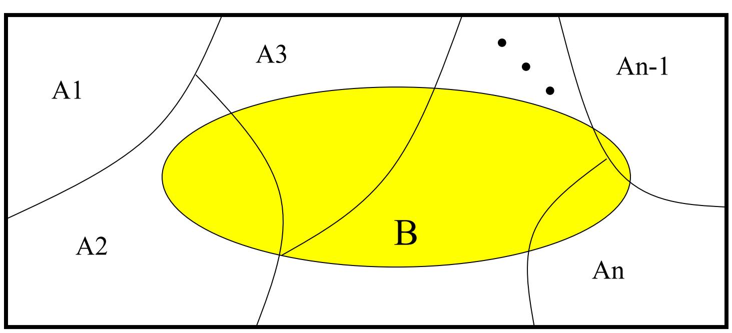
$$B = \bigcup_{i=1}^{n} A_i \cap B$$

Furthermore,

$$P(B) = \sum_{i=1}^{n} P(A_i \cap B)$$

Bayes Rule

- **Let A1, A2, …, An be a partition of a sample space S. Suppose the event A occurs**



A Partition of S into n disjoint sets

Bayes' Rule

- Using the total law of probability and applying it to the definition of the conditional probability, yields

$$P(A_i/B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i \cap B)}{\sum_{i=1}^{n} P(A_i \cap B)}$$

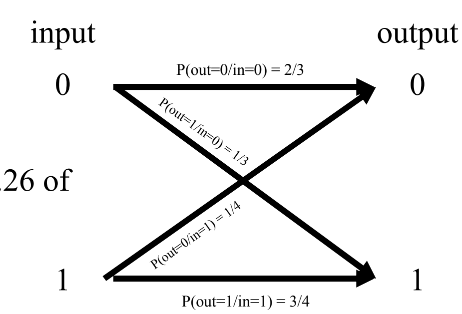
$$= \frac{P(A_i)P(B/A_i)}{\sum_{i=1}^{n} P(A_i)P(B/A_i)}$$

Example 1: Binary (Symmetric) Channel

- Given the binary symmetric channel depicted in figure, find P(input = j / output = i); i,j = 0,1. Given that P(input = 0) = 0.4, P(input = 1) = 0.6.

Solution:

Refer to examples 2.23 and 2.26 of Garcia's textbook



The Cumulative Distribution Function

- **The cumulative distribution function (cdf) of a random variable X is defined as the probability of the event $\{X \le x\}$:**

$$F_X(x) = \text{Prob}\{X \le x\} \quad \text{for } -\infty < x < \infty$$

i.e. it is equal to the probability the variable X takes on a value in the set $(-\infty, x]$

- **A convenient way to specify the probability of all semi-infinite intervals**

Properties of the CDF

- $0 \le F_X(x) \le 1$
- $\lim_{x \to \infty} F_X(x) = 1$
- $\lim_{x \to -\infty} F_X(x) = 0$
- $F_X(x)$ is a nondecreasing function → if $a < b$ → $F_X(a) \le F_X(b)$
- $F_X(x)$ is continuous from the right → for $h > 0$,
  $F_X(b) = \lim_{h \to 0} F_X(b+h) = F_X(b^+)$
- $\text{Prob}[a < X \le b] = F_X(b) - F_X(a)$
- $\text{Prob}[X = b] = F_X(b) - F_X(b^-)$

Example 2: Exponential Random Variable

- **Problem: The transmission time X of a message in a communication system obey the exponential probability law with parameter λ, that is**

  Prob [X > x] = $e^{-\lambda x}$ x > 0

Find the CDF of X. Find Prob [T < X ≤ 2T] where T = 1/ λ

Example 2: Exponential Random Variable – cont'd

- **Answer:**

The CDF of X is

$$F_X(x) = \text{Prob}\{X \le x\} = 1 - \text{Prob}\{X > x\}$$
$$= 1 - e^{-\lambda x} \quad x \ge 0$$
$$= 0 \quad x < 0$$

$$\text{Prob}\{T < X \le 2T\} = F_X(2T) - F_X(T)$$
$$= 1-e^{-2} - (1-e^{-1})$$
$$= 0.233$$

Example 3: Use of Bayes Rule

- **Problem: The waiting time W of a customer in a queueing system is zero if he finds the system idle, and an exponentially distributed random length of time if he finds the system busy. The probabilities that he finds the system idle or busy are p and 1-p, respectively. Find the CDF of W**

Example 3: cont'd

- **Answer:**

The CDF of W is found as follows:

$F_X(x) = \text{Prob}\{W \leq x\}$

$\quad = \text{Prob}\{W \leq x/\text{idle}\}p + \text{Prob}\{W \leq x/\text{busy}\}(1\text{-}p)$

Note $\text{Prob}\{W \leq x/\text{idle}\} = 1$ for any $x > 0$

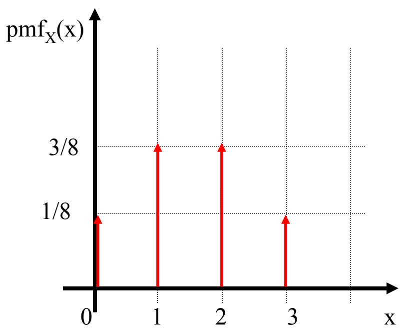
➔

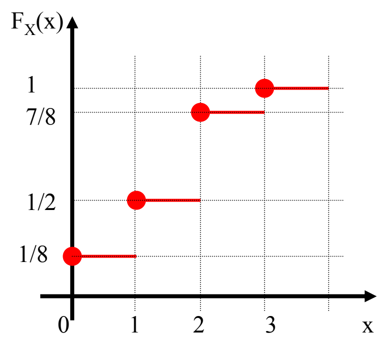
$F_X(x) = 0 \qquad x < 0$

$\quad = p+(1\text{-}p)(1\text{-}\ e^{-\lambda x}) \qquad x \geq 0$

Types of Random Variables

- **(1) Discrete Random Variables**
  - **CDF is right continuous, staircase function of x, with jumps at countable set x0, x1, x2, …**





Pmf: probability mass function

Types of Random Variables

- **(2) Continuous Random Variables**
  - **CDF is continuous for all values of x ➔ Prob { X = x} = 0 (recall the CDF properties)**
  - **Can be written as the integral of some non negative function**

$$F_X(x) = \int_{-\infty}^{\infty} f(t)dt$$

Or

$$f(t) = \frac{dF_X(x)}{dx}$$

f(t) is referred to as the probability density function or PDF

Types of Random Variables

- **(3) Random Variables of Mixed Types**

$$F_X(x) = p\,F_1(x) + (1-p)\,F_2(x)$$

Probability Density Function

- **The PDF of X, if it exists, is define as the derivative of CDF $F_X(x)$:**

$$f_x(x) = \frac{dF_X(x)}{dx}$$

Properties of the PDF

- $f_x(x) \geq 0$
- $$P\{a \le x \le b\} = \int_a^b f_x(x)dx$$
- $$F_X(x) = \int_{-\infty}^{x} f_x(t)dt$$
- $$1 = \int_{-\infty}^{\infty} f_x(t)dt$$

A valid pdf can be formed from any nonnegative, piecewise continuous function g(x) that has a finite integral:

$$\int_{-\infty}^{\infty} g(x)dx = c < \infty$$

By letting $f_X(x) = g(x)/c$, we obtain a function that satisfies the normalization condition.
This is the scheme we use to generate pdfs from simulation results!

Conditional PDFs and CDFs

- If some event A concerning X is given, then conditional CDF of X given A is defined by

$$F_X(x/A) = \frac{P\{[X \le x] \cap A\}}{P\{A\}} \quad \text{if } P\{A\} > 0$$

The conditional pdf of X given A is then defined by

$$f_X(x/A) = \frac{d}{dx} F_X(x/A)$$

Expectation of a Random Variable

- **Expectation of the random variable X can be computed by**

$$E[X] = \sum_{\forall i} x_i P[X = x_i]$$

for discrete variables, or

$$E[X] = \int_{-\infty}^{\infty} t f_x(t) dt$$

for continuous variables.

n^{th} Expectation of a Random Variable

- **The n^{th} expectation of the random variable X can be computed by**

$$E[X^n] = \sum_{\forall i} x^n{}_i P[X = x_i]$$

for discrete variables, or

$$E[X^n] = \int_{-\infty}^{\infty} t^n f_x(t) dt$$

for continuous variables.

The Characteristic Function

- **The characteristic function of a random variable X is defined by**

$$\Phi_x(\omega) = E[e^{j\omega X}]$$

$$= \int_{-\infty}^{\infty} f_X(x) e^{j\omega X} dx$$

- **Note that $\Phi_X(\omega)$ is simply the Fourier Transform of the PDF $f_X(x)$ (with a reversal in the sign of the exponent)**
- **The above is valid for continuous random variables only**

The Characteristic Function (2)

- **Properties:**

$$E[X^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_x(\omega) \bigg|_{\omega=0}$$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega) e^{-j\omega x} d\omega$$

The Characteristic Function (3)

- **For discrete random variables,**

$$\Phi_x(\omega) = E[e^{j\omega X}]$$
$$= \sum_{\forall k} p_X(x_k) e^{j\omega x_k}$$

- **For integer valued random variables,**

$$\Phi_x(\omega) = \sum_{k=-\infty}^{\infty} p_X(k) e^{j\omega k}$$

The Characteristic Function (4)

- **Properties**

$$p_X(k) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_x(\omega) e^{-j\omega k} d\omega$$

for k=0, ±1, ±2, …

Expectation of a Function of the Random Variable

- Let g(x) be a function of the random variable x, the expectation of g(x) is given by

$$E[g(x)] = \sum_{\forall i} g(x_i) P[X = x_i]$$

for discrete variables, or

$$E[g(x)] = \int_{-\infty}^{\infty} g(t) f_x(t) dt$$

for continuous variables.

Probability Generating Function

- A matter of convenience – compact representation
- The same as the z-transform
- If N is a non-negative integer-valued random variable, the probability generating function is defined as

$$\begin{aligned} G_N(z) &= E[z^N] \\ &= \sum_{k=0}^{\infty} p_N(k) z^k \\ &= p_N(0) + p_N(1) z + p_N(2) z^2 + \ldots. \end{aligned}$$

Probability Generating Function (2)

- **Properties:**
- **1** $$p_N(k) = \frac{1}{k!}\frac{d^k}{dz^k} G_N(z)\bigg|_{z=0}$$
- **2** $$E[N] = G'_N(1)$$
- **3** $$Var[N] = G''_N(1) + G'_N(1) - \left[G'_N(1)\right]^2$$

Probability Generating Function (3)

- **For non-negative continuous random variables, let us define the Laplace transform of the PDF**

$$X^*(s) = \int_0^\infty f_X(x)e^{-sx}dx$$

$$= E[e^{-sx}]$$

Properties:

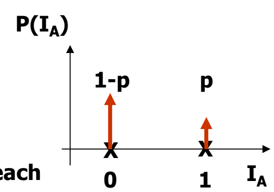
$$E[X^n] = (-1)^n \frac{d^n}{ds^n} X^*(s)\bigg|_{s=0}$$

Some Important Random Variables – Discrete Random Variables

- **Bernoulli**
- **Binomial**
- **Geometric**
- **Poisson**

Bernoulli Random Variable

- **Let A be an event related to the outcomes of some random experiment. The indicator function for A is defined as**

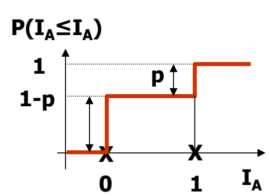


$$I_A(\zeta) = 0 \quad \text{if } \zeta \text{ not in A}$$
$$= 1 \quad \text{if } \zeta \text{ is in A}$$

- **I_A is random variable since it assigns a number to each outcome in S**
- **It is discrete r.v. that takes on values from the set {0,1}**
- **PMF is given by**

$$p_I(0) = 1\text{-}p, \; p_I(1) = p$$

where P{A} = p



- **Describes the outcome of a Bernoulli trial**
- **E[X] = p, VAR[X] = p(1-p)**
- **$G_X(z) = (1\text{-}p\text{+}pz)$**

Binomial Random Variable

- Suppose a random experiment is repeated n independent times; let X be the number of times a certain event A occurs in these n trials

  X = I1 + I2 + … + In

i.e. X is the sum of Bernoulli trials (X's range = {0, 1, 2, …, n})

- X has the following pmf

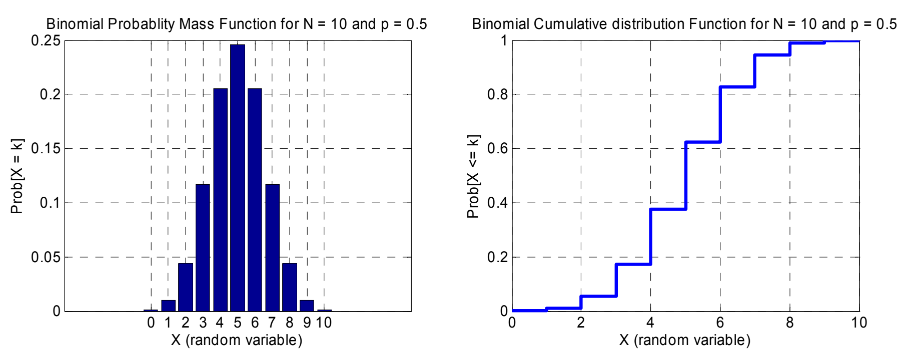
$$\Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

for k=0, 1, 2, …, n

- E[X] = np, Var[X] = np(1-p)
- $G_X(z) = (1-p + pz)^n$

Binomial Random Variable – cont'd

- **Example**



Geometric Random Variable

- Suppose a random experiment is repeated - We count the number of M of independent Bernoulli trials UNTIL the first occurrence of a success
- M is called geometric random variable
    - Range of M = 1, 2, 3, …
- X has the following pmf

$$\Pr[X = k] = (1-p)^{k-1} p$$

for k=1, 2, 3, …

- E[X] = 1/p, Var[X] = (1-p)/p^2
- $G_X(z)$ = pz/(1-(1-p)z))

Geometric Random Variable - 2

- Suppose a random experiment is repeated - We count the number of M of independent Bernoulli trials BEFORE the first occurrence of a success
- M is called geometric random variable
    - Range of M = 0, 1, 2, 3, …
- X has the following pmf

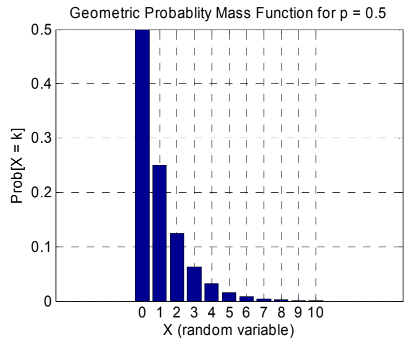
$$\Pr[X = k] = (1-p)^{k} p$$

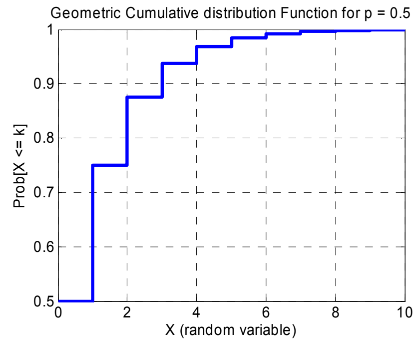
for k=0,1, 2, 3, …

- E[X] = (1-p)/p, Var[X] = (1-p)/p^2
- $G_X(z)$ = pz/(1-(1-p)z))

Geometric Random Variable – cont'd

- **Example: p = 0.5; X is number of failures BEFORE a success (2nd type)**
- **Note Matlab's version of geometric distribution is the 2nd type**





Poisson Random Variable

- **In many applications we are interested in counting the number of occurrences of an event in a certain time period**
- **The pmf is given by**

$$\Pr[X = k] = \frac{\alpha^k}{k!} e^{-\alpha}$$

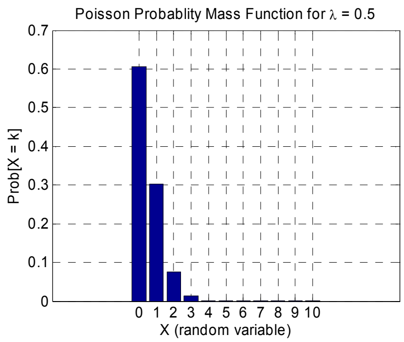
For k=0, 1, 2, … ; α is the average number of event occurrences in the specified interval

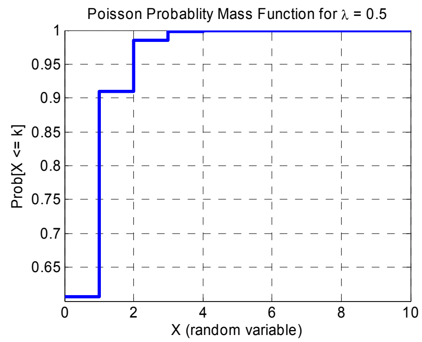
- **E[X] = α, Var[X] = α**
- **$G_X(z) = e^{\alpha(z-1)}$**

- **Remember: time between events is exponentially distributed!**
- **Poisson is the limiting case for Binomial as n→∞, p →0, such that np = α**

Poisson Random Variable – cont'd

- **Example:**





Matlab Code to Plot Distributions

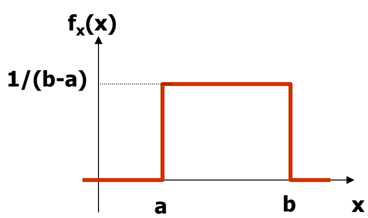
```
% plot distributions
% see "help stats"
clear all
FontSize = 14;
LineWidth = 3;
% Binomial
N = 10; X = [0:1:N]; P = 0.5;
ybp = binopdf(X, N, P); % get PMF
ybc = binocdf(X, N, P); % get CDF
figure(1); set(gca,'FontSize', FontSize);
bar(X, ybp);
title(['Binomial Probablity Mass Function for
    N = ' ...
    num2str(N) ' and p = ' num2str(P)]);
xlabel('X (random variable)');
ylabel('Prob[X = k]'); grid
figure(2); set(gca,'FontSize', FontSize);
stairs(X, ybc,'LineWidth', LineWidth);
title(['Binomial Cumulative distribution
    Function for N = ' ...
    num2str(N) ' and p = ' num2str(P)]);
xlabel('X (random variable)');
ylabel('Prob[X <= k]'); grid
% Geometric
P = 0.5; X = [0:10];
ygp = geopdf(X, P); % get pdf
ygc = geocdf(X, P); % get cdf
figure(3); set(gca,'FontSize', FontSize);
bar(X, ygp);
title(['Geometric Probablity Mass Function for
    p = ' num2str(P)]);
xlabel('X (random variable)');
ylabel('Prob[X = k]'); grid
figure(4); set(gca,'FontSize', FontSize);
stairs(X, ygc,'LineWidth', LineWidth);
title(['Geometric Cumulative distribution
    Function for p = ' num2str(P)]);
xlabel('X (random variable)');
ylabel('Prob[X <= k]'); grid
% Poisson
Lambda = 0.5; X     = [0:10];
ypp   = poisspdf(X, Lambda);
ypc   = poisscdf(X, Lambda);
figure(5); set(gca,'FontSize', FontSize);
bar(X, ypp);
title(['Poisson Probablity Mass Function for
    \lambda = ' num2str(Lambda)]);
xlabel('X (random variable)');
ylabel('Prob[X = k]'); grid
figure(6); set(gca,'FontSize', FontSize);
stairs(X, ypc,'LineWidth', LineWidth);
title(['Poisson Probablity Mass Function for
    \lambda = ' num2str(Lambda)]);
xlabel('X (random variable)');
ylabel('Prob[X <= k]'); grid
```

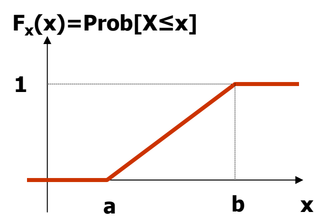
Some Important Random Variables – Continuous Random Variables

- **Uniform**
- **Exponential**
- **Gaussian (Normal)**
- **Rayleigh**
- **Gamma**
- **M-Erlang**
- **….**

Uniform Random Variables

- **Realizations of the r.v. can take values from the interval [a, b]**
- **PDF $f_X(x) = 1/(b-a) \qquad a \le x \le b$**
- **$E[X] = (a+b)/2, \quad Var[X] = (b-a)^2/12$**
- **$\Phi_X(\omega) = [e^{j\omega b} - e^{j\omega a}]/(j\omega(b-a))$**





Exponential Random Variables

- **The exponential r.v. X with parameter λ has pdf**

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

- **And CDF given by**

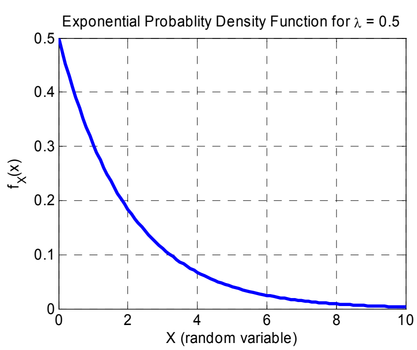
$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

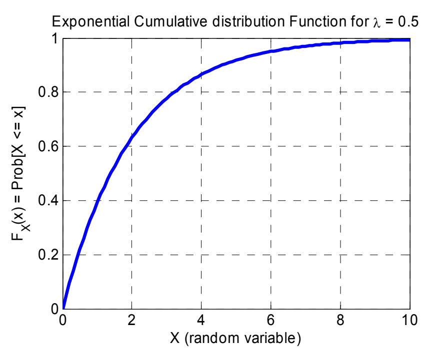
- **Range of X: [0, ∞)**
- **E[X] = 1/λ, Var[X] = 1/λ²**
- **$\Phi_X(\omega) = \lambda/(\lambda - j\omega)$**

This means:
Prob[X≤x] = 1-e$^{-\lambda x}$, or
Prob[X>x] = e$^{-\lambda x}$

Exponential Random Variables – cont'd

- **Example:**
  - **Note the mean is 1/λ = 2**





Exponential Random Variables – Memoryless Property

- The exponential r.v. is the only continuous r.v. with the memoryless property!!
- Memoryless Property:
  P[X>t+h/ X>t] = P[X>h]

i.e. the probability of having to wait h additional seconds given that one has already been waiting t second IS EXACTLY equal to the probability of waiting h seconds when one first begins to wait

<u>Proof</u>:

$$P[X>t+h/ X>t] = \frac{P[(X > t+h) \cap (X > t)]}{P[(X > t)]}$$

$$= \frac{P[(X > t+h)}{P[X > t]} = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}}$$

$$= e^{-\lambda h}$$

$$= P[X > h]$$

Gaussian (Normal) Random Variable

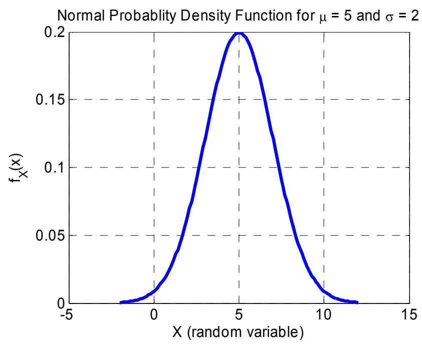
- Rises in situations where a random variable X is the sum of a large number of "small" random variables – central limit theorem
- PDF $$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

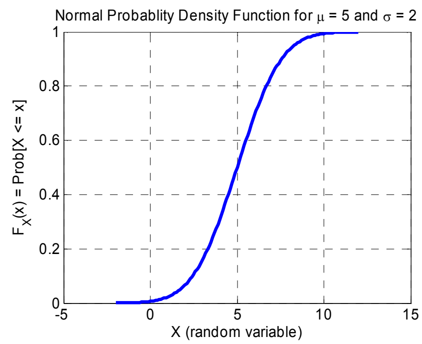
For $-\infty<x<\infty$; μ and $\sigma > 0$ are real numbers

- $E[X] = \mu$, $Var[X] = \sigma^2$
- $$\Phi_X(\omega) = e^{j\mu\omega - \sigma^2\omega^2/2}$$
- Under wide range of conditions X can be used to approximate the sum of a large number of independent random variables

Gaussian (Normal) Random Variable – cont'd

- **Example:**



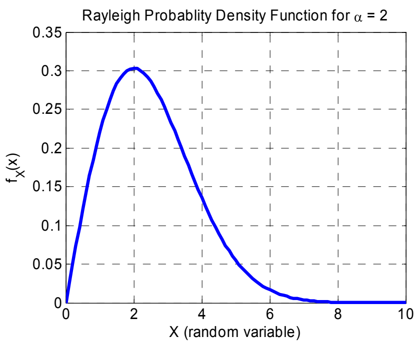


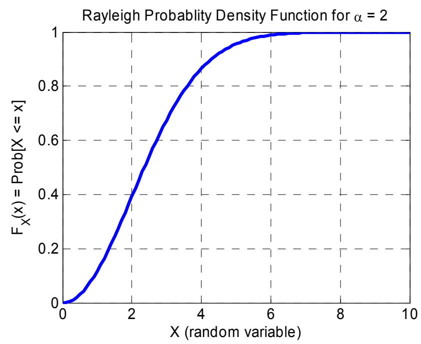
Rayleigh Random Variable

- **Arises in modeling of mobile channels**
- **Range: [0, ∞)**
- **PDF:** $$f_X(x) = \frac{x}{\alpha^2} e^{-x^2/(2\alpha^2)}$$
- **For x ≥ 0, α > 0**
- **E[X] = α√(π/2), Var[X] = (2-π/2)α²**

Rayleigh Random Variable – cont'd

- **Example:**
  - **Note that for Alpha = 2, the mean is 2√(π/2)**





Matlab Code to Plot Distributions

```
% plot distributions
% see "help stats"
clear all
FontSize = 14;
LineWidth = 3;
% exponential
X = [0:.1:10]; Lambda = 0.5;
yep = exppdf(X, 1/Lambda); % get PDF
yec = expcdf(X, 1/Lambda); % get CDF
figure(1); set(gca,'FontSize', FontSize);
plot(X, yep, 'LineWidth', LineWidth);
title(['Exponential Probablity Density
    Function for \lambda = ' ...
      num2str(Lambda)]);
xlabel('X (random variable)');
ylabel('f_X(x)'); grid
figure(2); set(gca,'FontSize', FontSize);
plot(X, yec,'LineWidth', LineWidth);
title(['Exponential Cumulative Distribution
    Function for \lambda = ' ...
      num2str(Lambda)]);
xlabel('X (random variable)');
ylabel('F_X(x) = Prob[X <= x]'); grid
% normal
X = [-2:.1:12]; Mu = 5; Sigma = 2;
ynp = normpdf(X, Mu, Sigma); % get PDF
ync = normcdf(X, Mu, Sigma); % get CDF
figure(3); set(gca,'FontSize', FontSize);
plot(X, ynp, 'LineWidth', LineWidth);
title(['Normal Probablity Density Function
    for \mu = ' ...
      num2str(Mu) ' and \sigma = '
    num2str(Sigma)]);
xlabel('X (random variable)');
ylabel('f_X(x)'); grid
figure(4); set(gca,'FontSize', FontSize);
plot(X, ync,'LineWidth', LineWidth);
title(['Normal Probablity Density Function
    for \mu = ' ...
      num2str(Mu) ' and \sigma = '
    num2str(Sigma)]);
xlabel('X (random variable)');
ylabel('F_X(x) = Prob[X <= x]'); grid
% Rayleigh
X = [0:.1:10]; Alpha = 2;
yrp = raylpdf(X, Alpha); % get PDF
yrc = raylcdf(X, Alpha); % get CDF
figure(5); set(gca,'FontSize', FontSize);
plot(X, yrp, 'LineWidth', LineWidth);
title(['Rayleigh Probablity Density Function
    for \alpha = ' ...
      num2str(Alpha)]);
xlabel('X (random variable)');
ylabel('f_X(x)'); grid
figure(6); set(gca,'FontSize', FontSize);
plot(X, yrc,'LineWidth', LineWidth);
title(['Rayleigh Probablity Density Function
    for \alpha = ' ...
      num2str(Alpha)]);
xlabel('X (random variable)');
ylabel('F_X(x) = Prob[X <= x]'); grid
```

Gamma Random Variable

- Versatile distribution ~ appears in modeling of lifetime of devices and systems
- Has two parameters: $\alpha > 0$ and $\lambda > 0$
- PDF: $$f_X(x) = \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$$
- For $0 < x < \infty$
- The quantity $\Gamma(z)$ is the gamma function and is specified by $$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$
- The gamma function has the following properties:
  - $\Gamma(1/2) = \sqrt{\pi}$
  - $\Gamma(z+1) = z\Gamma(z)$ for $z>0$
  - $\Gamma(m+1) = m!$ For m nonnegative integer
- $E[X] = \alpha/\lambda$, $Var[X] = \alpha/\lambda^2$
- $\Phi_X(\omega) = 1/(1-j\omega/\lambda)^\alpha$

If $\alpha = 1$ ➔ gamma r.v. becomes exponential

Joint Distributions of Random Variables

- Def: The joint probability distribution of two r.v.s X and Y is given by

$$F_{XY}(x,y) = P(X \leq x, Y \leq y)$$

where x and y are real numbers.

- This refers to the JOINT occurrence of $\{X \leq x\}$ AND $\{Y \leq y\}$
- Can be generalized to any number of variables

Joint Distributions of Random Variables - Properties

- $F_{XY}(-\infty, -\infty) = 0$
- $F_{XY}(\infty, \infty) = 1$
- $F_{XY}(x_1, y) \leq F_{XY}(x_2, y)$ for $x_1 \leq x_2$
- $F_{XY}(x, y_1) \leq F_{XY}(x, y_2)$ for $y_1 \leq y_2$
- The marginal distributions are given by
  - $F_X(x) = F_{XY}(x, \infty)$
  - $F_Y(y) = F_{XY}(\infty, y)$

Joint Distributions of Random Variables – Properties - 2

- **Density function:** $$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

  or $$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(\alpha,\beta)\,d\alpha\,d\beta$$

- **Marginal densities:** $$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

  and $$f_y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

Joint Distributions of Random Variables – Independence

- **Two random variables are independent if the joint distribution functions are products of the marginal distributions:**

$$F_{XY}(x,y) = F_X(x)F_Y(y)$$

or

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

Joint Distributions of Random Variables – Discrete Nonnegative Variables

- **Def:**

$$F_{XY}(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P(X = x_i, Y = y_i)U(x - x_i)U(y - y_i)$$

where

P(X=xi, Y=yi) is the joint probability for the r.v.s X and Y

U(x) is 1 for x ≥ 0 and 0 otherwise

Example 4: Packet Segmentation

- **Problem**: The number of bytes N in a message has a geometric distribution with parameter p. The message is broken into packets of maximum length M bytes. Let Q be the number of full packets in a message and let R be the number of bytes left over.

A) Find the join pmf for Q and R, and

B) Find the marginal pmfs of Q and R.

Example 4: Packet Segmentation - cont'd

- **Solution**:

N ~ geometric → $P(N=k) = (1-p)p^k$

Message of N bytes → Q full M-bytes packets + R remaining bytes

Therefore: $Q \in \{0, 1, 2, \ldots\}$, $R \in \{0, 1, 2, \ldots, M-1\}$

The join pmf is given by:

$P(Q=q, R=r) = P(N = qM+r) = (1-p)p^{(qM+r)}$

Example 4: cont'd

- **Solution:**

  The marginal pmfs:

$$P(Q=q)=\sum_{r=0}^{M-1}P(Q=q,R=r)$$
$$=\sum_{r=0}^{M-1}(1-p)p^{(qM+r)}$$
$$=\left(1-p^{M}\right)\left(p^{M}\right)^{q} \qquad q=0,1,2,\ldots$$

and

$$P(R=r)=\sum_{q=0}^{\infty}P(Q=q,R=r)$$
$$=\sum_{q=0}^{\infty}(1-p)p^{(qM+r)}$$
$$=\frac{(1-p)}{1-p^{M}}p^{r} \qquad r=0,1,\ldots,M-1$$

Verify the marginal pmfs add to ONE!!
P(R = r) is a truncated geometric r.v.

Independent Discrete R.V.s

- **For Discrete random variables:**

P(M=i, N=j) = P(M=i) P(N=j)

Example 5:

- **Problem: Are the Q and R random variables of Previous Example independent? Why?**

Conditional Distributions

- **Def: for continuous X and Y**

$$F_{Y/X}(y/x) = P(Y \le y / X \le x) = \frac{F_{XY}(x,y)}{F_X(x)}$$

Or

$$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

- **For discrete M and N**

$$P(M = i / N = j) = \frac{P(M = i, N = j)}{P(N = j)}$$

Conditional Distributions - 2

- **For mixed types:**

$$F_X(x) = \sum_{i=0}^{\infty} P(N = j, X \le x)$$

$$= \sum_{j=0}^{\infty} P(N = j) P(X \le x / N = j)$$

or

$$P(N = j) = \int_{-\infty}^{\infty} P(N = j / X = x) f_X(x) dx$$

Conditional Distributions - 3

- **For N random variables:**

$$F_{X_1, X_2, \ldots, X_N}(x_1, x_2, \ldots, x_N) =$$
$$= F_{X_1}(x_1) \times F_{X_2 / X_1}(x_2 / x_1) \times \ldots \times F_{X_N / X_1, \ldots, X_{N-1}}(x_N / x_1, \ldots, x_{N-1})$$

or

$$f_{X_1, X_2, \ldots, X_N}(x_1, x_2, \ldots, x_N) =$$
$$= f_{X_1}(x_1) \times f_{X_2 / X_1}(x_2 / x_1) \times \ldots \times f_{X_N / X_1, \ldots, X_{N-1}}(x_N / x_1, \ldots, x_{N-1})$$

Example 6:

- **Problem: The number of customers that arrive at a service station during a time t is a Poisson random variable with parameter βt. The time required to service each customer is exponentially distributed with parameter α. Find the pmf for the number of customers N that arrive during the service time T of a specific customer. Assume the customer arrivals are independent of the customer service time.**

Example 6: cont'd

- **Solution:**

The PDF for T is given by $f_T(t) = \alpha e^{-\alpha t} \quad t \geq 0$
Let N = number of arrivals during time t
➔ the arrivals conditional pmf is given by

$$P(N = j / T = t) = \frac{(\beta t)^j e^{-\beta t}}{j!} \qquad j = 0,1,\ldots \qquad t \geq 0$$

To find the arrivals pmf during service time T, we use:

$$P(N = j) = \int_{-\infty}^{\infty} P(N = j / T = t) f_T(t) dt$$

this reduces to:

$$= \int_{0}^{\infty} \alpha \frac{(\beta t)^j}{j!} e^{-\alpha t} e^{-\beta t} dt$$

Note that:

$$\Gamma(j+1) = \int_0^{\infty} t^j e^{-t} dt = j!$$

$$P(N = j) = \left(\frac{\alpha}{\alpha + \beta}\right)\left(\frac{\beta}{\alpha + \beta}\right)^j \qquad j = 0,1,\ldots$$

Thus N is geometrically distributed with probability of success equal to α /(β+ α)

This is NOT a thorough treatment of the subject. For a fair Treatment of the subject please refer to the textbook or to Garcia's textbook

Markov Processes

- Brief Introduction into Stochastic/Random Processes
- A random process X(t) is a Markov Process if the future of the process given the present is independent of the past.
- For arbitrary times: $t_1<t_2<\ldots<t_k<t_{k+1}$

Past or History

$$\text{Prob}[X(t_{k+1}) = x_{k+1}/X(t_k)=x_k, \underbrace{X(t_{k-1})=x_{k-1},\ldots, X(t_1)=x_1}_{\text{Past or History}}] = \text{Prob}[X(t_{k+1}) = x_{k+1}/X(t_k)=x_k]$$

Or (for discrete-valued)

$$\text{Prob}[a<X(t_{k+1})\le b/X(t_k)=x_k, \ldots, X(t_1)=x_1] = \text{Prob}[a<X(t_{k+1})\le b/X(t_k)=x_k]$$

Markov Property

Markov ≡ Memoryless

Continuous-Time Markov Chain

- An integer-valued Markov random process is called a Markov Chain
- The joint pmf for k+1 arbitrary time instances is given by:

$$\text{Prob}[X(t_{k+1}) = x_{k+1}, X(t_k)=x_k, \ldots, X(t_1)=x_1]$$

$$\begin{aligned} = {} & \text{Prob}[X(t_{k+1}) = x_{k+1}/X(t_k)=x_k] \times \\ & \text{Prob}[X(t_k) = x_k/X(t_{k-1})=x_{k-1}] \times \\ & \ldots \\ & \text{Prob}[X(t_2) = x_2/X(t_1)=x_1] \times \\ & \text{Prob}[X(t_1)=x_1] \end{aligned}$$

transition probabilities

← pmf of the initial time

Discrete-Time Markov Chains

- **Let X_n be a discrete-time integer values Markov Chain that starts at n = 0 with pmf**

$$p_j(0) = \text{Prob}[X_0 = j] \quad j=0,1,2, \ldots$$

$$\text{Prob}[X_n=i_n, X_{n-1}=i_{n-1},\ldots,X_0=i_0]$$
$$= \text{Prob}[X_n=i_n / X_{n-1}=i_{n-1}] \times$$
$$\text{Prob}[X_{n-1}=i_{n-1} / X_{n-2}=i_{n-2}] \times$$
$$\ldots.$$
$$\text{Prob}[X_1=i_1 / X_0=i_0] \times$$
$$\text{Prob}[X_0=i_0]$$

Same as the previous slide but for discrete-time

Discrete-Time Markov Chains – cont'd (2)

- **Assume the one-step state transition probabilities are fixed and do not change with time:**

$$\text{Prob}[X_{n+1}=j/X_n=i] = p_{ij} \quad \text{for all n}$$

➔ **X_n is said to be homogeneous in time**

- **The joint pmf for X_n, X_{n-1}, …, X_1,X_0 is then given by**

$$P[X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0]$$
$$= p_{i_{n-1},i_n} \times p_{i_{n-2},i_{n-1}} \times \ldots \times p_{i_0,i_1} \times p_{i_0}(0)$$

Discrete-Time Markov Chains – cont'd (3)

- **Thus X_n is completely specified by the initial pmf $p_i(0)$ and the matrix of one-step transition probabilities P:**

$$P=\begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0i} & p_{0i+1} & \cdots \\ p_{10} & p_{11} & \cdots & p_{1i} & p_{1i+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i0} & p_{i1} & \cdots & p_{ii} & p_{i,i+1} & \cdots \\ p_{i+1,0} & p_{i+1,1} & \cdots & p_{i+1i} & p_{i+1,i+1} & \cdots \\ \cdots & \cdots & \cdots & & & \end{bmatrix}$$

1-Step Transition Matrix, P (Transition Probabilities)

i.e. rows of P add to UNITY

$$1=\sum_j P[X_{n+1}=j/X_n=i]=\sum_j p_{ij}$$

Discrete-Time Markov Chains – cont'd (4)

- **The state probability at time n+1 is related to the state probabilities at time n as follows:**

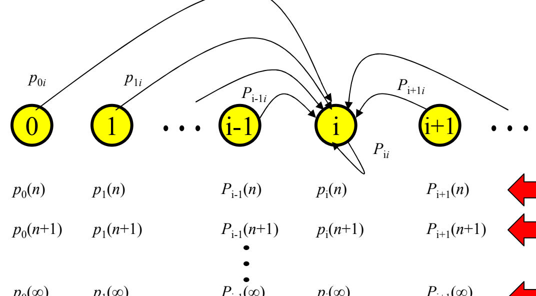
$$\left[\, p_0(n+1) \quad p_1(n+1) \quad \ldots \quad p_i(n+1) \quad p_{i+1}(n+1) \quad \ldots \,\right]=\left[\, p_0(n) \quad p_1(n) \quad \ldots \quad p_i(n) \quad p_{i+1}(n) \quad \ldots \,\right]\begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0i} & p_{0i+1} & \cdots \\ p_{10} & p_{11} & \cdots & p_{1i} & p_{1i+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i0} & p_{i1} & \cdots & p_{ii} & p_{i,i+1} & \cdots \\ p_{i+1,0} & p_{i+1,1} & \cdots & p_{i+1i} & p_{i+1,i+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

State probability at time n+1, p($n+1$)

State probability at time n, p(n)

The 1-step transition matrix P

Transition PROBABILTIES



p(n)

p(n+1)=p(n)P

$\Pi=\Pi P$

Discrete-Time Markov Chains – cont'd (5)

- Therefore, the vector p(n) representing the state probabilities at n is given by

$$p(n) = p(n-1)\,P$$

Remember P is the 1-step transition matrix

- The above also means that one can write

$$p(n) = p(0)\,P^n$$

Where P^n (P raised to the power n) is the n-step transition matrix

- Finally, the steady state distribution for the system, Π, is given by

$$\Pi = \Pi\,P$$

Π → is the steady state pmf
P → is the 1-step transition matrix

- This means at steady state – the state probabilities DO NOT change!

Markov Process versus Chains

- Continuous-Time Markov Process
- Continuous-Time Markov Chain
- Discrete-Time Markov Process
- Discrete-Time Markov Chain

- Process versus Chain → refers to the value of X(t)
- Continuous-time versus Discrete-time → refers to the instant when the variable (process) X(t) change

Markov Chains Models

- Model for an integer-values process
  - Buffer size
  - No of customers
- When change in process values occur at arbitrary (continuous) time values ➔ continuous time Markov chains
  - Length of queue at the bank teller – customer arrivals happen at any time instant
  - Size of input buffer of a router – packet arrivals at the port happen at any time instant
- When change in process values occur at specific (discrete) time values ➔ discrete-time Markov chains
  - The buffer size of a TDM multi-channel multiplexer – packet arrivals are restricted to slots (i.e. time-axis is slotted)

Examples 7: Continuous-Time Random Processes – Poisson Process

- **Problem:** Assume events (e.g. arrivals) occur at rate of λ events per second following a Poisson arrival process. Let N(t) be the number of occurrences in the interval [0,t]

A) Plot multiple realization of N(t)

B) Write the pmf for N(t)

C) Show that N(t) is a Markov chain

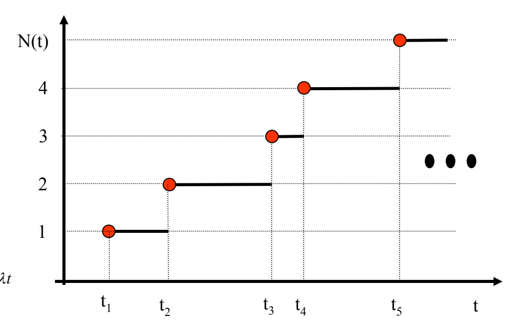
A) N(t) is non-decreasing integer-valued continuous-time random process – A plot for one realization is shown in figure – for other plots, choose different t1, t2, t3, …

Note the increments on Y-axis are in steps of 1 – while the arrival instants t_i i=1, 2, … are random

B) pmf for N(t) is given by

$$P\left(N(t)=k\right)=\frac{(\lambda t)^k}{k!}e^{-\lambda t}$$

for k=0,1, …



Example 7: cont'd

- We can show that N(t) has:
  - Independent increments
  - Stationary increments – the distribution for the number of event occurrences in ***ANY interval of length t*** is given by the previous pmf.

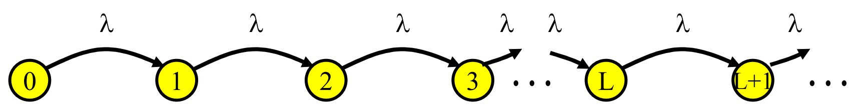
Example: $P(N(t_1)=i, N(t_2)=j] =$

$= P(N(t_1)=i)P(N(t_2)-N(t_1)=j-i]$

$= P(N(t_1)=i)P(N(t_2-t_1)=j-i]$

$$= \frac{(\lambda t_1)^i e^{-\lambda t_1}}{i!} \; \frac{(\lambda(t_2-t_1))^{j-i} e^{-\lambda t_2 - t_1}}{(j-i)!}$$

- If we select the value of N(t) as the STATE variable – one can draw the equivalent Markov model (below) – pure birth process



Example 8: Poisson Arrival Process is EQUIVALENT to Exponential Interarrival Times

- **<u>Problem:</u>** N(t) is a Poisson arrival process – Show that T, the time between event occurrences is exponentially distributed

- **<u>Solution:</u>**
  pmf of N(t) is given by

$$P(N(t)=k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad k = 0,1,\ldots$$

P(T > t) = P(ZERO events in t seconds)

$= e^{-\lambda t}$

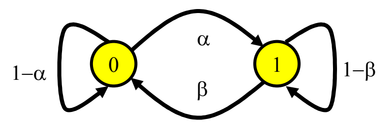
Therefore, $P(T \leq t) = F_T(t) = 1-e^{-\lambda t}$ – i.e. T is exponentially distributed with mean $1/\lambda$

Example 9: Speech Activity Model

Problem: A Markov model for packet speech assumes that if the nth packet contains silence then the probability of silence in the next packet is 1-α and the probability of speech activity is α. Similarly if the nth packet contains speech activity, then the probability of speech activity in next packet is 1-β and the probability of silence is β. Find the stationary state pmf.

Solution:

The state diagram is as shown:



State 0: silence
State 1: speech

The 1-step transition probability, P, is given by:

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Example 9: cont'd 2

Answer: The steady state pmf $\Pi = [\pi_0 \ \pi_1]$ can be solved for using

$$\Pi = \Pi \mathbf{P}$$

Or

$$\begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} = \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} \times \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Or

$$\pi_0 = (1-\alpha)\,\pi_0 + \beta \quad \pi_1$$
$$\pi_1 = \alpha \quad \pi_0 + (1-\beta)\,\pi_1$$

In addition to the constraint that $\pi_0 + \pi_1 = 1$
Therefore steady state pmf
$\Pi = [\pi_0 \ \pi_1]$ is given by:

$$\pi_0 = \beta / (\alpha + \beta)$$
$$\pi_1 = \alpha / (\alpha + \beta)$$

Note that sum of all π_i's should equal to 1!!
For $\alpha = 1/10$, $\beta = 1/5$ → $\Pi = [2/3 \ \ 1/3]$ – Refer to the matlab code to check convergence!!

Example 9: cont'd

Answer: Alternatively, one can find a general form for P^n and take the limit as $n \to \infty$.

P^n can be shown to be:

$$P^n = \frac{1}{\alpha+\beta}\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} + \frac{(1-\alpha-\beta)^n}{\alpha+\beta}\begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix}$$

Which clearly approaches:

$$\lim_{n\to\infty} P^n = \frac{1}{\alpha+\beta}\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix}$$

If the initial state pmf is $p_0(0)$ and $p_1(0) = 1-p_0(0)$

Then the n^{th} state pmf ($n \to \infty$) is given by:

$p(n)$ as $n \to \infty = [p_0(0) \;\; 1- p_0(0)]\, P^n$

$= [\beta/ (\alpha+ \beta) \quad \alpha/ (\alpha+ \beta)]$

Same as the solution obtained using the 1-step transition matrix!!

Example 9: cont'd

- This shows a simple Matlab code to determine p(n) for n=1, 2, 3, … given the 1-step probability matrix P and the initial condition p(0)
- Student must be convinced that the steady state distribution, if it exists, does not depend on p(0) but solely on P.

Main part of code

The rest is for initialization and presentation

```
clear all
LineWidth = 2; MarkerSize = 8;
FontSize = 14;
% 1-step probability transition matrix P
Alpha = 1/10;  Beta  = 1/5;
P = [ 1-Alpha Alpha; Beta 1-Beta];
% Set initial probability state distribution p(0)
p_0 = [1 0]; % i.e. system starts in  0
N = 11; % how many steps to predict (simulate)
p_n     = zeros(N,2); % p_n evolution of distribution
p_n(1,:)= p_0;        % insert initial condition
for n=2:N % the main loop in the code
    % find the state probability distribution after 1-step
    p_n(n,:) = p_n(n-1,:) * P;
end
% compare with analytical result - refer to class slides
Pi_vector = [Beta Alpha]./(Beta + Alpha);
% Show results graphically - JUST FOR PRESENTATION
n = 0:N-1; % define the x-axis for plotting
figure(1);
h = plot(n, p_n(:,1),'-xb',   ... % for state 0
          n, p_n(:,2), '--dr', ... % for state 1
          n, Pi_vector(1)*ones(size(n)), '-b', ...
          n, Pi_vector(2)*ones(size(n)), '--r');
set(h, 'LineWidth', LineWidth, 'MarkerSize', MarkerSize);
set(gca, 'FontSize', FontSize);
title({['Two state on/off discrete-time Markov chain'];
    ['\alpha = ' num2str(Alpha) ' and \beta = ' num2str(Beta) ...
    '- Initial condition p(0) = [' num2str(p_0(1)) ',' num2str(p_0(2)) ']']});
xlabel('time index, n');
ylabel('state probability distribution');
hl = legend('state 0 evolution', 'state 1 evolution', ...
        'state 0 steady state', 'state 1 steady state');
grid; set(hl, 'FontSize', 10); % for legend only
```

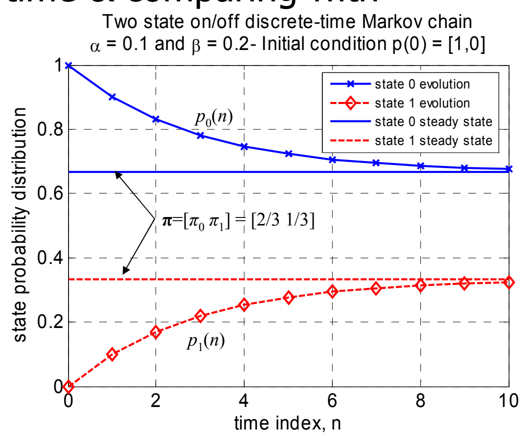
- Program Execution and results:

You can see that p(n) converges!

```
>> SimpleONOFFMarkovChain
>> p_n

p_n =
                              n
    1.0000         0     <--  0
    0.9000    0.1000          1
    0.8300    0.1700          2
    0.7810    0.2190          3
    0.7467    0.2533          4
    0.7227    0.2773          5
    0.7059    0.2941          6
    0.6941    0.3059          7
    0.6859    0.3141          8
    0.6801    0.3199          8
    0.6761    0.3239     <--  9

>> Pi_vector

Pi_vector =

    0.6667    0.3333

>>
```

Example 9: cont'd

- Plotting the state probability distribution p(n) as a function of time & comparing with analytical result

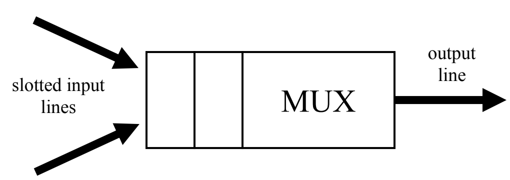


Example 10: Multiplexer

Problem: Data in the form of fixed-length packets arrive in slots on both of the input lines of a multiplexer. A slot contains a packet with probability p, independent of the arrivals during other slots or on the other line. The multiplexer transmits one packet per time slot and has the capacity to store two messages only. If no room for a packet is found, the packet is dropped.

a) Draw the state diagram and define the matrix P

b) Compute the throughput of the multiplexer for p = 0.3

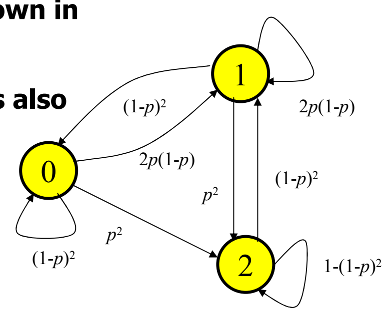


Example 10: Multiplexer – cont'd

<u>Solution</u>: In any slot time, the arrivals pmf is given by

P(j cells arrive) = (1-p)^2 j=0
2p(1-p) j=1
p^2 j=2

Let the state be the number of packets in the buffer, then the state diagram is shown in figure.



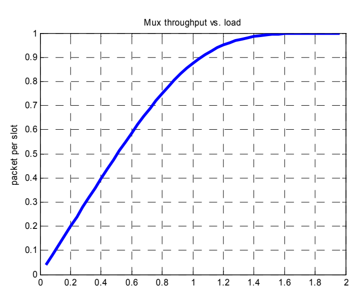
The corresponding transition matrix is also given below

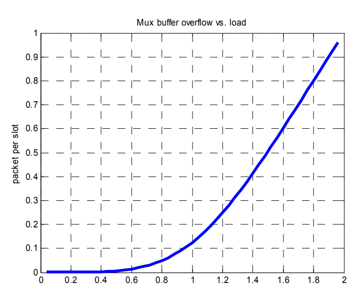
$$P = \begin{bmatrix} (1-p)^2 & 2p(1-p) & p^2 \\ (1-p)^2 & 2p(1-p) & p^2 \\ 0 & (1-p)^2 & 1-(1-p)^2 \end{bmatrix}$$

Example 10: Multiplexer – cont'd

<u>Solution-cont'd</u>:
Load: average arrivals = 2*p* packets/slot
Throughput: $\pi_1 + \pi_2$
Buffer overflow = Prob(two packet arrivals while in state 2)
= Prob(two arrivals) X π_2
= $p^2\ \pi_2$

The graphs below show the relation of load versus –throughput and buffer overflow for the MUX





Example 10: Multiplexer – cont'd

Solution-cont'd:

The matlab code used for plotted previous results is shown below.

Make sure you understand the matrix formulation and the solution for the steady state probability vector π

```
clear all
Step        = 0.02;
ArrivalProb = [Step:Step:1-Step];
A    = zeros(4,3);
E    = zeros(4,1);
E(4) = 1;
for i=1:length(ArrivalProb)
    p = ArrivalProb(i);
    P = [(1-p)^2  2*p*(1-p)  p^2; ...
         (1-p)^2  2*p*(1-p)  p^2; ...
          0         (1-p)^2    1-(1-p)^2];
   A(1:3,:) = (P - eye(3))';
   A(4,:)   = ones(1,3);
   E(4)     = 1;
   SteadyStateP = A\E;
   % Prob(packet is lost) = Prob(2 arrivals) X
   %                         Prob(being in state 2);
   DropProb(i) = p^2*SteadyStateP(3);
   Throughput(i) = sum(SteadyStateP(2:3));
end
```

```
% matlab code continued
figure(1),
h = plot(2*ArrivalProb, Throughput);
set(h, 'LineWidth', 3);
title('Mux throughput vs. load');
ylabel('packet per slot');
xlabel('packet per slot');
grid
figure(2),
h = plot(2*ArrivalProb, DropProb);
set(h, 'LineWidth', 3);
title('Mux buffer overflow vs. load');
ylabel('packet per slot');
xlabel('packet per slot');
grid
```


Continuous-Time Markov Chains -Steady State Probabilities and Global Balance Equations

- **What relation govern the state probabilities for continuous-time Markov chains?**
  - **Remember that probability for state i, $p_i(t)$, is now a function of time!**
- **The answer is given by Remember Chapman-Kolmogrov equations:**

$$p'_j(t) = \sum_i \gamma_{i,j}\, p_i(t) \quad \text{for all } j$$

Or in matrix form: $P'(t) = P(t)\Gamma$,
where $P(t) = [p_0(t), p_1(t), \dots, p_j(t), \dots]$

$$\Gamma = \begin{bmatrix} \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,j} & \gamma_{0\,j+1} & \cdots \\ \gamma_{1,0} & \gamma_{1,1} & \cdots & \gamma_{1,j} & \gamma_{1\,j+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{j,0} & \gamma_{j,1} & \cdots & \gamma_{j,j} & \gamma_{jj+1} & \cdots \\ \gamma_{j+1,0} & \gamma_{j+1,1} & \cdots & \gamma_{j+1,j} & \gamma_{j+1,j+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$\begin{bmatrix} p'_0(t) & p'_1(t) & \cdots & p'_j(t) & p'_{j+1}(t) & \cdots \end{bmatrix} = \begin{bmatrix} p_0(t) & p_1(t) & \cdots & p_j(t) & p_{j+1}(t) & \cdots \end{bmatrix} \begin{bmatrix} \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,j} & \gamma_{0\,j+1} & \cdots \\ \gamma_{1,0} & \gamma_{1,1} & \cdots & \gamma_{1,j} & \gamma_{1\,j+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{j,0} & \gamma_{j,1} & \cdots & \gamma_{j,j} & \gamma_{jj+1} & \cdots \\ \gamma_{j+1,0} & \gamma_{j+1,1} & \cdots & \gamma_{j+1,j} & \gamma_{j+1,j+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Continuous-Time Markov Chains -Steady State Probabilities and Global Balance Equations (2)

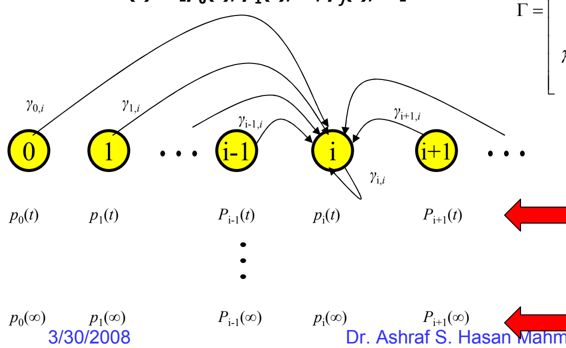
- **What relation govern the state probabilities for continuous-time Markov chains?**
  - **Remember that probability for state i, pi(t), is now a function of time!**
- **The answer is given by Remember Chapman-Kolmogrov equations:**

$$p'_j(t) = \sum_i \gamma_{i,j}\, p_i(t) \quad \text{for all } j$$

Or in matrix form: $P'(t) = P(t)\Gamma$,
where $P(t) = [p_0(t), p_1(t), \ldots, p_j(t), \ldots]$

$$\Gamma = \begin{bmatrix} \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,j} & \gamma_{0\,j+1} & \cdots \\ \gamma_{1,0} & \gamma_{1,1} & \cdots & \gamma_{1,j} & \gamma_{1\,j+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{j,0} & \gamma_{j,1} & \cdots & \gamma_{j,j} & \gamma_{jj+1} & \cdots \\ \gamma_{j+1,0} & \gamma_{j+1,1} & \cdots & \gamma_{j+1,j} & \gamma_{j+1,j+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Transition Rates



$$p_j'(t) = \sum_{\forall i} \gamma_{i,j}\, p_i(t) \qquad j = 0, 1, \cdots$$

$$p_j'(t) = 0 \qquad j = 0, 1, \cdots$$

Continuous-Time Markov Chains -Steady State Probabilities and Global Balance Equations (3)

- **If equilibrium exists, then $p'_j(t) = 0$ (i.e. no change in the state probabilities with time)**
- **Therefore, at steady state (if it exists), the following holds:**

$$0 = \sum_i \gamma_{ij}\, p_i(t) \quad \text{for all } j$$

- **These are referred to as the <u>GLOBAL BALANCE EQUATIONS</u>!!**
- **All flows (rate X probability) algebraically added for any state j equal to ZERO**

- It is given that γ_{01} is α, since the sum of row entries should be 0 → γ_{00} is $-\alpha$
- Same for the row corresponding to state 1.

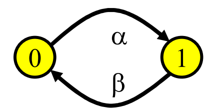
By Definition:
γ_{ii} **= - sum of all exit rates from state** ***i***

Example 11:

- **Problem: Consider the queueing system in Example 9 – find the steady state probabilities.**
- **Answer:**

$\gamma_{00} = -\alpha$ $\quad\quad$ $\gamma_{01} = \alpha$
$\gamma_{10} = \beta$ $\quad\quad$ $\gamma_{11} = -\beta$

$$[\pi_0 \quad \pi_1] = [\pi_0 \quad \pi_1] \times \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$



$$\Gamma = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

Applying the global balance equations, yields

$\alpha\pi_0 = \beta\pi_1$ **and** $\beta\pi_1 = \alpha\pi_0$

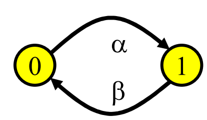
In addition to the constraints that: $\pi_0 + \pi_1 = 1$

Example 11: cont'd

- **Answer: Solving the previous simple equations leads to:**

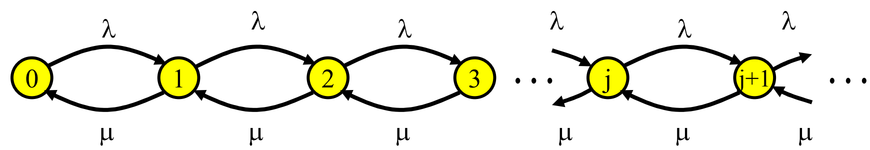
$\pi_0 = \beta/(\alpha+\beta)$

$\pi_1 = \alpha/(\alpha+\beta)$



Example 12:

- **<u>Problem</u>: The M/M/1 single-server queueing system**



The corresponding rate transition matrix is given by

$$\Gamma = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ 0 & \mu & -(\lambda+\mu) & \lambda & \cdots \\ 0 & 0 & \mu & -(\lambda+\mu) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example 12: cont'd

- **<u>Answer</u>: The state transition rates:**
  - **Customers arrive with rate λ ➔ $\gamma_{i,i+1} = \lambda$ for i = 0, 1, 2, …**
  - **When system is not empty, customers depart at rate μ ➔ $\gamma_{i,i-1} = \mu$ for i = 1, 2, 3, …**
- **The global balance equations:**

$$\lambda p_0 = \mu p_1 \qquad \text{for } j = 0$$
$$(\lambda + \mu)p_j = \lambda p_{j-1} + \mu p_{j+1} \qquad \text{for } j=1, 2, \ldots$$

➔ $\lambda p_j - \mu p_{j+1} = \lambda p_{j-1} - \mu p_j \qquad \text{for } j=1,2,\ldots$
$= \text{constant}$

Example 12: cont'd

- **Answer:**

For j = 1, we have

$$\lambda p_0 - \mu p_1 = \text{constant}$$

Therefore the constant is equal to zero.

Hence,

$$\lambda p_{j-1} = \mu p_j \text{ or}$$
$$p_j = (\lambda/\mu)\, p_{j-1} \quad \text{for } j=1,2, \ldots$$

By simple induction:

$$p_j = \rho^j\, p_0$$

where $\rho = \lambda/\mu$

Example 12: cont'd

- **Answer:**

To obtain p_0, we use the fact that

$$1 = \sum_j p_j = (1+\rho+\rho^2+\ldots)p_0$$

note the above series converges only for $\rho < 1$ or equivalently $\lambda < \mu$

Therefore, $p_0 = 1-\rho$

In general, the steady state pmf for the M/M/1 queue is given by

$$p_j = (1-\rho)\rho^j$$

References

- Alberto Leon-Garcia, Probability and Random Processes for Electrical Engineering, Addison Wesley, 1989