King Fahd University of Petroleum & Minerals Computer Engineering Dept

COE 200 – Fundamentals of Computer Engineering

Term 022

Dr. Ashraf S. Hasan Mahmoud

Rm 22-144

Ext. 1724

Email: ashraf@ccse.kfupm.edu.sa

Binary Logic

- Deals with *binary* variables that take one of two discrete values
- Values of variables are called by a variety of very different names
  - *high* or *low* based on voltage representations in electronic circuits
  - *true* or *false* based on their usage to represent logic states
  - *one* (1) or *zero* (0) based on their values in Boolean algebra
  - *open* or *closed* based on its operation in *gate* logic
  - *on* or *off* based on its operation in *switching* logic
  - *asserted* or *de-asserted* based on its effect in digital systems

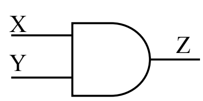
Basic Operations - AND

- Another Symbol is ".", e.g.

  Z = X AND Y or

  Z = X.Y or even

  Z = XY
- X and Y are inputs, Z is an output
- Z is equal to 1 if and only if X = 1 and Y = 1; Z = 0 otherwise (similar to the multiplication operation)
- Truth Table:
- Graphical symbol:



X	Y	Z=XY
0	0	0
0	1	0
1	0	0
1	1	1

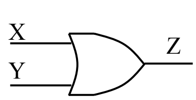
Basic Operations - OR

- Another Symbol is "+", e.g.

  Z = X OR Y or

  Z = X + Y
- X and Y are inputs, Z is an output
- Z is equal to 0 if and only if X = 0 and Y = 0; Z = 1 otherwise (similar to the addition operation)
- Truth Table:
- Graphical symbol:

X	Y	Z=X+Y
0	0	0
0	1	1
1	0	1
1	1	1



Basic Operations - NOT

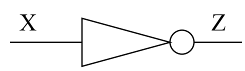
- Another Symbol is "¯", e.g.

$$Z = \overline{X} \text{ or } Z = X'$$

- X is the input, Z is an output
- Z is equal to 0 if X = 1; Z = 1 otherwise
- Sometimes referred to as the complement or invert operation
- Truth Table:

X	Z=X′
0	1
1	0

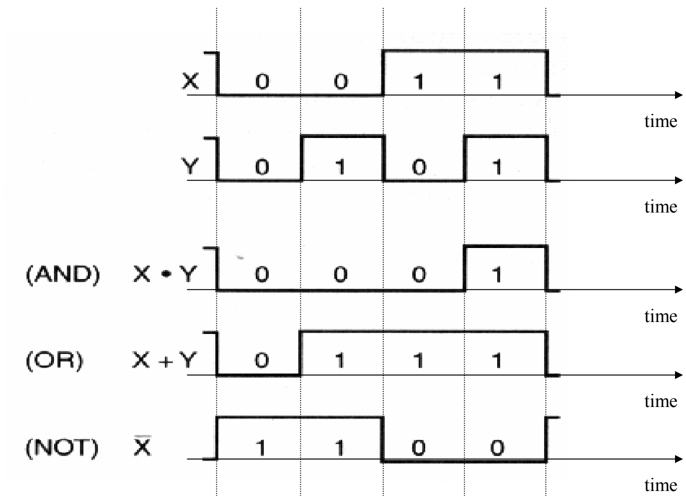
- Graphical symbol:



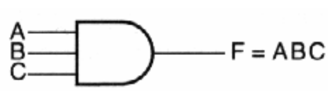
Exercise 1

- Build a NOT gate using one AND gate?

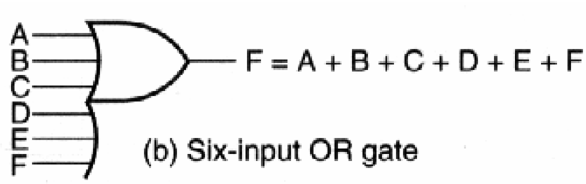
Time Diagrams



Multiple Input Gates



(a) Three-input AND gate



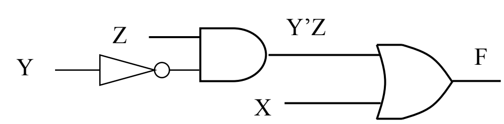
(b) Six-input OR gate

Boolean Algebra

- Consider the following *function*, F

$$F = X + Y'Z$$

- The function F is referred to as a BOOLEAN FUNCTION
- F has two terms: X and Y’Z
- The circuit diagram for F is as shown below



Boolean Algebra – cont’d

- The truth table for F is as follows
- Note:
  - In general, a truth table for an n-variable function, has 2^n rows to cover all possible input combinations
  - The table covers all possible combinations of the inputs
  - To arrive at the F’s column one could use an Y’Z column as follows

X	Y	Z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

The Y’Z column is computed using the Y and Z columns and then using the columns X and Y’Z, the column F is computed
The column Y’Z is **not** an essential part of truth table

X	Y	Z	Y’Z	F
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	0	0
1	0	0	0	1
1	0	1	1	1
1	1	0	0	1
1	1	1	0	1

Basic Identities

- For the AND operation

$X . 1 = X$

$X . 0 = 0$

$X . X = X$

$X . X' = 0$

- For the OR operation

$X + 0 = X$

$X + 1 = 1$

$X + X = X$

$X + X' = 1$

Notice the **duality**: start with one identity
-Replace the AND by OR
-Replace the 1 by 0 and vice versa
You end up with an identity from the other group

- For the NOT operation

$X'' = X$

Basic Identities (2)

For the AND operation	OR Operation
Cumulative: $X.Y=Y.X$	$X+Y=Y+X$
Associative: $X(YZ)=(XY)Z$	$X+(Y+Z)=(X+Y)+Z$
Distributive: $X+YZ=(X+Y)(X+Z)$	$X(Y+Z)=(XY)+(XZ)$
DeMorgan's: $(X.Y)'=X'+Y'$	$(X+Y)'=X'.Y'$

- All above properties can be generalized to n > 2 variables: e.g:
  - $(X_1+X_2+\ldots+X_n)' = X_1'.X_2'. \ldots .X_n'$, or
  - $(X_1.X_2. \ldots .X_n)' = X_1'+X_2'+ \ldots +X_n'$

Verifying Basic Identities

- Any identity (not only the basic ones) can be verified using the truth table
- Example: verify that (X+Y)′ = X′Y′

X	Y	X+Y	(X+Y)′	X′	Y′	X′.Y′
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

ALL possible combinations of inputs

Some columns to aid in calculations

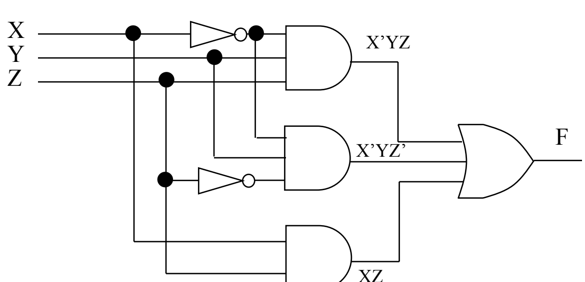
The two quantities in question

Algebraic Manipulation - Example

- Consider the following function, F

$$F = X'YZ + X'YZ' + XZ$$

- The function can be implemented using above expressions as in



We need:
-2 inverters
-3 AND gates
-1 OR gate

Algebraic Manipulation – Example – cont'd

- The function

$$F = X'YZ + X'YZ' + XZ$$

can be simplified "ALGEBRAICALLY" as follows:

$$\begin{aligned} F &= X'YZ + X'YZ' + XZ \\ &= X'Y(Z + Z') + XZ && \rightarrow \text{by the distributive property} \\ &= X'Y(1) + XZ && \rightarrow \text{by the properties of the OR operation} \\ &= X'Y + XZ && \rightarrow \text{by the properties of the AND operation} \end{aligned}$$

- Therefore F can be written as

$$F = X'Y + XZ$$

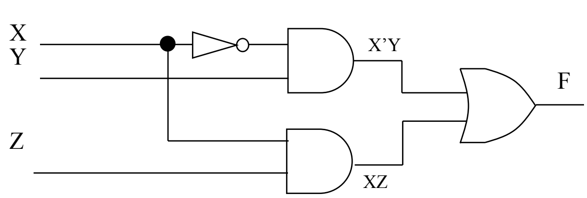
- Using this simpler form, one can implement the function as

Algebraic Manipulation – Example – cont'd

- Therefore F can be written as

$$F = X'Y + XZ$$

- Using this simpler form, one can implement the function as



We need:
-1 inverters
-2 AND gates
-1 OR gate

↓

Reduced hardware cost

- One can use the truth table method to show that F = X'YZ + X'YZ' + XZ is indeed equal to X'Y + XZ

More Notes on Function

- The function

$$F = X'Y + XZ$$

Can be written as

$$F(X,Y,Z) = X'Y + XZ$$

These are referred to as literals

These are referred to as terms

This is to emphasize the fact that the function has three inputs or variables

More Identities

- Page 37 in the text ➔ VERY IMPORTANT – make sure you can prove/verify all of these identities
- Listing
  1. $X + XY = X$
  2. $XY + XY' = X$
  3. $X + X'Y = X + Y$
  4. $X(X + Y) = X$
  5. $(X + Y)(X + Y') = X$
  6. $X(X' + Y) = XY$
  7. $XY + X'Z + YZ = XY + X'Z$ (the consensus theorem)

The proof/verification of these is in the textbook

More Identities - continued

- Using the duality principle (refer to slide XX) there are other equivalent 7 identities
- Example: The proof of the consensus theorem is as follows

The RHS $= XY + X'Z + YZ$
$= XY + X'Z + YZ(X + X')$
$= XY + X'Z + XYZ + X'YZ$
$= XY + XYZ + X'Z + X'YZ$
$= XY(1+Z) + X'Z(1+Y)$
$= XY + X'Z$
$=$ LHS

- The dual of the consensus theorem is given by

$$(X+Y)(X'+Z)(Y+Z) = (X+Y)(X'+Z)$$

Complement of a Function

- Using the truth table – complementing F means replacing every 0 with 1 and every 1 with 0 in the F column
- Algebraically, complementing F one can use DeMorgan's rule or the duality principle
- To use the duality principle
  - Replace Each AND with an OR and each OR with an AND
  - Complement each variable and constant

Example

- **Problem:** Find the complement of each of the following two functions $F_1 = X'YZ' + X'Y'Z$, and $F_2 = X(Y'Z' + YZ)$
- **Solution:**

For F_1, applying DeMorgan's rule as many times as necessary

$$
\begin{aligned}
F_1' &= (X'YZ' + X'Y'Z)' \\
&= (X'YZ')' \,.\, (X'Y'Z)' \\
&= (X + Y' + Z) \,.\, (X + Y + Z')
\end{aligned}
$$

Similarly for F_2:

$$
\begin{aligned}
F_2' &= (X(Y'Z' + YZ))' \\
&= X' + (Y'Z' + YZ)' \\
&= X' + (Y'Z')' \,.\, (YZ)' \\
&= X' + (Y + Z) \,.\, (Y' + Z')
\end{aligned}
$$

Examples

- **Problem 2-2**: Prove the identity of each of the following Boolean equations, using algebraic manipulations.
  a) $X'Y' + X'Y + XY = X' + Y$
  b) $A'B + B'C' + AB + B'C = 1$
- **Solution:**

a) LHS
$$
\begin{aligned}
&= X'Y' + X'Y + XY \\
&= X'Y' + X'Y + X'Y + XY \\
&= X'(Y'+Y) + Y(X + X') \\
&= X' + Y \\
&= RHS
\end{aligned}
$$

b) LHS
$$
\begin{aligned}
&= A'B + B'C' + AB + B'C \\
&= (A'+A)B + B'(C'+C) \\
&= B + B' \\
&= 1 \\
&= RHS
\end{aligned}
$$

Examples

- **Problem 2-6**: Simplify the following Boolean expressions to a minimum number of literals:
  a) ABC + ABC′ + A′B
  e) (A+B′+AB′)(AB+A′C+BC)
- **Solution:**

a) Expression = ABC + ABC′ + A′B
= AB(C + C′) + A′B
= (A+A′)B
= B

e) Expression = (A+B′+AB′)(AB+A′C+BC)
= (A+(1+A)B′)(AB + A′C)
= (A+B′)(AB+A′C)
= A(AB+A′C) + B′(AB+A′C)
= AB + A′B′C

Standard Forms of a Boolean Function

- A Boolean function can be written algebraically in a variety of ways
- Standard form: is an algebraic expression of the function that facilitates simplification procedures and frequently results in more desirable logic circuits (e.g. less number of gates)
- Standard form: contains product terms and sum terms
  - Product term: X′Y′Z
  - Sum term: X + Y + Z′

Standard Forms of a Boolean Function – cont’d

- A minterm: a product term in which all variables (or literals) of the function appear exactly once
- A maxterm: a sum term in which all the variables (or literals) of the function appear exactly once

- Example: for the function F(X,Y,Z),
  - the term X’Y is not a minterm, but XYZ’ is a minterm
  - The term X’+Z is not a maxterm, but X+Y’+Z’ is maxterm

More on Minterms and Maxterms

- A function of n variables – have 2^n possible minterms and 2^n possible maxterms
-

Naming Convention for Minterms

- Consider a function F(X, Y)

X	Y	Product Terms	Symbol	m_0	m_1	m_2	m_3
0	0	X'Y'	m_0	1	0	0	0
0	1	X'Y	m_1	0	1	0	0
1	0	XY'	m_2	0	0	1	0
1	1	XY	m_3	0	0	0	1

Variable complemented if 0
Variable not complemented if 1

m_i indicated the i^{th} minterm
For each binary combination of X and Y there is a minterm
The index of the minterm is specified by the binary combination
m_i is equal to 1 for ONLY THAT combination

Naming Convention for Maxterms

- Consider a function F(X, Y)

X	Y	Sum Terms	Symbol	M_0	M_1	M_2	M_3
0	0	X+Y	M_0	0	1	1	1
0	1	X+Y'	M_1	1	0	1	1
1	0	X'+Y	M_2	1	1	0	1
1	1	X'+Y'	M_3	1	1	1	0

Variable complemented if 1
Variable not complemented if 0

M_i indicated the i^{th} maxterm
For each binary combination of X and Y there is a maxterm
The index of the maxterm is specified by the binary combination
M_i is equal to 0 for ONLY THAT combination

More on Minterms and Maxterms

- In general, a function of n variables has
  - 2^n minterms: $m_0, m_1, \ldots, m_{2^n-1}$
  - 2^n maxterms: $M_0, M_1, \ldots, M_{2^n-1}$

- $m_i' = M_i$ or $M_i' = m_i$

Example: for F(X,Y):

$m_2 = XY' \rightarrow m_2' = X'+Y = M_2$

More on Minterms and Maxterms – cont'd

- A Boolean function can be expressed algebraically from a give truth table by forming the logical sum of ALL the minterms that produce 1 in the function

- **Example**:

Consider the function defined by the truth table

F(X,Y,Z) → 3 variables → 8 minterms

F can be written as

$F = X'Y'Z' + X'YZ' + XY'Z + XYZ$, or

$= m_0 + m_2 + m_5 + m_7$

$= \Sigma m(0,2,5,7)$

X	Y	Z	m	F
0	0	0	m_0	1
0	0	1	m_1	0
0	1	0	m_2	1
0	1	1	m_3	0
1	0	0	m_4	0
1	0	1	m_5	1
1	1	0	m_6	0
1	1	1	m_7	1

More on Minterms and Maxterms – cont'd

- A Boolean function can be expressed algebraically from a give truth table by forming the logical product of ALL the maxterms that produce 0 in the function

- **Example**:

Consider the function defined by the truth table F(X,Y,Z) ➔ in a manner similar to the previous example, F' can be written as

F' $= m_1 + m_3 + m_4 + m_6$

$= \Sigma m(1,3,4,6)$

Now apply DeMorgan's rule

$F = F'' = [m_1 + m_3 + m_4 + m_6]'$

$= m_1'.m_3'.m_4'.m_6'$

$= M_1.M_3.M_4.M_6$

$= \Pi\ M(1,3,4,6)$

Note the indices in this list are those that are missing from the previous list in $\Sigma m(0,2,5,7)$

X	Y	Z	M	F	F'
0	0	0	M_0	1	0
0	0	1	M_1	0	1
0	1	0	M_2	1	0
0	1	1	M_3	0	1
1	0	0	M4	0	1
1	0	1	M_5	1	0
1	1	0	M_6	0	1
1	1	1	M_7	1	0

Summary

- A Boolean function can be expressed algebraically as:
  - The logical sum of minterms
  - The logical product of maxterms
- Given the truth table, writing F as
  - Σm_i – for all minterms that produce 1 in the table, or
  - ΠM_i – for all maxterms that produce 0 in the table
- Another way to obtain the Σm_i or ΠM_i is to use ALGEBRA – see next example

Example:

- Write E = Y′ + X′Z′ in the form of Σm_i and ΠM_i?

- Solution: **Method1**

First construct the Truth Table as shown

Second:

E = $\Sigma m(0,1,2,4,5)$, and

E = $\Pi M(3,6,7)$

X	Y	Z	m	M	E
0	0	0	m_0	M_0	1
0	0	1	m_1	M_1	1
0	1	0	m_2	M_2	1
0	1	1	m_3	M_3	0
1	0	0	m_4	M4	1
1	0	1	m_5	M_5	1
1	1	0	m_6	M_6	0
1	1	1	m_7	M_7	0

Example: cont'd

- Solution: **Method2_a**

E = Y′ + X′Z′
= Y′(X+X′)(Z+Z′) + X′Z′(Y+Y′)
= (XY′+X′Y′)(Z+Z′) + X′YZ′+X′Z′Y′
= XY′Z+X′Y′Z+XY′Z′+X′Y′Z′+ X′YZ′+X′Z′Y′
= $m_5 + m_1 + m_4 + m_0 + m_2 + m_0$
= $m_0 + m_1 + m_2 + m_4 + m_5$
= $\Sigma m(0,1,2,4,5)$

To find the form ΠMi, consider the remaining indices

E = $\Pi M(3,6,7)$

- Solution: **Method2_b**

E = Y′ + X′Z′
E′ = Y(X+Z)
= YX + YZ
= YX(Z+Z′) + YZ(X+X′)
= XYZ+XYZ′+X′YZ
E = (X′+Y′+Z′)(X′+Y′+Z)(X+Y′+Z′)
= $M_7 . M_6 . M_3$
= $\Pi M(3,6,7)$

To find the form Σm_i, consider the remaining indices

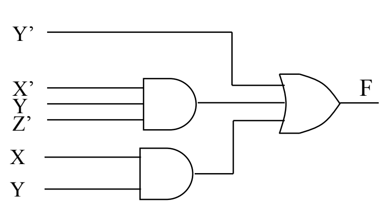
E = $\Sigma m(0,1,2,4,5)$

Exercise

- What is G(X,Y) = Σm(0,1,2,3) equal to?

Implementation – Sum of Products

- Consider F = Y' + X'YZ' + XY
  - Three products: Y' (one literal), X'YZ' (three literals), and XY (two literals)
- The logic diagram

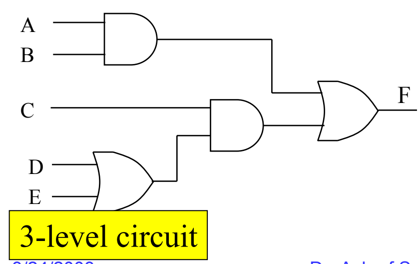


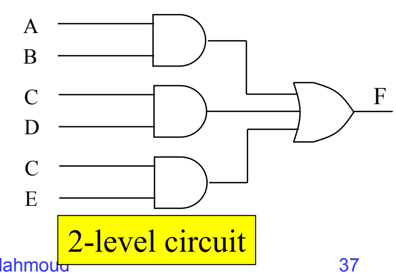
- Two-level implementation:
  - AND-OR
- Each product term requires an AND gate (except one literal terms)
- Logic diagram requires ONE OR gate

Implementation – Sum of Products – cont'd

- Consider F = AB + C(D+E)
- This expression is NOT in the sum-of-products form
- Use the identities/algebraic manipulation to convert to a standard form (sum of products), as in

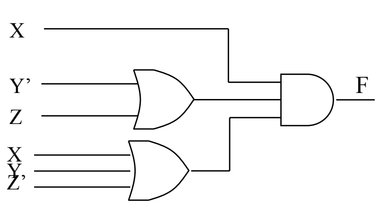
  F = AB + CD + CE
- Logic Diagrams:





Implementation – Product of Sums

- Consider F = X(Y'+Z)(X+Y+Z')
- This expression is in the product-of-sums form:
  - Thee summation terms: X (one literal), Y'+Z (two literals), and X+Y+Z' (three literals)
- Logic Diagrams:



- Two-level implementation:
  - OR-AND
- Each sum term requires an OR gate (except one literal terms)
- Logic diagram requires ONE AND gate