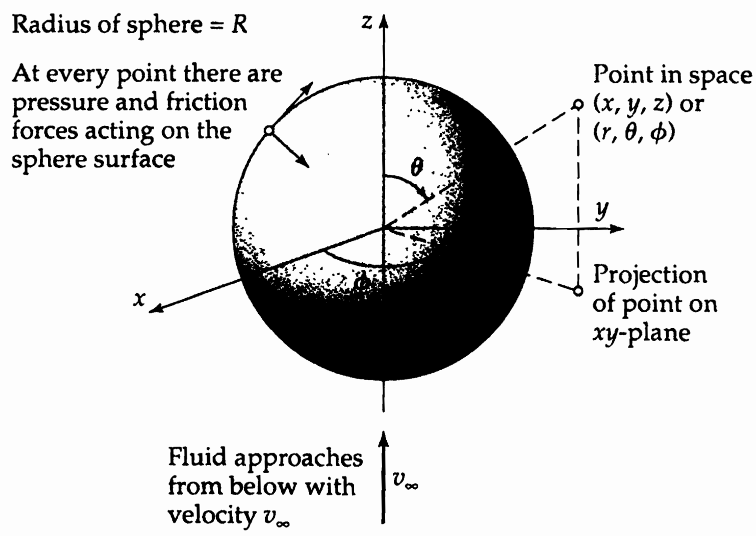
Example: Creeping Flow around A Sphere

Use Table 4.2-1 to set up the differential equation for the stream function for the flow of a Newtonian fluid around a stationary sphere of radius R at Re << 1. Obtain the velocity and pressure distributions when the fluid approaches the sphere in the positive z direction, as in Fig. 2.6-1.



Solution:

In this problem we consider the flow of an incompressible fluid about a solid sphear of radius R. The fluid approches the fixed sphere vertically upwards in the z-direction with a uniform velocity V_∞. The flow considered here is "creeping flow" means that the Reynolds number $Re = \frac{\rho V_\infty D}{\mu} < 0.1$.

For the flow considered here we have two velocity component close to the sphere: V_r and V_θ. $V_\phi = 0$, moreover, the flow is axisymmetric i.e. independent of ϕ.

For creeping flow (laminar flow, Stokes flow) the ② L.H.S. of Equation D of table 4.2-1 cancels out and hence

$$E^4 \psi = 0$$

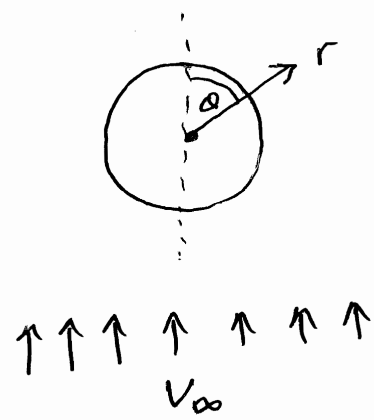
where $E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\right)$

$$\Rightarrow \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\right)\right]^2 \psi = 0 . \quad \text{---} ①$$

as shown in table 4.2-1 :

$$V_r = - \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}$$

$$V_\theta = \frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}$$



Boundary Conditions:

BC1 $\quad r = R \quad V_r = 0 \quad \Rightarrow \quad - \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} = 0 .$

BC2 $\quad r = R \quad V_\theta = 0 \quad \Rightarrow \quad \frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r} = 0 .$

BC3 $\quad r \to \infty \quad V_r = V_\infty \cos\theta, \quad V_\theta = -V_\infty \sin\theta \quad \Rightarrow \quad \psi = -\frac{1}{2} V_\infty r^2 \sin^2\theta$

BC3 suggests that we can postulate a solution for the stream function of the form:

$$\psi(r, \theta) = f(r) \sin^2(\theta)$$

substitution in equation 1 yields:

$$\left[\frac{d^2}{dr^2} - \frac{2}{r^2}\right]^2 f = 0$$

$$\Rightarrow \frac{d^4 f}{dr^4} - \frac{4}{r^2}\frac{d^2 f}{dr^2} + \frac{8}{r^3}\frac{df}{dr} - 8\frac{f}{r^4} = 0$$

The above equation is similar to C.1-14 but in this case it is fourth order. It admits a solution

$$f(r) = \frac{c_1}{r} + c_2 r + c_3 r^3 + c_4 r^4$$

BC3

$r \to \infty \qquad \psi = f \sin^2\theta = -\frac{1}{2} V_\infty r^2 \sin^2\theta$

$$\Rightarrow \left(\frac{c_1}{r} + c_2 r + c_3 r^2 + c_4 r^4\right)\Big|_{r=\infty} \cancel{\sin^2\theta} = -\frac{1}{2} V_\infty r^2 \Big|_{r=\infty} \cancel{\sin^2\theta} \quad \div r^2$$

$$\Rightarrow \left(\overset{0}{\cancel{\frac{c_1}{r^3}}} + \overset{0}{\cancel{\frac{c_2}{r}}} + c_3 + c_4 r^2\right)_{r=\infty} = -\frac{1}{2} V_\infty$$

$$\Rightarrow c_4 = 0 \quad \text{and} \quad c_3 = -\frac{1}{2} V_\infty$$

B.6-7 and

$$\Rightarrow \psi(r,\theta) = \left(\frac{c_1}{r} + c_2 r - \frac{1}{2} v_\infty r^2\right) \sin^2(\theta)$$

recall,

$$v_r = -\frac{1}{r^2 \sin(\theta)} \frac{\partial \psi}{\partial \theta} = -\left(\frac{c_1}{r^3} + \frac{c_2}{r} - \frac{1}{2} v_\infty\right) \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta}\left(\sin^2(\theta)\right)$$

$$= -\left(\frac{c_1}{r^3} + \frac{c_2}{r} - \frac{1}{2} v_\infty\right) \frac{1}{\cancel{\sin(\theta)}} 2\cancel{\sin(\theta)} \cos(\theta)$$

$$\Rightarrow v_r = \left(v_\infty - 2\frac{c_2}{r} - 2\frac{c_1}{r^3}\right) \cos(\theta)$$

$$v_\theta = \frac{1}{r\sin(\theta)} \frac{\partial \psi}{\partial r} = \frac{1}{r} \sin(\theta) \left(-\frac{c_1}{r^2} + c_2 + v_\infty r\right)$$

$$\Rightarrow v_\theta = \left(-v_\infty + \frac{c_2}{r} - \frac{c_1}{r^3}\right) \sin\theta$$

BC 1 & 2 $\Rightarrow$

$$v_\infty - 2\frac{c_2}{R} - 2\frac{c_1}{R^3} = 0$$

$$-v_\infty + \frac{c_2}{R} - \frac{c_1}{R^3} = 0$$

$$\Rightarrow c_1 = -\frac{1}{4} v_\infty R^3 \quad \& \quad c_2 = \frac{3}{4} v_\infty R$$

$$\Rightarrow \quad v_r = v_\infty\left[1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3\right] \cos(\theta)$$

$$v_\theta = -v_\infty\left[1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3\right] \sin(\theta)$$

substitution of v_r and v_θ into equation B.6-7 and B.6-8 yields:

(5)

Creeping flow $\rightarrow 0$

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right) + \cancelto{0}{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}}\right] + \rho g_r \qquad \text{(B.6-7)}^a$$

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2\cot\theta}{r}}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta)\right) + \cancelto{0}{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \cancelto{0}{\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}}\right] + \rho g_\theta \qquad \text{(B.6-8)}$$

$$\frac{\partial P}{\partial r} = \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right)\right]$$

$$= 3\left(\frac{\mu v_\infty}{R^2}\right)\left(\frac{R}{r}\right)^3\cos(\theta) \quad \cdots \quad (2)$$

$$\frac{\partial P}{\partial \theta} = r\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta) + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]$$

$$= \frac{3}{2}\left(\frac{\mu v_\infty}{R}\right)\left(\frac{R}{r}\right)^2\sin(\theta) \quad \cdots \quad (3)$$

Try to do the above manupulations usining Mathematical software!

Integration of (2) & (3) with the condition $r \longrightarrow \infty$ $P = P_0$

$$\Rightarrow P = P_0 - \rho g z - \frac{3}{2}\left(\frac{\mu v_\infty}{R}\right)\left(\frac{R}{r}\right)^2\cos(\theta).$$

The kinetic force F_K is evaluated by calculation of work done on sphere surface

$$work = F_K V_\infty = force * velocity$$

$$F_K V_\infty = -\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty} (\tau : \nabla v)\, r^2\, dr \sin\theta\, d\theta\, d\phi$$

(A·8−6)

$$= -\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty} \Big[2\left(\frac{\partial v_r}{\partial r}\right)^2 + 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 +$$
$$2\left(\frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right)^2 +$$
$$\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)^2 \Big] r^2\, dr \sin\theta\, d\theta\, d\phi$$

(B·7−3)

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⋮

$$F_K = 6\pi\mu V_\infty R$$