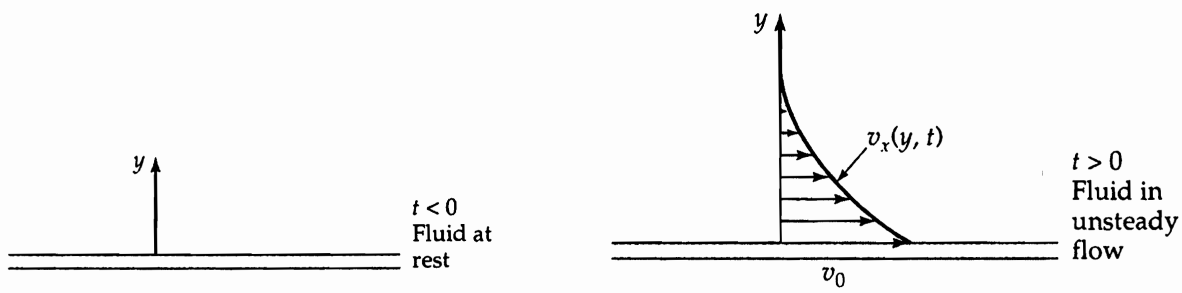
Chapter 4: Velocity Distribution with More That One Independent Variable

Example 1: Unsteady Flow near a Moving Wall, Semi-Infinite Domain

A semi-infinite body of liquid with constant density and viscosity is bounded below by a horizontal surface (the xz-plane). Initially the fluid and the solid are at rest. Then at time $t = 0$, the solid surface is set in motion in the positive x direction with velocity v_0 as shown in Fig. 4.4-1. Find the velocity v_x as a function of y and t. There is no pressure gradient or gravity force in the x direction, and the flow is presumed to be laminar.



Solution:

$V_y = V_z = 0$ $\qquad$ $V_x = f(y,t)$

$$\Rightarrow \quad \rho \frac{\partial V_x}{\partial t} = \mu \frac{\partial^2 V_x}{\partial y^2}$$

IC $\quad t = 0 \quad V_x = 0 \quad$ for all y

BC1 $\quad y = 0 \quad V_x = V_0 \quad$ for $t > 0$

BC2 $\quad y = \infty \quad V_x = 0 \quad$ for $t > 0$

For this PDE the domain y is semi-infinite and the IC is similar to BC2, hence, it can be solved by method of similarity solution (or combination of variables).

First of all non-dimentionalize velocity

$$\phi = \frac{V}{V_0} \quad , \quad \nu = \frac{\mu}{\rho}$$

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial y^2}$$

$t = 0 \qquad \phi = 0 \qquad$ for all y

$y = 0 \qquad \phi = 1 \qquad$ for $t > 0$

$y = \infty \qquad \phi = 0 \qquad$ for $t > 0$.

Similarity Variable:

scaling: $\frac{1}{t} = \nu \frac{1}{y^2} \quad \Rightarrow \quad \eta = \frac{y}{\sqrt{\nu t}}$

$$\frac{\partial \phi}{\partial t} = \frac{d\phi}{d\eta}\frac{d\eta}{\partial t} = -\frac{1}{2}\frac{t^{-3/2}}{\sqrt{\nu}}\, y \,\frac{d\phi}{d\eta}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{d^2\phi}{d\eta^2}\left(\frac{d\eta}{dy}\right)^2 = \frac{1}{\nu t}\frac{d^2\phi}{d\eta^2}$$

substitute in PDE:

$$-\frac{1}{2}\frac{t^{-3/2}}{\sqrt{\nu}}\, y \,\frac{d\phi}{d\eta} = \frac{1}{t}\frac{d^2\phi}{d\eta^2}$$

rearranging $\quad \frac{d^2\phi}{d\eta} + \frac{1}{2}\eta \frac{d\phi}{d\eta} = 0$.

BC1 $\quad \eta = 0 \qquad \phi = 1$

BC2 $\quad \eta = \infty \qquad \phi = 0$

Appendix C.1-8: $\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 0$

$$y = C_1 \int_0^x e^{-\bar{x}^2}\, d\bar{x} + C_2 .$$

let $x = \dfrac{\eta}{2}$ $\qquad \dfrac{d}{d\eta} = \dfrac{d}{dx}\dfrac{dx}{d\eta} = \dfrac{1}{2}\dfrac{d}{dx}.$

$$\Rightarrow \quad \frac{1}{4}\frac{d^2\phi}{dx^2} + \frac{1}{2}(2x)\,\frac{1}{2}\frac{d\phi}{dx} = 0 .$$

$$\vdots$$

$$\frac{d^2\phi}{dx^2} + 2x\frac{d\phi}{dx} = 0$$

BC1 $\quad x = 0 \quad \phi = 1$

BC2 $\quad x = \infty \quad \phi = 0$

$$\Rightarrow \quad \phi = C_1 \int_0^x e^{-\bar{x}^2}\, d\bar{x} + C_2$$

BC1 $\Rightarrow$ $C_2 = 1$.

BC2 $\Rightarrow$ $C_1 = \dfrac{-1}{\int_0^\infty e^{-\bar{x}^2}\, d\bar{x}}$

$$\Rightarrow \quad \phi = 1 - \frac{\int_0^x e^{-\bar{x}^2}\, dx}{\int_0^\infty e^{-\bar{x}^2}\, dx} = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-\bar{x}^2}\, dx = 1 - \mathrm{erf}(x)$$

see C.6-1

Example 2: Unsteady Flow near a Moving Wall, Bounded Domain

It is desired to re-solve the preceding illustrative example, but with a fixed wall at a distance b from the moving wall at $y = 0$. This flow system has a steady-state limit as $t \to \infty$, whereas the problem in Example 4.1-1 did not.

Solution: This problem is similar to previous one but the domain y is bounded between 0 and b.

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

IC $\quad t = 0 \quad v_x = 0 \quad 0 \leq y \leq b$

BC1 $\quad y = 0 \quad v_x = v_0 \quad t > 0$

BC2. $\quad y = b \quad v_x = 0. \quad t > 0.$

non-dimensionalization:

$$\phi = \frac{v_x}{v_0}, \quad z = \frac{y}{b}, \quad \tau = \frac{t}{b^2/\nu}$$

time scaling for momentum diffusion analogous to
b^2/α for heat transfer
b^2/D_{AB} " mass transfer

$$\Rightarrow \quad \frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \eta^2}$$

$$\tau = 0 \qquad \phi = 0 \qquad 0 \le \eta \le 1$$

$$\eta = 0 \qquad \phi = 1 \qquad \tau > 0$$

$$\eta = 1 \qquad \phi = 0 \qquad \tau > 0$$

This problem reaches steady state as $\tau \to \infty$

$$\Rightarrow \quad \left.\begin{array}{ll} \dfrac{d^2 \phi^{ss}}{d\eta^2} = 0 & \\ \eta = 0 & \phi^{ss} = 1 \\ \eta = 1 & \phi^{ss} = 0 \end{array}\right\} \Rightarrow \phi^{ss} = (1 - \eta)$$

We can write $\phi(\tau, \eta) = \phi^{ss}(\eta) - \phi^{t}(\tau, \eta)$

where ϕ^t is the transient part of the solution which fades out as $\tau \to \infty$ $\phi^t \to 0$.

Substituting $\phi = \phi^{ss} + \phi^t$ into PDE and BC's

$$\frac{\partial \phi^t}{\partial \tau} = \frac{\partial^2 \phi^t}{\partial \eta^2}.$$

$$\tau = 0 \qquad \phi^t = \phi^{ss} \quad (\phi = \phi^{ss} - \phi^t = 0)$$

$$\eta = 0 \qquad \phi^t = 0 \quad \left(\phi = \phi^{ss} - \phi^t = 1 \atop = 1 - \phi^t = 1\right)$$

$$\eta = 1 \qquad \phi^t = 0 \quad \left(\phi = \phi^{ss} - \phi^t = 0 \atop = 0 - \phi^t = 0\right)$$

Solution by separation of variables

let $\phi^t = f(\eta)\, g(\tau)$

$$f \frac{dg}{d\tau} = g \frac{d^2 f}{d\eta^2} \qquad (\div\, fg)$$

$$\frac{1}{g}\frac{dg}{d\tau} = \frac{1}{f}\frac{d^2 f}{d\eta^2}$$

The L.H.S is $f(\tau)$ and R.H.S. is $f(\eta)$

this means :

$$\frac{1}{g}\frac{dg}{d\tau} = -c^2 \quad \Rightarrow \quad \frac{dg}{d\tau} = -c^2 g.$$

$$\frac{1}{f}\frac{d^2 f}{d\eta^2} = -c^2 \quad \Rightarrow \quad \frac{d^2 f}{d\eta^2} + c^2 f = 0.$$

where $-c^2$ is constant and -ve sign is arbitrary and selected by experience to simlify subsequent mathematical manupulation.

integrating : $g = A\, e^{-c^2 \tau}$

$$f = B \sin(c\eta) + C \cos(c\eta) \qquad (C.1-3)$$

$\eta = 0 \quad \phi^t = 0 \Rightarrow f = 0 = 0 + C \Rightarrow \boxed{C = 0}$

$\eta = 1 \quad \phi^t = 0 \Rightarrow f = 0 = B \sin(c) \Rightarrow c_n = 0, \pm\pi, \pm 2\pi \ldots$

$$\Rightarrow f_n = B_n \sin(n\pi\eta) \qquad n = 0, \pm 1, \pm 2, \dots$$

similarly $\quad g_n = A_n e^{-n^2\pi^2\tau} \qquad n = 0, \pm 1, \pm 2, \dots$

$$\Rightarrow \phi^t = \underbrace{\sum_{n=-\infty}^{\infty} \underset{\substack{\uparrow \\ = A_n B_n}}{D_n} e^{-n^2\pi^2\tau} \sin(n\pi\eta)}$$

The product $f_n g_n$ satisfy PDE and so well any superposition of such products.

<u>note</u> $\quad \sin(-n\pi\eta) = -\sin(n\pi\eta)$

$\quad \sin(0) = 0$

$$\Rightarrow \phi^t = \sum_{n=1}^{\infty} D_n e^{-n^2\pi^2\tau} \sin(n\pi\eta)$$

apply I.C. $\qquad \tau = 0 \qquad \phi^t = \phi^{ss} = (1-\eta)$

$$\Rightarrow (1-\eta) = \sum_{n=1}^{\infty} D_n \sin(n\pi\eta)$$

use orthogonality of sinusodal functions to find D_n.

$$\int_0^1 (1-\eta)\sin(m\pi\eta)\, d\eta = \sum_{n=1}^{\infty} D_n \int_0^1 \sin(n\pi\eta)\sin(m\pi\eta)\, d\eta$$

evaluation of $\int_0^1 (1-\eta) \sin(m\pi\eta)\, d\eta$

integration by parts $\quad u = 1-\eta \quad dv = \sin(m\pi\eta)\,d\eta$

$du = -d\eta \quad v = \frac{-1}{m\pi}\cos(m\pi\eta)$

$uv - \int v\, du$

$$-\frac{(1-\eta)}{m\pi}\cos(m\pi\eta)\Big|_0^1 - \frac{1}{m\pi}\int_0^1 \cos(n\pi\eta)$$

$$\frac{1}{m\pi} - \frac{1}{m\pi}\cancel{\sin(n\pi\eta)}\Big|_0^1 = \frac{1}{m\pi}$$

evaluation of $\int_0^1 \sin(n\pi\eta)\sin(m\pi\eta)\,d\eta = \begin{cases} 0 & n \neq m \\ \frac{1}{2} & n = m \end{cases}$

$$\Rightarrow D_n = \frac{2}{n\pi}$$

$$\Rightarrow \phi^t = \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-n^2\pi^2\tau} \sin(n\pi\eta)$$

recall $\phi = \phi^{ss} - \phi^t$

$$\phi = (1-\eta) - \sum_{n=1}^{\infty}\left(\frac{2}{n\pi} e^{-n^2\pi^2\tau} \sin(n\pi\eta)\right)$$

Example 3: Unsteady Flow near an Oscillating Plate

A semi-infinite body of liquid is bounded on one side by a plane surface (the xz-plane). Initially the fluid and solid are at rest. At time $t = 0$ the solid surface is made to oscillate sinusoidally in the x direction with amplitude X_0 and (circular) frequency ω. That is, the displacement X of the plane from its rest position is

$$X(t) = X_0 \sin \omega t \tag{4.1-41}$$

and the velocity of the fluid at $y = 0$ is then

$$v_x(0, t) = \frac{dX}{dt} = X_0\,\omega \cos \omega t \tag{4.1-42}$$

We designate the amplitude of the velocity oscillation by $v_0 = X_0\omega$ and rewrite Eq. 4.1-42 as

$$v_x(0, t) = v_0 \cos \omega t = v_0 \Re\{e^{i\omega t}\} \tag{4.1-43}$$

where $\Re\{z\}$ means "the real part of z."

For oscillating systems we are generally not interested in the complete solution, but only the "periodic steady state" that exists after the initial "transients" have disappeared. In this state all the fluid particles in the system will be executing sinusoidal oscillations with frequency ω, but with phase and amplitude that are functions only of position. This "periodic steady state" solution may be obtained by an elementary technique that is widely used. Mathematically it is an asymptotic solution for $t \to \infty$.

Recall, $e^{i\omega t} = \cos(\omega t) - i\sin(\omega t)$

Solution:

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}.$$

$t = 0 \qquad v_x = 0$ for all y.

$y = 0 \qquad v_x = v_0 \, Re\{e^{i\omega t}\} \qquad t > 0$

$y = \infty \qquad v_x = 0 \qquad t > 0$

we postulate an oscillatory solution of the form:

$$v_x(t, y) = Re\{v^0(y)\, e^{i\omega t}\}$$

substitution the trial solution in PDE:

$$Re\left\{ v^{\circ}\, i\omega\, e^{i\omega t} \right\} = \gamma\, Re\left\{ \frac{d^2 v^{o}}{dy^2}\, e^{i\omega t} \right\}$$

<u>note</u> $Re\left\{ z_1\, w \right\} = Re\left\{ z_2\, w \right\} \Rightarrow z_1 = z_2$.

$$\Rightarrow \quad v^{o}(i\omega)\, \cancel{e^{i\omega t}} = \gamma\, \frac{d^2 v^{o}}{dy^2}\, \cancel{e^{i\omega t}}$$

$$\frac{d^2 v^{o}}{dy^2} - \frac{i\omega}{\gamma} v^{o} = 0.$$

BC1 $\quad y = 0 \qquad v^{o} = v_o$

BC2 $\quad y = \infty \qquad v^{o} = 0$.

$$\Rightarrow \quad v^{o} = C_1\, e^{\sqrt{\frac{i\omega}{\gamma}}\, y} + C_2\, e^{-\sqrt{\frac{i\omega}{\gamma}}\, y}$$

<u>note</u> $\sqrt{i} = \pm \frac{1}{\sqrt{2}}(i+1)$

$$\rightarrow \quad v^{o} = C_1\, e^{\sqrt{\frac{\omega}{2\gamma}}\,(i+1)\, y} + C_2\, e^{-\sqrt{\frac{\omega}{2\gamma}}\,(i+1)\, y}$$

<u>BC2</u> $\quad y = \infty \Rightarrow v^{o} = 0 = C_1 + C_2(0) \quad \Rightarrow \boxed{C_1 = 0}$

BC1 $\quad y = 0 \quad v^{o} = v_o = C_2 \quad \Rightarrow \boxed{C_2 = v_o}$

$$\Rightarrow \quad v^{o} = v_o\, e^{-\sqrt{\frac{\omega}{2\gamma}}\,(i+1)\, y}$$

recall $v_x(t,y) = Re\left\{ v^0(y)\, e^{i\omega t} \right\}$.

$$\Rightarrow v_x = v_0\, e^{-\sqrt{\frac{\omega}{2\gamma}}\, y}\, Re\left\{ e^{-i \frac{\omega}{2\gamma}(y - \omega t)} \right\}$$

$$v_x = v_0\, e^{-\sqrt{\frac{\omega}{2\gamma}}\, y} \cos\left(\omega t - \underbrace{\sqrt{\frac{\omega}{2\gamma}}\, y}_{\text{phase shift}}\right)$$