Temperature Distribution in Solids and Laminar Flow

The energy equation for constant μ, ρ & k is written in three coordinate systems as follows:

Cartesian coordinates (x, y, z):

$$\rho \hat{C}_p\left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z}\right) = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \mu \Phi_v \qquad \text{(B.9-1)}^b$$

Cylindrical coordinates (r, θ, z):

$$\rho \hat{C}_p\left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z}\right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right] + \mu \Phi_v \qquad \text{(B.9-2)}^b$$

Spherical coordinates (r, θ, ϕ):

$$\rho \hat{C}_p\left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial T}{\partial \phi}\right) = k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right] + \mu \Phi_v \qquad \text{(B.9-3)}^b$$

where Φ_v is the viscous dissipation function given by

Cartesian coordinates (x, y, z):

$$\Phi_v = 2\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] + \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]^2 + \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right]^2 + \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right]^2 - \frac{2}{3}\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right]^2 \qquad \text{(B.7-1)}$$

Cylindrical coordinates (r, θ, z):

$$\Phi_v = 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]^2 + \left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]^2$$

$$- \frac{2}{3}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right]^2 \qquad \text{(B.7-2)}$$

Spherical coordinates (r, θ, ϕ):

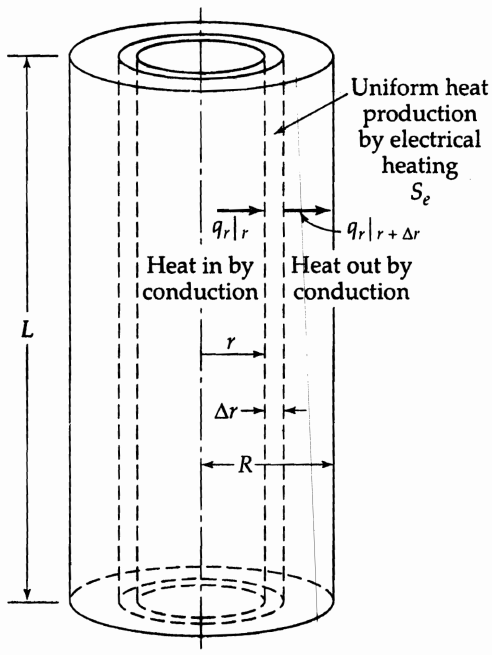
$$\Phi_v = 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r + v_\theta \cot\theta}{r}\right)^2\right]$$

$$+ \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r \sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]^2 + \left[\frac{1}{r \sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]^2$$

$$- \frac{2}{3}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi}\right]^2 \qquad \text{(B.7-3)}$$

Example 1 Heat conduction with electrical heat source:

Consider and electrical wire with circular cross section of Radius R. The rate of heat generation per unit volume is given by S_e.



In this problem there is no flow, hence, the L.H.S. of eq B.9 is negligible. Also $\Phi_v = 0$. In this problem it is assumed

1) steady state
2) one dimensional conduction in r-direction
3) uniform function for energy production by electrical heating S_e to be added to R.H.S.

Therefor, B.9-2 can be simplified ③

$$k \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \underset{\uparrow \atop \frac{J}{m^3 \cdot s}}{S_e} = 0 .$$

re-arrange and integrate once:

$$\Rightarrow r\frac{dT}{dr} \doteq -\frac{S_e}{k}\frac{r^2}{2} + C_1$$

⋮

$$T = -\frac{S_e}{k}\frac{r^2}{4} + C_1 \ln(r) + C_2$$

Boundary conditions

BC1 $r = 0$ $\quad \frac{dT}{dr} = 0$ (symmetry)

BC2 $r = R$ $\quad T = T_0$ (Assumed to have const. wall Temp.) (see Example 10.2-2 of textbook)

BC1 $\Rightarrow$ $C_1 = 0$.

BC2 $\Rightarrow$ $C_2 = T_0 + \frac{S_e}{k}\frac{R^2}{4}$

$$\Rightarrow T - T_0 = \frac{S_e R^2}{4k}\left[1 - \left(\frac{r}{R}\right)^2\right]$$ parabolic temp. rise.

$(T - T_0)$ represents the temperature rise in the wire due to heating.

Maximum Temperature Rise:

occurs at $r = 0$: $(T_{max} - T_0) = \dfrac{S_e R^2}{4k}$.

Average Temperature Rise:

$$\langle T - T_0 \rangle = \frac{\int_0^{2\pi}\int_0^R (T - T_0)\, r\,dr\,d\theta}{\int_0^{2\pi}\int_0^R r\,dr\,d\theta}$$

$$= \frac{\frac{S_e R^2}{4k}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R}{\frac{R^2}{2}}$$

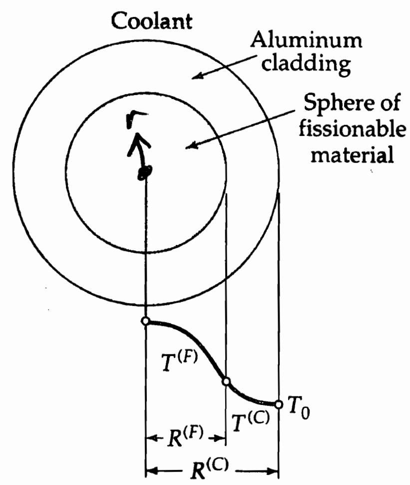
$$= \frac{S_e R^2}{8k}$$

Rate of heat generated at wire surface:

$$Q\Big|_{r=R} = 2\pi RL \cdot \underbrace{q_r}_{\text{radial flux}}\Big|_{r=R} = 2\pi RL\left(-k\frac{dT}{dr}\right)$$

$$= 2\pi RL\left(\frac{S_e \cancel{R^2}}{2\cancel{4}}\;\frac{\cancel{2}r}{\cancel{R^2}}\right)\Bigg|_{r=R} = \pi R^2 L S_e$$

Example 2 Heat generation in a sphere due to nucular heat source.



In this problem the heat generation takes place in the fissionable sphere from $r=0$ to $r=R^F$. There is no generation in the Aluminum cladding. Therefor, equation B.9-3 is simplified for steady state one dimensional conduction

For $0 \le r \le R^F$

$$k_F \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT_F}{dr}\right) + S_{n_0}\left[1 + b\left(\frac{r}{R^F}\right)^2\right] = 0 \quad \text{---} \; (1)$$

For $R^F < r \le R^C$

$$k_C \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT_C}{dr}\right) = 0 \quad \text{---} \; (2)$$

Integrating 1 :

$$r^2 \frac{dT_F}{dr} = -\frac{S_{no}}{K_F}\left[\frac{r^3}{3} + 6\,\frac{r^5}{5(R^F)^2}\right] + C_1$$

$$T_F = -\frac{S_{no}}{K_F}\left[\frac{r^2}{6} + 6\,\frac{r^4}{20(R^F)^2}\right] - \frac{C_1}{r} + C_2 \quad \text{---- } (3).$$

Integrating 2 :

$$r^2 \frac{dT_C}{dr} = C_3$$

$$T_C = -\frac{C_3}{r} + C_4 \quad \text{---} \quad (4).$$

Boundary Conditions :

$r = 0$ $\quad \frac{dT_F}{dr} = 0$. (symmery)

$r = R^F$ $\quad T_F = T_C$ (continuity of T)

$r = R^F$ $\quad -K_F \frac{dT_F}{dr} = -K_C \frac{dT_C}{dr}$ (Continuity of flux)

$r = R^C$ $\quad T_C = T_0$ assuming const. wall Temperature.

BC1 $\Rightarrow$ $\boxed{C_1 = 0}$.

BC2 $\Rightarrow$ $-\frac{S_{no}}{k_F}\left[\frac{R_F^2}{6} + b\,\frac{R_F^2}{20}\right] + C_2 = -\frac{C_3}{R_F} + C_4$

BC3 $\Rightarrow$ $+S_{no}\left[\frac{R_F}{3} + b\,\frac{R_F}{5}\right] = -k_c\,\frac{C_3}{R_F^2}$.

BC4 $\Rightarrow$ $T_0 = -\frac{C_3}{R_c} + C_4$

BC3 $\Rightarrow$ $\boxed{C_3 = -\frac{S_{no} R_F^3}{k_c}\left[\frac{1}{3} + \frac{b}{5}\right]}$.

BC4 $\Rightarrow$ $\boxed{C_4 = T_0 - \frac{S_{no} R_F^3}{k_c R_c}\left(\frac{1}{3} + \frac{b}{5}\right)}$

BC2 $\Rightarrow$ $C_2 = \frac{S_{no} R_F^2}{k_c}\left(\frac{1}{3} + \frac{b}{5}\right) + T_0 - \frac{S_n R_F^3}{k_c R_c} *$

$\left(\frac{1}{3} + \frac{b}{5}\right) + \frac{S_{no} R_F^2}{2k_F}\left(\frac{1}{3} + \frac{b}{5}\right)$

$$\boxed{C_2 = \frac{S_{no} R_F^2}{k_c}\left(\frac{1}{3} + \frac{b}{5}\right)\left[1 - \frac{R_F}{R_c} + \frac{k_c}{2k_F}\right] + T_0}$$

$$T_F = -\frac{S_{no} R_F^2}{6 K_F}\left[\left(\frac{r}{R_F}\right)^2 + \frac{3}{10} b \left(\frac{r}{R_F}\right)^4\right] + \frac{S_{no} R_F^2}{3 k_c}\left(1 + \frac{3}{5} b\right)\left[1 - \frac{R_F}{R_c} + \frac{k_c}{2 k_F}\right] + T_0 .$$

re - arrange:

$$T_F = \frac{S_{no} R_F^2}{6 K_F}\left[\left(1 - \left(\frac{r}{R_F}\right)^2\right) + \frac{3}{10} b \left(1 - \left(\frac{r}{R_F}\right)^4\right)\right]$$

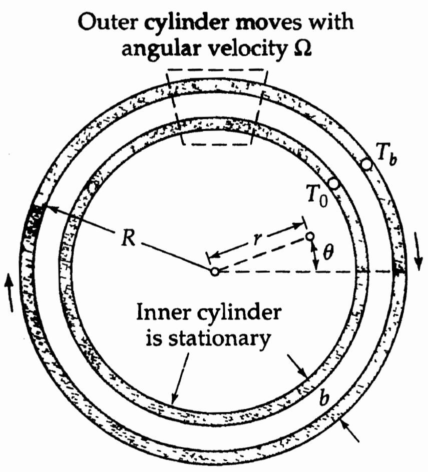
$$+ \frac{S_{no} R_F^2}{3 k_c}\left(1 + \frac{3}{5} b\right)\left(1 - \frac{R_F}{R_c}\right) + T_0 .$$

$$T_c = \frac{S_{no} R_F^2}{k_c}\left(\frac{R_F}{r}\right)\left(\frac{1}{3} + \frac{b}{5}\right) + T_0 - \frac{S_{no} R_F^3}{K_c R_c}\left(\frac{1}{3} + \frac{b}{5}\right)$$

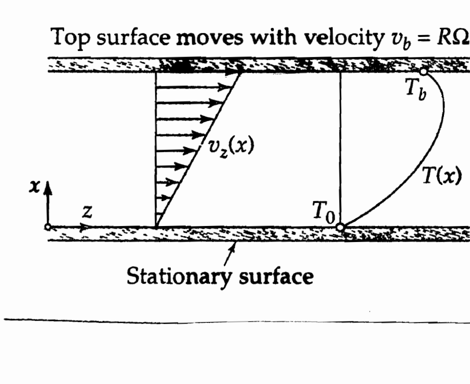
re-arrange:

$$T_c = \frac{S_n R_F^2}{3 K_c}\left(1 + \frac{3}{5} b\right)\left(\frac{R_F}{r} - \frac{R_F}{R_c}\right) + T_0 .$$

Example 3 Flow with dissipation between concentric cylinders.



In this problem we consider energy generation between co-axial concentric cylinders due to viscous dissipation. First, we assume thin gap between cylinders available for flow, hence, the problem can be solved in chartesian coordinates which offers a mathematical symplification:



B.6-3 $\Rightarrow$ $0 = \dfrac{d^2 V_z}{dx^2}$ --- (1)

B.9-1 $\Rightarrow$ $0 = k \dfrac{d^2 T}{dx^2} + \mu \Phi_v$ -- (2)

B.7-1 $\Rightarrow$ $\Phi_v = \left(\dfrac{dV_z}{dx}\right)^2$ --- (3)

solving (1) $\Rightarrow$ $V_z = C_1 x + 6$

BC's $x = 0$ $V_z = 0$

$x = b$ $V_z = R\Omega$

$\Rightarrow$ $V_z = \dfrac{R\Omega}{b} x$ substituting in (2)

$$0 = k \frac{d^2 T}{dx^2} + \mu \frac{R^2 \Omega^2}{b^2}$$

Integrating $\Rightarrow$ $T = -\dfrac{\mu}{k} \dfrac{R^2 \Omega^2}{b^2} \dfrac{x^2}{2} + C_1 x + C_2$

BC's $x = 0$ $T = T_0$

$x = b$ $T = T_b$

$$T_0 = -\frac{\mu}{K}\frac{\ell^2\Omega^2}{b^2}\frac{0}{2} + C_1(0) + C_2 \Rightarrow \boxed{C_2 = T_0}$$ (11)

$$T_b = -\frac{\mu}{K}\frac{\ell^2\Omega^2}{\cancel{b^2}}\frac{\cancel{b^2}}{2} + C_1 b + T_0$$

$$\Rightarrow C_1 = \frac{T_b - T_0}{b} + \frac{\mu}{bK}\frac{\ell^2\Omega^2}{2}$$

$$\Rightarrow T = -\frac{\mu}{K}\frac{\ell^2\Omega^2}{b^2}\frac{x^2}{2} + \frac{T_b - T_0}{b}x + \frac{\mu}{K}\frac{\ell^2\Omega^2}{b2}x + T_0$$

re-arranging.

$$\frac{T - T_0}{T_b - T_0} = -\frac{\mu}{K}\frac{\ell^2\Omega^2}{2(T_b - T_0)}\frac{x^2}{b^2} + \frac{x}{b} + \frac{\mu}{K}\frac{\ell^2\Omega^2}{2(T_b - T_0)}\frac{x}{b}$$

$$= \frac{1}{2}\underbrace{\boxed{\frac{\ell^2\Omega^2\mu}{(T_b - T_0)K}}}_{\text{Brinkman Number}}\frac{x}{b}\left[1 - \frac{x}{b}\right] + \frac{x}{b}$$

$$\boxed{\frac{T - T_0}{T_b - T_0} = \frac{1}{2}Br\,\frac{x}{b}\left[1 - \frac{x}{b}\right] + \frac{x}{b}}$$

Example 4 Heat Conduction in a Fin

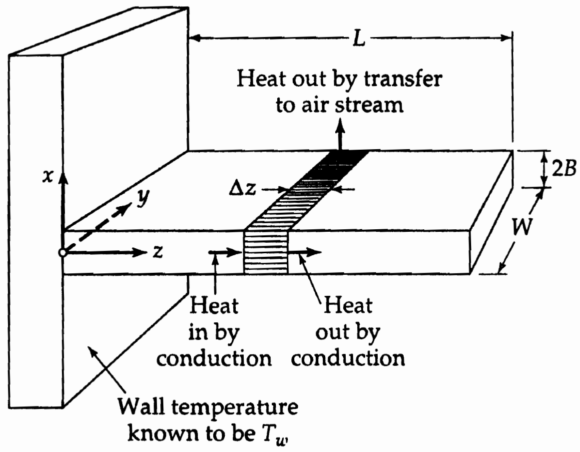
Consider the heat conduction problem for a cooling Fin. In this case heat is conducted along the z-direction and a same time is lost from upper and lower surfaces due to convection to ambient air at T_a. Recall equation B.9-1, and after assuming steady state one dimensional conduction one have:

$$0 = k\frac{d^2T}{dz^2} - \frac{hA}{V}(T - T_a)$$

In this case we have added an extra term ~~at~~ in the RHS of the equation which is heat loss per unit volume.

$$A = 2LW \qquad V = LW2B.$$

$$\Rightarrow \frac{d^2T}{dz^2} - \frac{h}{kB}(T - T_a) = 0.$$



introducing dimensionless variables:

$$\theta = \frac{T - T_a}{T_w - T_a} \qquad \xi = \frac{z}{L}$$

$$\Rightarrow \quad \frac{d^2\theta}{d\xi^2} - \frac{hL^2}{kB}\theta = 0 .$$

$$\frac{d^2\theta}{d\xi^2} - \frac{hL}{k}\frac{L}{B}\theta$$

$\frac{hL}{k} \rightarrow$ Bi = Biot number.

$$\Rightarrow \quad \frac{d^2\theta}{d\xi^2} - Bi\frac{L}{B}\theta = 0 .$$

$\xi = 0 \qquad T = T_w \Rightarrow \theta = 1$

$\xi = 1 \qquad q = 0 \Rightarrow \frac{dT}{dz} = \frac{d\theta}{d\xi} = 0 .$

Appendix

C.1-4 a $\Rightarrow \Theta = C_1 \cosh \sqrt{Bi \frac{L}{B}}\, z + C_2 \sinh \sqrt{Bi \frac{L}{B}}\, z$ (14)

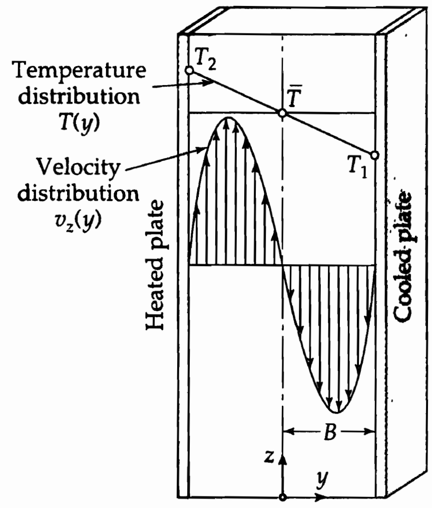
$\boxed{1 = C_1} + C_2 (0)$ BC1

$0 = \cancel{\sqrt{Bi \frac{L}{B}}} \sinh \left[\sqrt{Bi \frac{L}{B}} (1)\right] + \cancel{\sqrt{Bi \frac{L}{B}}}\, C_2 \cosh \left[\sqrt{Bi \frac{L}{B}} (1)\right]$

$\Rightarrow C_2 = -\tanh\left(\sqrt{Bi \frac{L}{B}}\right)$

$\Rightarrow \Theta = \cosh\left[\sqrt{Bi \frac{L}{B}}\, z\right] - \tanh\left[\sqrt{Bi \frac{L}{B}}\right] \sinh\left(\sqrt{Bi \frac{L}{B}}\, z\right)$

Example 5 Free Convection Between Two Vertical Plates.



Consider the case of a fluid confiend between two vertical plates separated by a distance $2B$. One plate is heated and the other is cooled. This induces buoyancy and ~~hot~~ fluid adjacent to hot plate ~~so~~ moves up and cold fluid goes down.

First, the velocity profile. Recall equation B.6. and after simplification:

$$0 = -\frac{dP}{dz} + \mu \frac{d^2 v_z}{dy^2} + \rho g_z .$$

$g_z = -g$ and re-arranging:

$$\mu \frac{d^2 V_z}{dy^2} = \frac{dp}{dz} + \rho g.$$

here viscosity is assumed constant but the density is not! Hence, we use

Boussienesq approximation, expand ρ as function of T in Taylor series expansion around average temperature $\bar{T} = \frac{T_1 + T_2}{2}$

$$\rho = \rho\big|_{T=\bar{T}} + \frac{d\rho}{dT}\bigg|_{T=\bar{T}} (T - \bar{T}) + \cdots$$

introduce the coefficient of thermal expansion:

$$\beta = \frac{1}{V}\left(\frac{dV}{dT}\right)_p = -\frac{1}{\rho}\frac{d\rho}{dT}$$

$$\Rightarrow \rho = \bar{\rho} - \bar{\rho}\beta(T - \bar{T})$$

re-write velocity equation:

$$\mu \frac{d^2 v_z}{dy^2} = \frac{dP}{dz} + \bar{\rho} g - \bar{\rho} g \bar{\beta} (T - \bar{T}) \quad \text{---- (1)}$$

now Consider the energy equation B.9-1 after simplification:

$$\frac{d^2 T}{dy^2} = 0.$$

$$\left.\begin{array}{ll} y = -B & T = T_2 \\ y = B & T = T_1 \end{array}\right\} \Rightarrow T = \bar{T} - \left(\frac{T_2 - T_1}{2}\right)\frac{y}{B} \quad \text{---- (2)}$$

(2) into (1)

$$\Rightarrow \frac{d^2 v_z}{dy^2} = \frac{1}{\mu}\left(\frac{dP}{dz} + \bar{\rho} g\right) - \frac{\bar{\rho} g \bar{\beta}}{2\mu} \Delta T \frac{y}{B}$$

$$y = -B \qquad v_z = 0$$

$$y = B \qquad v_z = 0.$$

integrating and apply BC's

$$\Rightarrow V_z = \frac{\bar{\rho} g \bar{\beta} \Delta T B^2}{12\mu}\left[\left(\frac{y}{B}\right)^3 - \left(\frac{y}{B}\right)\right] +$$

$$\frac{B^2}{12\mu}\left(\frac{dP}{dz} + \bar{\rho} g\right)\left[\left(\frac{y}{B}\right)^2 - 1\right]$$

now, how to find $\frac{dP}{dz} = ?$ this can be obtained by mass conservation

the net mass flow rate in z-direction is zero :

$$\int_{-B}^{B} \rho V_z \, dy = 0 .$$

This implied $\left(\frac{dP}{dz} + \bar{\rho} g\right) = 0$ do it yourself

$$\Rightarrow V_z = \frac{\bar{\rho} g \bar{\beta} \Delta T B^2}{12\mu}\left[\left(\frac{y}{B}\right)^3 - \frac{y}{B}\right]$$

$$\langle V_z \rangle = \frac{\int_0^W \int_{-B}^{B} V_z \, dy \, dx}{\int_0^W \int_{-B}^{B} dy \, dx} = \frac{\bar{\rho} g \bar{\beta} \Delta T B^2}{48\mu}$$