Chapter 7
Lecture # 2-3

- **Cash Flow Diagrams (CFD)**

Cash Flow Diagrams

- Shows money flow as a function of time
  - *x*-axis is time and *y*-axis is magnitude of money.
  - both positive and negative payments are shown as arrows with upward or downward directions .

Cash Flow Diagrams

- **CASH FLOW DIAGRAMS (CFD)**
  - **DISCRETE CFD**
  - **CUMULATIVE CFD**

Cash Flow Diagrams

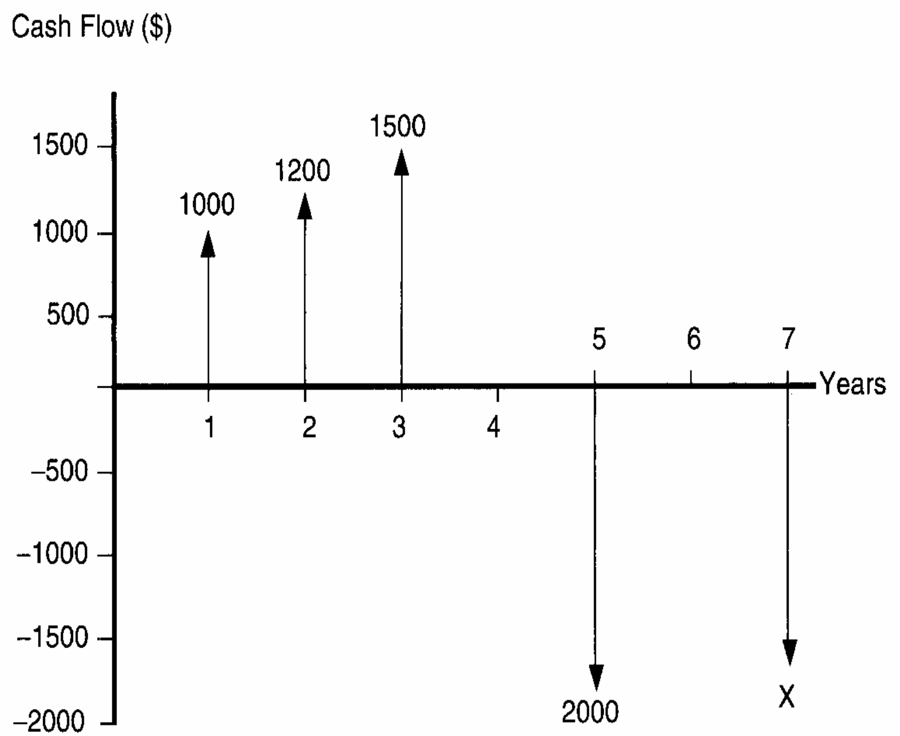
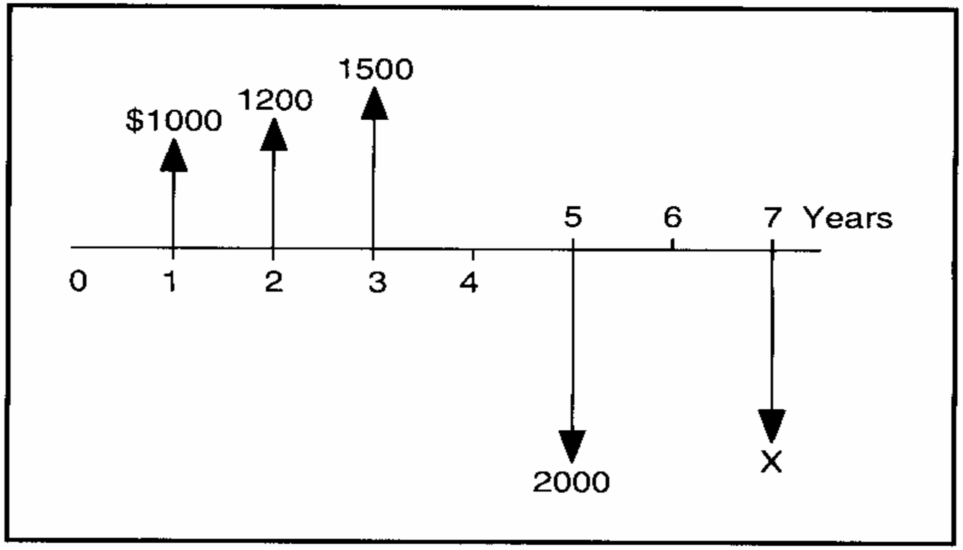


Figure 7.2 An Example of a Representative Discrete Cash Flow Diagram (CFD)

Cash Flow Diagrams

Example 7.10

I borrow $1000, $1200, and $1500 from a bank (at 8% p.a. effective interest rate) at the end of years 1, 2, and 3, respectively. At the end of year 5, I make a payment of $2000, and at the end of year 7, I pay off the loan in full. The CFD for this exchange from my point of view (producer) is given to the right.

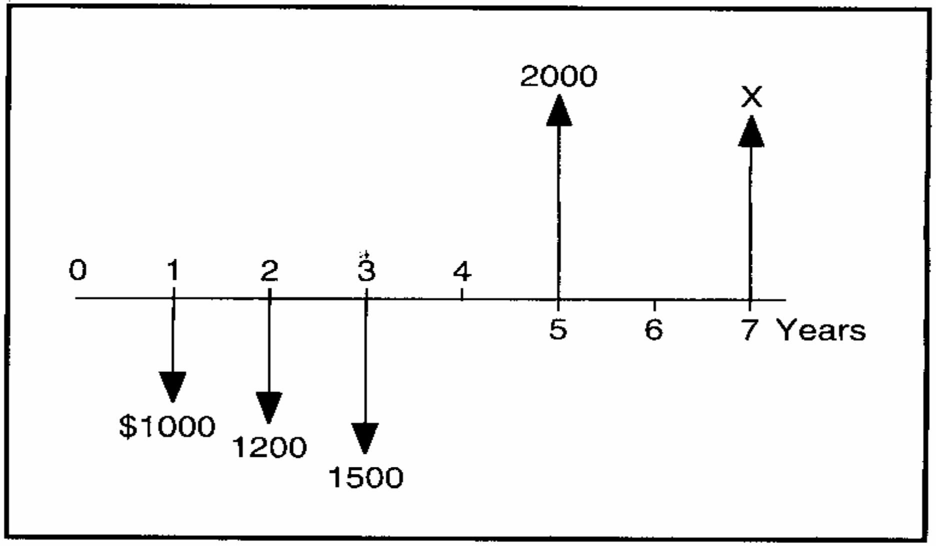


Note: This figure is the short-hand version of the one presented in Figure 7.2 used to introduce the CFD.

Draw a discrete cash flow diagram for the investor.

The bank represents the investor. From the investor's point of view, the initial three transactions are negative and the last two are positive.

The figure to the right represents the CFD for the bank. It is the mirror image of the one given above in the problem statement.

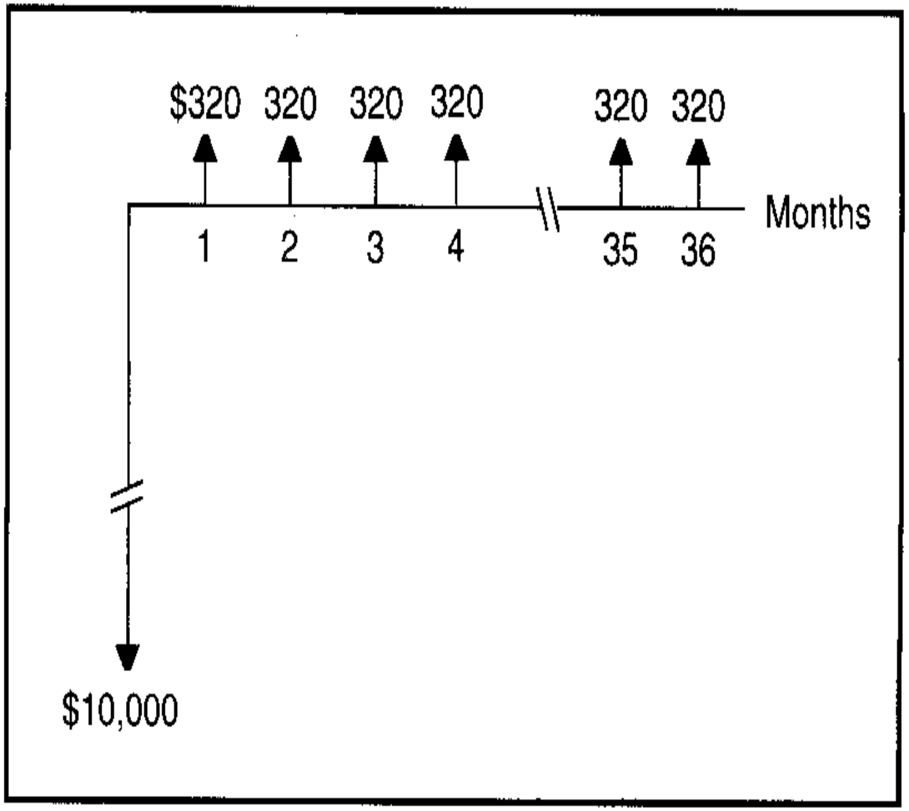


Cash Flow Diagrams

Example 7.11

You borrow $10,000 from a bank to buy a new car and agree to make 36 equal monthly payments of $320 each to repay the loan. Draw the discrete CFD for the investor in this agreement.

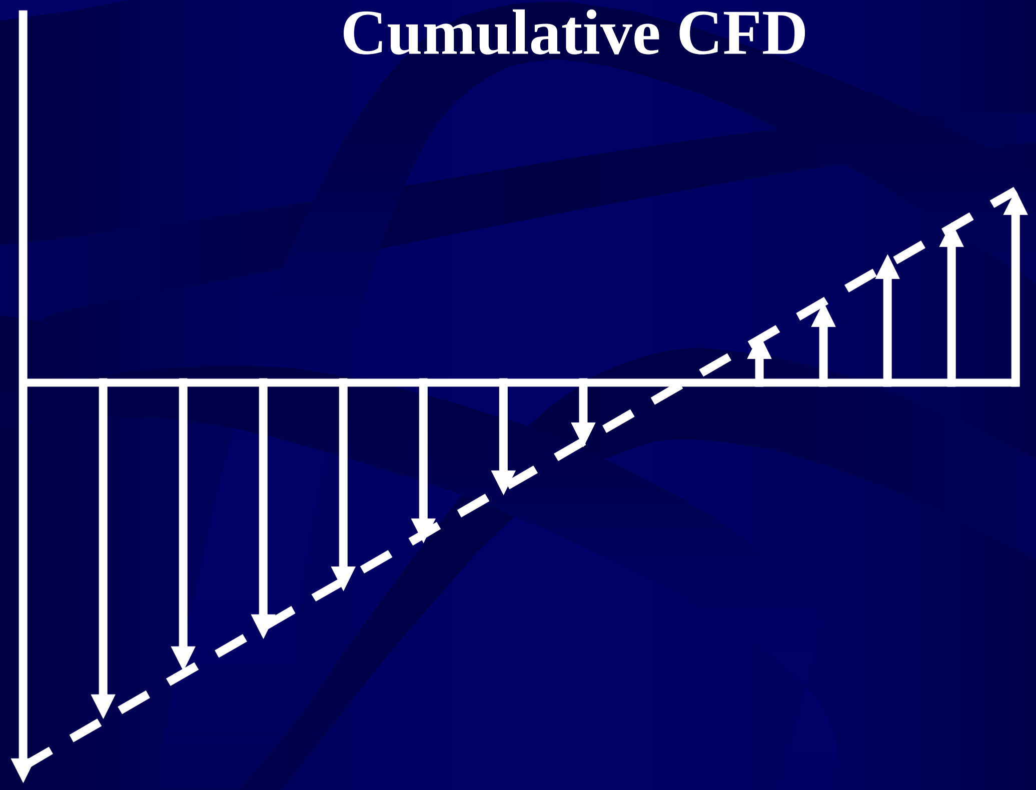
The bank is the investor. The discrete CFD for this investment is shown on the right.



Notes:

1. There is a break in both the time scale and in the investment at time = 0 (the initial investment).
2. From your point of view, the cash flow diagram would be the mirror image of the one shown.

Cash Flow Diagrams

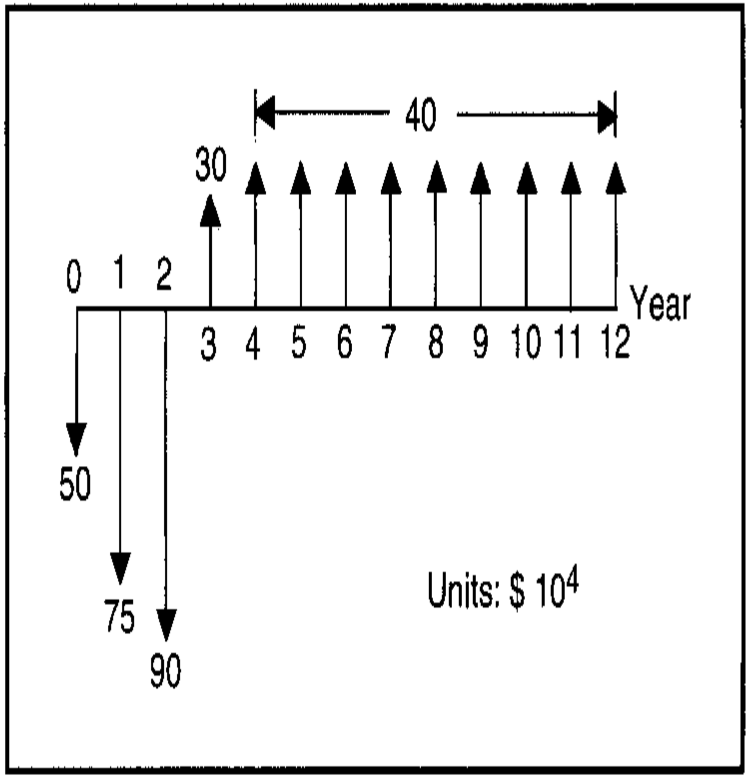


Cash Flow Diagrams

Example 7.12

The yearly cash flows estimated for a project involving the construction and operation of a chemical plant producing a new product are provided in the discrete CFD given on the right. Using this information, construct a cumulative CFD.

The numbers shown in the worksheet below were obtained from this diagram.

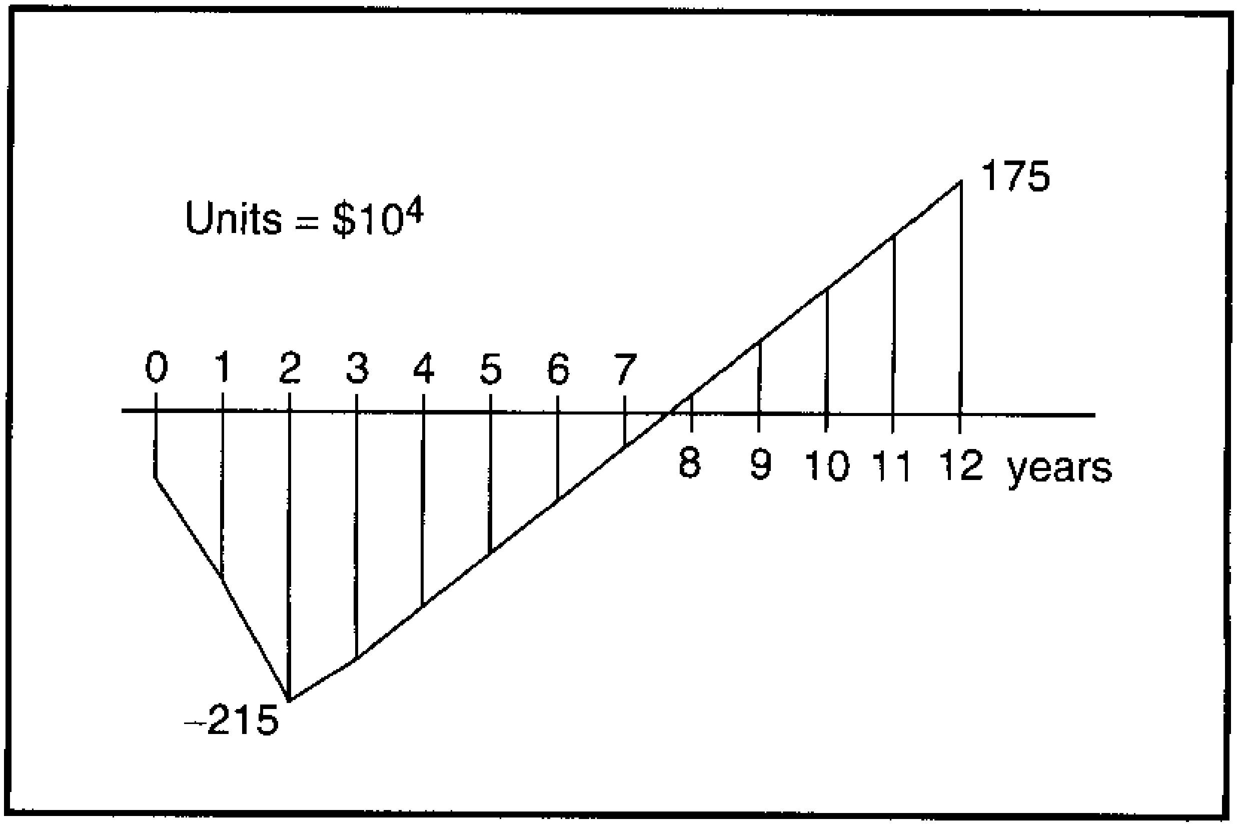


Cash Flow Diagrams

Year	Cash Flow ($) (from discrete CFD)	Cumulative Cash Flow (calculated)
0	–500,000	–500,000
1	–750,000	–1,250,000
2	–900,000	–2,150,000
3	300,000	–1,850,000
4	400,000	–1,450,000
5	400,000	–1,050,000
6	400,000	–650,000
7	400,000	–250,000
8	400,000	150,000
9	400,000	550,000
10	400,000	950,000
11	400,000	1,350,000
12	400,000	1,750,000

Cash Flow Diagrams

The cumulative cash flow diagram is plotted below.



Cash Flow Diagrams

When cash flows occur at different times, each cash flow must be brought forward (or backward) to the same point in time and then compared.

Cash Flow Diagrams

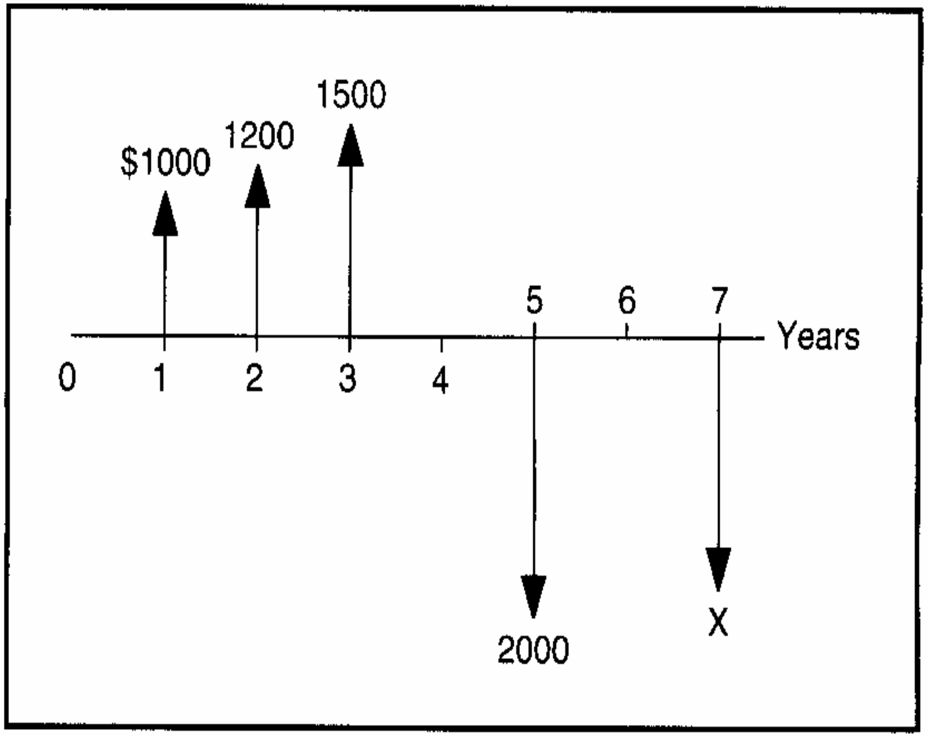
Example 7.13

The CFD obtained from Example 7.10 (for the borrower) is copied below. The annual interest rate paid on the loan is 8% p.a.

In year 7, the remaining money owed on the loan is paid off.

a. Determine the amount, *X*, of the final payment.

b. Compare the value of *X* with the value that would be owed if there were no interest paid on the loan.



Cash Flow Diagrams

With the final payment at the end of year 7, no money is owed on the loan. If we sum all the positive and negative cash flows adjusted for the time of the transactions, this adjusted sum must equal zero.

We select as the base time the date of the final payment.

a. From Equation (7.5) for $i = 0.08$ we obtain:

For withdrawals:

\$1000 end of year 1: $F_6 = (\$1000)(1 + 0.08)^6 = \$1586.87$

\$1200 end of year 2: $F_5 = (\$1200)(1 + 0.08)^5 = \$1763.19$

\$1500 end of year 3: $F_4 = (\$1500)(1 + 0.08)^4 = \$2040.73$

Total withdrawals = \$5390.79

Cash Flow Diagrams

For repayments:

$\$2000$ end of year 5: $F_2 = -(\$2000)(1+0.08)^2 = -\ \$2332.80$

$\$X$ end of year 7: $F_0 = -(\$X)(1+0.08)^0 = -\ \$X$

Total repayments $= -\ \$(2332.80 + X)$

Summing the cash flows and solving for X yields

$$0 = \$5390.79 - \$(2332.80 + X)$$
$$X = \$3057.99 \approx \$3058$$

Cash Flow Diagrams

b. For $i = 0.00$

$$\text{Withdrawals} = \$1000 + \$1200 + \$1500 = \$3700$$

$$\text{Repayments} = -\$(2000 + X)$$

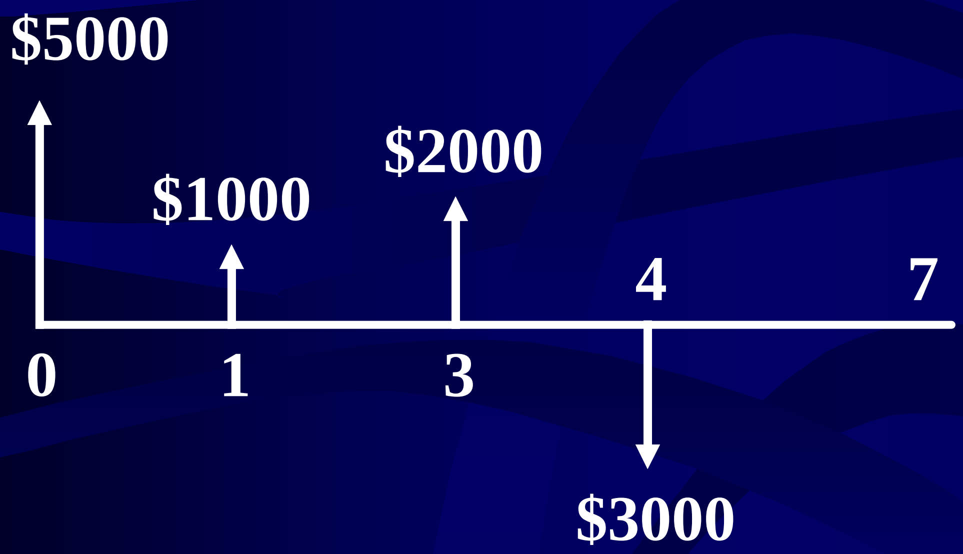
$$0 = \$3700 - \$(2000 + X)$$

$$X = \$1700$$

Note: Because of the interest paid to the bank, the borrower repaid a total of \$1358 (\$3058 – \$1700) more than was borrowed from the bank seven years earlier.

Cash Flow Diagrams

Example



- Invest 5K, 1K, 2K at End of Years 0, 1, 3, and take 3K at End of Year 4

Cash Flow Diagrams

- How much is in account at end of Year 7 if i = 8% p.a.

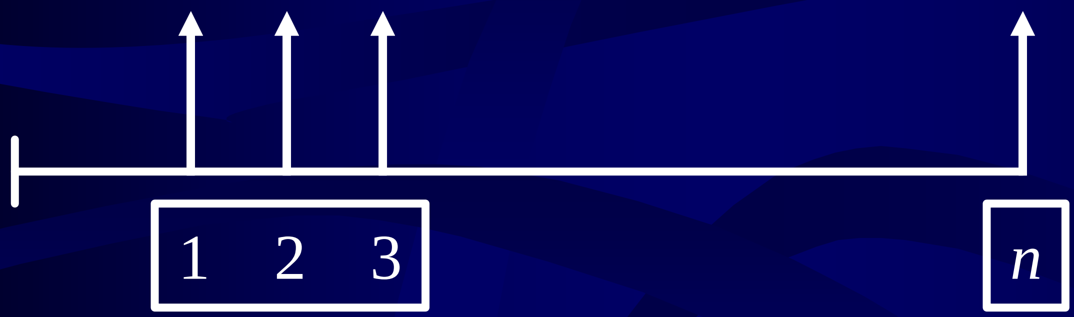
$$F_7 = 5{,}000(1+0.08)^7 + 1000(1+0.08)^6 + 2000(1+0.08)^4$$
$$-3000(1+0.08)^3$$
$$F_7 = \$9097.84$$

- What would investment at Year 0 be to get this amount at Year 7

$$P = \frac{9097.84}{(1.08)^7} = 5308.50$$

Cash Flow Diagrams

Annuities



Uniform series of equally spaced – equal value cash flows

Cash Flow Diagrams

- What is future value $F_n = ?$

$$F_n = A(1+i)^{n-1} + A(1+i)^{n-2} + \ldots.A$$

- Geometric progression

$$F_n = S_n = A\left[\frac{(1+i)^n - 1}{i}\right]$$

Cash Flow Diagrams

Discount Factors

Discount Factor for X/Y = (X/Y, i, n) = f (i, n)

X and Y could be F, P or A

Cash Flow Diagrams

$$P = \frac{F}{(1+i)^n} \Rightarrow \left(\frac{P}{F}, i, n\right) = \frac{1}{(1+i)^n}$$

$$\Rightarrow P = F\left(\frac{P}{F}, i, n\right) = F\left(\frac{1}{(1+i)^n}\right)$$

$$\Rightarrow A \rightarrow P \Rightarrow \left(\frac{P}{F}, i, n\right) = \frac{(1+i)^n - 1}{i(1+i)^n}$$

Table 7.1: List of Discount Factors

Cash Flow Diagrams

Example

What should my annual monthly car payment be if interest rate is 8% p.a. compounded monthly for 60 months?

Cash Flow Diagrams

$$F_{60} = A\left[\frac{\left(1+\frac{0.08}{12}\right)^{60} - 1}{\frac{0.08}{12}}\right] = 73.47A$$

$$F_{60} = -20,000\left[\left(1+\frac{0.08}{12}\right)^{60}\right] = -29796.90$$

$$73.47A - 29796.90 = 0$$

$$A = \$405.53$$

You could directly use the discount factor (A/P) from Table 7.1

Cash Flow Diagrams

Example 7.14

You have just won \$2,000,000 in the Texas Lottery as one of seven winners splitting up a jackpot of \$14,000,000. It has been announced that each winner will receive \$100,000/year for the next 20 years. What is the equivalent present value of your winnings if you have a secure investment opportunity providing 7.5% p.a.?

From Table 7.1, Equation 7.14, for $n = 20$ and $i = 0.075$

$$P = (\$100{,}000)[(1 + 0.075)^{20} - 1]/[(0.075)(1 + 0.075)^{20}]$$

$$P = \$1{,}019{,}000$$

A present value of \$1,019,000 is equivalent to a 20-year annuity of \$100,000/yr when the effective interest rate is 7.5%.

Cash Flow Diagrams

Example 7.15

Consider Example 7.11, involving a car loan. The discrete CFD from the bank's point of view was shown previously.

What interest rate is the bank charging for this loan?

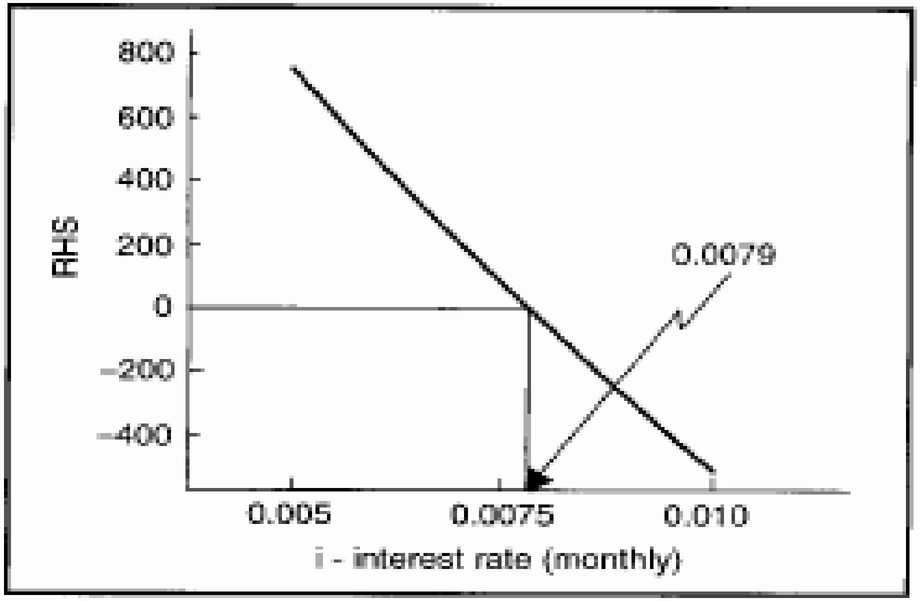
You have agreed to make 36 monthly payments of \$320. The time selected for evaluation is the time at which the final payment is made. At this time, the loan will be fully paid off. This means that the future value of the \$10,000 borrowed is equivalent to a \$320 annuity over 36 payments.

$$(\$10{,}000)(F/P, i, n) = (\$320)(F/A, i, n)$$

Substituting the equations for the discount factors given in Table 7.1, with $n = 36$ months, we get:

$$0 = -(10{,}000)(1+i)^{36} + (320)[(1+i)^{36} - 1]/i$$

This equation cannot be solved explicitly for i. We solve this equation by plotting the value of the right-hand side of



the equation shown above for various interest rates. This equation could also be solved using a numerical technique. From the graph, the interest rate that gives a value of zero represents the answer. From the graph on the previous page the rate of interest is i = 0.0079.

The nominal annual interest rate is (12)(0.00786) = 0.095 (9.5%).

Cash Flow Diagrams

Example 7.16

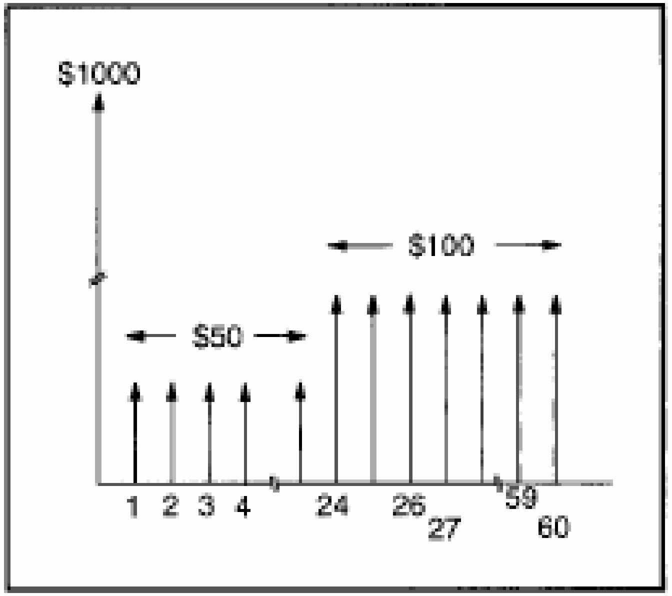
I invest money in a savings account that pays a nominal interest rate of 6% p.a. compounded monthly. I open the account with a deposit of $1000 and then deposit $50 at the end of each month for a period of two years followed by a monthly deposit of $100 for the following three years. What will the value of my savings account be at the end of the five-year period?

Cash Flow Diagrams

First, draw a discrete CFD (shown to the right).

Although this CFD looks rather complicated, we can break it down into 3 easy subproblems:

1. The initial investment
2. The 24 monthly investments of \$50
3. The 36 monthly investments of \$100



Each of these investments is brought forward to the end of month 60.

$F = (\$1000)(F/P, 0.005, 60) + (\$50)(F/A, 0.005, 24)(F/P, 0.005, 36) + (\$100)(F/A, 0.005, 36)$

Note: the effective monthly interest rate is $0.06/12 = 0.005$

$$F = (\$1000)(1.005)^{60} + (\$50)\frac{(1.005^{24} - 1)}{0.005}(1.005)^{36} + (\$100)\frac{(1.005^{36} - 1)}{0.005} = \$6804.16$$

Cash Flow Diagrams

Example 7.17

In Example 7.1, we introduced an investment plan for retirement. It involved investing \$5000/year for 40 years leading to retirement. The plan then provided \$67,468/year for twenty years of retirement income.

a. What yearly interest rate was used in this evaluation?

b. How much money was invested in the retirement plan before withdrawals began?

Cash Flow Diagrams

a. The evaluation is performed in two steps:

Step 1: Find the value of the \$5000 annuity investment at the end of the 40 years.

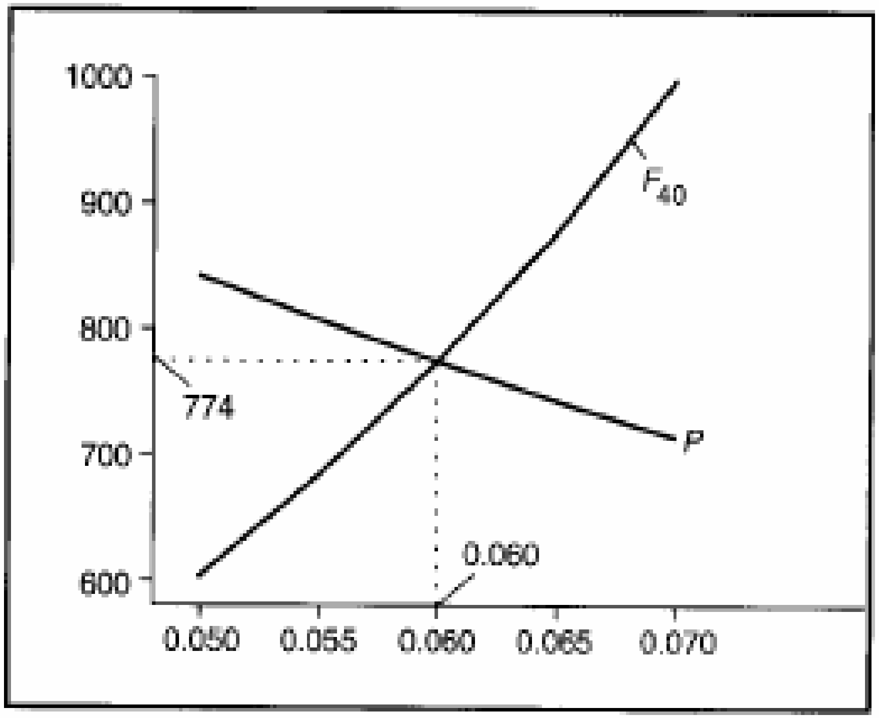
Step 2: Evaluate the interest rate of an annuity that will pay out this amount in 20 years at \$67,468/year.

Step 1: From Equation 7.11, Table 7.1, for $A = \$5000$ and $n = 40$,

$F_{40} = (A)(F/A, n, i) = (\$5000)[(1+i)^{40}-1]/i$

Step 2: From Equation 7.14, Table 7.1, for $A = \$67{,}468$ and $n = 20$,

$$P = (A)(P/A, n, i) = (\$67{,}468)[(1+i)^{20} - 1]/[(i)(1+i)^{20}]$$



Set $F_{40} = P$ and solve for i. From the graph above, we get $i = 0.060$

b. With $i = 0.060$, we have from the graph $F_{40} = \$774{,}000$