2-52. Express each force as a Cartesian vector and then determine the resultant force $\mathbf{F}_R$. Find the magnitude and coordinate direction angles of the resultant force and sketch this vector on the coordinate system.

2-68. Determine the magnitude and coordinate direction angles of $\mathbf{F}_2$ so that the resultant of the two forces acts along the positive x axis and has a magnitude of 350 N.

2-52

Unit vector:

$\mathbf{u}_{F1} = \cos 60^{o}\mathbf{i} + \cos 45^{o}\mathbf{j} + \cos 60^{o}\mathbf{k}$ $\quad \mathbf{u}_{F2} = -\mathbf{j}$

Force vector: $\quad \mathbf{F} = F(\mathbf{u}_F)$

$\mathbf{F}_1 = 5\,(\cos 60^{o}\mathbf{i} + \cos 45^{o}\mathbf{j} + \cos 60^{o}\mathbf{k})$
$= \{2.5\mathbf{i} + 3.536\mathbf{j} + 2.5\mathbf{k}\}$ kN
$= \{2.5\mathbf{i} + 3.54\mathbf{j} + 2.5\mathbf{k}\}$ kN Ans

$\mathbf{F}_2 = 2(-\mathbf{j}) = \{-2\mathbf{j}\}$ kN Ans

Resultant force vector:

$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$
$= \{2.5\mathbf{i} + 3.536\mathbf{j} + 2.5\mathbf{k}\} + -2\mathbf{j}\}$
$= \{2.5\mathbf{i} + 1.536\mathbf{j} + 2.5\mathbf{K}\}$ kN

Magnitude of $\mathbf{F}_R$:

$F_R = \sqrt{2.5^2 + 1.536^2 + 2.5^2} = 3.855\ kN = 3.85\ kN$ Ans

Coordinate direction angles :

$$\mathbf{u}_{F_R} = \frac{\mathbf{F}_R}{F_R} = \frac{2.5\mathbf{i} + 1.536\mathbf{j} + 2.5\mathbf{k}}{3.855} = 0.6486\mathbf{i} + 0.3984\mathbf{j} + 0.6486\mathbf{k}$$

$\cos\alpha = 0.6486$ $\quad \alpha = 49.6^{o}$ Ans

$\cos\beta = 0.3984$ $\quad \beta = 66.5^{o}$ Ans

$\cos\gamma = 0.6486$ $\quad \gamma = 49.6^{o}$ Ans

2-68

Force Vector

$\mathbf{F}_1 = 180\ (\cos 15^o\ \sin 60^o\mathbf{i} + \cos 15^o\ \cos 60^o\mathbf{j} - \sin 15^o\mathbf{k})$

$= \{150.573\mathbf{i} + 86.933\mathbf{j} - 46.587\mathbf{k}\}$ N

$\mathbf{F}_2 = F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k}$

$\mathbf{F}_R = \{350\mathbf{i}\}$ N

$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$$
$$350\mathbf{i} = \{150.573\mathbf{i} + 86.933\mathbf{j} - 46.587\mathbf{k}\} + F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k}$$

$$350\mathbf{i} = (150.573 + F_{2x})\mathbf{i} + (86.933 + F_{2y})\mathbf{j} + (F_{2z} - 46.587)\mathbf{k}$$

Equating **i**, **j**, **k** components yields :

$150.573 + F_{2x} = 350$ $\qquad F_{2x} = 199.427$ N

$86.933 + F_{2y} = 0$ $\qquad F_{2y} = -86.93$ N

$F_{2z} - 46.587 = 0$ $\qquad F_{2z} = 46.58$ N

$$\mathbf{F}_2 = \{199.427\mathbf{i} - 86.933\mathbf{j} + 46.587\mathbf{k}\}\ \text{N}$$

Magnitude of $\mathbf{F}_2$:

$$F_2 = \sqrt{199.427^2 + (-86.933)^2 + 46.587^2} = 222\ \text{N}$$ Ans

Coordinate direction angles :

$$\mathbf{u}_{F2} = \frac{\mathbf{F}_2}{F_2} = \frac{199.427\mathbf{i} - 86.933\mathbf{j} + 46.58\mathbf{k}}{222.48}$$

$$= 0.8964\mathbf{i} - 0.3907\mathbf{j} + 0.209\mathbf{k}$$

$\cos\alpha_2 = 0.8964$ $\qquad \alpha_2 = 26.3^o$ $\qquad$ Ans

$\cos\beta_2 = -0.3907$ $\qquad \beta_2 = 113^o$ $\qquad$ Ans

$\cos\gamma_2 = 0.2094$ $\qquad \gamma_2 = 77.9^o$ $\qquad$ Ans

2.7 Position Vectors:

- Concept
- Formulating Cartesian force vector directed between any two points in space.
- Uses for finding Moment

x, y, z Coordinates:

- R.H. coordinate system
- points in space are located relative to coordinate origin by successive measurements along x, y, & z axes.

Position Vector: $\bar{r}$

Def.- a fixed vector which locates a point in the space relative to a other point if $\bar{r}$ extends from origin to point P (x, y, z).

$$\therefore \quad \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

to arrive at P

$$0 \rightarrow x\hat{i} \rightarrow y\hat{j} \rightarrow z\hat{k}$$

head-to-tail vector addition

General Case : vector directed from point A to point B

$\vec{r}$ or r_{AB}

$\vec{r}_A = x_A\,\hat{i} + y_A\,\hat{j} + z_A\,\hat{k}$

$\vec{r}_B = x_B\,\hat{i} + y_B\,\hat{j} + z_B\,\hat{k}$

by head-to-tail vector addition or $\vec{r}_A + \vec{r} - \vec{r}_B = \vec{0}$

$\vec{r}_A + \vec{r} = \vec{r}_B$

$$\therefore \quad \vec{r} = \vec{r}_B - \vec{r}_B = \left(x_B\,\hat{i} + y_B\,\hat{j} + z_B\,\hat{k}\right) - \left(x_A\,\hat{i} + y_A\,\hat{j} + z_A\,\hat{k}\right)$$

$$= (x_B - x_A)\hat{i} + (y_B - y_A)\hat{j} + (z_B - z_A)\hat{k}$$

$\hat{i}, \hat{j}, \hat{k}$ components of the position vector r :

subtracting coordinates of the tail of the vector A (x_A , y_A , z_A)

from coordinates of the tail head of the vector B (x_B , y_B , Z_B)

2.8 Force Vector directed along a line

$$\vec{F} = F \cdot \vec{u}$$

$$\vec{u} \ : \ \text{unit vector} = \frac{\vec{r}}{r}$$

- F has units of forces
- F can't be scaled

Procedure:

To express $\vec{F}$ in Cartesian coordinates (along line A to B)

1. Determine the position vector $\vec{r}$ & compute its magnitude, r.
   directed from A to B

2. Determine the unit vector $\vec{u} = \frac{\vec{r}}{r}$
   $\Rightarrow$ direction & sense of $\vec{r}$ & $\vec{F}$

3. Determine force vector, $\vec{F}$, by combining mag. F & direction $\vec{u}$.

$$\vec{F} = F.\ \vec{u}$$

2-89. Two tractors pull on the tree with the forces shown. Represent each force as a Cartesian vector and then determine the magnitude and coordinate direction angles of the resultant force.

2-89

Force vector:

$\mathbf{r}_{BA} = (20 \cos 30^{o} - 0)\mathbf{i} + (-20 \sin 30^{o} - 0)\mathbf{j} + (2-30)\mathbf{k}$

$= \{17.32\mathbf{i} - 10\mathbf{j} - 28\mathbf{k}\}$ ft

$r_{BA} = \sqrt{17.32^2 + (-10)^2 + (-28)^2} = 34.409$ ft

$$\mathbf{F}_1 = F_1\left(\frac{\mathbf{r}_{AB}}{r_{AB}}\right) = 150\left(\frac{17.32\mathbf{i} - 10\mathbf{j} - 28\mathbf{k}}{34.409}\right)$$

$= \{75.505\mathbf{i} - 43.593\mathbf{j} - 122.060\mathbf{k}\}$ lb

$= \{75.5\mathbf{i} - 43.6\mathbf{j} - 122\mathbf{k}\}$ lb Ans

$\mathbf{r}_{BC} = (8 - 0)\mathbf{i} + (10 - 0)\mathbf{j} + (3 - 30)\mathbf{k}$

$= (8\mathbf{i} + 10\mathbf{j} - 27\mathbf{k})$ ft

$r_{BC} = \sqrt{8^2 + 10^2 + (-27)^2} = 29.883$ ft

$$\mathbf{F}_2 = F_2\left(\frac{\mathbf{r}_{BC}}{r_{BC}}\right) = 100\left(\frac{8\mathbf{i} + 10\mathbf{j} - 27\mathbf{k}}{29.833}\right)$$

$= \{26.771\mathbf{i} + 33.464\mathbf{j} - 90.352\mathbf{k}\}$ lb

$= \{26.8\mathbf{i} + 33.5\mathbf{j} - 90.4\mathbf{k}\}$ lb Ans

Resultant force vector:

$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$

$= (75.505\mathbf{i} - 43.593\mathbf{j} - 122.060\mathbf{k}) + (26.771\mathbf{i} + 33.464\mathbf{j} - 90.352\mathbf{k})$

$= \{102.276\mathbf{i} - 10.129\mathbf{j} - 212.412\mathbf{k}\}$ lb

Magnitude:

$F_R = \sqrt{102.276^2 + (-10.129)^2 + (-212.412)^2}$

$= 235.97$ lb $= 236$ lb Ans

Coordinate direction angles:

$$\mathbf{u}_R = \frac{\mathbf{F}_R}{F_R} = \frac{102.276\mathbf{i} - 10.129\mathbf{j} - 212.412\mathbf{k}}{235.97}$$

$= 0.4334\mathbf{i} - 0.04292\mathbf{j} - 0.9002\mathbf{k}$

$\cos \alpha = 0.4334$ $\alpha = 64.3^{o}$ Ans

$\cos \beta = -0.04292$ $\beta = 92.5^{o}$ Ans

$\cos \gamma = -0.9002$ $\gamma = 154^{o}$ Ans

2.9 Dot Product:

- angle between two lines
- components of a force // and ⊥ a line

in 2-D → by trigonometry, ∴ trigonometry is easy to visualize

in 3-D → trigonometry is difficult
└→ vector methods

Dot Product: a particular method of "multiplying" two vectors

Ex. dot product of $\vec{A}$ & $\vec{B}$ is $\vec{A} \cdot \vec{B}$

Def. $\vec{A} \cdot \vec{B}$ is the product of the magnitude of A & B and the cosine of the angle θ between their tails.

$$\vec{A} \cdot \vec{B} = AB \cos\theta \qquad 0^o \le \theta \le 180^o$$

Dot product = scalar product ⇒ result is a scalar and a vector

Laws of Operation

1. Cumulative Law $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

2. Multiplication by a Scalar $a(\vec{A} \cdot \vec{B}) = (a\vec{A}) \cdot \vec{B} = \vec{A} \cdot (a\vec{B})$

   $= (\vec{A} \cdot \vec{B})\, a$

3. Distributive Law $\vec{A} \cdot (\vec{B} + \vec{D}) = (\vec{A} \cdot \vec{B}) + (\vec{A} \cdot \vec{D})$

Cartesian Vector Formulation:

$\hat{i}\cdot\hat{i} = (1)(1)(\cos\ 0) = 1$, $\hat{i}\cdot\hat{j} = (1)(1)(\cos 90) = 0$, $\hat{i}\cdot\hat{k} = (1)(1)\cos 90 = 0$

$= \hat{j}\cdot\hat{i}$ $= \hat{k}\cdot\hat{i}$

$\hat{j}\cdot\hat{j} = 1$

$\hat{k}\cdot\hat{k} = 1$

Ex. $\vec{A}\cdot\vec{B} = (A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k})\cdot(B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k})$

$$= A_x B_x (\hat{i}\cdot\hat{i}) + A_x B_y (\hat{i}\cdot\hat{j}) + A_x B_z (\hat{i}\cdot\hat{k})$$

$$+ A_y A_x (\hat{j}\cdot\hat{i}) + A_y B_y (\hat{j}\cdot\hat{j}) + A_y B_z (\hat{j}\cdot\hat{k})$$

$$+ A_z B_x (\hat{k}\cdot\hat{i}) + A_z B_y (\hat{k}\cdot\hat{j}) + A_z B_z (\hat{k}\cdot\hat{k})$$

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

$\therefore$ dot product of two Cartesian vectors

$\Rightarrow$ multiply their corresponding x, y, z components & sum their products algebraically.

* Scalar

Applications:

1. Angle formed between two vectors or intersecting lines, θ

$$\vec{A}\cdot\vec{B} = AB\ \text{Cos}\ \theta \quad \rightarrow \quad \text{Cos}\ \theta = \frac{\vec{A}\cdot\vec{B}}{AB}$$

$$\theta = \text{Cos}^{-1}\frac{\vec{A}\cdot\vec{B}}{AB} \qquad 0^o \le \theta \le 180^o$$

if $\vec{A}\cdot\vec{B} = 0 \quad \Rightarrow \quad \theta = \text{Cos}^{-1}(0) = 90 \ \Rightarrow\ \vec{A} \perp \vec{B}$

2. Components of a vector // & ⊥ a line.

Components of $\vec{A}$ // or collinear *a-a′* line

$= \vec{A}_{//}$

$A_{//} = A\ \text{Cos}\ \theta$ projection of $\vec{A}$ onto line *a-a′*

$= A \cdot u$

direction of $\vec{A}_{//}$ $\qquad$ $\bar{u}$ unit vector

$$\vec{A}_{//} = \vec{A}_{//}\ \bar{u} = A\ \text{Cos}\,\theta\ \bar{u}$$

$$= (\vec{A}\cdot\bar{u})\ \bar{u}$$

Scalar projection of $\vec{A}$ along a line a-a $\Rightarrow$ dot product of $\vec{A}$ and the unit vector $\bar{u}$ which define the direction of the line.

A
- $+$: $\vec{A}_{//}$ same sense as $\bar{u}$
- -ve: $\vec{A}_{//}$ opposite sense of $\bar{u}$

o Component of $\vec{A} \perp$ *a-a′*

$$\vec{A} = \vec{A}_{//} + \vec{A}_{\perp} \quad \Rightarrow \quad \vec{A}_{\perp} = \vec{A} - \vec{A}_{//}$$

mag. $A_{\perp}$ $\rightarrow$ Find $\theta = \cos^{-1}\left(\frac{\vec{A}\cdot\bar{u}}{A}\right)$

$$A_{\perp} = A\sin\theta$$

$$A_{\perp} = \sqrt{A^2 - A_{//}^2}$$