

posites. The dependence of damage characteristics on the load history is believed to be related to the results of the complex interaction among the residual stress field, the residual strain fields, cumulative damage, and time-dependent nature of the matrix material. Since the current damage state depends on past load history, exercise caution when interpreting either acoustic imaging results or other nondestructive testing results.

Although destructive testing techniques can be used to detect damage in particulate composites, interpreting results in terms of physical damage parameters and evaluating the effect of damage state on crack growth behavior are problematic at this time. It is recommended that development of nondestructive testing techniques for damage detection and interpretation of test results proceed in a fashion which emphasizes concurrent investigation of nondestructive test methods and damage mechanics. Only by understanding both the effect of damage on a material's response and the nondestructive test methods can we hope to develop the required nondestructive test techniques.

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Damage Model for Monotonic and Fatigue Response of High Strength Concrete

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ABSTRACT: An anisotropic elasto-damage model for predicting the response of concrete subject to monotonic and fatigue loading is presented in this study. The model utilizes a concrete appropriate damage-effect tensor M in constructing the constitutive equations. The concept of multiple bounding surfaces is used, with a varying size limit fracture surface defining fatigue loading in contrast to a fixed size limit fracture surface for monotonic loading. The model after calibration is shown to predict the monotonic, compressive uniaxial stress-strain path for concretes of various strengths as well as S-N curves depicting the fatigue response of concrete.

INTRODUCTION

THE ADVANCEMENT AND growth in concrete construction industry in response to the demand for high quality concrete has led to the production of concrete of compressive strength of the order of 120 MPa through the use of selective admixtures and additives. In contrast, the understanding pertaining to behavior of high-strength concrete and failure mechanisms, especially under cyclic loading, has not kept pace. It is only in recent years that fatigue of concrete has received increased attention, being essentially stimulated by the increased use of slender concrete structures with repetitive live load possibly being the major portion of the total load [1].

Fatigue, unfortunately, is a property of concrete that is least understood, especially with regard to modeling. Continuum damage modeling (CDM) has given

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new impetus to the constant search for improvement in constitutive modeling of complex bimodular materials [2,3]. With sufficient insight gained into CDM for both brittle and ductile fracture [4-6], research has been initiated into incorporation of CDM into finite element models for solution of problems of engineering interest [7-9]. Suaris et al. [4] have developed a damage model for monotonic and cyclic behavior of concrete where elastic potential is introduced in terms of principal stresses and a damage dependent compliance tensor. The evolution of damage is calculated by tracking the movement of the loading surfaces in its approach towards the bounding surface, a concept originally introduced in elasto-plasticity by Dafalias and Popov [10]. The limit fracture surface (which sets the threshold of damage), the loading surface, and the bounding surface are all expressed in terms of the strain energy release rate vector R_i . Al-Gadhib, Rahman and Baluch [11] have extended the Suaris stress control model of Reference [4], making provisions for strain-control under proportional triaxial loading and incorporation into a finite element code to solve a variety of problems including strip loading of a rectangular prism and the Brazilian test.

The Suaris model [4] presents an elegant approach to the formulation of the elasto-damage problem, but has certain inherited deficiencies. The first is the underestimation of concrete strength by approximately 25% in biaxial compression as obtained from Kupfer experiments [12]. The second limitation is that the Suaris model when applied to cyclic loading of concrete simulates the experimental S-N curve only in the range of the low-cycle fatigue which corresponds to a high cycling stress ratio σ/f'_c . Further, the material damage parameters are calibrated for one specific concrete compressive strength, thus limiting the use of the model.

Nonlinearity in concrete is attributed to the development of microcracks and microvoids which tend to destroy the interface bond between the cement matrix and aggregate and/or destruction of the material grains themselves, affecting the elastic properties and imparting anisotropy to the material [13]. Continuum damage models have been developed to account for such anisotropic damage for an initially isotropic material [14]. Based on the use of damage-effect tensor M , constitutive equations of anisotropic damage have been developed [6], using the hypothesis of elastic strain energy equivalence introduced in Reference [15]. A symmetric stiffness matrix results in contrast to the postulate of equivalence of strain which leads to asymmetry of the stiffness matrix [16]. The concept of the M -tensor is an attractive one which has been applied to predict damage and behavior of metals [5-7]. It is only recently that state-of-the-art constitutive equations expressed in terms of the M -tensor for concrete have been developed by Khan, Al-Gadhib, and Baluch [17] where essential features of concrete such as degradation of elastic properties, strain softening, gain in strength under confinement and different behavior in tension and compression have been captured effectively.

This paper is an attempt to develop an anisotropic damage model capable of

predicting the fatigue life under compression for a range of concrete strengths through the adaptation of the constitutive model developed in Reference [17] for monotonic loading. This model is capable of predicting fatigue life for both low-cycle and high-cycle fatigue induced by a wide range of cycling stress ratios, with a model feature being the introduction of a bounding surface conjugate to the limit fracture surface.

DAMAGE EFFECT TENSOR VERSUS EFFECTIVE COMPLIANCE MATRIX

Damage variable may be considered as an internal state variable which characterizes the irreversible deterioration of a material point in accordance with the thermodynamic formulation [18]. Based on the theory of isotropic continuum damage mechanics, the effective Cauchy stress tensor $\bar{\sigma}$ is related to the usual Cauchy stress tensor σ by

$$\bar{\sigma} = \left(\frac{1}{1 - \omega} \right) \sigma \quad (1)$$

where ω is a scalar measure for damage and $0 \leq \omega \leq 1$. However, in general, the internal state variable may be portrayed through a damage effect tensor, as introduced by Leckie and Onat [19]. For anisotropic damage, the effective stress can be expressed in a generalized form as:

$$\bar{\sigma} = M(\omega) : \sigma \quad (2)$$

where the symbol $(:)$ means tensorial product contracted on two indices and $M(\omega)$, known as damage-effect tensor, is a linear symmetric operator represented by a fourth order tensor. There are many possible forms of the generalized damage-effect tensor M_{ij} . However, one obvious criterion for development of such a damage-effect tensor is that it should reduce to a scalar for isotropic damage. This reduction should be possible not only in a principal coordinate system but also in any coordinate system. There are many possible forms of the generalized damage-effect tensor which obey the stated criterion, with some of these forms defined in References [5-7] for analysis of metals. However, as far as concrete behavior is concerned, one form of the M tensor which satisfies the stipulated criterion has been introduced in Reference [17] and takes the following form:

$$M_{ij} = \begin{bmatrix} \frac{(1-\beta\omega_1)}{(1-\alpha\omega_1)(1-\beta\omega_2)(1-\beta\omega_3)} & 0 & 0 \\ 0 & \frac{(1-\beta\omega_2)}{(1-\alpha\omega_2)(1-\beta\omega_3)(1-\beta\omega_1)} & 0 \\ 0 & 0 & \frac{(1-\beta\omega_3)}{(1-\alpha\omega_3)(1-\beta\omega_1)(1-\beta\omega_2)} \end{bmatrix} \quad (3)$$

where $\omega_i, i = 1, 2, 3$, are the principal damage components. The parameters α and β are introduced as calibration parameters by matching experimentally measured peak strengths for various stress paths. It may be noted that for isotropic damage, $\omega_1 = \omega_2 = \omega_3 = \omega$ and Equation (3) readily reduces to a scalar.

This assumed form of M_{ij} has the constraint condition that principal axes of stress coincide with the principal damage directions. A more general form of M_{ij} valid for nonproportional loading has been presented in Reference [20], where the principal directions of stress do not necessarily coincide with those of damage.

For undamaged state, the linear elastic constitutive relation is

$$\epsilon^e = C : \sigma \quad (4)$$

where ϵ^e and σ are the elastic strain and stress tensor in the principal coordinate system and C is the elastic compliance tensor given by

$$[C] = \frac{1}{E_o} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \quad (5)$$

in which E_o, ν are the initial elastic modulus and Poisson ratio of the material, respectively. However, if the material is in a damaged state, then the elasto-damage

constitutive equation can be written as

$$\epsilon = \bar{C} : \sigma \quad (6)$$

where $\bar{C}$ is the effective compliance matrix and ϵ is the elasto-damage strain tensor. In the hypothesis of elastic energy equivalence stated in Reference [15], the complementary elastic energy for a damaged material is the same in form as that for an undamaged material, except that the Cauchy stress σ is replaced by the Cauchy effective stress $\bar{\sigma}$ in the energy formulation. Accordingly, the complementary energy per unit volume $\rho\Lambda$ (ρ = material mass density) for undamaged and damaged states may be written as

$$\rho\Lambda(\sigma, 0) = \frac{1}{2} \sigma^T : \epsilon^e = \frac{1}{2} \sigma^T : C : \sigma \quad (7)$$

$$\begin{aligned} \rho\Lambda(\sigma, \omega) &= \frac{1}{2} \bar{\sigma}^T : \bar{C} : \bar{\sigma} \\ &= \frac{1}{2} \sigma^T : (M^T : C : M) : \sigma \\ &= \frac{1}{2} \sigma^T : \bar{C} : \sigma \end{aligned} \quad (8)$$

where

$$\bar{C}^T = M^T : C : M \quad (9)$$

Upon substitution of Equations (3) and (5) into Equation (9), one may write the components of the effective compliance matrix explicitly as

$$\begin{aligned} \bar{C}_{11} &= \frac{(1-\beta\omega_1)^2}{E_o(1-\alpha\omega_1)^2(1-\beta\omega_2)^2(1-\beta\omega_3)^2} \\ \bar{C}_{22} &= \frac{(1-\beta\omega_2)^2}{E_o(1-\alpha\omega_2)^2(1-\beta\omega_3)^2(1-\beta\omega_1)^2} \\ \bar{C}_{33} &= \frac{(1-\beta\omega_3)^2}{E_o(1-\alpha\omega_3)^2(1-\beta\omega_1)^2(1-\beta\omega_2)^2} \end{aligned}$$

(continued)

$$\begin{aligned}\bar{C}_{12} = \bar{C}_{21} &= \frac{-v}{E_o(1 - \alpha\omega_1)(1 - \alpha\omega_2)(1 - \beta\omega_3)^2} \\ \bar{C}_{13} = \bar{C}_{31} &= \frac{-v}{E_o(1 - \alpha\omega_1)(1 - \alpha\omega_3)(1 - \beta\omega_2)^2} \\ \bar{C}_{23} = \bar{C}_{32} &= \frac{-v}{E_o(1 - \alpha\omega_2)(1 - \alpha\omega_3)(1 - \beta\omega_1)^2}\end{aligned}\quad (10)$$

From Equations (10) it is obvious that the thermodynamic constraint requirement $E_i v_{ji} = E_j v_{ij}$ is satisfied.

DAMAGE EVOLUTION LAW

In order to construct a rational model accounting for damage growth, concepts are borrowed from incremental theory of plasticity in general and the bounding surface plasticity model in particular as introduced by Dafalias and Popov [10]. Plasticity bounding surface model as proposed by Dafalias requires definition of multiple surfaces in stress space. However, the fundamental surfaces in the present work are best described in strain-energy release space, as proposed by Suaris et al. [4] and given by

$$f = (R_i R_i)^{1/2} - R_c / b = 0 \quad (11)$$

$$F = (\bar{R}_i \bar{R}_i)^{1/2} - R_c = 0 \quad (12)$$

$$f_o = (R_i R_i)^{1/2} - R_o = 0 \quad (13)$$

where, f is the loading surface (LS), F is the bounding surface (BS), and f_o is a limit fracture surface (LFS) as shown in Figure 1. The loading function surface f is defined in terms of thermodynamic-force conjugates, R_i , where,

$$R_i = \rho \frac{\partial \Lambda}{\partial \omega_i}(\sigma_{ij}, \omega_i) \quad (14)$$

$\bar{R}_i$ is an image point on $F = 0$ associated with a given point R_i on $f = 0$ defined by a mapping rule

$$\bar{R}_i = b R_i \quad (15)$$

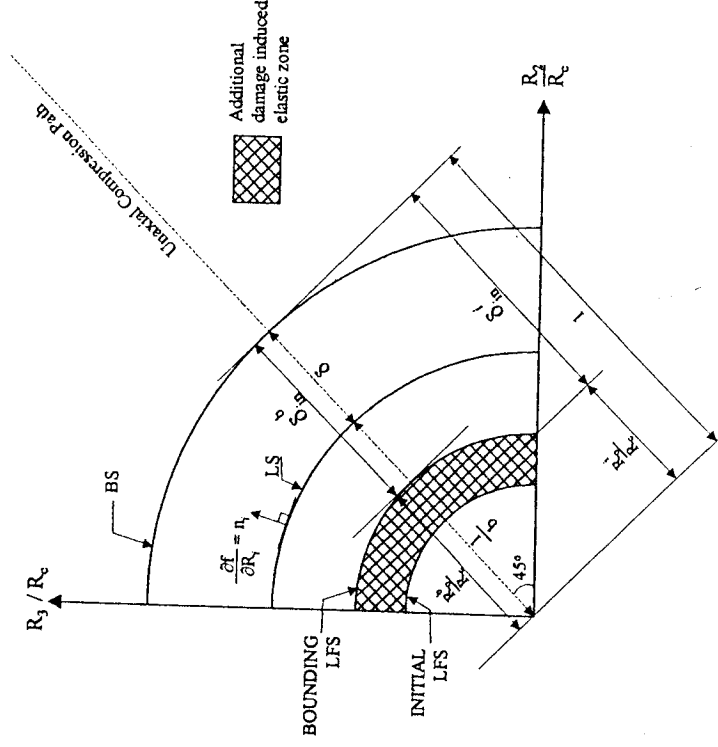


Figure 1. Limit fracture, loading and bounding surfaces.

$$b = R_c / (R_i R_i)^{1/2} \quad (16)$$

with the mapping parameter b ranging from an initial value of ∞ to a limiting value of 1 on growth of loading surface to eventual coalescence with bounding surface. R_c , critical energy release rate, is a parameter of the model and is calibrated to the standard uniaxial compression test. R_o is the size of the limit fracture surface, assumed to be constant for the case of monotonic loading and varying with the magnitude of damage for the case of fatigue loading. The limit fracture surface defines a threshold in R -space beyond which there is an onset of damage.

The damage growth is determined from the loading surface $f = 0$ where the damage increment vector is assumed to be coaxial with the gradient of f which itself is also the unit vector n_i to the loading surface as shown in Figure 1. Therefore the principal damage components may be written as

$$d\omega_i = d\lambda \frac{\partial f}{\partial R_i} = d\lambda n_i \quad (17)$$

with $k = R_c/b$, equation of loading surface becomes

$$f(R_i, k) = (R_i R_j)^{1/2} - k(\bar{\omega}_p) = 0 \quad (18)$$

where $\bar{\omega}_p$ is the norm of the accumulated damage and whose increment is defined by

$$d\bar{\omega}_p = [d\omega_i d\omega_j]^{1/2} \quad (19)$$

It can be shown readily from Equations (17) and (19) that the scalar magnitude of $d\bar{\omega}_p = d\lambda$. The satisfaction of the consistency condition $df = 0$ yields

$$\frac{\partial f}{\partial R_i} dR_i + \frac{\partial f}{\partial k} dk = 0 \quad (20)$$

From Equation (14) one may write

$$dR_i = \frac{\partial R_i}{\partial \sigma_k} d\sigma_k + \frac{\partial R_i}{\partial \omega_j} d\omega_j \quad (21)$$

Also from Equation (18), the incremental increase in the loading surface size may be written as

$$dk = \frac{\partial k}{\partial \bar{\omega}_p} d\bar{\omega}_p = \frac{\partial k}{\partial \bar{\omega}_p} d\lambda \quad (22)$$

Introducing $H = (\partial k / \partial \bar{\omega}_p)$ = damage modulus, it can be measured experimentally in a uniaxial compression test, and the same form assumed for a more general stress path. In the present work, H is expressed as a function of the distance between the loading and the bounding surface [4], given by

$$H = \frac{D\delta}{\langle \delta_m - \delta \rangle} \quad (23)$$

where D is a material parameter which needs to be calibrated in accordance with concrete strength. The role of D is to essentially simulate the post peak σ - ϵ response. $\langle \rangle$ are Macaulay brackets that set the quantity within to zero if the argument is negative. The normalized distance δ between the loading and bounding surface is given by

$$\delta = 1 - \frac{1}{b} \quad (24)$$

The $\delta = \delta_m$ corresponds to R_o when the loading surface first crosses the limit fracture surface (Figure 1). Substitution of Equations (21) and (22) into Equation (20) and solving for $d\lambda$ and then substituting the results into Equation (19) yields

$$d\omega_k = \left[\frac{\frac{\partial f}{\partial R_j} \frac{\partial R_j}{\partial \sigma_s} d\sigma_s}{H - \frac{\partial f}{\partial R_n} \frac{\partial R_n}{\partial \omega_m} \frac{\partial f}{\partial R_m}} \right] \frac{\partial f}{\partial R_k} \quad (25)$$

Equation (25) is convenient for stress control. However, for strain control one may adopt the strain energy density for damaged material ρW defined as

$$\rho W(\epsilon_i, \omega_k) = \frac{1}{2} \sigma^T(\epsilon_i, \omega_k) : \epsilon \quad (26)$$

The constitutive equation defined by Equation (6) may be inverted and expressed as

$$\sigma_i = \bar{D}_{ij} \epsilon_j \quad (27)$$

where $\bar{D}_{ij}$ are the components of the stiffness matrix defined by the inverse of the effective compliance matrix such that $\bar{D} = [\bar{C}]^{-1}$. Using Equation (27) into (26) yields

$$\rho W(\epsilon_i, \omega_k) = \frac{1}{2} \{\epsilon\}^T : \bar{D} : \{\epsilon\} \quad (28)$$

The energy release rate vector R_i may now be expressed alternatively to Equation (14) as

$$R_j = -\rho \frac{\partial W(\epsilon_i, \omega_k)}{\partial \omega_j} \quad (29)$$

whose increment may be written as

$$dR_i = \frac{\partial R_i}{\partial \omega_j} d\omega_j + \frac{\partial R_i}{\partial \epsilon_i} d\epsilon_i \quad (30)$$

Employing the same procedure as that used for derivation of Equation (25), except using Equation (30) instead of Equation (21) leads to the increment of damage

$$d\omega_i = \left[\frac{\frac{\partial f}{\partial R_j} \frac{\partial R_j}{\partial \epsilon_k} d\epsilon_k}{H - \frac{\partial f}{\partial R_n} \frac{\partial R_n}{\partial \omega_m} \frac{\partial f}{\partial R_m}} \right] \frac{\partial f}{\partial R_i} \quad (31)$$

which is convenient for the case of strain control.

INCREMENTAL STRESS-STRAIN RELATIONS

The incremental form of the elasto-damage constitutive equations of Equation (6) may be expressed in indicial notation as

$$d\epsilon_i = \bar{C}_{ij} d\sigma_j + \sigma_l \frac{\partial \bar{C}_{il}}{\partial \omega_k} d\omega_k \quad (32)$$

which upon substitution of Equation (25) yields

$$d\epsilon_i = \left[\bar{C}_{ij} + \sigma_l \frac{\partial \bar{C}_{il}}{\partial \omega_k} \left(\frac{\frac{\partial f}{\partial R_k} \frac{\partial R_k}{\partial \epsilon_s} \frac{\partial \sigma_j}{\partial R_s}}{H - \frac{\partial f}{\partial R_n} \frac{\partial R_n}{\partial \omega_m} \frac{\partial f}{\partial R_m}} \right) \right] d\sigma_j \quad (33)$$

The incremental form of Equation (27) in indicial notation may be written as

$$d\sigma_i = \bar{D}_{ij} d\epsilon_j + \epsilon_l \frac{\partial \bar{D}_{il}}{\partial \omega_k} d\omega_k \quad (34)$$

which upon substitution of Equation (31) results in

$$d\sigma_i = \left[\bar{D}_{ij} + \epsilon_l \frac{\partial \bar{D}_{il}}{\partial \omega_k} \left(\frac{\frac{\partial f}{\partial R_k} \frac{\partial R_k}{\partial \epsilon_s} \frac{\partial \epsilon_j}{\partial R_s}}{H - \frac{\partial f}{\partial R_n} \frac{\partial R_n}{\partial \omega_m} \frac{\partial f}{\partial R_m}} \right) \right] d\epsilon_j \quad (35)$$

The term within the square brackets in Equation (35) may be interpreted as being the elasto-damage stiffness D_{ij}^{ed} .

Uniaxial Compression—Stress Control

For uniaxial compression, the Cauchy stress tensor in the principal coordinate system reduces to a diagonal matrix or a stress vector given by

$$[-\sigma, 0, 0] \quad (36)$$

The complementary energy density $\rho\Lambda$ of Equation (25) takes the following form in indicial or in matrix notation

$$\rho\Lambda = \frac{1}{2} \sigma_i \bar{C}_{ij} \sigma_j = \frac{1}{2} \{\sigma\}^T [\bar{C}] \{\sigma\} \quad (37)$$

Substitution of Equations (10) and (36) into Equation (37) and setting $\alpha = 0$ (as it is used primarily as a parameter for matching peak strength in tension test), one obtains

$$\rho\Lambda = \frac{\sigma^2 (1 - \beta\omega_1)^2}{2E_o (1 - \beta\omega_2)^2 (1 - \beta\omega_3)^2} \quad (38)$$

Differentiating Equation (38) with respect to ω_i and substituting into Equation (14), yields

$$R_1 = \frac{-\beta\sigma^2 (1 - \beta\omega_1)}{E_o (1 - \beta\omega_2)^2 (1 - \beta\omega_3)^2} = 0 \quad (\text{since } R_i \neq 0) \quad (39a)$$

$$R_2 = \frac{\beta\sigma^2 (1 - \beta\omega_1)^2}{E_o (1 - \beta\omega_2)^3 (1 - \beta\omega_3)^2} \quad (39b)$$

$$R_3 = \frac{\beta\sigma^2 (1 - \beta\omega_1)^2}{E_o (1 - \beta\omega_2)^2 (1 - \beta\omega_3)^3} \quad (39c)$$

From symmetry, $\omega_2 = \omega_3 = \omega$ and $\omega_1 = 0$ [by virtue of Equation (39a)]. Thus

$$R_2 = R_3 = \frac{\beta\sigma^2}{E_o (1 - \beta\omega)^5} \quad (40)$$

and the loading surface of Equation (11) becomes

$$f = [R_2^2 + R_3^2]^{1/2} - R_c/b = 0 \quad (41)$$

whose gradient may be expressed as

$$\frac{\partial f}{\partial R_i} = \left[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right] \quad (42)$$

Differentiating R_j with respect to ω_i and σ_k and substituting the results along with Equation (42) into Equation (25) yields $d\omega_i = 0$ and $d\omega_2 = d\omega_3 = d\omega$, given by

$$d\omega = \frac{2\sigma\beta d\sigma/E_o(1-\beta\omega)^5}{H - [5\beta^2\sigma^2/E_o(1-\beta\omega)^6]} \quad (43)$$

Differentiating $\bar{D}_{ij}$ of Equation (10) with respect to ω_k and substituting the results along with Equations (10) and (43) into Equation (32), one obtains

$$d\epsilon_1 = \left\{ \frac{1}{E_o(1-\beta\omega)^4} + \frac{8\beta^2\sigma^2/E_o^2(1-\beta\omega)^{10}}{H - [5\beta^2\sigma^2/E_o(1-\beta\omega)^6]} \right\} d\sigma \quad (44)$$

as the elasto-damage constitutive equation for uniaxial compression with the damage parameter ω being obtained by the accumulation of its increment defined in Equation (43).

Uniaxial Compression—Strain Control

For uniaxial compression, the strain energy density ρW for damaged material may be expressed from Equations (26), (27), and (10) as

$$\rho W(\epsilon_1, \varphi_k) = \frac{1}{2} \sigma_1 \epsilon_1 = \frac{(\epsilon_1)^2 (E_o/2)(1-\beta\omega_2)^2 (1-\beta\omega_3)^2}{(1-\beta\omega_1)^2} \quad (45)$$

Note that for uniaxial compression $\sigma_2 = \sigma_3 = 0$ while $\epsilon_2 = \epsilon_3 \neq 0$.

Differentiating Equation (45) with respect to ω_i and substituting into Equation (29) yields

$$R_1 = 0 \quad (46a)$$

$$R_2 = \frac{\beta\epsilon_1^2 E_o(1-\beta\omega_2)(1-\beta\omega_3)^2}{(1-\beta\omega_1)^2} \quad (46b)$$

$$R_3 = \frac{\beta\epsilon_1^2 E_o(1-\beta\omega_2)^2(1-\beta\omega_3)}{(1-\beta\omega_1)^2} \quad (46c)$$

Again from symmetry $\omega_2 = \omega_3 = \omega$ and also $\omega_1 = 0$. Differentiating R_j with respect to ω_i and ϵ_k and substituting the results along with Equation (42) into Equation (31) yields $d\omega_1 = 0$ and $d\omega_2 = d\omega_3 = d\omega$, given by

$$d\omega = \frac{2\beta\epsilon_1 E_o(1-\beta\omega)^3 d\epsilon_1}{H + 3\beta^2\epsilon_1^2 E_o(1-\beta\omega)^2} \quad (47)$$

Differentiating $\bar{D}_{ij}$ with respect to ω_k and substituting the results along with Equation (47) into Equation (34) yields

$$d\sigma = \left\{ E_o(1-\beta\omega)^4 - \frac{8\beta^2\epsilon_1^2 E_o^2(1-\beta\omega)^6}{H + 3\beta^2\epsilon_1^2 E_o(1-\beta\omega)^2} \right\} d\epsilon_1 \quad (48)$$

MODEL FOR CYCLIC LOADING

The distinguishing feature of cyclic loading from monotonic loading is a rational updating of the threshold of damage with increasing number of stress cycles. For monotonic loading, the threshold of damage is identified by the limit fracture surface $f_o = 0$ and whose size is $R_o = R_o^i$ (Figure 1). However, for cyclic loading R_o is hypothesized to change and increase with each successive cycle and is represented as $R_o = R_o(\bar{\omega})$, where $\bar{\omega} = (\omega_i \omega_o)^{1/2}$ is the magnitude of the damage vector $\bar{\omega}_i$.

The limit fracture surface is the surface beyond which the material behaves inelastically due to initiation or propagation of crack damage. The size of the surface R_o is a function of the amount of the accumulated damage. Different functional forms describing the movement of the surface $f_o = 0$ were considered and that of an elliptical form was found to successfully predict the experimental results for cyclic loading in compression [21]. The form of the surface in R_o - $\bar{\omega}$ space may be expressed as

$$\frac{(R_o - R_o^i)^2}{(R_o^b - R_o^i)^2} + \frac{(\bar{\omega} - \bar{\omega}_b)^2}{(\bar{\omega}_i - \bar{\omega}_b)^2} = 1 \quad (49)$$

The parameters R_o^i and $\bar{\omega}_i$ correspond to the initial size of the limit fracture surface and the associated damage, respectively, with R_o^b and $\bar{\omega}_b$ corresponding to the bound or the limiting size of the limit fracture surface and the associated damage, respectively, as shown in Figure 2.

The function in Equation (49) represents an ellipse in R_o - $\bar{\omega}$ space, where R_o grows monotonically as damage $\bar{\omega}$ increases until it reaches a limiting value R_o^b , which is the size of the bounding limit fracture surface (Figure 1). At this stage

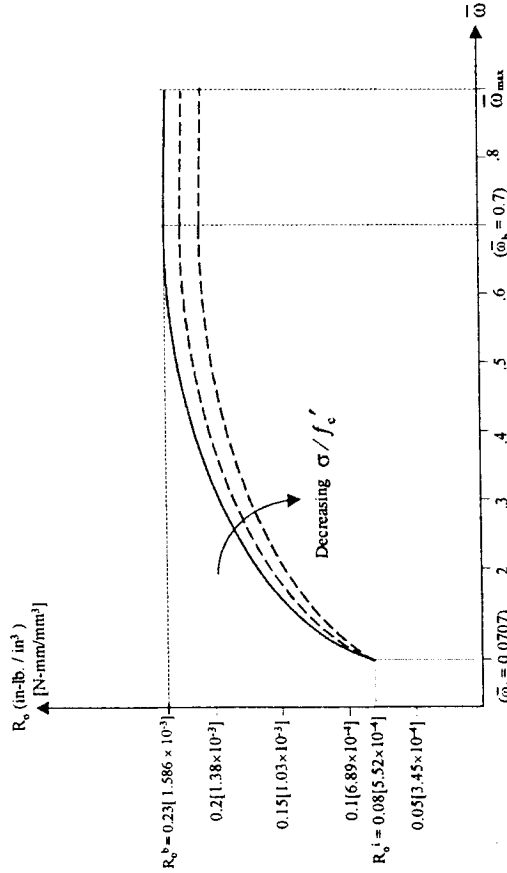


Figure 2. Size evolution of limit fracture surface in R_o - $\bar{\omega}$ space.

the damage will reach $\bar{\omega}_b$, as shown in Figure 2. It is emphasized that the limit fracture surface may reach its bounding surface while the loading surface $f = 0$ may still be remote from its own conjugate bounding surface $F = 0$. Consequently, further damage is deemed to occur at a fixed size of limit fracture surface (R_o^b) until damage reaches its limiting value $\bar{\omega}_m$, and the loading surface $f = 0$ reaches the bounding surface $F = 0$, defining incipient failure.

The experimental results of Suaris et al. [4] indicate that crack initiation in compression occurs at about 40% of the peak stress for the particular case of concrete strength $f'_c = 5600$ psi (38.6 MPa), assuming an inherent initial damage $\omega_i = 0.05$. The strain energy release rate components corresponding to this initial damage and stress level are used in Equation (13) in order to determine the initial size of the limit fracture surface R_o^i . However, the initial size R_o^i tends to increase with concrete strength as damage is initiated at higher levels of threshold stress [22].

Using $\bar{\omega}_i = \sqrt{2} \omega_i = 0.0707$ into Equation (49) yields the size of the limit fracture surface R_o as a function of $\bar{\omega}$:

$$R_o = R_o^i + \left\{ (R_o^b - R_o^i)^2 \left[1 - \frac{(\bar{\omega} - \bar{\omega}_b)^2}{(0.0707 - \bar{\omega}_b)^2} \right] \right\}^{1/2} \quad (50)$$

Equation (50) is a two parameter model in terms of R_o^b and $\bar{\omega}_b$ for describing the size of the limit fracture surface in R_o - $\bar{\omega}$ space. These two parameters remain to be calibrated in accordance with phenomenological data available from cyclic loading of concrete in uniaxial compression.

CALIBRATION OF MODEL PARAMETERS

The developed elasto-damage incremental laws given by Equations (33) and (35) and the associated damage evolution described by Equations (25) and (31) reflect a general form, valid for any monotonic loading state. Certain standard tests are utilized to calibrate the model.

In the model predicting monotonic response, there are basically five parameters that need to be calibrated using experimental data. These parameters are R_o^i (or δ_m), α , β , R_c , and D .

The parameter R_o^i defines the initiation of microcracking which occurs at about 40% of the peak stress under uniaxial compressive loading for concrete with $f'_c = 5600$ psi (38.6 MPa) (Reference [4]). This initiation of damage has been noted to vary with the compressive strength (Reference [22]) where the damage threshold has been noted to be almost 60% of the peak stress for concrete with $f'_c = 11,000$ psi (75.9 MPa). For concrete strength in the range of $3000 \leq f'_c \leq 7000$ psi (20.7 MPa $\leq f'_c \leq 48.3$ MPa), a median value of $R_o^i = 0.08$ in-lb/in³ (5.52×10^{-4} N-mm/m³) was computed; whereas for the range $7000 < f'_c \leq 9700$ psi (48.3 MPa $< f'_c \leq 66.9$ MPa), R_o^i was fixed at 0.16 in-lb/in³ (1.103×10^{-3} N-mm/m³). Once R_o^i is determined, then the corresponding δ_m can be obtained from Equations (16) and (24).

The α parameter is used primarily for matching peak strength of concrete in direct tension and which has been reported in Reference [17]. It has been found that α lies in the range $1.25 \leq \alpha \leq 6.6$ for compressive strength variation from 4000 (27.6) to 17,400 psi (120 MPa). Since the model is assessed through compressive fatigue life of concrete, the value of α its immaterial and it is outside the scope of the work presented herein.

The most important and critical model parameter is β which controls the damage growth rate and influences the pre-peak behavior as well as the level at which the peak stress is attained. Thus the behavior of concretes of varying compressive strength is simulated by different values of β . The variation of β (referred to as β_{aut} for monotonic loading) as a function of concrete strength f'_c is shown in Figure 3, where the higher the concrete strength, the lower is the value of β . This variation guarantees a slower accumulation of damage in order to attain levels of stress commensurate with increasing concrete strength.

The parameter R_c is the critical energy release rate and is the magnitude of the energy release rate vector R_i when the loading surface $f = 0$ reaches the bounding surface $F = 0$. Just as R_o is a function of concrete strength, R_c must also vary with f'_c . However, the introduction of the scaling parameters α and β obviates this requirement and R_c is chosen to be fixed at 1.29 in-lb/in³ (8.896×10^{-3} N-mm/m³).

The parameter D controls the softening phase of material response in σ - ϵ space. In order to simulate sharper softening gradients as depicted by concretes of increasing brittleness and higher strength [23], the variation of D with concrete strength was adopted as shown in Figure 3.

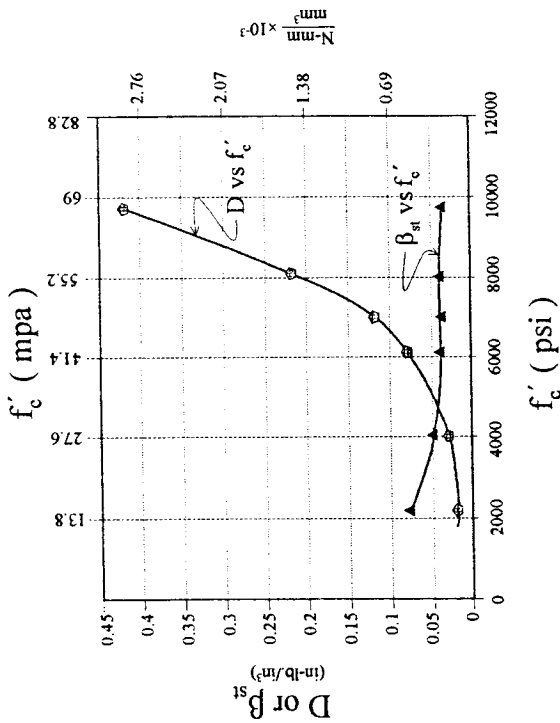


Figure 3. Variation of model parameters β_{st} and D with compressive strength f'_c .

MODEL RESPONSE UNDER MONOTONIC LOADING

In order to simulate the response under monotonic loading, the strain control model on page 68 was coded into a Fortran program. Input for four model parameters (R_o , R_c , β , D) was provided as a function of concrete strength.

Inasmuch as the model presented is elasto-damage, there is no account for any residual or plastic strain. In the σ - ϵ response depicted in Figure 4, a plastic strain component ϵ^p , assumed proportional to the damage strain ϵ^d with a constant of proportionality to be 1.5 (i.e., $\epsilon^p = 1.5 \epsilon^d$), has been superposed as adopted in Reference [4]. The total strain is thus calculated as $\epsilon = \epsilon^{el} + 1.5 \epsilon^d$, where $\epsilon^{el} = \epsilon^e + \epsilon^d$, the sum of elastic and damage strain components.

Predictions for Fatigue Life

The general problem of response of elasto-damage material to a prescribed loading is highly nonlinear. The use of incremental forms for description of state variables such as damage ω_i is necessary in order to describe their evolution. A Fortran 77 program coding has been written, the flow chart of which is shown in Figure 5. The coding computes cumulative damage as the number of cycles N is increased.

The flow chart consists of three main do-loops: the innermost loop accounts for

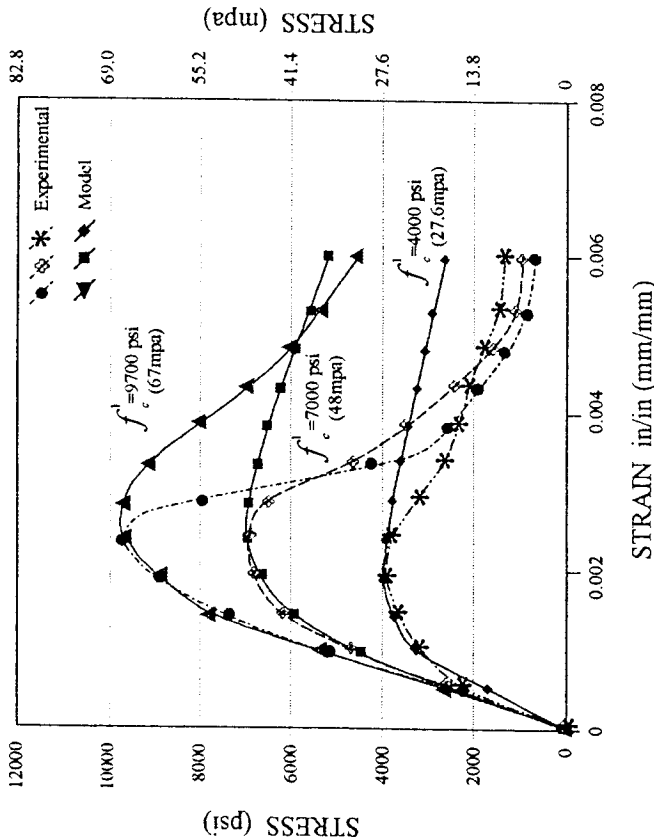


Figure 4. Model response in monotonic, uniaxial compression.

computation of damage increment based on the values of damage and strain energy release rate of previous increment as an initial guess. The iterative procedure is implemented until convergence is attained in terms of a consistent set $\omega_i - R_i$. The intermediate loop is related to the incrementation of stress starting from zero until the prescribed stress level is reached, followed by unloading to the origin; the outermost do-loop monitors the movement of the loading surface as it approaches the bounding surface. The latter effect is shown symbolically in σ - ϵ space in Figure 6.

The evolution of the limit fracture surface has been defined in terms of the parameters R_o^b and $\bar{\omega}_b$. In general these parameters are noted to be functions of the maximum amplitude of cycling load σ/f'_c . For the range of data presented in Reference [21], for concrete with $f'_c = 6100$ psi (42.1 MPa) it was noted that (assuming $\bar{\omega}_b = 0.7 = \text{constant for simplicity}$)

$$R_o^b = .294 \left(\frac{\sigma}{f'_c} \right) - 0.016 \quad 0.6 \leq \frac{\sigma}{f'_c} \leq .87 \quad (51)$$

$$R_o^b = 0.23 \quad \frac{\sigma}{f'_c} \geq .87$$

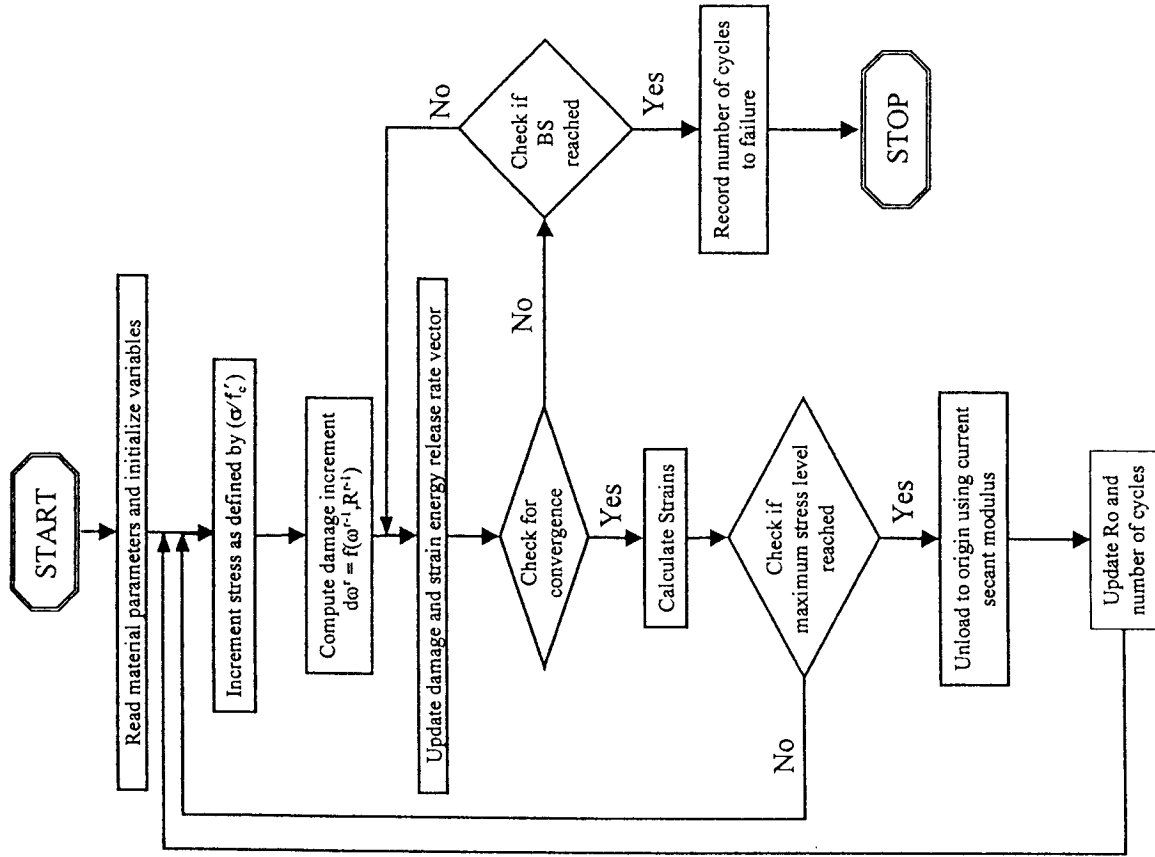


Figure 5. Flow chart for code predicting cycles to failure.

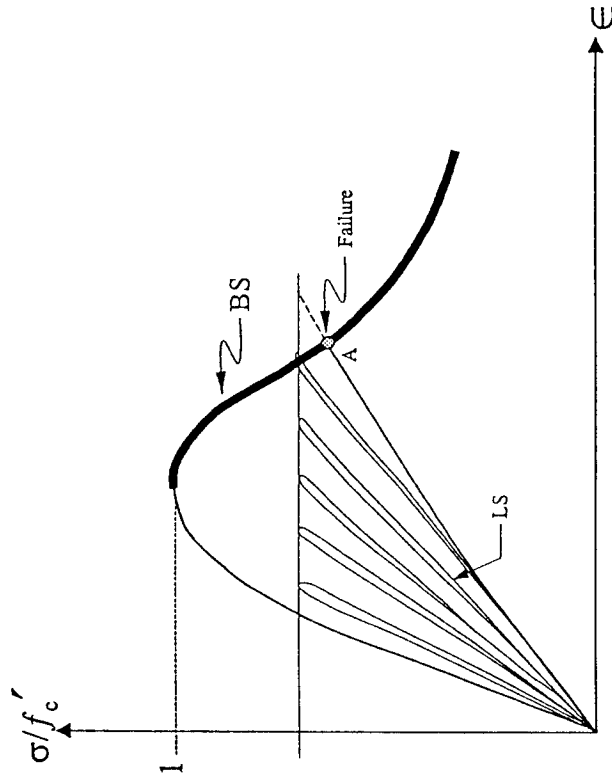


Figure 6. Loading and bounding surfaces in σ - ϵ space.

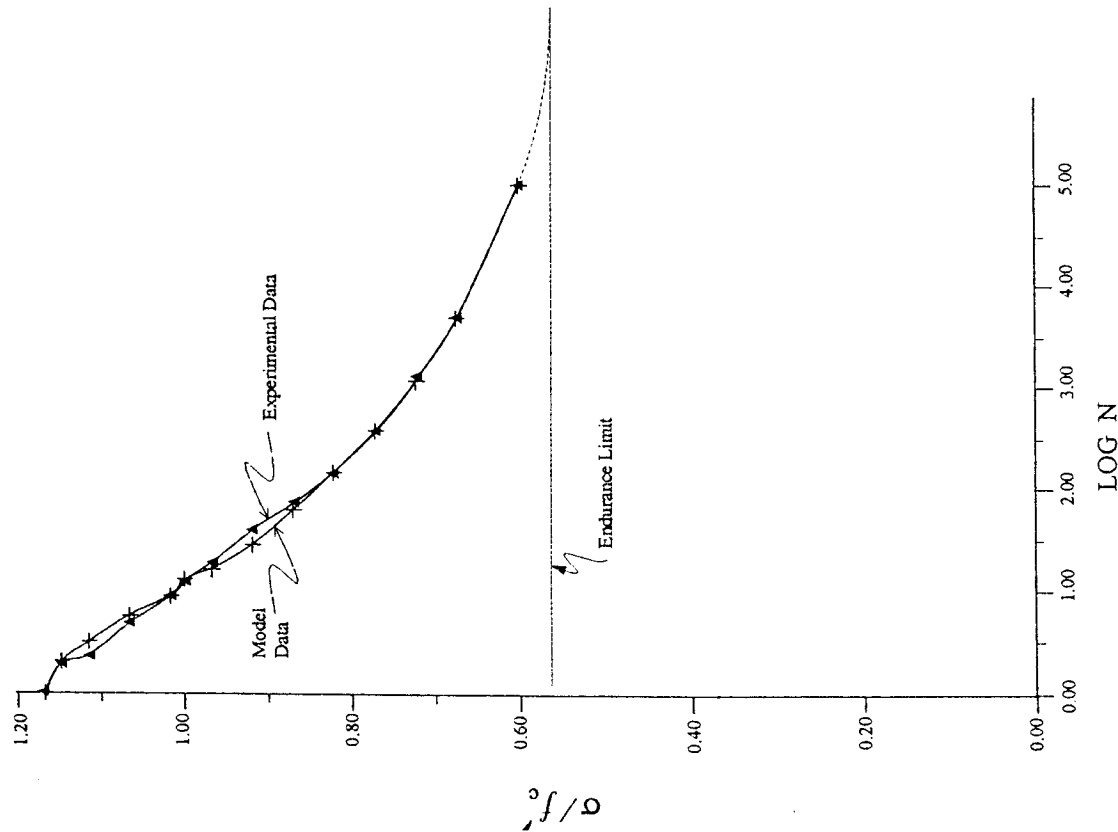


Figure 7. S-N space response of concrete under uniaxial compression.

The movement of the limit fracture surface is noted to be a function of σ/f_c' . The elastic core as defined by $(R_f R_c)^{1/2} \leq R_o$ is constrained so as to decrease with decreasing amplitude of cycling stress, resulting in a corresponding increase in the damage growth zone that allows for greater accumulation of damage. This transition allows for failure to occur at a number of cycles commensurate with experimental findings when cycling at lower σ/f_c' . The elastic core evolution also implies that the system is rendered more elastic or flexible when cycled at high σ/f_c' .

Figure 7 shows results of fourteen cyclic loading tests, conducted numerically, at various levels of σ/f_c' superimposed on an experimental diagram in S-N space for concrete of strength $f_c' = 6100$ psi (42.1 MPa) reported in Reference [4]. The analytical trend of the two parameter fatigue model juxtaposed on the five parameter monotonic model appears to yield predictions which are in close correlation with experimental values.

CONCLUSIONS

A damage evolution law for concrete has been derived with applicability for proportional monotonic loading conditions and the special case of uniaxial compression has been verified. The incremental elasto-damage constitutive equations have been developed not only for stress control but also for strain control, lending the formulation ease in adaptation for finite element application. The multi-surface continuum damage model is shown to simulate stress-strain curves for monotonic loading for a wide range of compressive strength of concrete.

Introduction of a moving limit fracture surface whose size R_o is governed by an elliptical dependence on cumulative damage $\bar{\omega}$ and appending it to the multi-surface continuum damage model is shown to simulate fatigue response of concrete subjected to cyclic loading under compression. The size of the limit fracture surface is noted to have a bound R_o^b which is governed by the amplitude of the cycling stress σ/f_c' . The model is also capable of simulating fatigue life for concrete of other strengths by calibrating the limit fracture surface bound size, R_o^b , with experimental data.

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Three-Dimensional Micromechanical Model for Quasi-Brittle Solids with Residual Strains under Tension

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ABSTRACT: Presented in this paper is a three-dimensional micromechanical model that enables the simulation of microcrack-weakened quasi-brittle materials with permanent residual strains subjected to tension. The microcracking damage of such a material is described by the concept of domain of microcrack growth. It is thought that the occurrence of residual strains is attributable to two reasons, namely, the release of microscopic residual stresses due to microcracking and the microscopic plastic deformation along microcrack front edges. By introducing the analytical results of the two physical mechanisms into the constitutive relation, a micromechanical damage model is established to describe the effective response of microcrack-weakened quasi-brittle materials such as ceramics and concrete under complex loading/unloading paths. The response of such a material is divided into four stages, namely, the stages of linear elasticity, pre-peak nonlinearity, post-peak stress drop, and strain softening before macroscopic fracture.

KEY WORDS: fracture mechanisms; microcracking; microstructures; constitutive behavior; inhomogeneous material; residual strain.

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