SE301: Numerical Methods

Topic 3:

Solution of Systems of Linear Equations

Lectures 12-17:

Dr. Samir Al-Amer
(Term 062)

Read Chapter 9 of the textbook

Lecture 12

Vector, matrices and linear equations

VECTORS

Vector : a one dimensional array of numbers

Examples :

row vector $\begin{bmatrix} 1 & 4 & 2 \end{bmatrix}$ column vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$Identity\ \ vectors\ \ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

MATRICES

Matrix : a two dimensional array of numbers

Examples :

zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\text{diagonal} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}, \quad \text{Tridiagonal} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

MATRICES

Examples:

$$\text{symmetric}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix}, \quad \text{upper triangular}\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Determinant of a MATRICES

Defined for square matrices only

Examples:

$$\det\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix} = 2\begin{vmatrix} 0 & 5 \\ 5 & 4 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 5 & 4 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 0 & 5 \end{vmatrix}$$

$$= 2(-25) - 1(12+5) - 1(15-0) = -82$$

Adding and Multiplying Matrices

The addition of two matrices A and B

- Defined only if they have the same size
- $C = A + B \Leftrightarrow c_{ij} = a_{ij} + b_{ij} \quad \forall i, j$

Multiplication of two matrices $A(n \times m)$ and $B(p \times q)$

- The product $C = AB$ is defined only if $m = p$
- $C = A\,B \Leftrightarrow c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} \quad \forall i, j$

Systems of linear equations

A system of linear equations can be presented in different forms

$$\left.\begin{array}{l} 2x_1 + 4x_2 - 3x_3 = 3 \\ 2.5x_1 - x_2 + 3x_3 = 5 \\ x_1 \qquad\quad - 6x_3 = 7 \end{array}\right\} \Leftrightarrow \begin{bmatrix} 2 & 4 & -3 \\ 2.5 & -1 & 3 \\ 1 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

Standard form Matrix form

Solutions of linear equations

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a solution to the following equations

$$x_1 + x_2 = 3$$
$$x_1 + 2x_2 = 5$$

Solutions of linear equations

- A set of equations is **inconsistent** if there exist no solution to the system of equations

$$x_1 + 2x_2 = 3$$
$$2x_1 + 4x_2 = 5$$

These equations are inconsistent

Solutions of linear equations

- Some systems of equations may have **infinite number of solutions**

$$x_1 + 2x_2 = 3$$

$$2x_1 + 4x_2 = 6$$

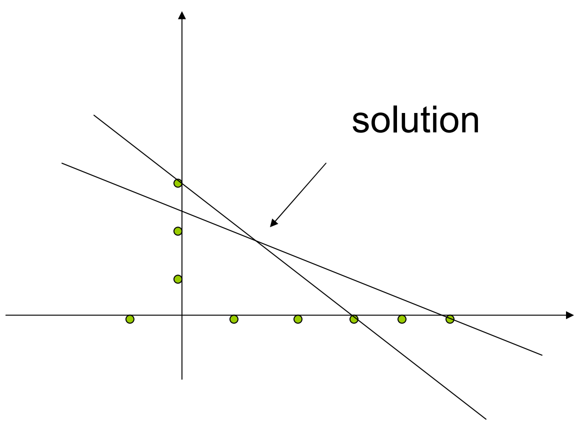
have infinite number of solutions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ 0.5(3-a) \end{bmatrix} \text{ is a solution for all } a$$

Graphical Solution of Systems of Linear Equations

$$x_1 + x_2 = 3$$

$$x_1 + 2x_2 = 5$$



Cramer's Rule is not practical

Cramer's Rule can be used to solve the system

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 1, \quad x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 2$$

Cramer's Rule is not practical for large systems.

A super computer needs 10^{17} years to solve a 30 by 30 system.

It can be used if the determinants are computed in efficient way

Lecture 13:
Naive Gaussian Elimination

- Naive Gaussian Elimination
- Examples

Naive Gaussian Elimination

- The method consists of two steps
  - **Forward Elimination**: the system is reduced to upper triangular form. A sequence of elementary operations is used.
  - **Backward substitution**: Solve the system starting from the last variable.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

Elementary Row operations

- Adding a multiple of one row to another
- Multiply any row by a non-zero constant

Example
Forward Elimination

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

Part 1: Forward Elimination

Step1: Eliminate x_1 from equations 2,3,4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -27 \\ -18 \end{bmatrix}$$

Example
Forward Elimination

Step2: Eliminate x_2 from equations 3,4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -21 \end{bmatrix}$$

Step3: Eliminate x_3 from equation 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Example

Forward Elimination

Summary of the Forward Elimination

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Example

Backward substitution

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

solve for x_4, then solve for x_3,...solve for x_1

$$x_4 = \frac{-3}{-3} = 1, \qquad x_3 = \frac{-9+5}{2} = -2$$

$$x_2 = \frac{-6-2(-2)-2(1)}{-4} = 1, \quad x_1 = \frac{16+2(1)-2(-2)-4(1)}{6} = 3$$

Forward Elimination

To eliminate x_1

$$\left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i1}}{a_{11}}\right) a_{1j} \quad (1 \le j \le n) \\ b_j &\leftarrow b_j - \left(\frac{a_{i1}}{a_{11}}\right) b_1 \end{aligned}\right\} 2 \le i \le n$$

To eliminate x_2

$$\left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i2}}{a_{22}}\right) a_{2j} \quad (2 \le j \le n) \\ b_j &\leftarrow b_j - \left(\frac{a_{i2}}{a_{22}}\right) b_2 \end{aligned}\right\} 3 \le i \le n$$

Forward Elimination

To eliminate x_m

$$\left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{im}}{a_{mm}}\right) a_{mj} \quad (m \le j \le n) \\ b_j &\leftarrow b_j - \left(\frac{a_{im}}{a_{mm}}\right) b_m \end{aligned}\right\} m+1 \le i \le n$$

continue until x_{n-1} is eliminated.

Backward substitution

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n}x_n - a_{n-2,n-1}x_{n-1}}{a_{n-2,n-2}}$$

$$x_m = \frac{b_m - \sum_{j=m+1}^{n} a_{m,j}x_j}{a_{m,m}}$$

Lecture 14:

Naive Gaussian Elimination

- Summary of the Naive Gaussian Elimination
- Example
- How do check a solution
- Problems with Naive Gaussian Elimination
  - Failure due to zero pivot element
  - Error

Naive Gaussian Elimination

- The method consists of two steps
  - **Forward Elimination**: the system is reduced to upper triangular form. A sequence of elementary operations is used.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

  - **Backward substitution**: Solve the system starting from the last variable. Solve for $x_n, x_{n-1}, \ldots x_1$.

Example 1

Solve using Naive Gaussian Elimination :

Part 1 : Forward Elimination ___ Step1 : Eliminate x_1 from equations 2,3

$$\begin{array}{ll} x_1 + 2x_2 + 3x_3 = 8 & eq1 \;\; unchainged \; (pivot\; equation) \\ 2x_1 + 3x_2 + 2x_3 = 10 & eq2 \leftarrow eq2 - \left(\frac{2}{1}\right) eq1 \\ 3x_1 + x_2 + 2x_3 = 7 & eq3 \leftarrow eq3 - \left(\frac{3}{1}\right) eq1 \end{array}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 8 \\ -\; x_2 - 4x_3 &= -6 \\ -5x_2 - 7x_3 &= -17 \end{aligned}$$

Example 1

Part 1: Forward Elimination Step2 : Eliminate x_2 from equations 3

$$x_1 + 2x_2 + 3x_3 = 8 \qquad eq1 \ \ unchainged$$

$$-\ x_2 - 4x_3 = -6 \qquad eq2 \ \ unchainged\ (pivot\ equation)$$

$$-5x_2 - 7x_3 = -17 \qquad eq3 \leftarrow eq3 - \left(\frac{-5}{-1}\right) eq2$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 8 \\ \quad -\ x_2 - 4x_3 = -6 \\ \qquad\quad 13x_3 = 13 \end{cases}$$

Example 1

Backward substitution

$$x_3 = \frac{b_3}{a_{3,3}} = \frac{13}{13} = 1$$

$$x_2 = \frac{b_2 - a_{2,3}x_3}{a_{2,2}} = \frac{-6 + 4x_3}{-1} = 2$$

$$x_1 = \frac{b_1 - a_{1,2}x_2 - a_{1,3}x_3}{a_{1,1}} = \frac{8 - 2x_2 - 3x_3}{a_{1,1}} = 1$$

$$\text{The solution is} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

How do we know if a solution is good or not

- Given AX=B
  - X is a solution if AX-B=0
  - Due to computation error AX-B may not be zero
  - Compute the residuals R=|AX-B|
- One possible test is ?????

$$\text{The solution is acceptable if } \max_i |r_i| \le \varepsilon$$

Determinant

The elementary operations does not affect the determinant

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{\text{Elementary operations}} A' = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 13 \end{bmatrix}$$

$$\det(A) = \det(A') = -13$$

How many solutions does a system of equations AX=B have?

Unique	No solution	infinte
$\det(A) \neq 0$	$\det(A) = 0$	$\det(A) = 0$
reduced matrix has no zero rows	reduced matrix has one or more zero rows corresponding B elements $\neq 0$	reduced matrix has one or more zero rows corresponding B elements $= 0$

Examples

Unique

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
solution:
$$X = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

No solution

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
No solution
$$0 = -1 \ \textit{impossible!}$$

infinte # of solutions

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$
Infinite # solutions
$$X = \begin{bmatrix} \alpha \\ 1 - .5\alpha \end{bmatrix}$$

Lectures 15-16:
Gaussian Elimination with Scaled Partial Pivoting

- Problems with Naive Gaussian Elimination
- Definitions and Initial step
- Forward Elimination
- Backward substitution
- Example

Problems with Naive Gaussian Elimination

- The Naive Gaussian Elimination may fail for very simple cases. (**The pivoting element is zero).**

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- **Very small pivoting element may result in , serious computation errors**

$$\begin{bmatrix} 10^{-10} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 2

Solve the following sytstem using Gaussian Elimination with Scaled Partial Pivoting

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Example 2

Initialization step

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Scale vector:

disregard sign

find largest in magnitude in each row

$$\text{Scale vector} \quad S = \begin{bmatrix} 2 & 4 & 8 & 5 \end{bmatrix}$$

$$\text{Index Vector} \quad L = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Why index vector?

- Index vectors are used because it is much easier to exchange a single index element compared to exchanging the values of a complete row.
- In practical problems with very large N, exchanging the contents of rows may not be practical since they could be stored at different locations.

Example 2

Forward Elimination-- Step 1: eliminate x1

Selection of the pivot equation

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} S = [2 \quad 4 \quad 8 \quad 5] \\ L = [1 \quad 2 \quad 3 \quad 4\,] \end{cases}$$

$$Ratios \ = \left\{ \frac{|a_{l_i,1}|}{S_{l_i}} \ i = 1,2,3,4 \right\} = \left\{ \frac{|1|}{2}, \frac{|3|}{4}, \frac{|5|}{8}, \boxed{\frac{|4|}{5}} \right\} \Rightarrow \text{max corresponds to } l_4$$

equation 4 is the first pivot equation Exchange l_4 and l_1

$L = [4 \quad 2 \quad 3 \quad 1\,]$

Example 2

Forward Elimination-- Step 1: eliminate x1

Update A and B

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \boxed{-1} \end{bmatrix}$$

First pivot equation

$$\Rightarrow \begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & 5.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ \boxed{-1} \end{bmatrix}$$

Example 2

Forward Elimination-- Step 2: eliminate x2

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & 5.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$\text{Ratios}: \left\{ \frac{0.5}{4} \quad \frac{5.5}{8} \quad \boxed{\frac{1.5}{2}} \right\}$$

$$L = [\ 4\ 1\ 3\ 2]$$

Example 2

Forward Elimination-- Step 2: eliminate x2

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & 5.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2 \;\; 4 \;\; 8 \;\; 5\,] \qquad L = [\;\; 4 \quad 2 \quad 3 \quad 1\,]$$

$$\text{Ratios}: \left\{ \frac{|a_{l_i,2}|}{S_{l_i}} \;\; i = 2,3,4 \right\} = \left\{ \frac{0.5}{4} \quad \frac{5.5}{8} \quad \boxed{\frac{1.5}{2}} \right\} \Rightarrow L = [\; 4\; 1 \;\; 3 \;\; 2 \;\;]$$

Example 2

Forward Elimination-- Step 3: eliminate x3

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & \boxed{-2.5 \quad 1.8333} \\ 0 & 0 & 0.25 & 1.6667 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 6.8333 \\ -1 \end{bmatrix}$$

Third pivot equation

$$L = [\; 4\; 1\; 2 \;\; 3]$$

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & -2.5 & 1.8333 \\ 0 & 0 & 0 & 2 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 9 \\ -1 \end{bmatrix}$$

Example 2

Backward substitution

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & -2.5 & 1.8333 \\ 0 & 0 & 0 & 2 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 9 \\ -1 \end{bmatrix} \qquad L = [\ 4\ 1\ 2\ \ 3]$$

$$x_4 = \frac{b_{l_4}}{a_{l_4,4}} = \frac{9}{2} = 4.5, \quad x_3 = \frac{b_{l_3} - a_{l_3,4} x_4}{a_{l_3,3}} = \frac{2.1667 - 1.8333 x_4}{-2.5} = 2.4327$$

$$x_2 = \frac{b_{l_2} - a_{l_2,4} x_4 - a_{l_2,3} x_3}{a_{l_2,2}} = \frac{1.25 - 0.25 x_4 - 0.75 x_3}{-1.5} = 1.1333$$

$$x_1 = \frac{b_{l_1} - a_{l_1,4} x_4 - a_{l_1,3} x_3 - a_{l_1,2} x_2}{a_{l_1,1}} = \frac{-1 - 3x_4 - 5x_3 - 2x_2}{4} = -7.2333$$

Example 3

Solve the following sytstem using Gaussian Elimination with Scaled Partial Pivoting

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Example 3

Initialization step

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\text{Scale vector} \quad S = \begin{bmatrix} 2 & 4 & 8 & 5 \end{bmatrix}$$

$$\text{Index Vector} \quad L = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Example 3

Forward Elimination-- Step 1: eliminate x1

Selection of the pivot equation

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} S = [2 \;\; 4 \;\; 8 \;\; 5] \\ L = [1 \;\; 2 \;\; 3 \;\; 4\,] \end{cases}$$

$$Ratios = \left\{ \frac{|a_{l_i,1}|}{S_{l_i}} \;\; i = 1,2,3,4 \right\} = \left\{ \frac{|1|}{2}, \frac{|3|}{4}, \frac{|5|}{8}, \boxed{\frac{|4|}{5}} \right\} \Rightarrow \text{max corresponds to } l_4$$

equation 4 is the first pivot equation Exchange l_4 *and* l_1

$L = [4 \;\; 2 \;\; 3 \;\; 1\,]$

Example 3

Forward Elimination-- Step 1: eliminate x1

Update A and B

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 3 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

Example 3

Forward Elimination-- Step 2: eliminate x2

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$\text{Ratios}: \left\{ \left|\frac{0.5}{4}\right| \quad \left|\frac{-10.5}{8}\right| \quad \left|\frac{-1.5}{2}\right| \right\} = \left\{ \frac{0.5}{4} \quad \frac{10.5}{8} \quad \frac{1.5}{2} \right\}$$

$$L = [\ 4\ 3\ 2\ 1]$$

Example 3

Forward Elimination-- Step 2: eliminate x2

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2 \;\; 4 \;\; 8 \;\; 5\,] \qquad L = [\;\; 4 \quad 2 \quad 3 \quad 1\;]$$

$$\text{Ratios}: \left\{ \frac{\left|a_{l_i,2}\right|}{S_{l_i}} \;\; i = 2,3,4 \right\} = \left\{ \frac{0.5}{4} \;\; \boxed{\frac{10.5}{8}} \;\; \frac{1.5}{2} \right\} \Rightarrow L = [\;4 \; 3 \; 2 \; 1]$$

Example 3

Forward Elimination-- Step 2: eliminate x2

Updating A and B

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & \boxed{-10.5 \quad -0.25 \quad -0.75} \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$L = [\;4 \; 1 \; 3 \; 2]$$

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

Example 3

Forward Elimination-- Step 3: eliminate x3

Selection of the third pivot equation

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

$S = [2 \ \ 4 \ \ 8 \ \ 5\ ] \qquad L = [\ \ 4 \quad 3 \quad 2 \quad 1\ ]$

$$\text{Ratios}: \left\{ \frac{|a_{l_i,3}|}{S_{l_i}} \ \ i = 3,4 \right\} = \left\{ \boxed{\frac{2.7619}{4}} \quad \frac{0.7857}{2} \right\} \Rightarrow L = [\ 4\ 3\ 2\ 1]$$

Example 3

Forward Elimination-- Step 3: eliminate x3

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

$L = [\ 4\ 3\ 2\ \ 1]$

$$\begin{bmatrix} 0 & 0 & 0 & 0.8448 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.4569 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

Example 2

Backward substitution

$$\begin{bmatrix} 0 & 0 & 0 & 0.8448 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.4569 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix} \qquad L = [\ 4\ 3\ 2\ 1]$$

$$x_4 = \frac{b_{l_4}}{a_{l_4,4}} = \frac{1.4569}{0.8448} = 1.7245, \quad x_3 = \frac{b_{l_3} - a_{l_3,4} x_4}{a_{l_3,3}} = \frac{1.8571 - 1.7143 x_4}{-2.7619} = 0.3980$$

$$x_2 = \frac{b_{l_2} - a_{l_2,4} x_4 - a_{l_2,3} x_3}{a_{l_2,2}} == -0.3469$$

$$x_1 = \frac{b_{l_1} - a_{l_1,4} x_4 - a_{l_1,3} x_3 - a_{l_1,2} x_2}{a_{l_1,1}} = \frac{-1 - 3x_4 - 5x_3 - 2x_2}{4} = -1.8673$$

How good is the solution?

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad solution \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1.8673 \\ -0.3469 \\ 0.3980 \\ 1.7245 \end{bmatrix}$$

$$\text{Residues}: \ R = \begin{bmatrix} 0.005 \\ 0.002 \\ 0.003 \\ 0.001 \end{bmatrix}$$

Remarks:

- We use index vector to avoid the need to move the rows which may not be practical for large problems.
- If you order equation as in the last value of the index vector, we have triangular form.
- Scale vector is formed by taking maximum in magnitude in each row.
- Scale vector do not change.
- The original matrices A and B are used in Checking the residuals.

Lecture 17: Tridiagonal & Banded Systems and Gauss-Jordan Method

- Tridiagonal systems
- diagonal dominance
- Tridiagonal Algorithm
- Examples
- Gauss-Jordan algorithm

Tridigonal Systems

Tridigonal Systems:

- The non-zero elements are in the main diagonal, super diagonal and subdiagoal.
- $a_{ij}=0 \ \text{if} \ |i-j| > 1$

$$\begin{bmatrix} 5 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

Tridigonal Systems

- Occur in many applications
- Needs less storage (4n-2 compared to n^2 +n for the general cased)
- Selection of pivoting rows is unnecessary
  (under some conditions)
- Efficiently solved by Gaussian elimination

Algorithm to solve Tridigonal Systems

- Based on Naive Gaussian elimination.
- As in previous Gaussian elimination algorithms
  - Forward elimination step
  - Backward substitution step
- Elements in the super diagonal are not affected.
- Elements in the main diagonal, and B need updating

Tridiagonal System

We need to update d and a, d and b

$$\begin{bmatrix} d_1 & c_1 & & & \\ a_1 & d_2 & c_2 & & \\ & a_2 & d_3 & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ & & & a_{n-1} & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_5 \end{bmatrix} \Rightarrow \begin{bmatrix} d_1 & c_1 & & & \\ & d_2 & c_2 & & \\ & & d_3 & \ddots & \\ & & & \ddots & c_{n-1} \\ & & & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_5 \end{bmatrix}$$

Diagonal Dominance

A matrix A is diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1,\\ j \neq i}}^{n} |a_{ij}| \qquad for\ (1 \leq i \leq n)$$

The magnitude of each diagonal element is larger than the sum of elements in the corresponding row.

Diagonal Dominance

Examples:

$$\begin{bmatrix} 3 & 0 & 1 \\ 1 & 6 & 1 \\ 1 & 2 & -5 \end{bmatrix} \qquad \begin{bmatrix} -3 & 0 & 1 \\ \boxed{2 \quad 3 \quad 2} \\ \boxed{1 \quad 2 \quad 1} \end{bmatrix}$$

Diagonally dominant Not Diagonally dominant

Diagonally Dominant Tridiagonal System

- A tridiagonal system is diagonally dominant if

$$|d_i| > |c_i| + |a_{i-1}| \qquad (1 \le i \le n)$$

- Forward Elimination preserves diagonal dominance

Solving Tridiagonal System

Forward Elimination

$$d_i \leftarrow d_i - \left(\frac{a_{i-1}}{d_{i-1}}\right) c_{i-1}$$

$$b_i \leftarrow b_i - \left(\frac{a_{i-1}}{d_{i-1}}\right) b_{i-1} \qquad 2 \le i \le n$$

Backward substitution

$$x_n = \frac{b_n}{d_n}$$

$$x_i = \frac{1}{d_i}\left(b_i - c_i x_{i+1}\right) \qquad \text{for } i = n-1, n-2, \ldots, 1$$

Example

Forward Elimination

$$d_i \leftarrow d_i - \left(\frac{a_{i-1}}{d_{i-1}}\right)c_{i-1}$$

$$b_i \leftarrow b_i - \left(\frac{a_{i-1}}{d_{i-1}}\right)b_{i-1} \qquad 2 \le i \le n$$

Backward substitution

$$x_n = \frac{b_n}{d_n}$$

$$x_i = \frac{1}{d_i}\left(b_i - c_i x_{i+1}\right) \qquad \text{for } i = n-1, n-2, \ldots, 1$$

Example

Solve

$$\begin{bmatrix} 5 & 2 & & \\ 1 & 5 & 2 & \\ & 1 & 5 & 2 \\ & & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix} \quad \Rightarrow D = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix}$$

Forward Elimination

$$d_i \leftarrow d_i - \left(\frac{a_{i-1}}{d_{i-1}}\right)c_{i-1}, \quad b_i \leftarrow b_i - \left(\frac{a_{i-1}}{d_{i-1}}\right)b_{i-1} \qquad 2 \le i \le 4$$

Backward substitution

$$x_n = \frac{b_n}{d_n}, \quad x_i = \frac{1}{d_i}\left(b_i - c_i x_{i+1}\right) \qquad \text{for } i = 3,2,1$$

Example

$$D=\begin{bmatrix}5\\5\\5\\5\end{bmatrix}, A=\begin{bmatrix}1\\1\\1\\ \end{bmatrix}, C=\begin{bmatrix}2\\2\\2\\ \end{bmatrix}, B=\begin{bmatrix}12\\9\\8\\6\end{bmatrix}$$

Forward Elimination

$$d_2=d_2-\left(\frac{a_1}{d_1}\right)c_1=5-\frac{1\times2}{5}=4.6,\quad b_2=b_2-\left(\frac{a_1}{d_1}\right)b_1=9-\frac{1\times12}{5}=6.6$$

$$d_3=d_3-\left(\frac{a_2}{d_2}\right)c_2=5-\frac{1\times2}{4.6}=4.5652,\quad b_3=b_3-\left(\frac{a_2}{d_2}\right)b_2=8-\frac{1\times6.6}{4.6}=6.5652$$

$$d_4=d_4-\left(\frac{a_3}{d_3}\right)c_3=5-\frac{1\times2}{4.5652}=4.5619,\quad b_4=b_4-\left(\frac{a_3}{d_3}\right)b_3=6-\frac{1\times6.5652}{4.5652}=4.5619$$

Example
Backward substitution

After the Forward Elimination

$$D^T=[5\quad 4.6\quad 4.5652\quad 4.5619],\ B^T=[12\quad 6.6\quad 6.5652\quad 4.5619]$$

Backward substitution

$$x_4=\frac{b_4}{d_4}=\frac{4.5619}{4.5619}=1,$$

$$x_3=\frac{b_3-c_3x_4}{d_3}=\frac{6.5652-2\times1}{4.5652}=1$$

$$x_2=\frac{b_2-c_2x_3}{d_2}=\frac{6.6-2\times1}{4.6}=1$$

$$x_1=\frac{b_1-c_1x_2}{d_1}=\frac{12-2\times1}{5}=2$$

Gauss-Jordan Method

- The method reduced the general system of equations AX=B to IX=B where I is an identity matrix.
- Only Forward elimination is done and no substitution is needed.
- It has the same problems as Naive Gaussian elimination and can be modified to do partial scaled pivoting.
- It takes 50% more time than Naive Gaussian method.

Gauss-Jordan Method

Example

$$\begin{bmatrix} 2 & -2 & 2 \\ 4 & 2 & -1 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

Step 1 Eleminate x_1 from *equations* 2*and* 3

$$\left.\begin{aligned} eq1 &\leftarrow eq1/2 \\ eq2 &\leftarrow eq2 - \left(\frac{4}{1}\right) eq1 \\ eq3 &\leftarrow eq3 - \left(\frac{2}{1}\right) eq1 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

Gauss-Jordan Method
Example

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

$Step\ \ 2\ \ \text{Eleminate}\ \ \mathrm{x}_2\ \ \text{from}\ \ equations\ 1\ and\ 3$

$$\left.\begin{matrix} eq2 \leftarrow eq2/6 \\ eq1 \leftarrow eq1 - \left(\dfrac{-1}{1}\right) eq2 \\ eq3 \leftarrow eq3 - \left(\dfrac{0}{1}\right) eq2 \end{matrix}\right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0.1667 \\ 0 & 1 & -0.8333 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.1667 \\ 1.1667 \\ 2 \end{bmatrix}$$

Gauss-Jordan Method
Example

$$\begin{bmatrix} 1 & 0 & 0.1667 \\ 0 & 1 & -0.8333 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.1667 \\ 1.1667 \\ 2 \end{bmatrix}$$

$Step\ \ 3\ \ \text{Eleminate}\ \ \mathrm{x}_3\ \ \text{from}\ \ equations\ 1\ and\ 2$

$$\left.\begin{matrix} eq3 \leftarrow eq3/2 \\ eq1 \leftarrow eq1 - \left(\dfrac{0.1667}{1}\right) eq3 \\ eq2 \leftarrow eq2 - \left(\dfrac{-0.8333}{1}\right) eq3 \end{matrix}\right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Gauss-Jordan Method

Example

$$\begin{bmatrix} 2 & -2 & 2 \\ 4 & 2 & -1 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

$$\textit{is transformed to}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \Rightarrow \textit{solution is} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- Check WebCT for HW problems and due date
- First Major Exam covers Topics 1,2 and 3
- No formula sheet is allowed.
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