SE207 Modeling and Simulation Unit 4

Standard Forms for System Models

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Term 072

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Lesson 1: State Variable Models

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Read Section 3.1

Outlines

- Two standard forms for models will be used.
  - State variable model
  - Input-output model

States (state Variables)

- For each dynamical system, there exist a set of variables (states) such that if the values of the states are known an any reference time t_0 and if the input is known for $t \geq t_0$ then the output and the states can be determined for $t \geq t_0$

States (State Variables)

- States are not unique.
- The value of states summarize effect of past inputs on the system.
- Usually the number of states is equal to the number of energy storing elements (Masses and springs)
- In some cases we may have redundant states (the number of states is less than the number of energy storing elements)

State Variable Models

Example (1-input,1-output and 2-states)

$$\dot{q}_1 = 2q_1 + 3q_2 \quad + b_1 u$$

$$\dot{q}_2 = q_1 + 0.5q_2 \quad + b_2 u$$

$$y = q_1 + 0.5u$$

- State Equations
- Output Equation

- u: input,
- y: output,
- q_i :states (or state variables)

State Variable Models

$$\dot{q}_1 = a_{11}q_1 + a_{12}q_2 + a_{13}q_3 \quad + b_{11}u_1 + b_{12}u_2$$
$$\dot{q}_2 = a_{21}q_1 + a_{22}q_2 + a_{23}q_3 \quad + b_{21}u_1 + b_{22}u_2$$
$$\dot{q}_3 = a_{31}q_1 + a_{32}q_2 + a_{33}q_3 \quad + b_{31}u_1 + b_{32}u_2$$

State Equations

$$y_1 = c_{11}q_1 + c_{12}q_2 + c_{13}q_3 \quad + d_{11}u_1 + d_{12}u_2$$
$$y_2 = c_{21}q_1 + c_{22}q_2 + c_{23}q_3 \quad + d_{21}u_1 + d_{22}u_2$$

Output Equations

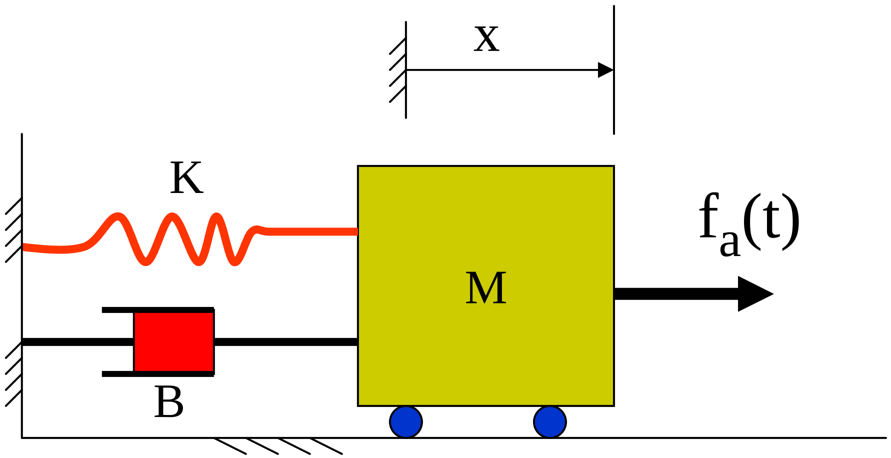
- State Equations: set of first order ODEs.
  - The derivative of a state = algebraic function of states and inputs
- Output Equation:
  - Output = algebraic function of states and inputs

Typical Choice of the States

Typical choice of states for translational Mechanical systems

- the number of states is equal to the number of energy storing elements (Masses and springs)
- Some times we have less than that number
- **Select velocity of a mass as a state**
- **Select elongation of a spring as a state**
- States are not unique.

Example 1



Input: fa(t)

Outputs :

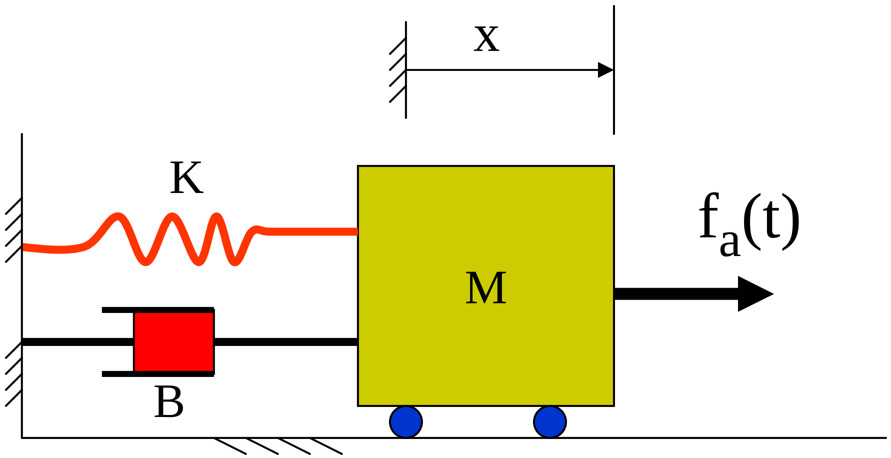
- **Tensile force in spring**
- **Velocity of mass**
- **Acceleration of mass**

# of states = 2

Velocity of mass v

Elongation of spring x (other choices are possible)

Example



$$f_k = K\,x$$

$$f_B = B\,v$$

$$f_I = M\dot{v}$$

$$K\,x + B\,v + M\,\dot{v} = f_a(t)$$

State Equations:

$$\dot{x} = v$$

$$\dot{v} = \frac{1}{M}\left[-Bv - Kx + f_a(t)\right]$$

Output Equation:

$$y_1 = kx$$

$$y_2 = v$$

$$y_3 = \frac{-B}{M}v - \frac{-K}{M}x + \frac{1}{M}f_a(t)$$

Example
State Variable Model

$$\dot{x} = v$$

$$\dot{v} = \frac{1}{M}\left[-Bv - Kx + f_a(t)\right]$$

State equations

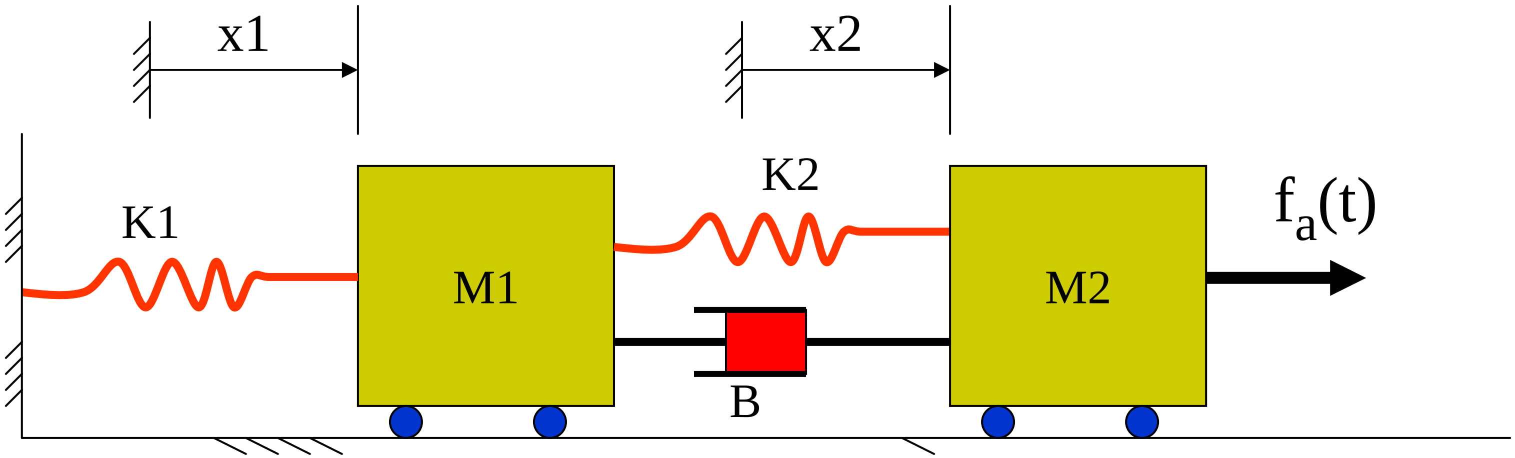
$$y_1 = kx$$

$$y_2 = v$$

$$y_3 = \frac{-B}{M}v - \frac{K}{M}x + \frac{1}{M}f_a(t)$$

Output equations

Example 2

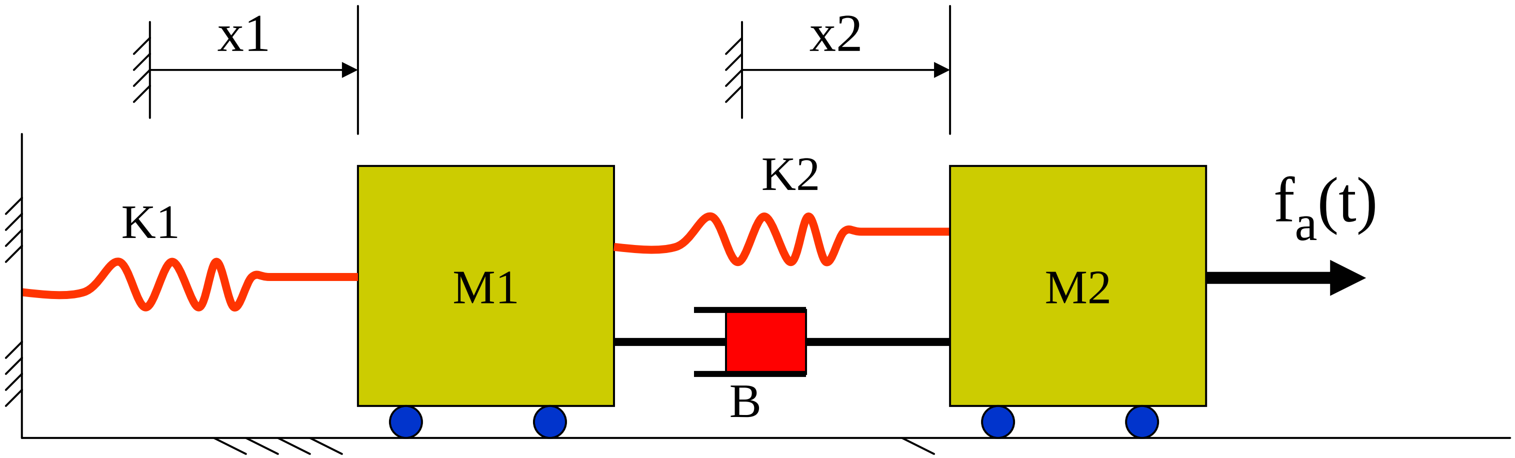


Input: fa(t)

Outputs: Tensile force in spring 2

Total momentum of the masses

Example 2



# of States = # of energy storing elements =4 (2 springs & 2 masses)

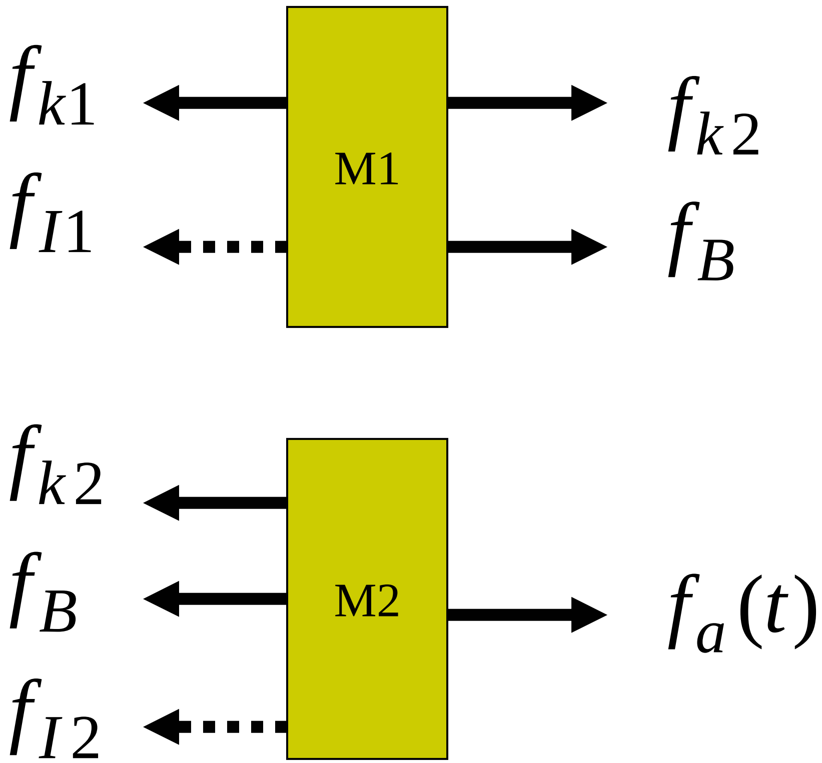
One possible choice of states : x1,x2,v1,v2

Elongation of spring 1= x1; Elongation of spring 2= x2-x1

Velocity of mass 1 =v1; Velocity of mass 2 =v2;

Example 2

Draw freebody diagrams

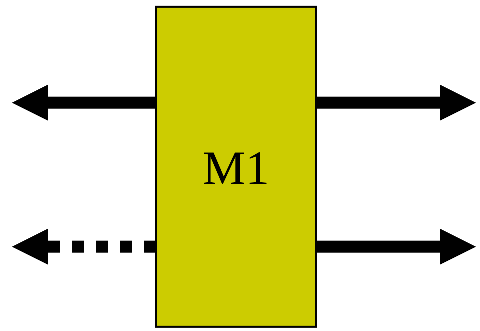


Example 2

Express all forces in terms of states (x1,x2,v1,v2)

$$f_{k1} = K_1 x_1$$

$$f_{I1} = M_1 \dot{v}_1$$



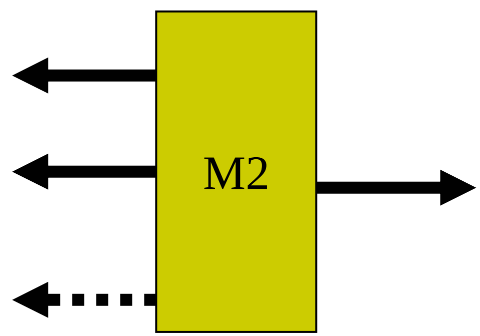
$$f_{k2} = K_2 (x_2 - x_1)$$

$$f_B = B(v_2 - v_1)$$

$$f_{k2} = K_2 (x_2 - x_1)$$

$$f_B = B(v_2 - v_1)$$

$$f_{I2} = M_2 \dot{v}_2$$



$$f_a(t)$$

Example

State Variable Model

$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = \frac{1}{M_1}\left[-(K_1 + K_2)x_1 - Bv_1 + K_2 x_2 + Bv_2\right]$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_1 = \frac{1}{M_2}\left[K_2 x_1 + Bv_1 - K_2 x_2 - Bv_2 + f_a(t)\right]$$

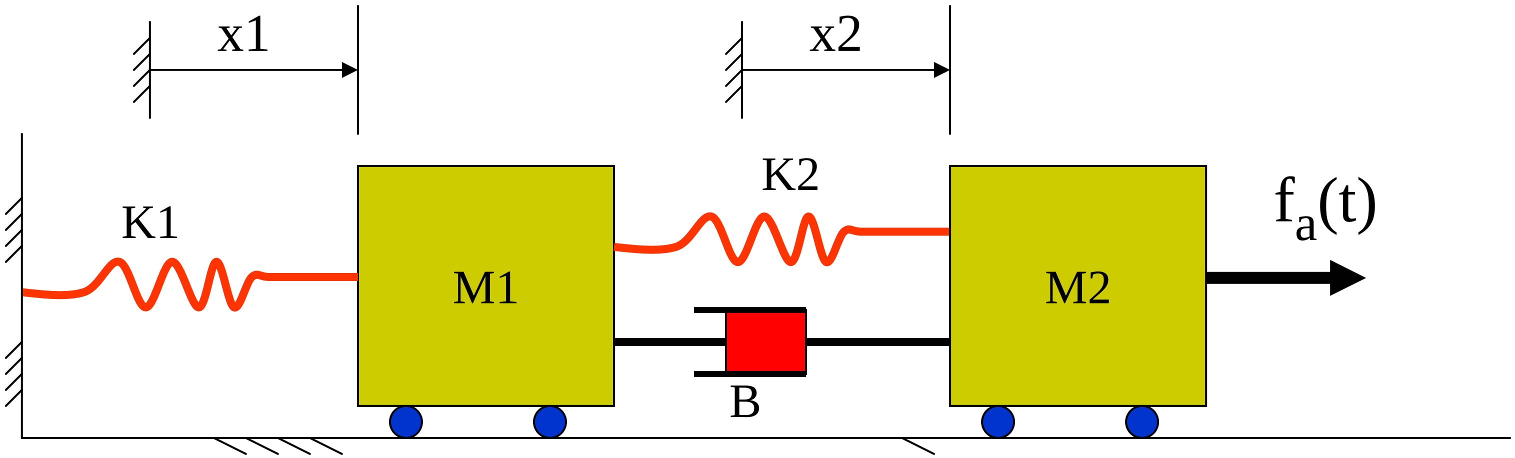
State equations

$$y_1 = K_2(x_2 - x_1)$$

$$y_2 = M_1 v_1 + M_2 v_2$$

Output equations

Example 2
Alternative choice of states



# of States = # of energy storing elements =4 (2 springs & 2 masses)

Alternative choice One possible choice of states : x1,xr=x2-x1 ,v1,vr=v2-v1

Elongation of spring 1= x1; Elongation of spring 2= xr

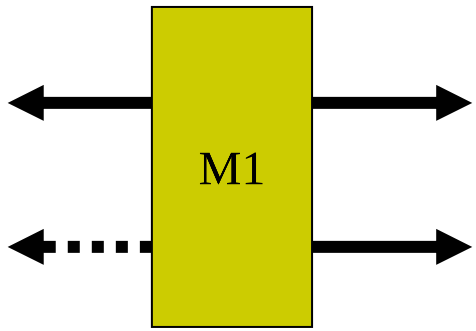
Velocity of mass 1 =v1; Velocity of mass 2 =vr+v1;

Example 2

Express all forces in terms of states (x1,x2,v1,v2)

$$f_{k1} = K_1 x_1$$

$$f_{I1} = M_1 \dot{v}_1$$



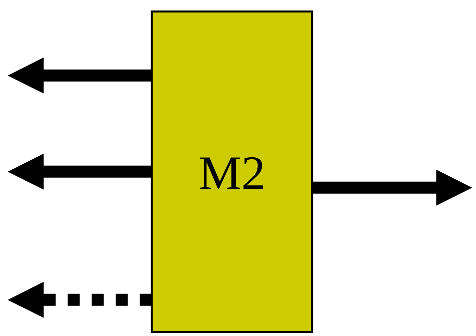
$$f_{k2} = K_2 x_R$$

$$f_B = B v_R$$

$$f_{k2} = K_2 x_R$$

$$f_B = B v_R$$

$$f_{I2} = M_2 (\dot{v}_2 + \dot{v}_R)$$



$$f_a(t)$$

Example

State Variable Model

$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = \frac{1}{M_1}\left[-K_1 x_1 + K_2 x_R + B v_R\right]$$

$$\dot{x}_R = v_R$$

$$\dot{v}_R = \frac{1}{M_1 M_2}\left[K_1 M_2 x_1 - K_2 (M_1 + M_2) x_R - B(M_1 + M_2) v_R + M_1 f_a(t)\right]$$

State equations

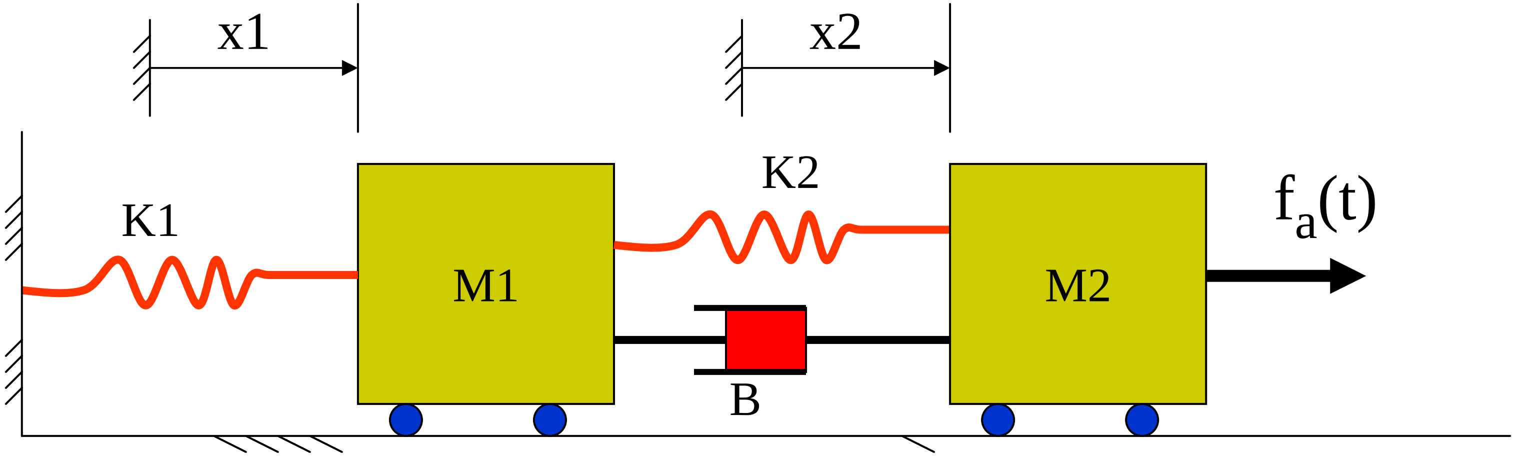
$$y_1 = K_2 x_R$$

$$y_2 = (M_1 + M_2) v_1 + M_2 v_R$$

Output equations

Example 2

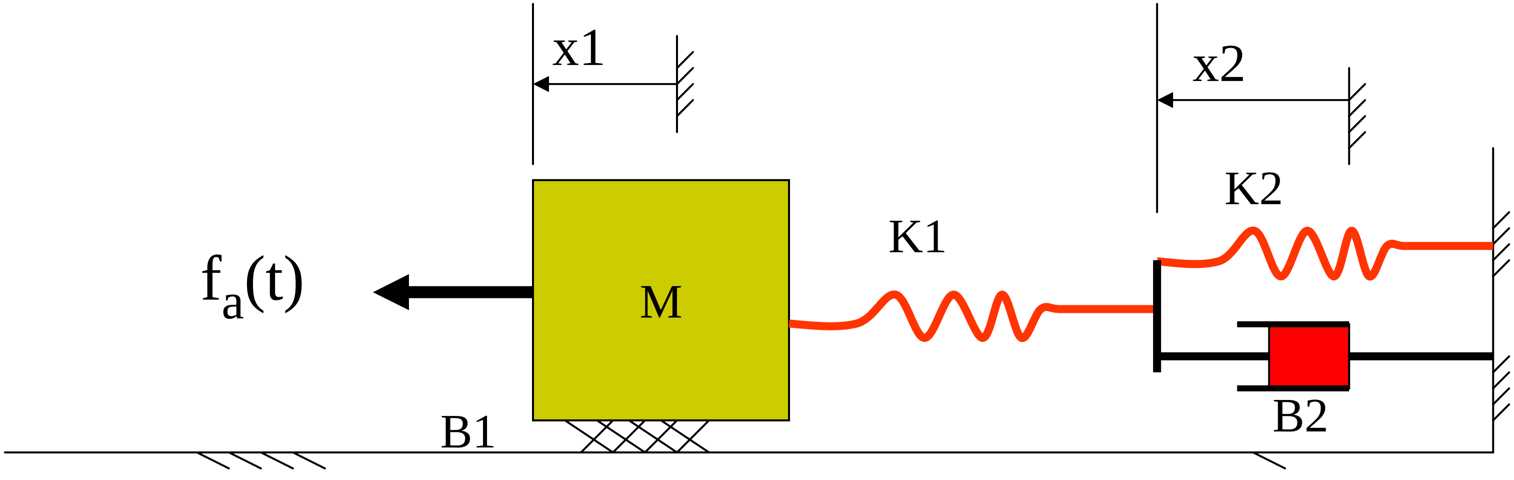
Two possible state variable models for the system



Two state variable models were obtained for the same system

They are different but they represent the same relationship between inputs and outputs.

Example 3

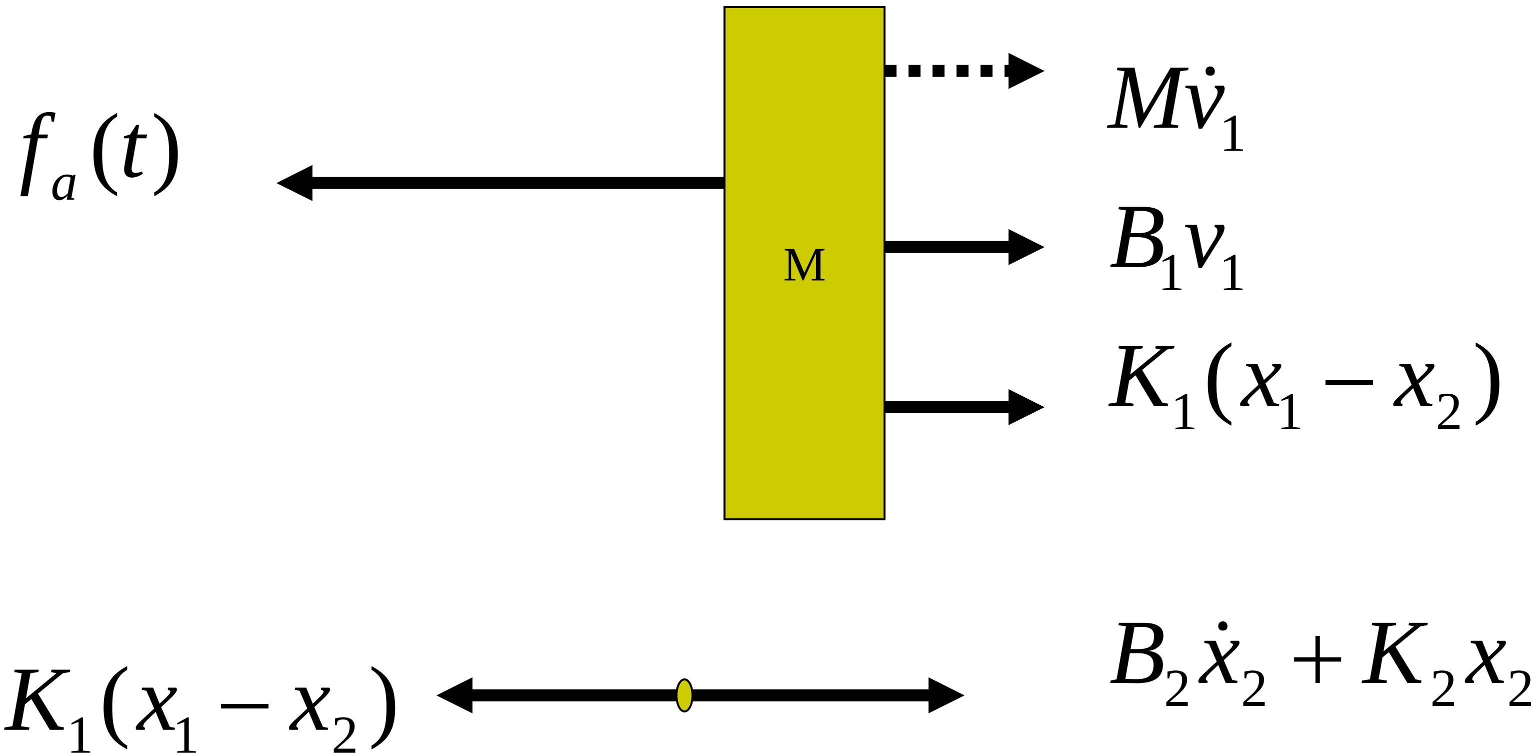


Input fa(t)

Output : displacement of the mass-less junction

Example 2

Express all forces in terms of states (x1,x2,v1,v2)



Example

State Variable Model

$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = \frac{1}{M}\left[-K_1 x_1 - B_1 v_1 + K_1 x_2 + f_a(t)\right]$$

$$\dot{x}_2 = \frac{1}{B_2}\left[K_1 x_1 - K_2 x_2 - K_1 x_2\right]$$

State equations

$$y_1 = x_2$$

Output equations

State Variable Models

- States are not unique.
- The value of states summarize effect of past inputs on the system.
- Usually the number of states is equal to the number of energy storing elements (Masses and springs)
- In some cases we may have redundant states (the number of states is less than the number of energy storing elements)
- Typical choice of states: velocity of mass and elongation of spring.
- State variable model:
  - State equations (set of first order ODE, involves states and inputs)
  - Output equations (algebraic equations , involves states and inputs)

SE207 Modeling and Simulation Unit 4

Lesson 2: Input-Output (I/O) Models

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Read Section 3.2

Outlines

- Two standard forms for models will be used.
  - State variable model
  - Input-output model

States (State Variables)

- States are not unique.
- The value of states summarize effect of past inputs on the system.
- Usually the number of states is equal to the number of energy storing elements (Masses and springs)
- In some cases we may have redundant states (the number of states is less than the number of energy storing elements)

State Variable Models

Example (1-input,1-output and 2-states)

$$\dot{q}_1 = 2q_1 + 3q_2 \quad + b_1 u$$
$$\dot{q}_2 = q_1 + 0.5q_2 \quad + b_2 u$$

$$y = q_1 + 0.5u$$

- State Equations
- Output Equation

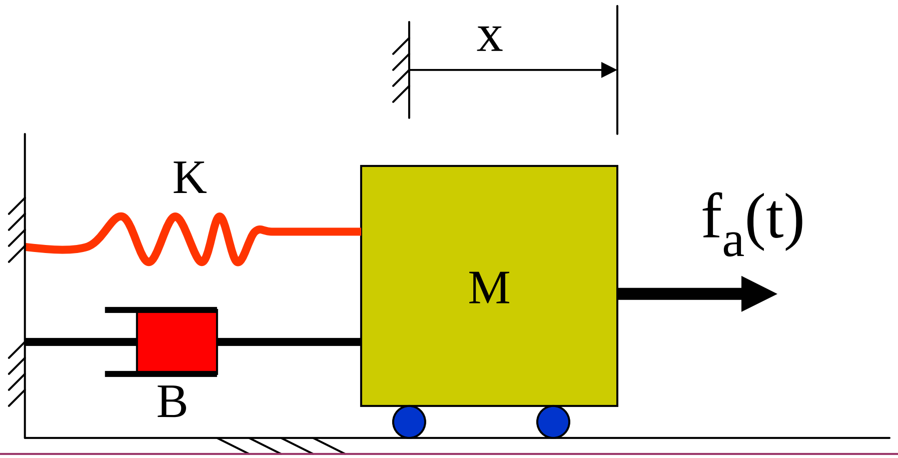
- u: input,
- y: output,
- q_i :states (or state variables)

Input-Output Models

- ODE that contains the **inputs**, **the output** and **their derivatives** and no other variables.
- $$M\ddot{x} + Kx + B\dot{x} = f_a(t)$$

Output and its derivatives   input

State Variable and Input-output Models



Input: fa(t)

Output : displacement *x*

$$\dot{x} = v$$

$$\dot{v} = \frac{1}{M}\left[-Bv - Kx + f_a(t)\right]$$

$$y = x$$

State Variable Model

$$M\ddot{x} + Kx + B\dot{x} = f_a(t)$$

input-output model

Example
Input-Output Models

$$\dot{x} = v$$

$$\dot{v} = \frac{1}{M}\left[-Bv - Kx + f_a(t)\right]$$

$$y_1 = x$$

State variable model

$$M\ddot{x} + Kx + B\dot{x} = f_a(t)$$

Input-Output Model

Example
Input-Output Models

For simple problems, you may be able to obtain the I/O model directly by expressing all forces (in the free body diagram) in terms of inputs, outputs and their derivatives.

Input-Output Models

$$M\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - K_1x_2 = f_a(t)$$

$$B_2\dot{x}_2 + (K_1 + K_1)x_2 - K_1x_2 = 0$$

$$input \quad : \quad f_a(t)$$

$$output : \quad x_1$$

Eliminate $x_2, \dot{x}_2$

Input-Output Models

$$M\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - K_1x_2 = f_a(t)$$

$$B_2\dot{x}_2 + (K_1 + K_1)x_2 - K_1x_2 = 0$$

$$solve\ \ for\ \ x_2 \Rightarrow x_2 = \left(\frac{1}{K_1}\right)\left(M\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - f_a(t)\right)$$

$$differentiate\ \ x_2 \Rightarrow \dot{x}_2 = \left(\frac{1}{K_1}\right)\left(M\dddot{x}_1 + B_1\ddot{x}_1 + K_1\dot{x}_1 - \dot{f}_a(t)\right)$$

$$\text{subsitute}\ \ x_2\ \ and\ \ \dot{x}_2\ \ \text{and simplify to get input} - \text{output model}$$

$$MB_2\dddot{x}_1 + \left(B_1B_2 + K_1M + K_2M\right)\ddot{x}_1 + \left(B_2K_1 + K_1B_1 + K_2B_1\right)\dot{x}_1$$

$$+ K_1K_2x_1 = B_2\dot{f}_a + (K_1 + K_2)f_a$$

Input-Output Models

$$MB_2\dddot{x}_1 + (B_1B_2 + K_1M + K_2M)\ddot{x}_1 + (B_2K_1 + K_1B_1 + K_2B_1)\dot{x}_1$$
$$+ K_1K_2x_1 = B_2\dot{f}_a + (K_1 + K_2)f_a$$

This equation involves the input and its derivatives $(\dddot{x}_1, \ddot{x}_1, \dot{x}_1, x_1)$

and the output and its derivatives $(\dot{f}_a, f_a)$

and no other variables

Reduction using differentiator operator

Define p : differentiator operator

px means $\frac{d}{dt}x$ *and* p^2x means $\frac{d^2}{dt^2}x$

$\left(p^2+2p+4\right)x$ means $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+4x$

Apply the p operator to

$2\ddot{x}_1+3\dot{x}_1+5x_1=f_a(t)$

$2p^2x+3px+5x=f_a$ *or* $(2p^2+3p+5)x=f_a$

Reduction using differentiator operator

The differentiator operator converts the differential equations into algebraic equations in the p operator which are easier to manipulate.

$$2\ddot{x}_1 + 3\dot{x}_1 + 5x_1 = f_a(t)$$

$$\Downarrow$$

$$2p^2x + 3px + 5x = f_a$$

Input-Output Models

$$M\ddot{x}_1 + B_1\dot{x}_1 + K_1 x_1 - K_1 x_2 = f_a(t)$$

$$B_2\dot{x}_2 + (K_1 + K_1)x_2 - K_1 x_2 = 0$$

$$input \quad : \quad f_a(t), \qquad output: \quad x_1$$

$$Mp^2 x_1 + B_1 p x_1 + K_1 x_1 - K_1 x_2 = f_a(t)$$

$$B_2 p\, x_2 + (K_1 + K_1)x_2 - K_1 x_2 = 0$$

$$\Rightarrow \left(B_2 p + K_1 + K_1\right)x_2 - K_1 x_1 = 0$$

$$\Rightarrow M\,(Mp^2 x_1 + B_1 p + K_1)x_1 - K_1 x_2 = f_a(t)$$

Input-Output Models

$$M\ddot{x}_1 + B_1\dot{x}_1 + K_1 x_1 - K_1 x_2 = f_a(t)$$

$$B_2\dot{x}_2 + (K_1 + K_1)x_2 - K_1 x_2 = 0$$

$$input \;:\quad f_a(t), \qquad output : \quad x_1$$

Remove $x_2, \dot{x}_2$

The final equation contains f_a, x_1 and their derivatives only

Input-Output Models

To eliminate $x_2, \ \dot{x}_2$

$$\left(B_2 p + K_1 + K_1\right) x_2 = K_1 x_1$$

$$muliply \ both \ sides \ by \ \left(B_2 p + K_1 + K_1\right)$$

$$\left(B_2 p + K_1 + K_1\right) M \left(Mp^2 x_1 + B_1 p + K_1\right) x_1 - \left(B_2 p + K_1 + K_1\right) K_1 x_2$$

$$= \left(B_2 p + K_1 + K_1\right) f_a(t)$$

$$Now \ \ replace \left(B_2 p + K_1 + K_1\right) x_2 \ \ by \ \ K_1 x_1$$

$$M \left(B_2 p + K_1 + K_1\right) \left(Mp^2 x_1 + B_1 p + K_1\right) x_1 - K_1 x_1$$

$$= \left(B_2 p + K_1 + K_1\right) f_a(t)$$

Input-Output Models

Replace p operator by $\dfrac{d}{dt}$ and simplify

$$MB_2\dddot{x}_1 + \left(B_1B_2 + K_1M + K_2M\right)\ddot{x}_1$$

$$+ \left(B_2K_1 + K_1B_1 + K_2B_1\right)\dot{x}_1 + K_1K_2x_1$$

$$= B_2\dot{f}_a + (K_1 + K_2)f_a$$

State space versus input/output models

- The order of the I/O model = number of states (unless some states are redundant

  or some states have no effect on output)
- State variable models are more convenient for solution and for modeling MIMO (multi-input-multi-output) systems