SE207 Modeling and Simulation
Unit 4
Standard Forms for System Models

Dr. Samir Al-Amer
Term 072
SE207 Modeling and Simulation
Unit 4
Lesson 1: State Variable Models

Dr. Samir Al-Amer
Term 072

Read Section 3.1
Outlines

- Two standard forms for models will be used.
  - State variable model
  - Input-output model
States (state Variables)

- For each dynamical system, there exist a set of variables (states) such that if the values of the states are known at any reference time \( t_0 \) and if the input is known for \( t \geq t_0 \) then the output and the states can be determined for \( t \geq t_0 \).
States (State Variables)

- States are not unique.
- The value of states summarize effect of past inputs on the system.
- Usually the number of states is equal to the number of energy storing elements (Masses and springs)
- In some cases we may have redundant states (the number of states is less than the number of energy storing elements)
State Variable Models

Example (1-input, 1-output and 2-states)

\[ \begin{align*}
\dot{q}_1 &= 2q_1 + 3q_2 + b_1 u \\
\dot{q}_2 &= q_1 + 0.5q_2 + b_2 u \\
y &= q_1 + 0.5u
\end{align*} \]

- **State Equations**
- **Output Equation**

- **u**: input,
- **y**: output,
- **\( q_i \)**: states (or state variables)
State Variable Models

\[ \dot{q}_1 = a_{11} q_1 + a_{12} q_2 + a_{13} q_3 + b_{11} u_1 + b_{12} u_2 \]
\[ \dot{q}_2 = a_{21} q_1 + a_{22} q_2 + a_{23} q_3 + b_{21} u_1 + b_{22} u_2 \]
\[ \dot{q}_3 = a_{31} q_1 + a_{32} q_2 + a_{33} q_3 + b_{31} u_1 + b_{32} u_2 \]

\[ y_1 = c_{11} q_1 + c_{12} q_2 + c_{13} q_3 + d_{11} u_1 + d_{12} u_2 \]
\[ y_2 = c_{21} q_1 + c_{22} q_2 + c_{23} q_3 + d_{21} u_1 + d_{22} u_2 \]

- State Equations: set of first order ODEs.
  - The derivative of a state = algebraic function of states and inputs

- Output Equation:
  - Output = algebraic function of states and inputs
Typical Choice of the States

Typical choice of states for translational Mechanical systems

- the number of states is equal to the number of energy storing elements (Masses and springs)
- Some times we have less than that number
- **Select velocity of a mass as a state**
- **Select elongation of a spring as a state**
- States are not unique.
Example 1

Input: \( f_a(t) \)

Outputs:
- Tensile force in spring
- Velocity of mass
- Acceleration of mass

# of states = 2

Velocity of mass \( v \)

Elongation of spring \( x \) (other choices are possible)
Example

State Equations:
\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= \frac{1}{M} \left[ -Bv - Kx + f_a(t) \right]
\end{align*}
\]

Output Equation:
\[
\begin{align*}
y_1 &= kx \\
y_2 &= v \\
y_3 &= -\frac{B}{M}v - \frac{K}{M}x + \frac{1}{M}f_a(t)
\end{align*}
\]

\[
f_k = Kx \\
f_B = Bv \\
f_I = M\dot{v}
\]

\[
Kx + Bv + M\dot{v} = f_a(t)
\]
Example
State Variable Model

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= \frac{1}{M} \left[-B v - K x + f_a(t)\right] \\
y_1 &= k x \\
y_2 &= v \\
y_3 &= \frac{-B}{M} v - \frac{K}{M} x + \frac{1}{M} f_a(t)
\end{align*}
\]
Example 2

Input: \( f_a(t) \)

Outputs: Tensile force in spring 2
          Total momentum of the masses
Example 2

# of States = # of energy storing elements = 4  (2 springs & 2 masses)

One possible choice of states: x1, x2, v1, v2

Elongation of spring 1 = x1;   Elongation of spring 2 = x2 - x1

Velocity of mass 1 = v1;   Velocity of mass 2 = v2;
Example 2
Draw freebody diagrams

\[ f_{k1} \quad \quad \quad \quad f_{k2} \]
\[ f_{I1} \quad \quad \quad \quad f_B \]
\[ f_{k2} \quad \quad \quad \quad f_{a(t)} \]
\[ f_B \quad \quad \quad \quad f_{I2} \]
Example 2
Express all forces in terms of states \((x_1,x_2,v_1,v_2)\)

\[
\begin{align*}
  f_{k1} &= K_1 x_1 \\
  f_{I1} &= M_1 \dot{v}_1 \\
  f_{k2} &= K_2 (x_2 - x_1) \\
  f_B &= B (v_2 - v_1) \\
  f_{I2} &= M_2 \dot{v}_2 \\
  f_B &= B (v_2 - v_1) \\
  f_a &= f_a(t)
\end{align*}
\]
Example
State Variable Model

\[ \dot{x}_1 = v_1 \]
\[ \dot{v}_1 = \frac{1}{M_1} \left[ -(K_1 + K_2)x_1 - Bv_1 + K_2x_2 + Bv_2 \right] \]
\[ \dot{x}_2 = v_2 \]
\[ \dot{v}_1 = \frac{1}{M_2} \left[ K_2x_1 + Bv_1 - K_2x_2 - Bv_2 + f_a(t) \right] \]
\[ y_1 = K_2(x_2 - x_1) \]
\[ y_2 = M_1 v_1 + M_2 v_2 \]
Example 2
Alternative choice of states

# of States = # of energy storing elements = 4 (2 springs & 2 masses)

Alternative choice One possible choice of states: $x_1, x_r = x_2 - x_1, v_1, v_r = v_2 - v_1$

Elongation of spring 1 = $x_1$; Elongation of spring 2 = $x_r$

Velocity of mass 1 = $v_1$; Velocity of mass 2 = $v_r + v_1$;
Example 2
Express all forces in terms of states \((x_1,x_2,v_1,v_2)\)

\[
\begin{align*}
f_{k1} &= K_1 x_1 \\
f_{I1} &= M_1 \dot{v}_1 \\
f_{k2} &= K_2 x_R \\
f_B &= B v_R \\
f_{I2} &= M_2 (\dot{v}_2 + \dot{v}_R) \\
\end{align*}
\]
Example
State Variable Model

\[ \dot{x}_1 = v_1 \]

\[ \dot{v}_1 = \frac{1}{M_1} \left[ -K_1 x_1 + K_2 x_R + B v_R \right] \]

\[ \dot{x}_R = v_R \]

\[ \dot{v}_R = \frac{1}{M_1 M_2} \left[ K_1 M_2 x_1 - K_2 (M_1 + M_2) x_R - B (M_1 + M_2) v_R + M_1 f_a (t) \right] \]

\[ y_1 = K_2 x_R \]

\[ y_2 = (M_1 + M_2) v_1 + M_2 v_R \]
Example 2

Two possible state variable models for the system

Two state variable models were obtained for the same system

They are different but they represent the same relationship between inputs and outputs.
Example 3

Input $f_a(t)$

Output: displacement of the mass-less junction
Example 2
Express all forces in terms of states \((x_1, x_2, v_1, v_2)\)

\[
f_a(t) \quad \begin{cases} 
    \begin{array}{c}
    \rightarrow Mv_1 \\
    \rightarrow B_1v_1 \\
    \rightarrow K_1(x_1 - x_2)
    \end{array}
\end{cases}
\]

\[
K_1(x_1 - x_2) \quad \begin{cases} 
    \begin{array}{c}
    \rightarrow B_2\ddot{x}_2 + K_2x_2
    \end{array}
\end{cases}
\]
Example
State Variable Model

\[ \dot{x}_1 = v_1 \]
\[ \dot{v}_1 = \frac{1}{M} \left[ -K_1 x_1 - B_1 v_1 + K_1 x_2 + f_a(t) \right] \]
\[ \dot{x}_2 = \frac{1}{B_2} \left[ K_1 x_1 - K_2 x_2 - K_1 x_2 \right] \]

\[ y_1 = x_2 \]
State Variable Models

- States are not unique.
- The value of states summarize effect of past inputs on the system.
- Usually the number of states is equal to the number of energy storing elements (Masses and springs)
- In some cases we may have redundant states (the number of states is less than the number of energy storing elements)
- Typical choice of states: velocity of mass and elongation of spring.

State variable model:

- State equations (set of first order ODE, involves states and inputs)
- Output equations (algebraic equations, involves states and inputs)
SE207 Modeling and Simulation
Unit 4
Lesson 2: Input-Output (I/O) Models

Dr. Samir Al-Amer
Term 072

Read Section 3.2
Two standard forms for models will be used.
- State variable model
- Input-output model
States (State Variables)

- States are not unique.
- The value of states summarize effect of past inputs on the system.
- Usually the number of states is equal to the number of energy storing elements (Masses and springs).
- In some cases we may have redundant states (the number of states is less than the number of energy storing elements).
State Variable Models
Example (1-input, 1-output and 2-states)

\[
\begin{align*}
\dot{q}_1 &= 2q_1 + 3q_2 + b_1 u \\
\dot{q}_2 &= q_1 + 0.5q_2 + b_2 u \\
y &= q_1 + 0.5u
\end{align*}
\]

- **State Equations**
- **Output Equation**

- \( u \): input,
- \( y \): output,
- \( q_i \): states (or state variables)
Input-Output Models

- ODE that contains the inputs, the output and their derivatives and no other variables.

\[ M\ddot{x} + Kx + B\dot{x} = f_a(t) \]

Output and its derivatives \quad \text{input}
**State Variable and Input-output Models**

Input: \( f_a(t) \)

Output: displacement \( x \)

\[
M \ddot{x} + Kx + B\dot{x} = f_a(t)
\]

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= \frac{1}{M} [-Bv - Kx + f_a(t)] \\
y &= x
\end{align*}
\]

State Variable Model

input-output model
Example
Input-Output Models

\[ \dot{x} = v \]
\[ \dot{v} = \frac{1}{M} \left[ -Bv - Kx + f_a(t) \right] \]
\[ y_1 = x \]

\[ M\ddot{x} + Kx + B\dot{x} = f_a(t) \]

State variable model
Input-Output Model
Example
Input-Output Models

For simple problems, you may be able to obtain the I/O model directly by expressing all forces (in the free body diagram) in terms of inputs, outputs and their derivatives.
Input-Output Models

\[ M\ddot{x}_1 + B_1\dot{x}_1 + K_1 x_1 - K_1 x_2 = f_a(t) \]
\[ B_2 \dot{x}_2 + (K_1 + K_1)x_2 - K_1 x_2 = 0 \]

input : \( f_a(t) \)
output : \( x_1 \)
Eliminate \( x_2, \dot{x}_2 \)
Input-Output Models

\[ M\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - K_1x_2 = f_a(t) \]
\[ B_2\dot{x}_2 + (K_1 + K_1)x_2 - K_1x_2 = 0 \]

**solve for** \( x_2 \) \( \Rightarrow \)
\[ x_2 = \left(\frac{1}{K_1}\right)(M\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - f_a(t)) \]

**differentiate** \( x_2 \) \( \Rightarrow \)
\[ \dot{x}_2 = \left(\frac{1}{K_1}\right)(M\dddot{x}_1 + B_1\ddot{x}_1 + K_1\dot{x}_1 - \dot{f}_a(t)) \]

**substitute** \( x_2 \) and \( \dot{x}_2 \) **and simplify** to get input–output model
\[ MB_2\ddot{x}_1 + (B_1B_2 + K_1M + K_2M)\dddot{x}_1 + (B_2K_1 + K_1B_1 + K_2B_1)\dot{x}_1 + K_1K_2x_1 = B_2\dot{f}_a + (K_1 + K_2)f_a \]
Input-Output Models

\[ MB_2 \ddot{x}_1 + \left( B_1 B_2 + K_1 M + K_2 M \right) \dot{x}_1 + \left( B_2 K_1 + K_1 B_1 + K_2 B_1 \right) \dot{x}_1 + K_1 K_2 x_1 = B_2 \dot{f}_a + (K_1 + K_2) f_a \]

This equation involves the input and its derivatives \((\ddot{x}_1, \dot{x}_1, \dot{x}_1, x_1)\) and the output and its derivatives \((\dot{f}_a, f_a)\) and no other variables.
Reduction using differentiator operator

Define \( p \) : differentiator operator

\[ px \text{ means } \frac{d}{dt} x \quad \text{and} \quad p^2x \text{ means } \frac{d^2}{dt^2} x \]

\[ (p^2 + 2p + 4)x \text{ means } \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x \]

Apply the \( p \) operator to

\[ 2\ddot{x} + 3\dot{x} + 5x = f_a(t) \]

\[ 2p^2x + 3px + 5x = f_a \quad \text{or} \quad (2p^2 + 3p + 5)x = f_a \]
Reduction using differentiator operator

The differentiator operator converts the differential equations into algebraic equations in the p operator which are easier to manipulate.

\[ 2\ddot{x}_1 + 3\dot{x}_1 + 5x_1 = f_a(t) \]

\[ \downarrow \]

\[ 2p^2x + 3px + 5x = f_a \]
Input-Output Models

\[ M\ddot{x}_1 + B_1 \dot{x}_1 + K_1 x_1 - K_1 x_2 = f_a(t) \]
\[ B_2 \dot{x}_2 + (K_1 + K_1) x_2 - K_1 x_2 = 0 \]

input : \( f_a(t) \), output : \( x_1 \)

\[ M p^2 x_1 + B_1 p x_1 + K_1 x_1 - K_1 x_2 = f_a(t) \]
\[ B_2 p x_2 + (K_1 + K_1) x_2 - K_1 x_2 = 0 \]
\[ \Rightarrow (B_2 p + K_1 + K_1) x_2 - K_1 x_1 = 0 \]
\[ \Rightarrow M (M p^2 x_1 + B_1 p + K_1) x_1 - K_1 x_2 = f_a(t) \]
Input-Output Models

\[ M\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - K_1x_2 = f_a(t) \]
\[ B_2\dot{x}_2 + (K_1 + K_1)x_2 - K_1x_2 = 0 \]

input : \( f_a(t) \), output : \( x_1 \)

Remove \( x_2, \dot{x}_2 \)

The final equation contains \( f_a, x_1 \) and their derivatives only
Input-Output Models

To eliminate \( x_2, \dot{x}_2 \)

\[
(B_2 p + K_1 + K_1)x_2 = K_1 x_1
\]

multiply both sides by \((B_2 p + K_1 + K_1)\)

\[
(B_2 p + K_1 + K_1)M(Mp^2 x_1 + B_1 p + K_1)x_1 - (B_2 p + K_1 + K_1)K_1 x_2
\]

\[
= (B_2 p + K_1 + K_1)f_a(t)
\]

Now replace \((B_2 p + K_1 + K_1)x_2\) by \(K_1 x_1\)

\[
M(B_2 p + K_1 + K_1)(Mp^2 x_1 + B_1 p + K_1)x_1 - K_1 x_1
\]

\[
= (B_2 p + K_1 + K_1)f_a(t)
\]
Replace \( p \) operator by \( \frac{d}{dt} \) and simplify

\[
MB_2 \ddot{x}_1 + \left( B_1 B_2 + K_1 M + K_2 M \right) \ddot{x}_1 \\
+ \left( B_2 K_1 + K_1 B_1 + K_2 B_1 \right) \dot{x}_1 + K_1 K_2 x_1 \\
= B_2 \dot{f}_a + (K_1 + K_2) f_a
\]
State space versus input/output models

- The order of the I/O model = number of states (unless some states are redundant or some states have no effect on output)

- State variable models are more convenient for solution and for modeling MIMO (multi-input-multi-output) systems