CISE302: Linear Control Systems

4. Inverse Laplace Transform

Outlines:
- Introduction
- Partial Fraction Expansion
  - Simple poles case
  - Complex poles case
  - Repeated pole case
- Inverse transform of non-strictly proper functions

Learning Objectives:
- To be able to obtain inverse Laplace transform of rational functions

4.1 Introduction

If $F(s)$ is the Laplace transform of $f(t)$ then we can say that $f(t)$ is the inverse Laplace transform of $F(s)$. The following notation is used

$$F(s) = L\{f(t)\} \quad \Leftrightarrow \quad f(t) = L^{-1}\{F(s)\}$$

We have seen in Chapter 3 that

$$L\{e^{-2t}\} = \frac{1}{s+2} \quad \Leftrightarrow \quad e^{-2t} = L^{-1}\left\{\frac{1}{s+2}\right\}$$

The inverse transform of $F(s)$ can be obtained using the formula

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\alpha - j\infty}^{\alpha + j\infty} F(s)e^{st}ds$$

Where α is a real number that is greater than the real part of any singularity of F(s). The complex integral above is usually difficult to evaluate. Instead the approach that is considered here is to express $F(s)$ as the sum of simple terms that are usually available in the Laplace transform Table. Before discussing the way to do this, the following property of inverse Laplace transform are listed. These properties can be derived from the properties of Laplace transform listed in Chapter 1 and therefore no proof for these properties will be given here.

The Laplace transform is a linear transformation

$$L^{-1}\{aF(s) + bG(s)\} = aL^{-1}\{F(s)\} + bL^{-1}\{G(s)\}$$

This is simple to prove from the definition of the inverse transform but it has a major impact. It allows us to simplify the computation of the inverse transform. If we can express $F(s)$ as the sum of simple functions that can be inverted easily then the inverse transform is simply the sum of the individual inverses.

Example 4.1

Obtain the inverse Laplace transform of $\dfrac{1}{s(s+1)}$

Simple algebraic manipulation allows us to write

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

The linearity property allows us to

$$L^{-1}\left\{\frac{1}{s(s+1)}\right\} = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\} = 1 - e^{-t}.$$

Details of the procedure to do this are illustrated in the next section.

4.2 Partial Fraction Expansion

When the function F(s) is a rational function then it can be expanded as the sum of simple terms whose inverse Laplace transform is easy to obtain. Three special cases are discussed in this section. The following definitions are essential to the remaining part of this chapter.

Definition

A complex function $F(s)$ is said to be a **rational function** if can be expressed as a ratio of two polynomials. $F(s) = \frac{N(s)}{D(s)}$ where N(s) and D(s) are polynomials in the complex variable s.

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Definition

The function F(s) is said to be **proper** if it is rational and degree of N(s) ≤ degree of D(s). It is **strictly proper** if degree of N(s) is strictly less than degree of D(s).

Definition

A complex valued function $F(s)$ is said to have a **singularity** at a point in the s-plane if the function or some of its derivatives does not exist at that point.

The most common type of singularity is the **pole** which is defined next.

Definition

A complex valued function $F(s) = \frac{N(s)}{D(s)}$ is said to have a pole of order r at $s = p$ if $\lim_{s=p}\left((s-p)^r F(s)\right)$ Has a finite non-zero value. If r is one then the pole is called a **simple pole** if r >1 then it is a **repeated pole**.

Example 4.2

$F(s)=\dfrac{s-3}{s^2(s+1)}$ has a simple pole at $s=-1$ and double (repeated) pole as $s=0$.

$F(s)=\dfrac{s-3}{s(s^2+1)}$ has three simple poles (at $s=0,\ \sqrt{-1},\ -\sqrt{-1}$).

An important step in partial fraction expansion is to factor the denominator of F(s) into factors. See Appendix B for review of factoring polynomials.

4.2.1 Distinct Pole Case

In this subsection we consider the case when all the poles of the system are simple. Assume F(s) is strictly proper with n simple poles (real or complex)

$$F(s)=\frac{N(s)}{(s-p_1)(s-p_2)....(s-p_n)}$$

then F(s) can be expressed as

$$F(s)=\frac{N(s)}{(s-p_1)(s-p_2)....(s-p_n)}=\frac{a_1}{(s-p_1)}+\frac{a_2}{(s-p_2)}+...+\frac{a_n}{(s-p_n)}$$

The coefficients a_i can obtained in different ways. A simple and convenient way is

$$a_1=\lim_{s=p_1}(s-p_1)F(s)$$

$$a_2=\lim_{s=p_1}(s-p_2)F(s)$$

....

$$a_n=\lim_{s=p_n}(s-p_n)F(s)$$

Once the coefficients are obtained, the inverse Laplace transform is given by

$$F(s)=\frac{N(s)}{(s-p_1)(s-p_2)....(s-p_n)}=a_1e^{p_1t}+a_2e^{p_2t}+...+a_ne^{p_nt}$$

Example 4.3

Obtain the inverse Laplace transform of $F(s)=\dfrac{2}{s(s+1)}$

Solution:

F(s) has two simple poles (at $s=0,\ s=-1$)

$$F(s)=\frac{2}{s(s+1)}=\frac{a_1}{s}+\frac{a_2}{s+1}$$

$$a_1=s\left.\frac{2}{s(s+1)}\right|_{s=0}=\left.\frac{2}{(s+1)}\right|_{s=0}=2$$

$$a_2=(s+1)\left.\frac{2}{s(s+1)}\right|_{s=-1}=\left.\frac{2}{s}\right|_{s=-1}=-2$$

$$F(s)=\frac{2}{s(s+1)}=\frac{2}{s}+\frac{-2}{s+1}$$

$$f(t)=2e^{0t}-2e^{-t}=2-2e^{-t}$$

Example 4.4

Obtain the inverse Laplace transform of $F(s) = \dfrac{1}{s^2 + 5s + 4}$

Solution:

The first step is to factor $s^2 + 5s + 4$ =(s+4)(s+1) which means that F(s) has two simple poles.

$$F(s) = \frac{1}{(s+4)(s+1)} = \frac{a_1}{s+4} + \frac{a_2}{s+1}$$

$$a_1 = (s+4) \left. \frac{1}{(s+4)(s+1)} \right|_{s=-4} = \left. \frac{1}{(s+1)} \right|_{s=-4} = \frac{1}{-3}$$

$$a_2 = (s+1) \left. \frac{1}{(s+4)(s+1)} \right|_{s=-1} = \left. \frac{1}{(s+4)} \right|_{s=-1} = \frac{1}{3}$$

$$F(s) = \frac{1}{(s+4)(s+1)} = -\frac{\frac{1}{3}}{s+4} + \frac{\frac{1}{3}}{s+1}$$

$$f(t) = -\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t}$$

Example 4.5

Obtain the inverse Laplace transform of $F(s) = \dfrac{1}{s(s^2+4)}$

Solution:

F(s) has three simple poles ($s = 0, +2i \;\; and \;\; -2i$).

$$F(s) = \frac{a_1}{s} + \frac{a_2}{s+2i} + \frac{a_3}{s-2i}$$

$$a_1 = s \left. \frac{1}{s(s^2+4)} \right|_{s=0} = \left. \frac{1}{(s^2+4)} \right|_{s=0} = \frac{1}{4}$$

$$a_2 = (s+2i) \left. \frac{1}{s(s+2i)(s-2i)} \right|_{s=-2i} = \frac{1}{-2i(-4i)} = \frac{-1}{8}$$

$$a_3 = (s-2i) \left. \frac{1}{s(s+2i)(s-2i)} \right|_{s=2i} = \frac{1}{2i(4i)} = \frac{-1}{8}$$

$$F(s) = \frac{0.25}{s} + \frac{-0.125}{s+2i} + \frac{-0.125}{s-2i}$$

$$f(t) = 0.25 - 0.125e^{-2it} - 0.125e^{2it}$$

Using Euler identity $\cos(\omega t) = \dfrac{e^{j\omega t} + e^{-j\omega t}}{2}$, the expression of f(t) is simplified as

$$f(t) = 0.25 - 0.125e^{-2it} - 0.125e^{2it} = 0.25 - 0.25\cos(2t) .$$

4.2.2 Complex Poles Case

When F(s) has distinct poles that are complex, the same technique discussed in the previous section can be applied. In this section we consider an alternative approach for distinct complex poles that are more convenient to compute.

It is a fact that if the coefficients of a polynomial are real then the complex roots occur as pairs of complex conjugate roots. Keeping the factors that correspond to complex conjugate pairs as second order factor allows us to avoid using complex arithmetic and the resulted inverse transform is easy to obtain. The previous Example 5.4 is solved using the alternative approach.

Example 4.6

Obtain the inverse Laplace transform of $F(s)=\dfrac{1}{s(s^2+4)}$

Solution:
F(s) has three simple poles ($s=0,+2i \ \ and \ -2i$).

$$F(s)=\frac{a_1}{s}+\frac{a_2s+a_3}{s^2+4}$$

$$a_1 = s \ \ \frac{1}{s(s^2+4)}\bigg|_{s=0} = \frac{1}{(s^2+4)}\bigg|_{s=0} = \frac{1}{4}$$

$$\frac{1}{s(s^2+4)}=\frac{0.25}{s}+\frac{a_2s+a_3}{s^2+4}=\frac{0.25(s^2+4)+s(a_2s+a_3)}{s(s^2+4)}$$

Matching the coefficients of equal powers of s we have the following equations

$$\frac{1}{s(s^2+4)}=\frac{0.25}{s}+\frac{a_2s+a_3}{s^2+4}=\frac{0.25(s^2+4)+s(a_2s+a_3)}{s(s^2+4)}$$

$$0.25s^2+a_2s^2=0 \Rightarrow a_2=-0.25$$

$$a_3s=0 \Rightarrow a_3=0$$

$$F(s)=\frac{0.25}{s}+\frac{-0.25s}{s^2+4}$$

$$f(t)=0.25-0.25\cos(2t)$$

It may be convenient to use completing the square. This is an easy way to find the real and imaginary parts of the roots. A second order polynomial in the form

$$s^2+cs+d=0$$

Can be expressed as

$$s^2+2as+a^2+\omega^2=0$$

$$(s+a)^2+\omega^2=0$$

The roots are $-a \pm j\omega$

Example 4.7

Obtain the inverse Laplace transform of $F(s)=\dfrac{s+1}{s^2+4s+13}$

Solution:
F(s) has complex conjugate poles. We apply completing the square to denominator

$$F(s)=\frac{s+1}{s^2+4s+13}=\frac{s+1}{(s^2+4s+4)+9}=\frac{s+1}{(s+2)^2+3^2}=\frac{s+2}{(s+2)^2+3^2}+\frac{-1}{(s+2)^2+3^2}$$

$$F(s)=\frac{s+2}{(s+2)^2+3^2}+\frac{-1}{3}\frac{3}{(s+2)^2+3^2}$$

$$f(t)=e^{-2t}\cos(3t)-\frac{1}{3}e^{-2t}\sin(3t)$$

4.2.3 Repeated Poles Case

In this section we consider the case when some of the poles occur at multiplicity more than one. The coefficients corresponding to simple poles will be obtained in the same way discussed earlier. If p is a pole of *F(s)* with **multiplicity** *m* then the partial expansion of F(s) will contain terms like

$$\frac{a_m}{(s-p)^m}+\frac{a_{m-1}}{(s-p)^{m-1}}+\ldots+\frac{a_2}{(s-p)^2}+\frac{a_1}{(s-p)}$$

The coefficients are obtained as follows

$$a_m=(s-p)^m F(s)\Big|_{s=p}$$

$$a_{m-1}=\frac{d}{ds}\Big((s-p)^m F(s)\Big)\Big|_{s=p}$$

$$a_{m-2}=\frac{1}{2!}\;\frac{d^2}{ds^2}\Big((s-p)^m F(s)\Big)\Big|_{s=p}$$

$$a_{m-3}=\frac{1}{3!}\;\frac{d^3}{ds^3}\Big((s-p)^m F(s)\Big)\Big|_{s=p}$$

and so on. Note that the derivative is obtained before substituting the value *s=p*.

Example 4.8

Obtain the inverse Laplace transform of $F(s)=\dfrac{1}{s(s+1)^2}$

$$\frac{1}{s(s+1)^2}=\frac{A}{s}+\frac{B}{(s+1)^2}+\frac{C}{s+1}$$

$$A=s\,\frac{1}{s(s+1)^2}\Bigg|_{s=0}=1$$

$$B=(s+1)^2\,\frac{1}{s(s+1)^2}\Bigg|_{s=-1}=\frac{1}{s}\Bigg|_{s=-1}=-1$$

$$C=\frac{d}{ds}\left((s+1)^2\,\frac{1}{s(s+1)^2}\right)\Bigg|_{s=-1}=\frac{d}{ds}\left(\frac{1}{s}\right)\Bigg|_{s=-1}=-1$$

$$f(t)=1-te^{-t}-e^{-t}$$

Example 4.9

Obtain the inverse Laplace transform of $F(s) = \dfrac{1}{(s+1)(s+2)^3}$

Solution:

$$\frac{1}{(s+1)(s+2)^3} = \frac{A}{s+1} + \frac{B}{(s+2)^3} + \frac{C}{(s+2)^2} + \frac{D}{s+2}$$

$$A = (s+1)\frac{1}{(s+1)(s+2)^3}\Bigg|_{s=-1} = 1$$

$$B = (s+2)^3\frac{1}{(s+1)(s+2)^3}\Bigg|_{s=-2} = \frac{1}{s+1}\Bigg|_{s=-2} = -1$$

$$C = \frac{d}{ds}\left(\frac{1}{s+1}\right)\Bigg|_{s=-2} = \frac{-1}{(s+1)^2}\Bigg|_{s=-2} = -1$$

$$D = \frac{1}{2!}\,\frac{d^2}{ds^2}\left(\frac{1}{s+1}\right)\Bigg|_{s=-2} = \frac{1}{2!}\,\frac{d}{ds}\left(\frac{-1}{(s+1)^2}\right)\Bigg|_{s=-2} = -1$$

$$\frac{1}{(s+1)(s+2)^3} = \frac{1}{s+1} + \frac{-1}{(s+2)^3} + \frac{-1}{(s+2)^2} + \frac{-1}{s+2}$$

$$f(t) = e^{-t} - t^2e^{-2t} - te^{-2t} - e^{-2t}$$

4.2.4 Inverse of Non-Strictly Proper Functions

So far we considered finding the inverse Laplace transform of rational functions that are strictly proper $F(s) = \dfrac{N(s)}{D(s)}$ with degree of *D(s)*> degree of *N(s)*. In this section we consider the case when degree of *D(s)* is the same as the degree of *N(s)*. Two steps are done. First *F(s)* is expressed as the sum of a constant number and a strictly proper function. The inverse of a constant is the same constant multiplied by a **Dirac impulse** function and inverse of a strictly proper function is done as usual.

Example 4.10

Obtain the inverse Laplace transform of $F(s) = \dfrac{2s^2+s+5}{s(s+1)}$

Solution:

Using long division one can express $F(s) = \dfrac{2s^2+s+5}{s(s+1)}$ as

$$F(s) = \frac{2s^2+s+5}{s(s+1)} = 2 + \frac{-s+5}{s(s+1)}.$$

The strictly proper part has two distinct poles and F(s) can be expressed as

$$F(s) = 2 + \frac{-s+5}{s(s+1)} = 2 + \frac{5}{s} + \frac{-6}{s+1}$$

The inverse Laplace transform is $f(t) = 2\delta(t) + 5 - 6e^{-t}$

Solved Problems

Problem SP4.1

Obtain the inverse Laplace transform of $\frac{s+1}{s(s+3)^2}$

Solution:

$$\frac{s+1}{s(s+3)^2} = \frac{A}{s} + \frac{B}{(s+3)^2} + \frac{C}{s+3}$$

$$A = s\frac{s+1}{s(s+3)^2}\bigg|_{s=0} = \frac{1}{9}$$

$$B = (s+3)^2 \frac{s+1}{s(s+3)^2}\bigg|_{s=-3} = \frac{s+1}{s}\bigg|_{s=-3} = \frac{2}{3}$$

$$C = \frac{d}{ds}\left((s+3)^2 \frac{s+1}{s(s+3)^2}\right)\bigg|_{s=-3} = \frac{d}{ds}\left(\frac{s+1}{s}\right)\bigg|_{s=-3} = -\frac{1}{9}$$

$$\frac{s+1}{s(s+3)^2} = \frac{\frac{1}{9}}{s} + \frac{\frac{2}{3}}{(s+3)^2} + \frac{\frac{-1}{9}}{s+3}$$

$$f(t) = \frac{1}{9} + \frac{2}{3}te^{-3t} - \frac{1}{9}e^{-3t}$$

Problem SP4.2

Obtain the inverse Laplace transform of $\frac{s+1}{s(s+3)(s+2)}$

Solution:

$$\frac{s+1}{s(s+3)(s+2)} = \frac{A}{s} + \frac{B}{(s+3)} + \frac{C}{s+2}$$

$$A = s\frac{s+1}{s(s+3)(s+2)}\bigg|_{s=0} = \frac{1}{6}$$

$$B = (s+3)\frac{s+1}{s(s+3)(s+2)}\bigg|_{s=-3} = \frac{s+1}{s(s+2)}\bigg|_{s=-3} = -\frac{2}{3}$$

$$C = (s+2)\frac{s+1}{s(s+3)(s+2)}\bigg|_{s=-2} = \frac{s+1}{s(s+3)}\bigg|_{s=-2} = \frac{1}{2}$$

$$f(t) = \frac{1}{6} - \frac{2}{3}e^{-3t} + \frac{1}{2}e^{-2t}$$

Problem SP4.3

Obtain the inverse Laplace transform of $\dfrac{3}{s^2+4s+3.91}$

Solution:

$$\frac{3}{s^2+4s+3.91}=\frac{A}{s+2.3}+\frac{B}{s+1.7}$$
$$A=(s+2.3)\left.\frac{3}{(s+2.3)(s+1.7)}\right|_{s=-2.3}=-5$$
$$B=(s+1.7)\left.\frac{3}{(s+2.3)(s+1.7)}\right|_{s=-1.7}=5$$
$$\frac{3}{s^2+4s+3.91}=\frac{-5}{s+2.3}+\frac{5}{s+1.7}$$
$$f(t)=-5e^{-2.3t}+5e^{-1.7t}$$

Summary

In this chapter, the inverse Laplace transform was considered. Inverse Laplace transform is simply obtaining the time domain function f(t) that corresponds to the given frequency domain function F(s). Computing f(t) using the definition directly is not simple. An easier approach is to use partial fraction expansion in which the function is expressed as the sum of simple function for which the inverse Laplace transforms are known. Three special cases were considered: simple poles, repeated poles and complex poles. Inverse Laplace transform of proper functions that are not strictly proper was also considered.

Simple poles	Repeated poles	multiplicity	residue
Partial fraction	Completing the square		

Review Questions

Problems

4.1 Find the inverse Laplace transform of the following functions

a) $F(s)=\dfrac{s}{s^2+16}$ ANS: $f(t)=\cos(4t)$

b) $F(s)=\dfrac{1}{(s+5)^2}$. ANS: $f(t)=te^{-5t}$

.

c) $F(s)=\dfrac{2}{s^2+5s+2}$ ANS: $f(t)=-0.4851e^{-4.562t}+0.4851e^{-0.4384t}$

d) $F(s)=\dfrac{s-1}{s^3+5s^2+4s}$ ANS: $f(t)=\dfrac{-1}{4}-\dfrac{5}{12}e^{-4t}+\dfrac{2}{3}e^{-t}$

e) $F(s)=\dfrac{1}{4s-3}$

f) $F(s)=\dfrac{1}{s-3}-\dfrac{1}{s^2}$ ANS: $f(t)=e^{3t}-t^2$

g) $F(s)=\dfrac{s^2+4s-4}{s^3-4s}$

4.2 Find the inverse Laplace transform of the following functions

a) $F(s)=\dfrac{2}{\left(s^2-s\right)^2}$

b) $F(s)=\dfrac{2s}{\left(s^2+1\right)^2}$

c) $F(s)=\dfrac{2s}{\left(s+1\right)^2+4}$

4.3 Find the inverse Laplace transform of the following functions

a) $F(s)=\dfrac{s^2-s}{(s+4)(s+2)}$

b) $F(s)=\dfrac{4s^2-2}{s^2-s}$

4.4 The Laplace transform of a signal is given by $X(s)=\dfrac{1}{2s^2+s}$. Obtain the signal x(t).