CISE302: Linear Control Systems

3. Introduction to Laplace Transform

Outlines:	Learning Objectives:
• Introduction • Definition of the Laplace transform • Laplace transform of simple functions • Existence of Laplace transform	• To state the definition of Laplace transform. • To obtain Laplace transform of simple functions using the Laplace transform definition. • To give sufficient conditions for existence of Laplace transform.

3.1 Introduction

Many mathematical problems are solved using transformations. The idea is to transform the problem into another problem that is easier to solve. Once a solution is obtained, the inverse transform is used to obtain the solution to the original problem. The Laplace transform is an important tool that makes solution of linear constant coefficient differential equations much easier. The Laplace transform transforms the differential equations into algebraic equations which are easier to manipulate and solve. Once the solution is obtained in the Laplace transform domain is obtained, the inverse transform is used to obtain the solution to the differential equation. Laplace transform is an essential tool for the study of linear time-invariant systems.

In this chapter, the definition of the Laplace transform is presented. The basic definition is used to obtain the Laplace transforms of simple commonly used functions. The inverse Laplace transform is covered in Chapter 4. Chapter 5 discusses several properties of Laplace transform and their applications. Solving the differential equations and determining the response of systems are covered in Chapters 6 and 7.

Laplace transform allows us to obtain transient and steady state response of linear systems in a simple way. In addition, the concept of transfer functions, which is simply a ratio of Laplace transforms of the outputs and the inputs of a system is an essential concept in the analysis and design of linear systems.

3.2 Definition of The Laplace Transform

The Laplace transform converts a function of real variable $f(t)$ into a function of complex variable $F(s)$. The Laplace transform is defined[1] as

$$F(s) = L\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\, dt$$

The variable s is a complex variable that is commonly known as the **Laplace operator**. The Laplace operator is usually expressed as $s = \sigma + j\omega$ where σ and ω are the real part and the imaginary part of s respectively. The unit of s is (1/time) and this is the reason for saying that $F(s)$ is in the frequency domain. In many textbooks the lower limit is 0_- rather than 0. There is no difference between them except when an impulse

[1] This is known as the one sided Laplace transform. A more general two sided Laplace transform is defined as $\int_{-\infty}^{\infty} f(t)e^{-st}\, dt$. In this course only the one-sided Laplace transform is used.

function at t = 0 is used.[2]

The Laplace transform is a function of the complex variable s. The following notation will be used throughout the book. Lower case letters are used to denote time domain functions while capital letters are used to denote their Laplace transform.

3.2 Laplace Transform of Simple Functions

In the following discussions we present Laplace transforms of simple functions that are commonly used in Control Engineering applications.

3.2.1 Step Function

Consider the **unit step** function

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

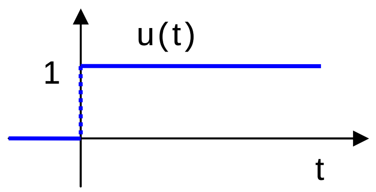


Figure 3.1 Unit Step Function

A step function of magnitude M is expressed as $M\,u(t)$. Physically a step function corresponds to a constant signal suddenly applied to the system. Throughout the course, if u(t) is not specified it will be used to mean unit step function.

Theorem:

The Laplace transform of the unit step is $\dfrac{1}{s}$.

Proof:

$$L\,\{u(t)\} = \int_0^\infty u(t)e^{-st}\;dt = \int_0^\infty e^{-st}\;dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}$$

3.2.2 Ramp Function

A **ramp** function is a linearly increasing function. A **unit ramp** is defined as

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

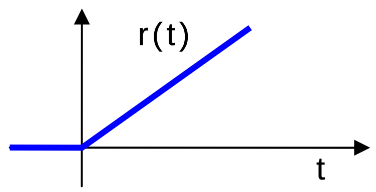


Figure 3.2 Ramp Function

[2] Both definitions are identical $\int_{0_-}^\infty f(t)e^{-st}\;dt = \int_0^\infty f(t)e^{-st}\;dt$ except when $\int_{0-}^0 f(t)\,dt \neq 0$

Theorem:

The Laplace transform of the unit ramp is $\frac{1}{s^2}$.

Proof:

$$L\{r(t)\} = \int_0^\infty r(t)e^{-st}\, dt = \int_0^\infty t\, e^{-st}\, dt = t\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, dt = \frac{1}{s^2}$$

3.2.3 Pulse Function

Consider the following function

$$f(t) = \begin{cases} A & 0 \le t \le a \\ 0 & elsewhere \end{cases}$$

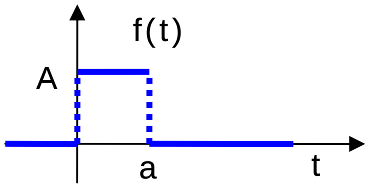


Figure 3.3 Pulse Function

This is commonly known as the **rectangular pulse** function.

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}\, dt = \int_0^a A e^{-st}\, dt = \frac{Ae^{-st}}{-s}\bigg|_0^a = \frac{A}{s}(1 - e^{-sb})$$

1.2.4 Exponential Function

Consider the following function

$$f(t) = \begin{cases} e^{at} & t \ge 0 \\ 0 & elsewhere \end{cases}$$

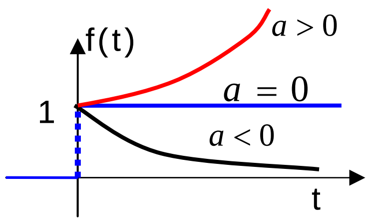


Figure 1.4 Exponential Function

From Figure 1.4, one can see that when $a = 0$, the **exponential function** reduces to the step function. The magnitude of the exponential function is getting closer to zero as t increases for $a < 0$ while the magnitude of the function increases when $a > 0$.

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}\, dt = \int_0^\infty e^{at}\; e^{-st}\, dt = \left.\frac{-e^{-(s+a)t}}{s+a}\right|_0^\infty = \frac{1}{s+a}$$

3.2.3 Impulse Function

The **impulse** function, often called the **Dirac Delta** is an important function. It is defined as

$$\delta(t) = 0 \;\; if\; t \neq 0 \;\; and \;\; \int_{-\varepsilon}^{\varepsilon} \delta(t)\; dt = 1, \;\; \varepsilon > 0$$

The impulse is infinite at t=0 and zero elsewhere and the integral under the impulse curve over any non-zero interval containing zero is equal to one. The Laplace transform of a unit impulse is 1. ($L\{\delta(t)\} = 1$).

3.2.5 Sinusoidal Functions

Sinusoidal functions often appear is studying control Systems.

$$f(t) = \begin{cases} A\sin(\omega t) & t \geq 0 \\ 0 & elsewhere \end{cases}$$

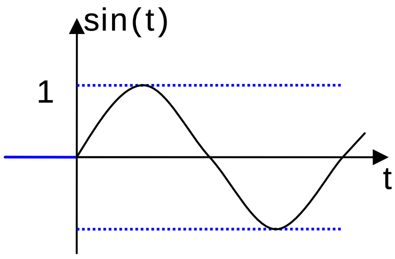


Figure 3.5 Sine Function

Where A is the magnitude and ω is the frequency. The Laplace transform of sinusoidal function is given by

$$L\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}.$$

The Laplace transform of cosine function is given by

$$L\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}$$

To prove this relationships we can use the following identities

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$L\{\sin\omega t\} = L\left\{\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right\} = \frac{1}{2j}\int_0^\infty \left(e^{j\omega t} - e^{-j\omega t}\right)e^{-st}dt$$

$$= \frac{1}{2j}\int_0^\infty \; e^{j\omega t - st}dt - \frac{1}{2j}\int_0^\infty \; e^{-j\omega t - st}dt = \frac{\omega}{s^2 + \omega^2}$$

f(t)	F(s)		f(t)	F(s)
$\delta(t)$	1		$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
1	$\frac{1}{s}$		$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
t	$\frac{1}{s^2}$		$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
t^2	$\frac{2}{s^3}$		$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$		$t\,e^{-a\,t}$	$\frac{1}{(s+a)^2}$
$e^{-a\,t}$	$\frac{1}{s+a}$		$t^n\,e^{-a\,t}$	$\frac{n!}{(s+a)^{n+1}}$

Table 3.1 Laplace transform of simple functions

3.3 Existence of Laplace Transform

The Laplace transform is an **improper function** since one of the limits is infinity. The Laplace transform is said to exist if the integral converges to a finite value. Consider the Laplace transform of exponential function for example, the integral $\int_0^\infty e^{at}\;e^{-st}\;dt$ converges to $\frac{1}{s+a}$ for $\mathrm{Re}\{s\} > a$.

In the following discussion, we will present a result that can be used to proof existence of most commonly used functions.

Definition:

A function f(t) is **piecewise continuous** on a finite interval $a \le t \le b$ if this interval can be divided into a finite number of subintervals such that

a) The function is continuous in the interior of each subinterval,
b) The function approaches finite limit as t approaches either limits from its interior.

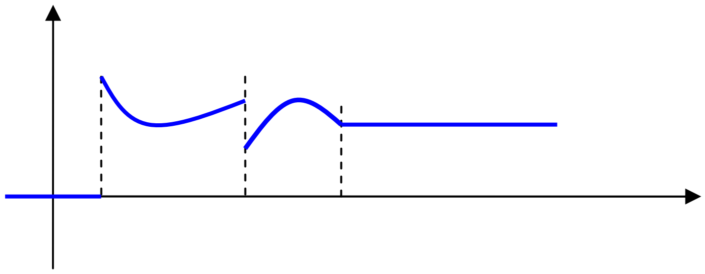


Figure 3.5 Piecewise-Continuous Function

Definition:
A function is said to be of **exponential order** if there exits a constant α and positive constants T and M such that

$$|f(t)| < M\, e^{\alpha t} \quad for\ all\ \ t > T$$

A function is of exponential order if its magnitude is bounded by an exponential function of the form $M\, e^{\alpha t}$.

Now we are ready to state the main result in this section.

Theorem:
Let f(t) be a piecewise continuous and of exponential order for constants M and α then the Laplace transform $L\{f(t)\}$ exists for $s > \alpha$.

The above theorem gives sufficient conditions for existence and there are some functions that do not satisfy the conditions and have Laplace transforms.

Solved Problems

S3.1 Obtain the Laplace transform of $f = \begin{cases} 1-2t & 0 \le t < 2 \\ 0 & 2 \le t < 3 \\ 1 & 3 \le t \end{cases}$

Solution:

$$F(s) = \int_0^2 (1-2t)e^{-st}dt + \int_2^3 0dt + \int_3^\infty e^{-st}dt = \frac{(e^{-2s}-1)}{s} - \frac{2(e^{-2s}-1)}{s^2} + \frac{e^{-3s}}{s}$$

Summary

Laplace transform is an important tool. In addition to its use in solving differential equations, it is used in some key concepts in control engineering. This chapter covers the definition of Laplace transform and way to compute it for some functions. Conditions for existence of Laplace transform were discussed.

Review Questions

1. State the mathematical definition of the Laplace transform.

2. Give conditions for existence of Laplace transform of a function f(t).
3. Does $f(t)=\frac{2}{t-1}$ has a Laplace transform?
4. Is $f(t)=2e^{t^2}$ of exponential order?
5. Can we find a Laplace transform of $f(t)=\sin(t^3)$?

Problems

3.1 Find the Laplace Transform of the following functions

a) $f(t)=t^2$ ANS: $F(s)=\frac{2}{s^3}$

b) $f(t)=t-1$ ANS: $F(s)=\frac{1}{s^2}-\frac{1}{s}$

c) $f(t)=t^{2.5}$

d) $f(t)=t\sin(2t)$

e) $f(t)=\cos(4t)$. ANS: $F(s)=\frac{s}{s^2+16}$

f) $f(t)=te^{-5t}$. ANS: $F(s)=\frac{1}{(s+5)^2}$

g) $f(t)=1+2e^{-4t}$. ANS: $F(s)=\frac{3s+4}{s(s+4)}$

h) $f(t)=\begin{cases}1 & 0\le t\le 2\\ e^{-2t} & t>2\end{cases}$

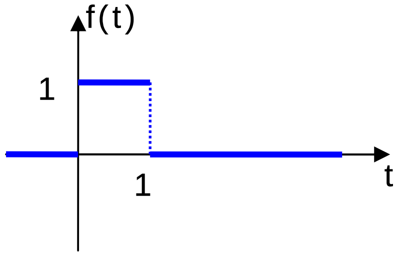
i) $f(t)=\begin{cases}0 & t<1\\ 1 & 1\le t\le 2\\ e^{-2t} & t>2\end{cases}$.

j) $f(t)=2e^{-4t}-5e^{-3t}$. ANS: $F(s)=\frac{-2s+14}{s^2+7s+12}$

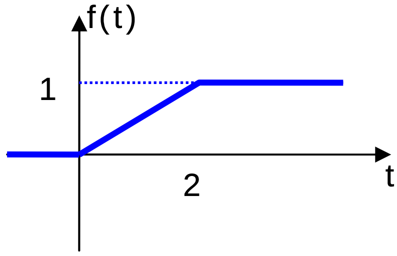
k) $f(t)=\cos\left(2t+\frac{\pi}{3}\right)$

l) $f(t)=\cos^2(2t)$

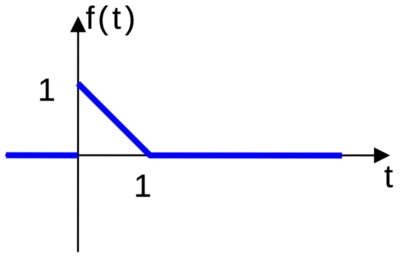
3.2 Find the Laplace transform of the functions shown



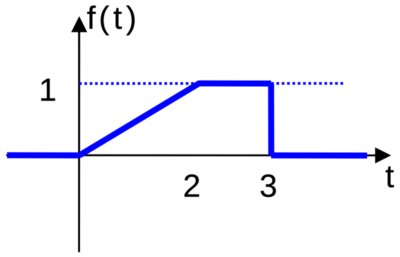
Problem 3.2 (a)



Problem 3.2 (b)



Problem 3.2 (c)



Problem 3.2 (d)

3.12 Find the Laplace transform of

3.13 Find the Laplace transform of

3.14 Prove the following functions are of exponential order

a) $f(t) = t$

b) $f(t) = t^2$

c) $f(t) = t^3$

d) $f(t) = \cos^2(2t)$

e) $f(t) = \sin(3t)$

f) $f(t) = e^{2t}\sin(5t)$