King Fahd University of Petroleum and Minerals

Prep-Year Math Program

Math 002 Class Test II
Textbook Sections: 6.3 to 8.3
Term 142
Time Allowed: 90 Minutes

Student's Name: …………… ………………………………………………
ID #: ………………………. **Section:** ……. **Serial Number:** …….

Provide neat and complete solutions.
Show all necessary steps for full credit and write the answer in simplest form.

No Calculators, Cameras, or Mobiles are allowed during this exam.

Question	Points	Student's Score
1	7	
2	7	
3	8	
4	8	
5	7	
6	8	
7	7	
8	8	
9	7	
10	10	
11	7	
12	8	
13	8	
Total	**100**	___ / **100**

Q1. (7 points):

(a): Graph the function $f(x) = -\frac{3}{2}\cos\left(\frac{3}{4}x\right)$ over the interval $\left[-\frac{8\pi}{3}, \frac{8\pi}{3}\right]$

(b): Determine the intervals where the graph of the function is above the x-axis.

(c): Determine the intervals where the function is decreasing.

Solution (a):

Q8. For $-\frac{8\pi}{3} \le x \le \frac{8\pi}{3}$, the graph of the function $f(x) = -\frac{3}{2}\cos\left(\frac{3}{4}x\right)$ is above the x-axis on the intervals

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A) $\left(-2\pi, -\frac{2\pi}{3}\right) \cup \left(\frac{2\pi}{3}, 2\pi\right)$

B) $\left(-\frac{4\pi}{3}, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{3}, 2\pi\right)$

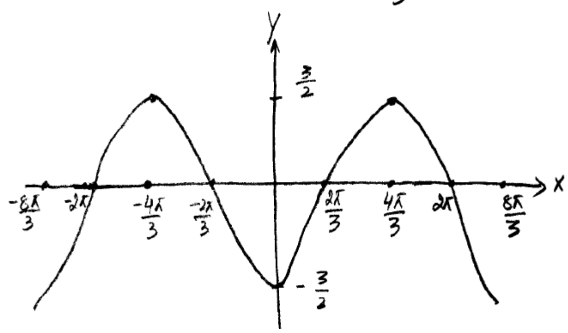
C) $\left(-\pi, -\frac{\pi}{3}\right) \cup \left(\pi, \frac{8\pi}{3}\right)$

D) $\left(-2\pi, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{3}, 2\pi\right)$

E) $\left(-\frac{5\pi}{3}, 0\right) \cup \left(\frac{4\pi}{3}, \frac{8\pi}{3}\right)$

$0 \le \frac{3}{4}x \le 2\pi$

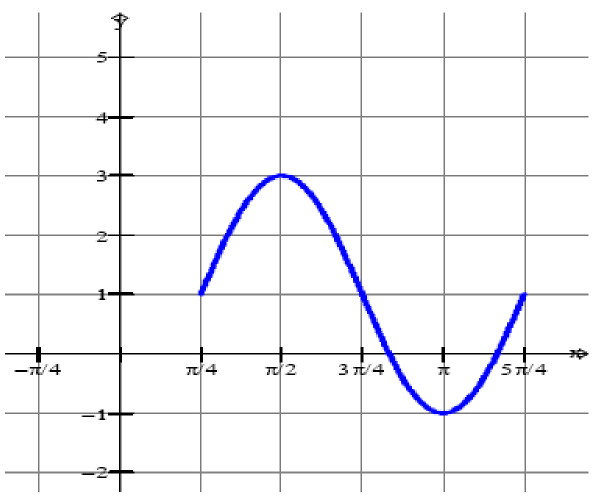
$0 \le x \le \frac{8\pi}{3}$



(b): The graph is above the x-axis on $\left(-2\pi, -\frac{2\pi}{3}\right) \cup \left(\frac{2\pi}{3}, 2\pi\right)$

(c): The function is decreasing on $\left[-\frac{4\pi}{3}, 0\right]$ and $\left[\frac{4\pi}{3}, \frac{8\pi}{3}\right]$

Q2. (7 points): The graph given below represents the graph of a sine function of the form $y = a\sin(bx + c) + d$. Find the values of $a, b, c,$ and d.



Solution:

$|a| = \text{Amplitude} = \frac{\text{Maximum} - \text{minimum}}{2} = \frac{3-(-1)}{2} = \frac{4}{2} = 2 \Rightarrow |a| = 2 \Rightarrow a = -2,\ \boxed{a = 2}$

$\text{Period} = \frac{5\pi}{4} - \frac{\pi}{4} = \pi$

$$Period = \frac{2\pi}{|b|}$$

$$\pi = \frac{2\pi}{|b|}$$

$$b = -2 \text{ , } \boxed{b = 2}$$

$$Maximum = |a| + d$$

$$3 = 2 + d$$

$$\boxed{d = 1}$$

$$\text{Phase shift} = \frac{\pi}{4} \text{ , } \quad bx + c = 0 \quad \Rightarrow \quad 2\left(\frac{\pi}{4}\right) + c = 0 \quad \Rightarrow \quad \Rightarrow \quad \boxed{c = -\frac{\pi}{2}}$$

OR:

$$\text{Phase shift} = -\frac{c}{b}$$

$$\frac{\pi}{4} = -\frac{c}{2} \quad \Rightarrow \quad \boxed{c = -\frac{\pi}{2}}$$

$$y = a\sin(bx + c) + d = 2\sin\left(2x - \frac{\pi}{2}\right) + 1$$

Q3. (8 points): (6.5 Recitation Q#1): Consider the function $f(x) = -2\tan(2x - \frac{\pi}{4})$, find the equation of all vertical asymptotes over the interval $[-2\pi, 2\pi]$

Solution:

Find the equation of all vertical asymptotes over the interval $[-2\pi, 2\pi]$.

$$-\frac{\pi}{2} < 2x - \frac{\pi}{4} < \frac{\pi}{2}$$

$$-2\pi < 8x - \pi < 2\pi$$

$$-\pi < 8x < 3\pi$$

$$-\frac{\pi}{8} < x < \frac{3\pi}{8}$$

The equations of the vertical asymptotes over the interval $[-2\pi, 2\pi]$ are:

$$\boxed{x = -\frac{\pi}{8}}, \quad \boxed{x = \frac{3\pi}{8}}$$

$$x = -\frac{\pi}{8} - \frac{\pi}{2}, \quad x = \frac{3\pi}{8} + \frac{\pi}{2} \quad \Rightarrow \quad \boxed{x = -\frac{5\pi}{8}}, \quad \boxed{x = \frac{7\pi}{8}}$$

$$x = x = -\frac{5\pi}{8} - \frac{\pi}{2}, \quad x = \frac{7\pi}{8} + \frac{\pi}{2} \quad \Rightarrow \quad \boxed{x = -\frac{9\pi}{8}}, \quad \boxed{x = \frac{11\pi}{8}}$$

$$x = -\frac{9\pi}{8} - \frac{\pi}{2}, \quad x = \frac{11\pi}{8} + \frac{\pi}{2} \quad \Rightarrow \quad \boxed{x = -\frac{13\pi}{8}}, \quad \boxed{x = \frac{15\pi}{8}}$$

Q4. (8 points): (6.6 Textbook Exercise15): Given $y = 2 + 3\sec(2x - \pi)$, $\frac{\pi}{4} < x < \frac{7\pi}{4}$.

(a): Graph the function over the interval $\left(\frac{\pi}{4}, \frac{7\pi}{4}\right)$.

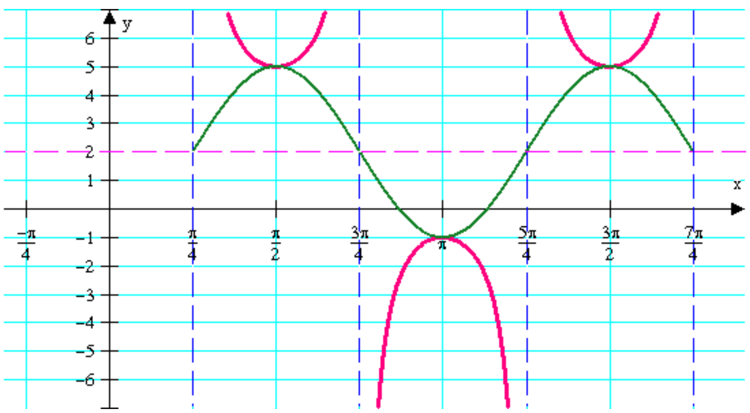
(b): Find the intervals where the function is decreasing.
(c): Find the intervals where the function is above the x-axis.
(d): Find the coordinates of points where the function has local minimum.

Solution: $0 \le 2x - \pi \le 2\pi \Rightarrow \pi \le 2x \le 3\pi \Rightarrow \frac{\pi}{2} \le x \le \frac{3\pi}{2}$

Phase shift is $\frac{\pi}{2}$.

The next key point is $\frac{\pi}{2} + \frac{1}{4}P = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

First, graph $y = 2 + 3\cos(2x - \pi)$, then graph $y = 2 + 3\sec(2x - \pi)$.



(b): The function is decreasing on $\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$, $\left[\pi, \frac{5\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right]$

(c): The graph is above the x-axis on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

(d): The points are $\left(\frac{\pi}{2}, 5\right)$ and $\left(\frac{3\pi}{2}, 5\right)$

Q5. (7 points): (7.1 Recitation Q3):

Write $\csc t$ in terms of $\tan t$, where $\pi < t < \frac{3\pi}{2}$..

Solution: $$\csc t = \frac{1}{\sin t} = \frac{\frac{1}{\cos t}}{\frac{\sin t}{\cos t}} = \frac{\sec t}{\tan t} = \frac{-\sqrt{1+\tan^2 t}}{\tan t}$$

Another Method:

$\csc^2 t = 1 + \cot^2 t$

$$\csc t = -\sqrt{1+\cot^2 t} = -\sqrt{1+\frac{1}{\tan^2 t}} = -\sqrt{\frac{\tan^2 t+1}{\tan^2 t}} = -\frac{\sqrt{\tan^2 t+1}}{\sqrt{\tan^2 t}} = -\frac{\sqrt{\tan^2 t+1}}{|\tan t|}$$

$$= -\frac{\sqrt{\tan^2 t+1}}{\tan t}$$ (because $\tan t$ is positive for t in Quadrant III)

Q6. (8points): (7.2 Exercises 20 and 21): Factor:

(a): $\cot^4 x + 3\cot^2 x + 2$

(b): $\sin^3 x - \cos^3 x$

Solution:

(a):

20. $\cot^4 x + 3\cot^2 x + 2$

Let $\cot^2 x = a.$

$$\begin{aligned}\cot^4 x + 3\cot^2 x + 2 &= a^2 + 3a + 2\\ &= (a+2)(a+1)\\ &= (\cot^2 x + 2)(\cot^2 x + 1)\\ &= (\cot^2 x + 2)(\csc^2 x)\\ &= \csc^2 x(\cot^2 x + 2)\end{aligned}$$

(b):

21. $\sin^3 x - \cos^3 x$

Let $\sin x = a$ and $\cos x = b.$

$\sin^3 x - \cos^3 x$

$$\begin{aligned}&= a^3 - b^3 = (a-b)(a^2 + ab + b^2)\\ &= (\sin x - \cos x)(\sin^2 x + \sin x\cos x + \cos^2 x)\\ &= (\sin x - \cos x)\left[(\sin^2 x + \cos^2 x) + \sin x\cos x\right]\\ &= (\sin x - \cos x)(1 + \sin x\cos x)\end{aligned}$$

Q7. (7 points): (7.3 Exercise 65): $\tan\frac{11\pi}{12} = ?$

$$\begin{aligned}\tan\frac{11\pi}{12} &= -\tan\left(\pi - \frac{11\pi}{12}\right)\\ &= -\tan\left(\frac{\pi}{12}\right)\\ &= \tan\left(\frac{-\pi}{12}\right)\\ &= \tan\left(\frac{3\pi}{12} - \frac{4\pi}{12}\right)\\ &= \tan\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\end{aligned}$$

$$= \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{3}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{3}} = \frac{1-\sqrt{3}}{1+\sqrt{3}} = \frac{1-\sqrt{3}}{1+\sqrt{3}}\cdot\frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{1-2\sqrt{3}+3}{1-3} = \frac{4-2\sqrt{3}}{-2} = -2+\sqrt{3}$$

Q8. (8 points): (7.4 Exercise 30): Write $\cos 3x$ in terms of $\cos x$ as:
$\cos 3x = A\cos x + B\cos^3 x$. Determine the value of $A = ?$ and $B = ?$

Solution:

$$\cos 3x = \cos(x+2x)$$
$$= \cos x \cos 2x - \sin x \sin 2x$$
$$= \cos x\left(2\cos^2 x - 1\right) - \sin x\left(2\sin x \cos x\right)$$
$$= 2\cos^3 x - \cos x - 2\cos x \sin^2 x$$
$$= 2\cos^3 x - \cos x - 2\cos x\left(1-\cos^2 x\right)$$
$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$
$$-3\cos x + 4\cos^3 x$$

$\boxed{A=-3}$ and $\boxed{B=4}$

Another Method:

30.
$$\cos 3x = \cos(2x+x)$$
$$= \cos 2x \cos x - \sin 2x \sin x$$
$$= \left(1-2\sin^2 x\right)\cos x - \left(2\sin x \cos x\right)\sin x$$
$$= \cos x - 2\sin^2 x \cos x - 2\sin^2 x \cos x$$
$$= \cos x - 4\sin^2 x \cos x$$
$$= \cos x\left(1-4\sin^2 x\right)$$
$$= \cos x\left[1-4\left(1-\cos^2 x\right)\right]$$
$$= \cos x\left(-3+4\cos^2 x\right)$$
$$= -3\cos x + 4\cos^3 x$$

$\boxed{A=-3}$ and $\boxed{B=4}$

Q9. (7 points): (7.4 Exercise 60): Given $\sin\theta = -\frac{4}{5}$, with $180° < \theta < 270°$. Find $\cos\frac{\theta}{2}$

60. Find $\cos\frac{\theta}{2}$, if $\sin\theta = -\frac{4}{5}$, with $180° < \theta < 270°$.
Since $180° < \theta < 270°$, θ is in quadrant III, Thus, $\cos\theta < 0$.

$$\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\left(-\frac{4}{5}\right)^2}$$
$$= -\sqrt{1-\frac{16}{25}} = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$$

Since $180° < \theta < 270° \Rightarrow 90° < \frac{\theta}{2} < 135°$, $\frac{\theta}{2}$ is in quadrant II. Thus, $\cos\frac{\theta}{2} < 0$.

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}}$$
$$= -\sqrt{\frac{\frac{2}{5}}{2}} = -\sqrt{\frac{2}{10}} = -\sqrt{\frac{1}{5}}$$
$$= -\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

Q10. (10 points): (7.5 Textbook Exercises 42, 43, 46 and 48): Give the degree measure of θ if it exists.

(a) $\theta = \sec^{-1}(-2)$ **Answer:** $\boxed{120°}$

(b) $\theta = \cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ **Answer:** $\boxed{120°}$

(c) $\theta = \csc^{-1}(-1)$ **Answer:** $\boxed{-90°}$

(d) $\theta = \cot^{-1}(-1)$ **Answer:** $\boxed{135°}$

(e) $\theta = \cos^{-1}(-2)$ **Answer:** undefined or it does not exist

Solution: (a):

42. $\theta = \sec^{-1}(-2)$

$\sec\theta = -2,\ 0° \le \theta \le 180°, \theta \ne 90°$

θ is in quadrant II. The reference angle is $60°$. $\theta = 180° - 60° = 120°$.

(b):

43. $\theta = \cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

$\cot\theta = -\frac{\sqrt{3}}{3},\ 0° < \theta < 180°$

θ is in quadrant II. The reference angle is $60°$. $\theta = 180° - 60° = 120°$

(c):

46. $\theta = \csc^{-1}(-1)$

$\csc\theta = -1,\ -90° \le \theta \le 90°,\ \theta \ne 0°$

Since the terminal side of θ lies on the y-axis, there is no reference angle. $\theta = -90°$.

(d): $\theta = \cot^{-1}(-1)$

$\cot\theta = -1,\quad 0 < \theta < 180°$

θ is in quadrant II. The reference angle is 45°,

Since $\cot 135° = -1,\quad \theta = 135°$

(e):

48. $\theta = \cos^{-1}(-2)$

$\cos\theta = -2, 0° \le \theta \le 180°$

There is no angle θ such that $\cos\theta = -2$.

Q11. (7 points)(7.6 Exercise): Solve $4\sin 3\theta\cos 3\theta - 2\sqrt{3}\sin 3\theta - 2\sqrt{2}\cos 3\theta + \sqrt{6} = 0$, where $0^\circ \le \theta < 180^\circ$

Solution:

$4\sin 3\theta\cos 3\theta - 2\sqrt{3}\sin 3\theta - 2\sqrt{2}\cos 3\theta + \sqrt{2}\sqrt{3} = 0$

$2\sin 3\theta(2\cos 3\theta - \sqrt{3}) - \sqrt{2}(2\cos 3\theta - \sqrt{3}) = 0$

$(2\cos 3\theta - \sqrt{3})(2\sin 3\theta - \sqrt{2}) = 0$

$2\cos 3\theta - \sqrt{3} = 0$, $2\sin 3\theta - \sqrt{2} = 0$

$\cos 3\theta = \dfrac{\sqrt{3}}{2}$, $\sin 3\theta = \dfrac{\sqrt{2}}{2}$

$3\theta = 30^\circ + k\,360^\circ$, $3\theta = 330^\circ + k\,360^\circ$, $3\theta = 45^\circ + k\,360^\circ$, $3\theta = 135^\circ + k\,360^\circ$ where k is an integer.

$\theta = 10^\circ + k\,120^\circ$, $\theta = 110^\circ + k\,120^\circ$, $\theta = 15^\circ + k\,120^\circ$, $\theta = 45^\circ + k\,120^\circ$

$k = 0 \Rightarrow \theta = 10^\circ , 110^\circ , 15^\circ , 45^\circ$

$k = 1 \Rightarrow \theta = 130^\circ , 230^\circ , 135^\circ , 165^\circ$

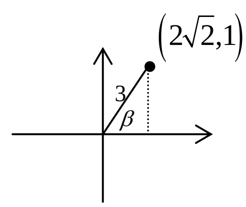
$SS = \{10^\circ, 15^\circ, 45^\circ, 110^\circ, 130^\circ, 135^\circ, 165^\circ\}$

Q12. (8 points): (7.7 Recitation 4): If $\cos^{-1} x - \tan^{-1}\sqrt{3} = \sin^{-1}\dfrac{1}{3}$, then $x =$

Solution: $\cos^{-1} x - \tan^{-1}\sqrt{3} = \sin^{-1}\dfrac{1}{3} \Rightarrow \cos^{-1} x - \dfrac{\pi}{3} = \sin^{-1}\dfrac{1}{3} \Rightarrow \cos^{-1} x = \dfrac{\pi}{3} + \sin^{-1}\dfrac{1}{3}$

$\cos\left(\cos^{-1} x\right) = \cos\left(\dfrac{\pi}{3} + \sin^{-1}\dfrac{1}{3}\right)$

$\Rightarrow x = \cos\left(\dfrac{\pi}{3} + \sin^{-1}\dfrac{1}{3}\right)$. Let $\beta = \sin^{-1}\dfrac{1}{3}$, where $-\dfrac{\pi}{2} \le \beta \le \dfrac{\pi}{2}$.



$x = \cos\left(\dfrac{\pi}{3} + \sin^{-1}\dfrac{1}{3}\right) = \cos\left(\dfrac{\pi}{3} + \beta\right)$

$= \cos\dfrac{\pi}{3}\cos\beta - \sin\dfrac{\pi}{3}\sin\beta$

$= \dfrac{1}{2}\cdot\dfrac{2\sqrt{2}}{3} - \dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{3} = \dfrac{2\sqrt{2}-\sqrt{3}}{6} \Rightarrow SS = \left\{\dfrac{2\sqrt{2}-\sqrt{3}}{6}\right\}$

Q13. (8 points): If θ is an angle between the vectors $\boldsymbol{v} = -\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{w} = 2\boldsymbol{i} - \boldsymbol{j}$, where $0 \le \theta \le \pi$, then find the following:

(a): $\|\boldsymbol{v}\| = ?$ $\|\boldsymbol{w}\| =$

(b): $\dfrac{\boldsymbol{v}.\boldsymbol{w}}{\|\boldsymbol{v}\|.\|\boldsymbol{w}\|} = ?$

(c): $\sin\theta. = ?$

Solution: (a): $\|\boldsymbol{v}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$ and $\|\boldsymbol{w}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

(b): $\cos\theta = \dfrac{\boldsymbol{v}.\boldsymbol{w}}{\|\boldsymbol{v}\|.\|\boldsymbol{w}\|} = \dfrac{<-1,2>\cdot<2,-1>}{\|<-1,2>\|\cdot\|<2,-1>\|} = \dfrac{(-1)2+2(-1)}{\sqrt{5}\cdot\sqrt{5}} = \dfrac{-4}{5}$

(c): Since $0 \le \theta \le \pi$, then $\sin\theta = +\sqrt{1-\cos^2\theta} = \sqrt{1-\left(\dfrac{-4}{5}\right)^2} = \sqrt{1-\dfrac{16}{25}} = \sqrt{\dfrac{25-16}{25}} = \sqrt{\dfrac{9}{25}} = \dfrac{3}{5}$