2.2 INTRODUCTION TO FUNCTIONS

Objectives:

In this section, we will learn about:

- Relations.
- Functions.
- Function Notation.
- Identifying Functions.
- Graphs of Functions.
- The Greatest Integer Function.(Floor Function)

Definition of a Relation

A ***relation*** is any set of ordered pairs.

- The set of all the first coordinate of the ordered pairs is called the ***domain*** **(المجال)** of the relation,
- The set of all the second coordinate of the ordered pairs is called the ***range*** **(المدى)** of the relation.

Ex1 Consider the function $\{(0,0)\ , (-1,1), (2,3)\ \}$

The domain is $\{0,-1,2\}$

The range is $\{0,1,3\}$

Definition of a Function **(دالة)**

A ***function*** is a set of ordered pairs in which no two ordered pairs have the same first coordinate and different second coordinates.

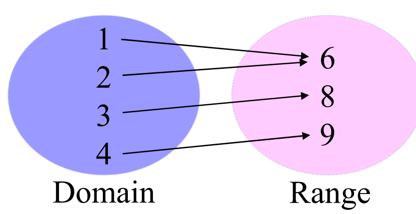
- Such that each element in the domain corresponds to ***exactly one*** element in the range.

Ex 2 Determine whether each relation defines y as a function of x :

a) $\{(1, 6), (2, 6), (3, 8), (4, 9)\}$

Notice that every element in the domain corresponds to exactly one element in the range.

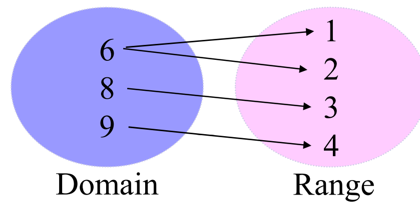
So, it is a function.



b) {(6, 1), (6, 2), (8, 3), (9, 4)}

Notice that 6 in the domain corresponds to both 1 and 2 in the range.

So, it is **not** a function



Function Notation

Functions are usually given in terms of **equations** rather than as sets of ordered pairs where:

- A variable *y* is said to be a function of the variable *x* if for **each value of *x***, there is **one and only one value of *y***.
- The variable *x* is called the ***independent variable*** because it can be assigned **any value** from the **domain**.
- The variable *y* is called the ***dependent variable*** because its value depends on *x*.

Illustration:

y and *f(x)* can be used interchangeably

$$y = f(x) = 3x + 2$$

Name of the function: f

Name of the independent variable: x

Value of the function (Name of the dependent Variable): y

Defining expression: $3x + 2$

f (x) is read as: " the value of f at x" Or " f of x" Or " f evaluated at x"

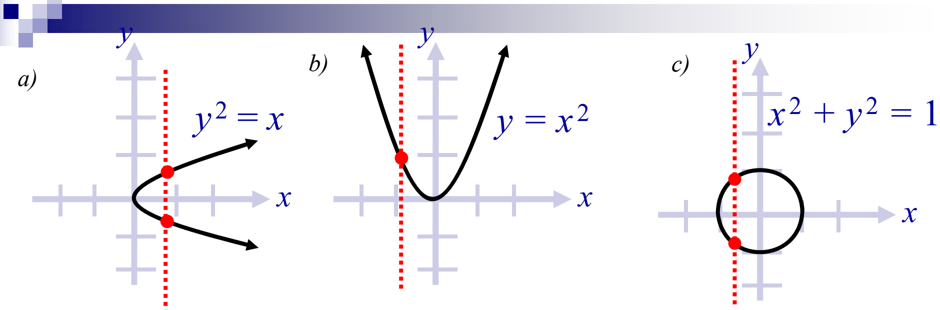
How to identify if an equation represents a function or not?

Case1: Algebraically (جبرياً)

Solve for y:
- one value of y ⇒ function
- two values of y ⇒ not function

Case 2: Graphically (من الرسم)

The vertical (العمودي) line test : A graph is the graph of a function if and only if no vertical line intersects the graph at more than one point.



Not function — function — Not function

Ex 3 Determine whether each equation defines y as a function of x :

Sol:

a) $x^2 + y = 4$

$x^2 + y - x^2 = 4 - x^2$

$y = 4 - x^2$

For each value of x there is one and only one value of y.
Therefore, the equation defines y as a function of x.

b) $x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$|y| = \sqrt{4 - x^2} \quad \Rightarrow y = \pm\sqrt{4 - x^2}$

The ± in the resulting equation shows that for certain values of x (all values between – 2 and 2), there are two values of y. Therefore, the equation does not define y as a function of x.

c) $x^2 + y^2 = 4$, if $y < 0$

$y^2 = 4 - x^2$

$|y| = \sqrt{4 - x^2}$

$y = -\sqrt{4 - x^2}$ *only*

For each value of x there is one and only one value of y, Therefore, the equation defines y as a function of x.

d) $y^3 + x = 4$

$y^3 = 4 - x$

$y = \sqrt[3]{4 - x}$

For each value of x there is one and only one value of y.
Therefore, the equation defines y as a function of x.

e) $|x| + y = 3$

$y = 3 - |x|$

For each value of x there is one and only one value of y.
Therefore, the equation defines y as a function of x.

To ***evaluate*** a function $f(x)$ at $x = a$, **substitute** the specified value a for x into the given function.

Ex 4 If $f(x) = x^2 + 3x + 5$, evaluate:

a) $f(2)$

b) $f(x + 3)$

c) $f(-x)$

d) $f(2a)$

Sol: Given $f(x) = x^2 + 3x + 5$

a) $f(2) = 2^2 + 3 \cdot 2 + 5$

$f(2) = 15$

b) $f(x + 3)$ " the value of f at ($x+3$)"

$= (x + 3)^2 + 3 \cdot (x + 3) + 5$

$= x^2 + 6x + 9 + 3x + 9 + 5$

$= x^2 + 9x + 23$

c) $f(-x)$ " the value of f at $-x$"

$f(-x) = (-x)^2 + 3(-x) + 5$

$= x^2 - 3x + 5$

Sol: Given $f(x) = x^2 + 3x + 5$

d) $f(2a)$ " the value of f at $2a$ "

$= (2a)^2 + 3 \cdot (2a) + 5$

$= 4a^2 + 6a + 5$

Piecewise-Defined Functions :

are functions represented by **more** than one expression.

Ex 5 Given the function:

$$f(x) = \begin{cases} 2x, & x < -2 \\ x^2, & -2 \le x < 1 \\ 4 - x, & x \ge 1 \end{cases}$$

Find:

a) $f(-4)$

b) $f(-1)$

c) $f(7)$.

d) Graph the function.

Sol

a) $f(-4)$ Since $-4 < -2$, substitute in the first piece

$= 2(-4) = -8$

$$f(x)=\begin{cases} 2x, & x<-2 \\ x^2, & -2\le x<1 \\ 4-x, & x\ge 1 \end{cases}$$

b) $f(-1)$ Since $-2 < -1 < 1$, substitute in the second piece

$= (-1)^2 = 1$

c) $f(7)$ Since $7 \geq 1$, substitute in the third piece

$= 4 - 7 = -3$

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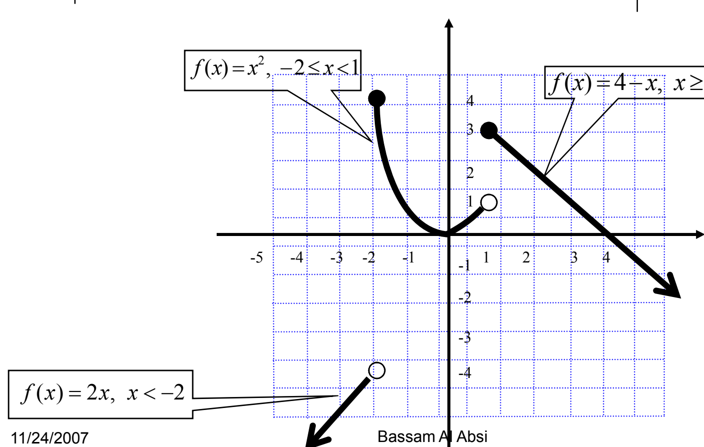
d) Make data table of some sample points for each equation(piece)

x	$f(x) = x^2,\ -2 \le x < 1$
−2	4
−1	1
0	0
1	1 (not included)

x	$f(x) = 4 - x,\ x \ge 1$
1	3 (included)
2	2
3	1

x	$f(x) = 2x,\ x < -2$
−2	− 4 (not included)
−3	− 6
−4	−8



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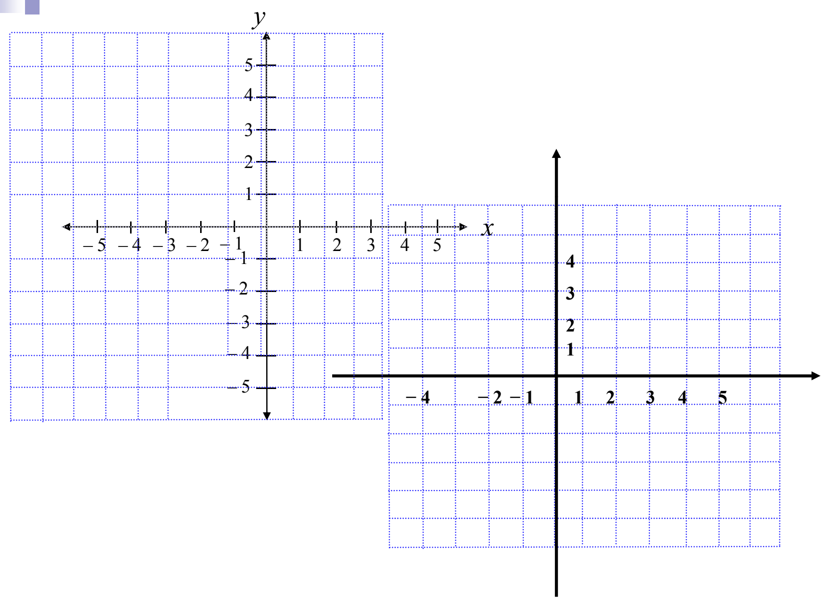


The domain of a function *f* :

is the set of all real numbers for which the function formula yields real number function values.

Hints on Finding a Function’s Domain:

- **Exclude** from a function’s domain real numbers that cause ***division by zero.***
- **Exclude** from a function’s domain **real numbers** that ***cause an even root of a negative number.***

Ex 6 Find the domain of each function:

a) $f(x) = x^2 - 7x$ contains neither division nor an even root.

The domain of f is the set of all real numbers.

b) $g(x) = \dfrac{6x}{x^2 - 9}$ we exclude values that cause $x^2 - 9$ to equal 0.

The domain of g is $\{x|\ x$ is any real number, $x \neq -3, x \neq 3\}$.

c) $h(x) = \sqrt{3x + 12}$ contains an even root

$$3x + 12 \geq 0$$

$$3x \geq -12$$

$$x \geq -4$$

The domain of h is $\{x|\ x$ is any real number, $x \geq -4\}$ or the interval $[-4, \infty)$.

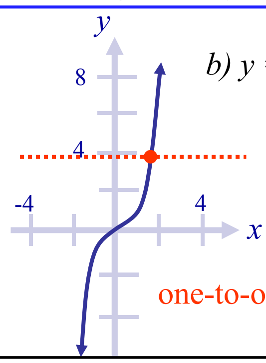
The **graph of a function** is the graph of its ordered pairs.

- The **zeros of a function** f are the x-values for which $f(x) = 0$.
- A **function can have more than one x-intercept but at most one y-intercept**, $f(0) = b$.

Horizontal (أفقي) Line Test

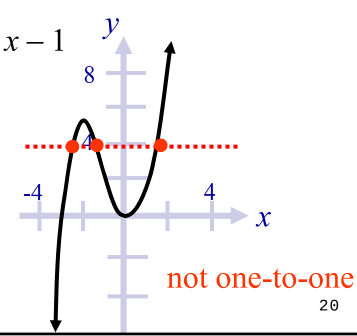
A function $y = f(x)$ is **one-to-one** if and only if no horizontal line intersects the graph of $y = f(x)$ in more than one point.

a) $y = x^3$



one-to-one

b) $y = x^3 + 3x^2 - x - 1$

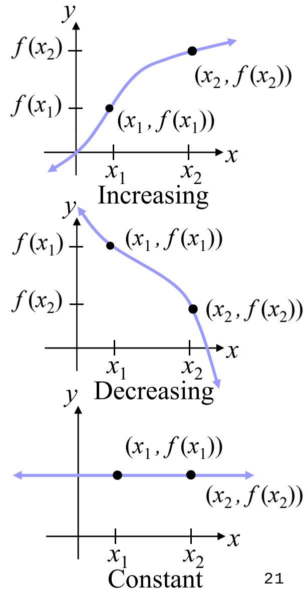


not one-to-one

Increasing, Decreasing, and Constant Functions

تزايد تناقص ثابت

1. A function is ***increasing*** on an interval
   - if for any x_1 and x_2 in the interval,
   - where $x_1 < x_2$,
   - then $f(x_1) < f(x_2)$.
2. A function is ***decreasing*** on an interval
   - if for any x_1 and x_2 in the interval,
   - where $x_1 < x_2$,
   - then $f(x_1) > f(x_2)$.
3. A function is ***constant*** on an interval
   - if for any x_1 and x_2 in the interval,
   - where $x_1 < x_2$,
   - then $f(x_1) = f(x_2)$.



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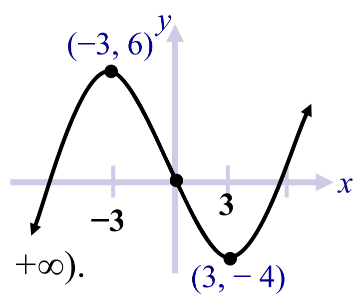
Ex 7 Find the intervals on which the adjacent graph is:

increasing or decreasing

Hints:

1) Start from left to right
2) Follow the graph
3) Read the intervals from the *x*-axis



- **Increasing** on $(-\infty, -3) \cup (3, +\infty)$.
- **Decreasing** on $(-3, 3)$

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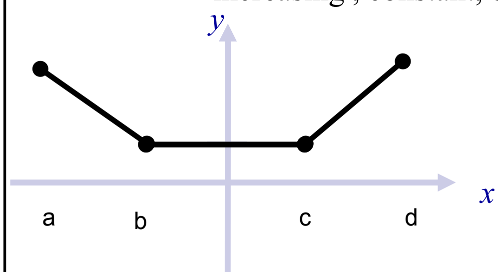
Ex 8 Find the intervals on which the adjacent graph is:

increasing , constant, or decreasing.



Hints:

1) Start from left to right
2) Follow the graph
3) Read the intervals from the x-axis

- **Increases** on (c, d)
- **Decreases** on (a, b)
- **Constant** on (b, c)

The Greatest Integer Function

$f(x) = \text{int}(x)$ or $[[x]]$ greatest integer less than or equal to x.

$$[\![x]\!] = \begin{cases} x & \text{, if } x \text{ is an integer} \\ \text{greates integer less than } x & \text{, if } x \text{ is not an integer} \end{cases}$$

Ex 9 Evaluate

$\text{int}(4.5) = 4$

$\text{int}(3) = 3$

$\text{int}(0.7) = 0$

$\text{int}(-3.5) = -4$

In general, If $n \le x < n+1$, then $\text{int}(x) = n$, where n is an integer,

If $[\![f(x)]\!] = n \iff n \le f(x) < n+1$, n: integer

If $f(x) = [\![x]\!]$, then

$D_f = (-\infty, \infty)$, R_f = the set of all integers

Ex 10 Find all values of x that satisfy the equation $[\![2x-1]\!] = \frac{1}{2}$

Sol

$[\![2x-1]\!] = \frac{1}{2}$ **Contradiction ?**

No solution

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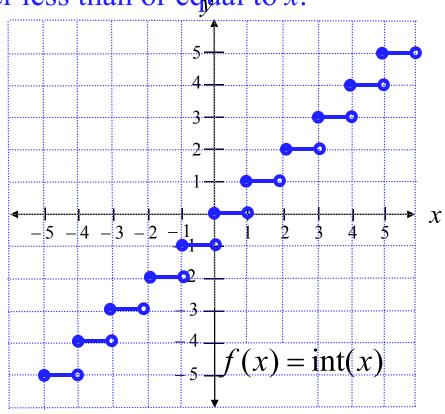
The Greatest Integer Function

$f(x) = \text{int}(x)$ or $[[x]]$ greatest integer less than or equal to x.

x	$y = \text{int}(x)$	
If $2 \le x < 3$,	2	, $\text{int}(x) = 2$
If $1 \le x < 2$,	1	, $\text{int}(x) = 1$
If $0 \le x < 1$,	0	, $\text{int}(x) = 0$
If $-1 \le x < 0$,	-1	, $\text{int}(x) = -1$
If $-2 \le x < -1$	-2	, $\text{int}(x) = -2$
If $-3 \le x < -2$	-3	, $\text{int}(x) = -3$



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Ex 11 If $f(x)=\begin{cases}\left[\!\left[\frac{1}{3}x\right]\!\right] & if \quad x<0\\ |4-3x| & if \quad x\geq 0\end{cases}$

, then evaluate $f(-5)+f(5)$

Sol $f(-5)+f(5)=\left[\!\left[\frac{1}{3}(-5)\right]\!\right]+|4-3(5)|$

$=\left[\!\left[\frac{-5}{3}\right]\!\right]+|-11|$

$=-2+11$

$=9$

Ex 12 Find the domain of the function $f(x)=\dfrac{1}{[\![x-1]\!]}$

Sol

$[\![x-1]\!]\neq 0 \Rightarrow x-1<0$ and $x-1\geq 1$

$x<1$ and $x\geq 2$ thus, the domain is $(-\infty,1)\cup[2,\infty)$

Ex 13 Find the x-intercept and y-intercept of $f(x)=\left[\!\left[\frac{2x+1}{3}\right]\!\right]$

Sol

y-intercept : set $x=0$ $\Rightarrow y=\left[\!\left[\frac{2(0)+1}{3}\right]\!\right]\Rightarrow y=\left[\!\left[\frac{1}{3}\right]\!\right] \Rightarrow y=0$

$\Rightarrow$ *the* y-intercept $(0,0)$

x-intercept : set $y=0$

$0=\left[\!\left[\frac{2x+1}{3}\right]\!\right] \Rightarrow 0\leq\frac{2x+1}{3}<1 \Rightarrow 0\leq 2x+1<3 \Rightarrow \frac{-1}{2}\leq x<1$

The x-intercept is [− ½ , 1)

Ex 12 Let $f(x) = \begin{cases} 2 & if \quad x < -2 \\ |x| & if \quad -2 \le x < 1 \\ x-1 & if \quad 1 \le x < 5 \end{cases}$

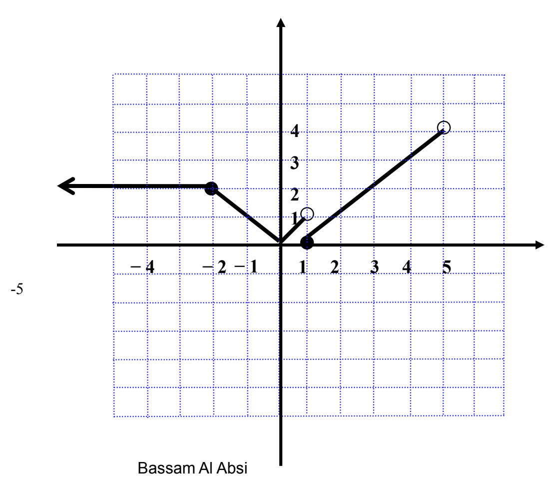
1) sketch the graph of $f(x)$
2) find $f(-2),\ f(0),\ f(3),\ f(6)$
3) determine the $x-$intercept and the $y-$intercept
4) find the interval (s) of increasing, decreasing, and constant
5) find the domain and the range of $f(x)$
6) find all values of x such that $f(x) = 3$

Sol Make data table of some sample points for each equation(piece) based on the **domain of each piece**.

x	$f(x) = 2,\ x < -2$
-2	2 (not included)
-3	2
-4	2

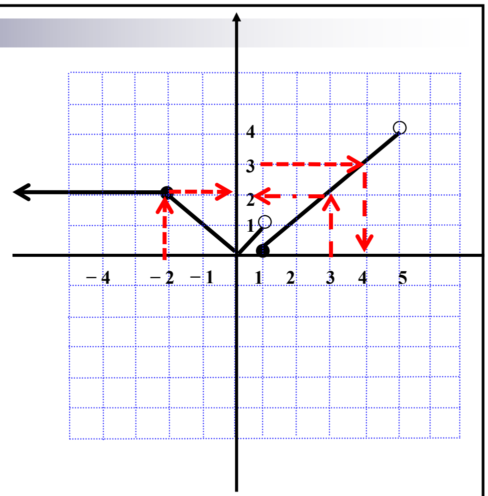
x	$f(x) = \lvert x \rvert,\ -2 \le x < 1$
-2	2
-1	1
0	0
1	1 (not included)

x	$f(x) = x - 1,\ 1 \le x < 5$
1	0
2	1
4	3
5	4 (not included)



2) $f(-2)=2$, $f(0)=0$, $f(3)=2$

$f(6)$ *is undefined* ?



3) x-intercept (s) : (0, 0), (1, 0)
y-intercept (s) : (0, 0)

4) The graph is:

Increasing on (0, 5)
Decreasing on (−2, 0)
Constant on $(-\infty, -2)$

5) The domain is $(-\infty, 5)$
The range is [0,4)

6) $f(x) = 3$ when $x = 4$

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Ex 13 Let $f(x)=\begin{cases} -2x-8 & if \quad x<-3 \\ 3x-1 & if \quad -3\le x<2 \\ -4x & if \quad x>2 \end{cases}$

Find all x-intercept(s) and y-intercept(s) without sketching the graph.

Sol

y-intercept : set $x = 0$. Notice that $x = 0$ *lies* in the **second** piece only.

So, substituting $x = 0$ *in the* ***second expression*** yields : y = 3 (**0**) − 1 , or $y = -1$, thus the **y-intercept** is (0, − 1).

x-intercept : set $y = 0$ **,** this puts us in front of three possibilities, which are:

$-2x-8=0, x<-3$, or $3x-1=0, -3\le x<2$, or $-4x=0, \ x>2$

$x=-4$ | $x=1/3$ | $x=0$ *rejected ?*

Thus**,** the **x-intercept(s)** are**:** (− 4, 0) and (1/3, 0) .

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