4.3 LOGARITHMIC FUNCTIONS AND THEIR APPLICATIONS

Logarithmic function is the inverse of an exponential function.
If an exponential function is of the form $g(x) = b^x$ then its inverse (logarithmic function) $f(x) = \log_b x$ $x > 0$, $b > 0$ and $b \neq 1$ is such that $fog(x) = gof(x) = x$. b is called the base.

Changing Exponential to logarithmic and Logarithmic to exponential form

1. If $y = \log_b x$ then $x = b^y$ is its exponential form
2. If $y = b^x$ then $x = \log_b y$ is its logarithmic form

Example 1

Write each of the following equations in exponential form
a) $2 = \log_5(x+7)$ b) $\log_{(x+2)} 3 = y - 7$ c) $\log_x x^5 = 5$

Solution
a) $5^2 = x + 7$ b) $3 = (x+2)^{(y-7)}$ c) $x^5 = x^5$

Example 2

Write each of the following equations in logarithmic form
a) $11 = 11$ b) $\left(\frac{1}{8}\right)^3 = 2k + 5$ c) $b^{\log_b 7} = 7$ c) $2^3 = 8$

Solution
a) $\log_{11} 11 = 1$ b) $3 = \log_{\frac{1}{8}}(2k+5)$ c) $3 = \log_2 8$

Basic Logarithmic Properties

1. $\log_a a = 1$

2. $\log_a 1 = 0$

3. $\log_a a^k = k$

4. $a^{\log_a x} = x$

5. $\log_{\frac{1}{a}} a^k = -k$

Example 3

Evaluate each of the following;

a) $\log_{13} 1 + \log_2 \frac{1}{16}$ b) $\log_{\frac{1}{2}} 4 - 4^{\log_4 5}$ c) $\dfrac{\log_2 8}{\log_3 27}$

Solution

a) $\log_{13} 1 + \log_2 \frac{1}{16} = 0 + -4\log_2 2 = -4$

b) $\log_{\frac{1}{2}} 4 - 4^{\log_4 5} = -2 - 5 = -7$

c) $\dfrac{\log_2 8}{\log_3 27} = \dfrac{\log_2 2^3}{\log_3 3^3} = \dfrac{3\log_2 2}{3\log_3 3} = 1$

Example 4

Find $f^{-1}(x)$ if $f(x) = 3^{-2x+3} + 1$

Solution

$y = 3^{-2x+3} + 1 \Leftrightarrow y - 1 = 3^{-2x+3}$

$\Rightarrow \log_3(y-1) = -2x + 3$

$\Rightarrow 2x = 3 - \log_3(y-1)$

$\Rightarrow x = \frac{3}{2} - \frac{1}{2}\log_3(y-1)$

$\Rightarrow f^{-1}(x) = \frac{3}{2} - \frac{1}{2}\log_3(x-1)$

Example 5

Find $f^{-1}\left(\frac{1}{4}\right)$ if $f(x)$ is an exponential function such that $f(2) = 16$.

Solution

$f(x) = b^x \Rightarrow f^{-1}(x) = \log_b x$

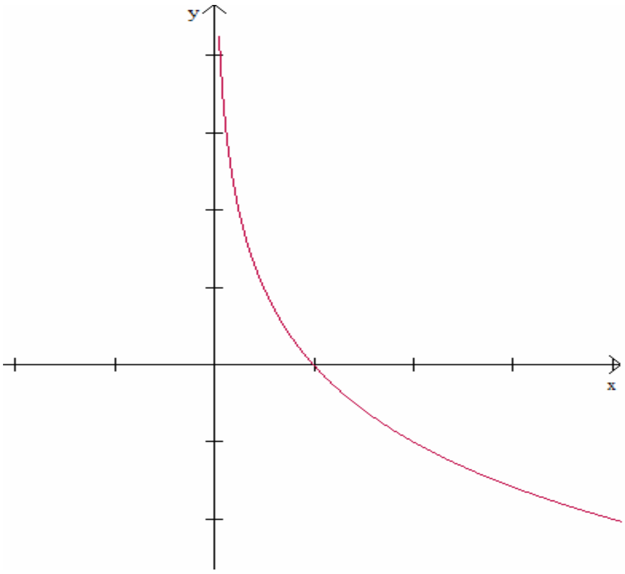
$f(2) = 16 \Rightarrow b^2 = 16$

$\Rightarrow b = 4$

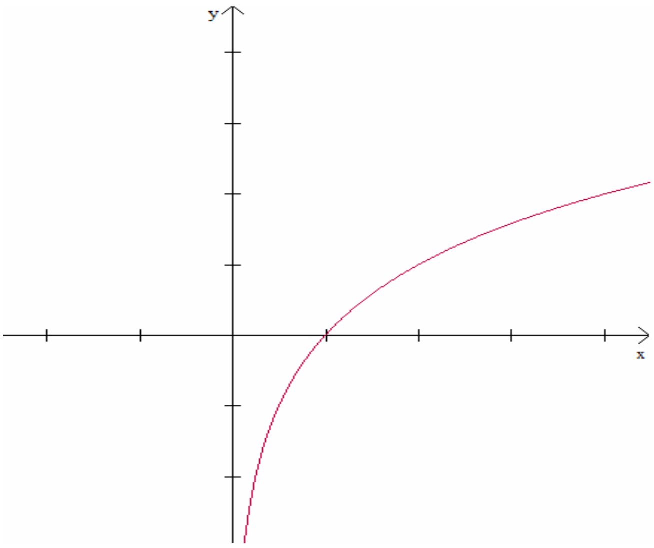
$\therefore f^{-1}\left(\frac{1}{4}\right) = \log_4 \frac{1}{4} = -1$

Graphs of logarithmic functions.

a) $0 < b < 1$



b) $b > 1$



Properties of graphs of Exponential functions:

1. $y-$ intercept at (1,0).
2. Domain is $(0,+\infty)$, Range is $(-\infty,+\infty)$
3. $x-$axis is an asymptote

4. Behaviour of the graph when

   a) $b > 1$

   as $y \to +\infty$, $x \to +\infty$

   as $y \to -\infty$, $x \to 0$

   An increasing function.

   b) $0 < b < 1$

   as $y \to +\infty$, $x \to 0$

as $y \to -\infty$, $x \to +\infty$

A decreasing function

5. One-to one function.
6. No symmetry with respect to x, y or origin

Example 6

Sketch the graph of ;

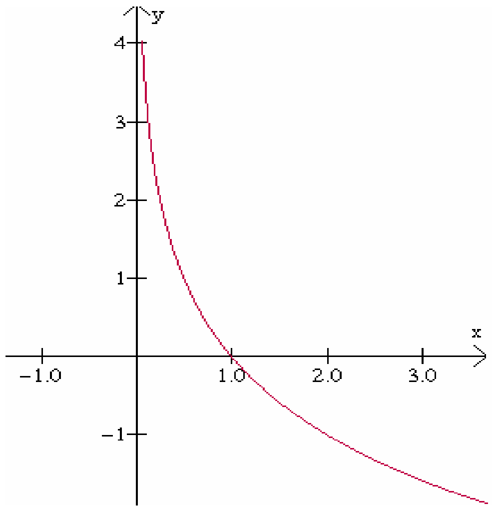
a) $f(x) = \log_{\frac{1}{2}} x$ b) $f(x) = -\log_2(x+1)$

c) $f(x) = \log_{\frac{1}{3}}(-x)$ d) $f(x) = 4 + \log_2(x-1)$ e) $f(x) = \log_2\left[-(x+1)\right]$

Solution

a) Method 1

x	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
y	-2	-1	0	1	2



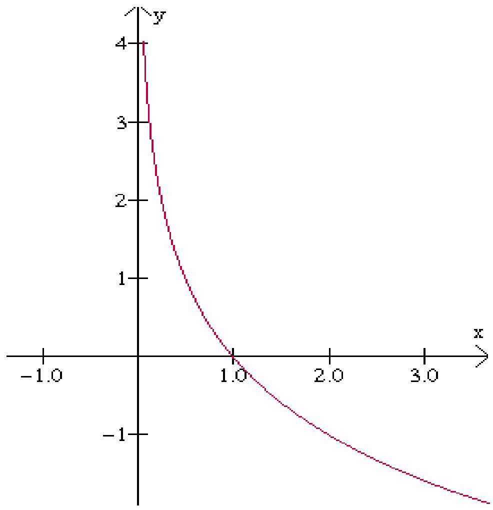
Method 2

$y = \log_{\frac{1}{2}} x \Leftrightarrow \left(\frac{1}{2}\right)^y = x$

as $y \to +\infty$, $x \to \left(\frac{1}{2}\right)^{\infty} = 0$

as $y \to -\infty$, $x \to \left(\frac{1}{2}\right)^{-\infty} = \infty$

$x -$ Intercept $(1,0)$



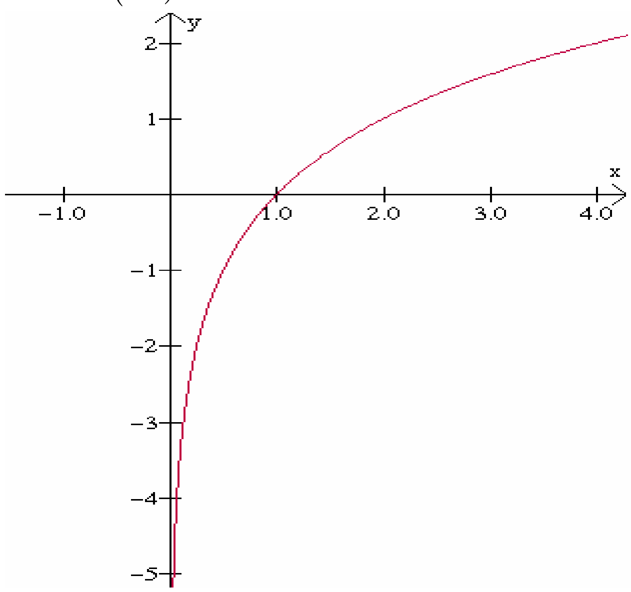
b) *Method 1*

$y = \log_2 x \Leftrightarrow 2^y = x$

as $y \to +\infty$, $x \to 2^{\infty} = \infty$

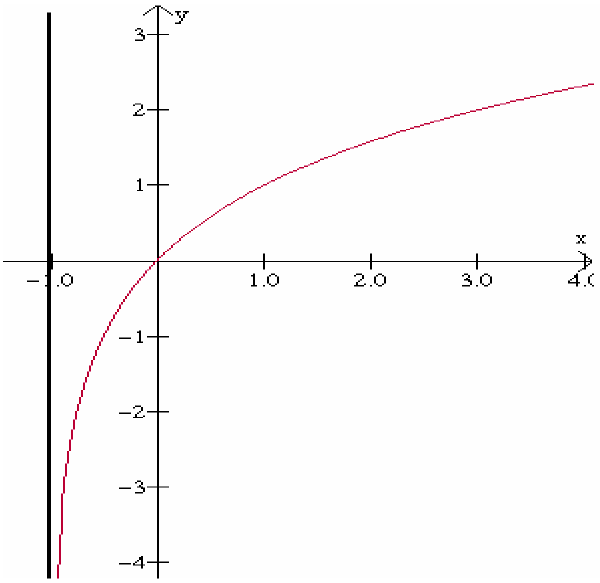
as $y \to -\infty$, $x \to 2^{-\infty} = 0$

$x-$ Intercept $(1,0)$



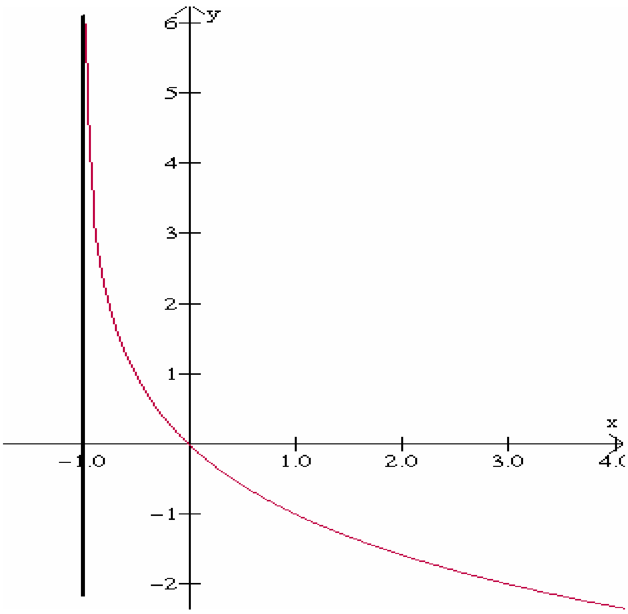
$f(x) = \log_2 x$

The graph of $f(x) = \log_2(x+1)$ is the same as the graph of $f(x) = \log_2 x$ shifted horizontally 1 unit to the left.



$f(x) = \log_2(x+1)$

The graph of $f(x) = -\log_2(x+1)$ is the same as the graph $f(x) = \log_2(x+1)$ reflected in the $x-$axis



$f(x) = -\log_2(x+1)$

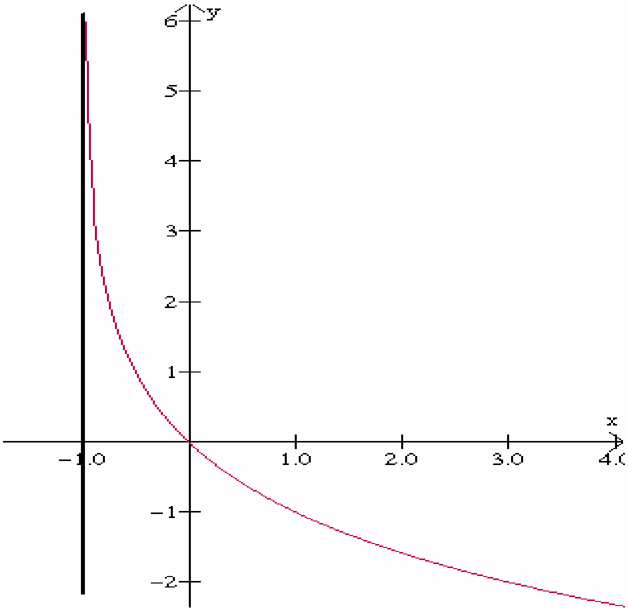
Method 2

$y = -\log_2(x+1) \Leftrightarrow 2^{-y} - 1 = x$

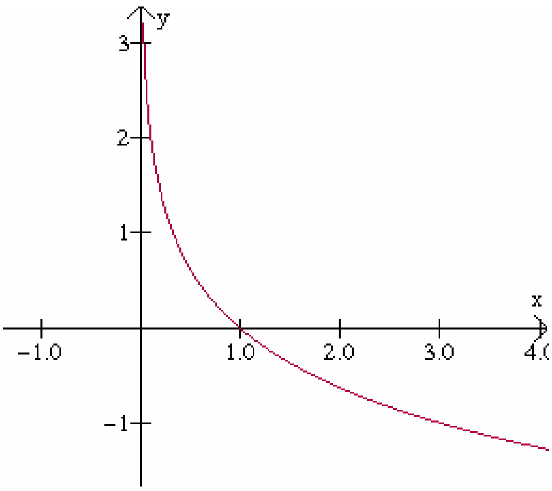
as $y \to +\infty$, $x \to 2^{-\infty} - 1 = -1$

as $y \to -\infty$, $x \to 2^{\infty} - 1 = \infty$

$x-$Intercept $(0,0)$

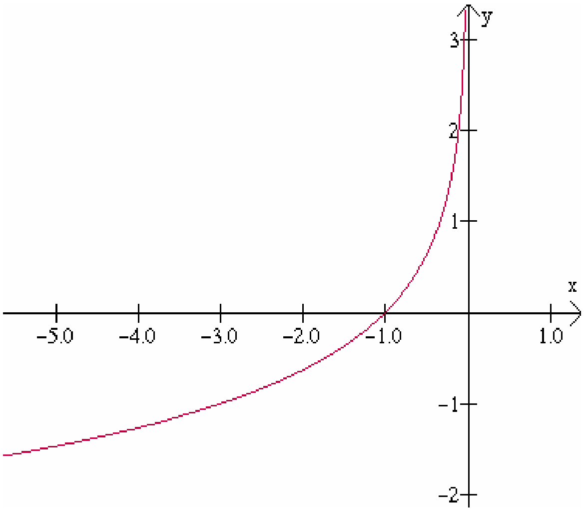


c)



$f(x) = \log_{\frac{1}{3}} x$

The graph of $f(x) = \log_{\frac{1}{3}}(-x)$ is the same as the graph of $f(x) = \log_{\frac{1}{3}} x$ reflected in the $y-$axis

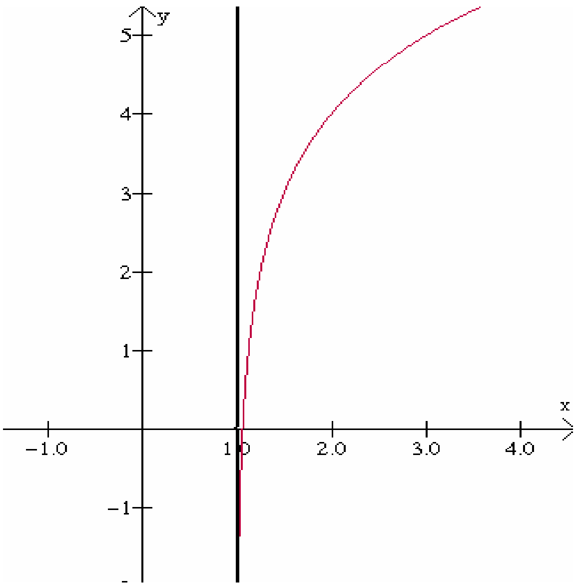


d) $y = 4 + \log_2(x-1) \Leftrightarrow 2^{y-4} + 1 = x$

as $y \to +\infty$, $x \to 2^{\infty-4} + 1 = +\infty$

as $y \to -\infty$, $x \to 2^{-\infty-4} + 1 = 1$

$x-$ Intercept $\left(\frac{17}{16}, 0\right)$

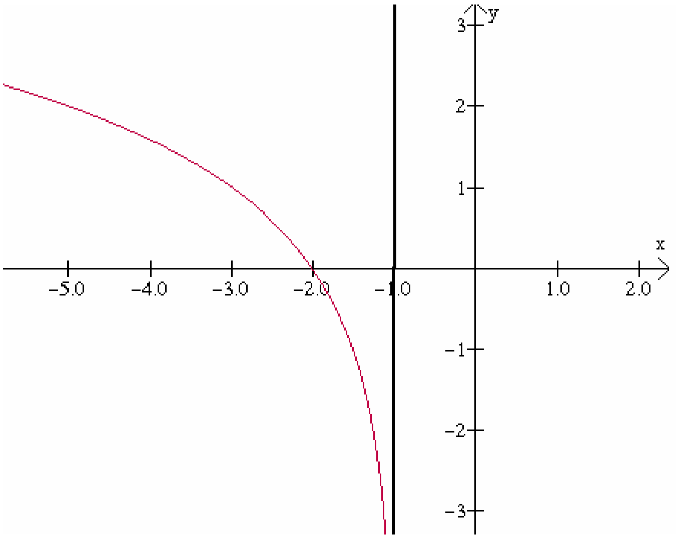


e) $y = \log_2\left[-\left(x+1\right)\right] \Leftrightarrow -2^y - 1 = x$

as $y \to +\infty$, $x \to -2^{\infty} - 1 = -\infty$

as $y \to -\infty$, $x \to -2^{-\infty} - 1 = -1$

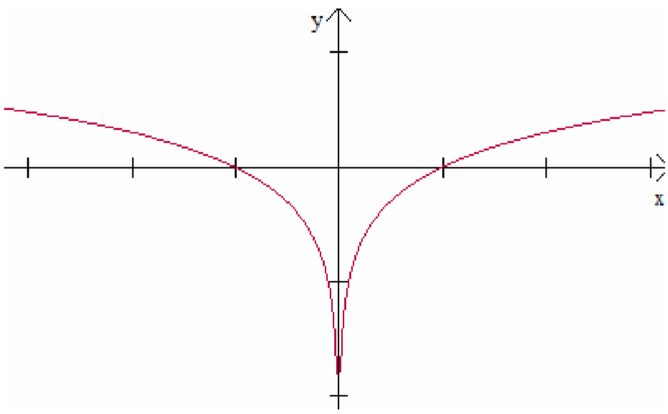
$x-$ Intercept $\left(-2, 0\right)$



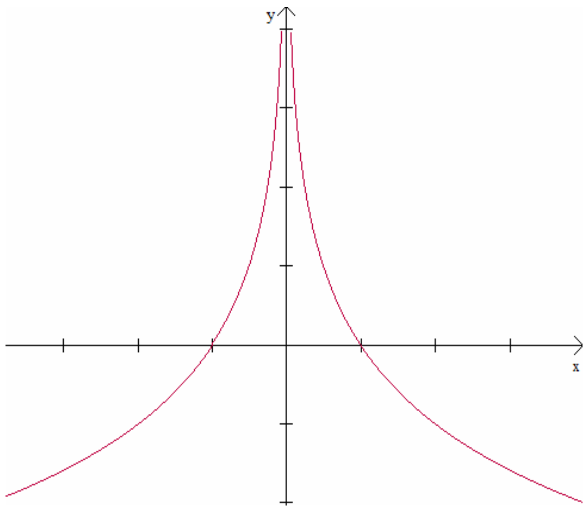
Effect of Absolute Value and Squaring

1. $f(x) = \log_b |x|$ or $f(x) = \log_b x^2$ produces a symmetric graph with y intercept at $(1,0)$.

   a) If $b > 1$ the graph of $f(x)$ has the shape shown below;

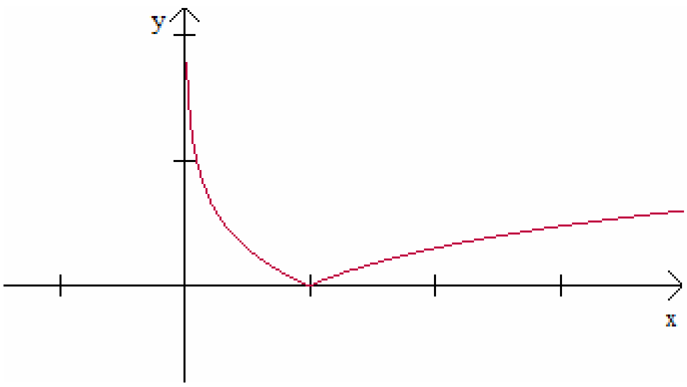


   b) If $0 < b < 1$ the graph of $f(x)$ has the shape shown below

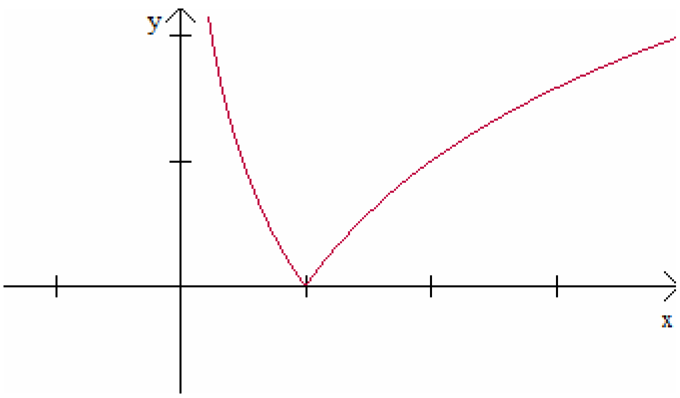


2. $f(x)=|\log_b x|$ produces a non symmetric graph with y intercept at $(1,0)$ with shape as shown below;

a) if $b>1$



b) If $0<b<1$



Example 7:
Sketch the graphs of the equations below and each case state the;

a) domain b) range c) asymptote

a) $f(x)=\log_5|x|$ b) $f(x)=|\log_2 x|+1$

c) $f(x)=2-\left|\log_{\frac{1}{2}} x\right|$ d) $f(x)=\ln(x-2)^2$

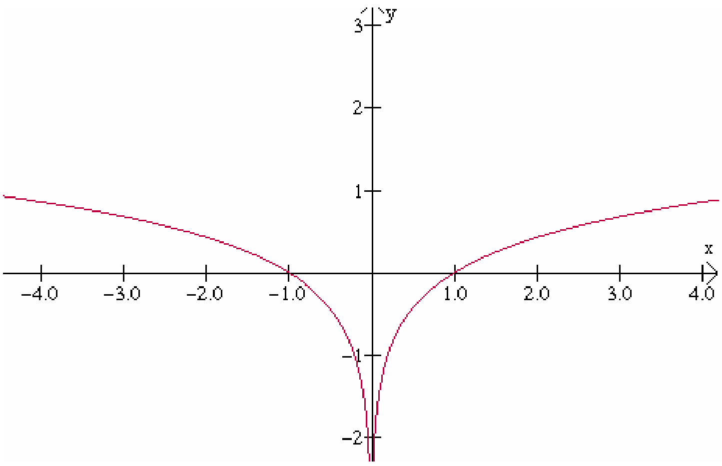
Solution

a) $y=\log_5|x| \Leftrightarrow 5^y=|x|$

as $y\to+\infty$, $x\to\pm\infty$

as $y\to-\infty$, $x\to 0$

$x-$Intercept $(\pm1,0)$



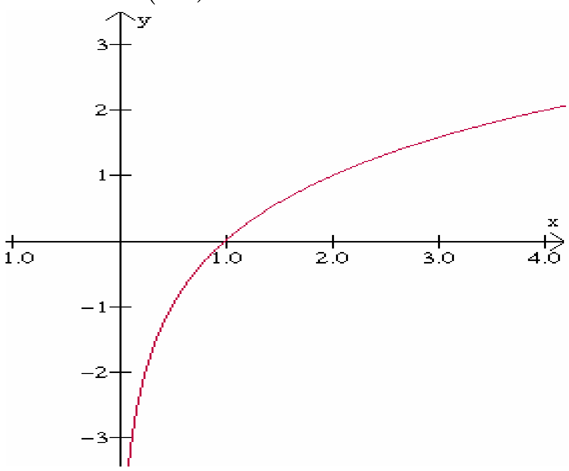
Domain is $(-\infty,0)\cup(0,+\infty)$, Range is $(-\infty,+\infty)$ and asymptote is the line $x=0$

b) $y=\log_2 x \Leftrightarrow 2^y=x$

as $y\to+\infty$, $x\to+\infty$

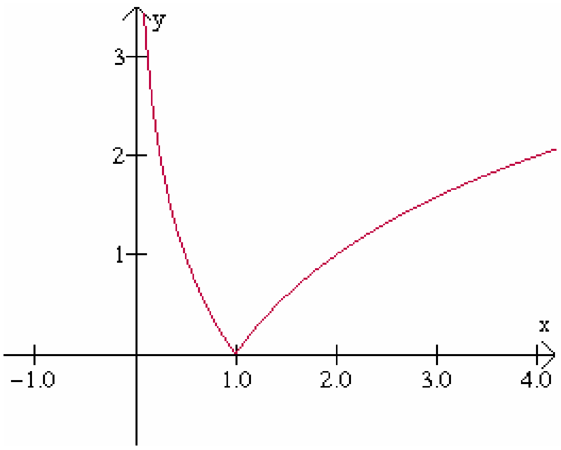
as $y\to-\infty$, $x\to 0$

$x-$Intercept $(1,0)$



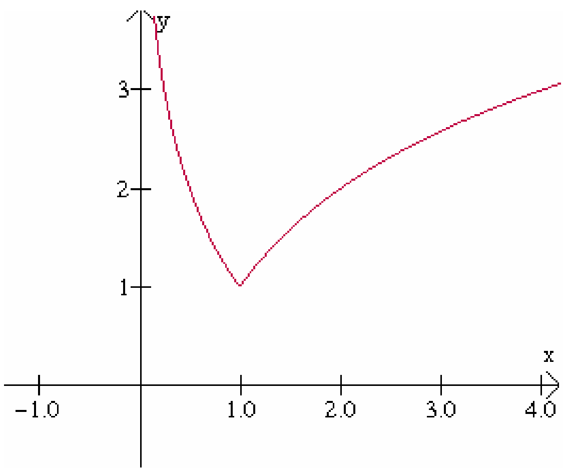
$f(x) = \log_2 x$

The graph of $f(x) = |\log_2 x|$ is the same as the graph of $f(x) = \log_2 x$ with $f(x) < 0$ reflected in the $x-$axis and $f(x) \geq 0$ unchanged



$f(x) = |\log_2 x|$

The graph of $f(x) = |\log_2 x| + 1$ is the same as the graph of $f(x) = |\log_2 x|$ shifted upwards 1 unit.



$f(x) = |\log_2 x| + 1$

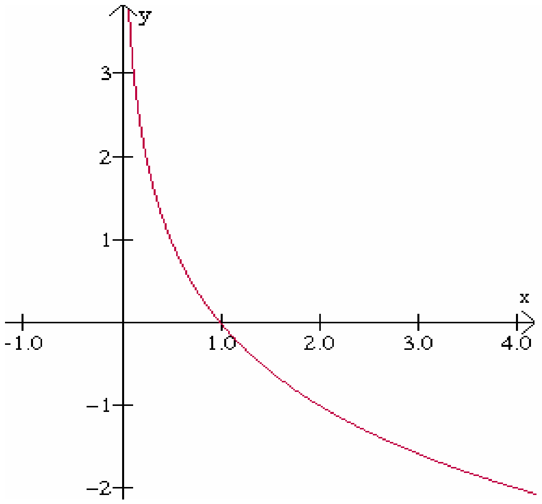
Domain is $(0, +\infty)$, Range is $[1, +\infty)$ and asymptote is the line $x = 0$

c) $y = \log_{\frac{1}{2}} x \Leftrightarrow \left(\frac{1}{2}\right)^y = x$

as $y \to +\infty$, $x \to 0$

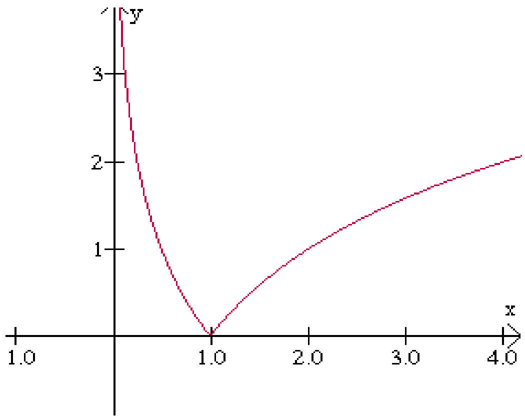
as $y \to -\infty$, $x \to +\infty$

$x-$Intercept $(1, 0)$



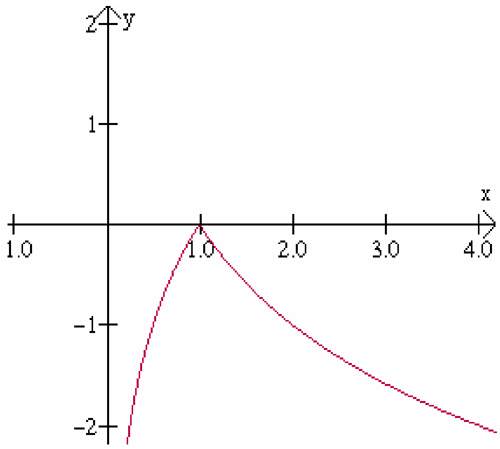
$f\left(x\right)=\log_{\frac{1}{2}} x$

The graph of $f(x)=\left|\log_{\frac{1}{2}} x\right|$ is the same as the graph of $f(x)=\log_{\frac{1}{2}} x$ $f\left(x\right)<0$ reflected in the $x-$axis and $f\left(x\right)\geq 0$ unchanged



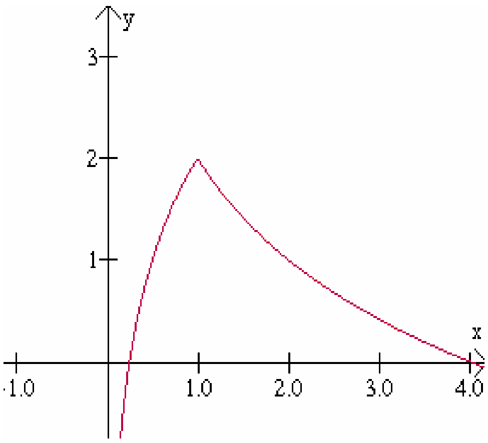
$f(x)=\left|\log_{\frac{1}{2}} x\right|$

The graph of $f(x)=-\left|\log_{\frac{1}{2}} x\right|$ is the same as the graph of $f(x)=\left|\log_{\frac{1}{2}} x\right|$ reflected in the $x-$axis



$f(x) = -\left|\log_{\frac{1}{2}} x\right|$

The graph of $f(x) = 2 - \left|\log_{\frac{1}{2}} x\right|$ is the same as the graph of $f(x) = -\left|\log_{\frac{1}{2}} x\right|$ shifted upwards 2 units.

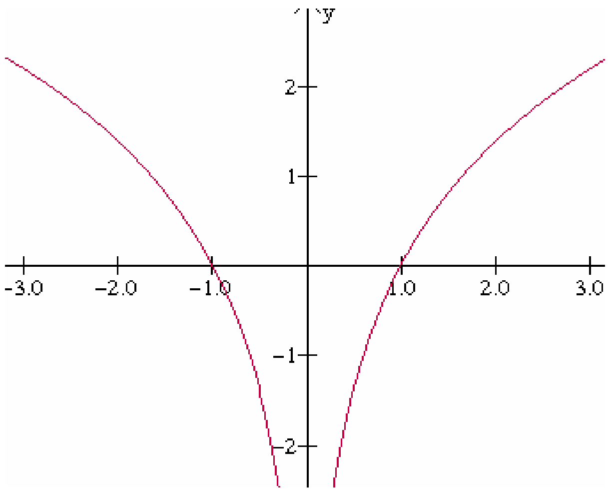


$f(x) = 2 - \left|\log_{\frac{1}{2}} x\right|$

Domain is $(0, +\infty)$, Range is $(-\infty, 2]$ and asymptote is the line $x = 0$

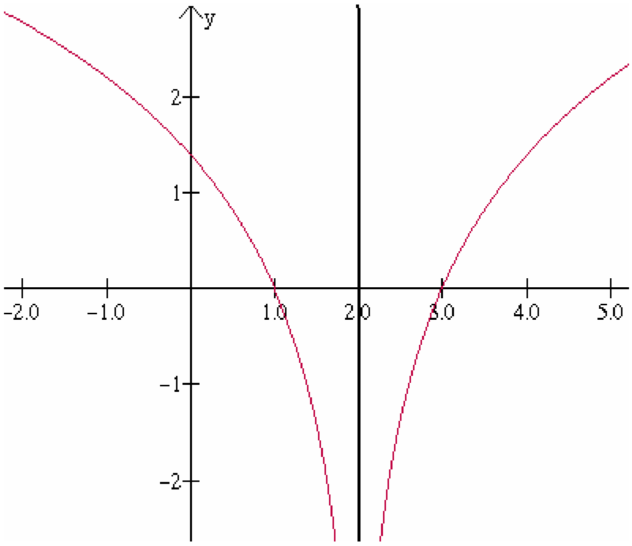
d) $y = \ln x^2 \Leftrightarrow e^y = x^2$

as $y \to +\infty$, $x \to \pm\infty$
as $y \to -\infty$, $x \to 0$
$x-$Intercept $(\pm 1, 0)$



$f(x) = \ln x^2$

The graph of $f(x) = \ln(x-2)^2$ is the same as the graph of $f(x) = \ln x^2$ shifted horizontally 2 units to the right.



$f(x)=\ln(x-2)^2$

Domain is $(-\infty,2)\cup(2,+\infty)$, Range is $(-\infty,+\infty)$ and asymptote is the line $x=2$

Domain of Logarithmic function

The domain is given by the expression;

a) $x>0$ if $f(x)=\log_b x$

b) $g(x)>0$ if $f(x)=\log_b g(x)$

c) $\dfrac{g(x)}{h(x)}>0$ if $f(x)=\log_b \frac{g(x)}{h(x)}$

d) $\log_a \dfrac{g(x)}{h(x)}>0$ if $f(x)=\log_b \log_a \frac{g(x)}{h(x)}$

Common and Natural logarithms

1. A logarithmic function with the base e is called a natural logarithm and is written as $f(x)=\ln x$
2. A logarithmic function with the base 10 is called a common logarithm and is written as $f(x)=\log x$

Example 8

Find the domain of

a) $f(x)=\log x^2$ b) $f(x)=\log\ln(x-1)$ c) $f(x)=\ln\sqrt{(x^2-1)}$

d) $f(x)=\log_7 \frac{3x}{2-x}$ e) $f(x)=\log_5 \log_3\left(\frac{x-2}{x^2-1}\right)$

Solution

a) $f(x)=\log x^2 \Rightarrow x^2>0 \Rightarrow$ Domain is $(-\infty,0)\cup(0,+\infty)$

b) $f(x)=\log\ln(x-1)\Rightarrow \ln\left(x-1\right)>0\Rightarrow x-1>e^0\Rightarrow x>2$

Domain is $\left(2,\infty\right)$

c) $f(x)=\ln\sqrt{(x^2-1)}\Rightarrow (x^2-1)>0\Rightarrow$ Domain is $\left(-\infty,-1\right)\cup\left(1,\infty\right)$

d) $f(x)=\log_7\frac{3x}{2-x}\Rightarrow \frac{3x}{2-x}>0\Rightarrow$ Domain is $\left(0,2\right)$

e) $f(x)=\log_5\log_3\left(\frac{x-2}{x^2-1}\right)\Rightarrow \log_3\left(\frac{x-2}{x^2-1}\right)>0\Rightarrow\left(\frac{x-2}{x^2-1}\right)>3^0\Rightarrow\dfrac{x-x^2-1}{x^2-1}>0$

$\Rightarrow$ Domain is $\left(-1,1\right)$

Logarithmic Inequalities

Theorem:

If $\log_b x\le\log_b y$;

a) $x\ge y$ when $0<b<1$

b) $x\le y$ when $b>1$

Example 9:

Solve the exponential equations;

a) $\log(x-1)\le\log(\frac{1}{2}x+3)$ b) $\log_{\frac{1}{5}}(2x+1)\ge\log_{\frac{1}{5}}(x-2)$

c). $10^{\log(\ln|2x-1)|}\le\ln 2$

Solution

a) $\log(x-1)\le\log(\frac{1}{2}x+3)\Rightarrow\left(x-1\right)\le\left(\frac{1}{2}x+3\right)\Rightarrow$ Domain is $\left(-\infty,8\right]$

b) $\log_{\frac{1}{5}}(2x+1)\ge\log_{\frac{1}{5}}(x-2)\Rightarrow(2x+1)\le(x-2)\Rightarrow$ Domain is $\left(-\infty,-3\right]$

c). $10^{\log(\ln|2x-1|)}\le\ln 2\Rightarrow\ln\left|2x-1\right|\le\ln 2\Rightarrow\left|2x-1\right|\le 2\Rightarrow$ Domain is $\left[-\frac{1}{2},\frac{3}{2}\right]$