— 1.3. Complex Numbers —

Objectives.

1) Define i, a complex nbr, square root of negative nbrs.
2) Learn to do the operations & to simplify.
3) Learn to compute the powers of i.

Complex Nbrs:

$x^2+1=0$ has no real solutions.

We want to extend the real nbr system to include a solution of this equation.

We define $\boxed{i=\sqrt{-1}}$ $\Rightarrow$ $\boxed{i^2=-1}$

Square Root of a Negative Number

If $a>0$, we define

$\sqrt{-a}=i\sqrt{a}$ eg: $\sqrt{-9}=3i$, $\sqrt{-5}=i\sqrt{5}$

Caution if $b<0$, $c<0$, $\sqrt{b}\sqrt{c}\neq\sqrt{bc}$

A complex number is an expression of form

$a+bi$ for any $a, b \in \mathbb{R}$

a: real part of this nbr $\quad$ b = imaginary part

Exp

	real nbr	Complex nbr	Real part	Imaginary part
$1+2i$	No	✓	1	2
-1	✓	✓	-1	0
$2i$	✗	✓	0	2

So any <u>real</u> <u>number</u> is a complex nbr.

A nbr is <u>pure imaginary</u> if its real part is 0.

eg. $6i$, $-2i$

Operations on complex Numbers

$(a+bi)+(c+di) = (a+c)+(b+d)i$ Addition

$(a+bi)-(c+di) = (a-c)+(b-d)i$ Subtraction

$(a+bi)\cdot(c+di) = ac + adi + bci + \underbrace{bdi^2}_{-bd}$

$= (ac-bd)+(ad+bc)i$

Exp 1 Perform & write in standard position

a) $(3+5i)+(4-2i)$

b) $(3+5i)-(5-4i)$

c) $(3+5i)(4-3i)$

a) $3+5i+4-2i = 7+3i$

b) $3+5i-5+4i = -2+9i$

c) $(3+5i)(4-3i) = 12-9i+20i-15\underbrace{i^2}_{=-1}$

$= 12+11i+15 = \boxed{27+11i}$

Division of complex is like rationalizing the denominator of a radical expression.

The complex conjugate of $z = a + bi$ is $\bar{z} = a - bi$

Eg: $z = 2 + 3i \Rightarrow \bar{z} = 2 - 3i$

$w = 3 - 5i \Rightarrow \bar{z} = 3 + 5i$

$s = 2 \Rightarrow \bar{s} = 2$

$t = 4i \Rightarrow \bar{t} = -i$

Property:

If $z = a + bi$ ~~then~~

$$z \cdot \bar{z} = (a+bi)(a-bi) = a^2 + b^2$$

To simplify the quotient $\dfrac{a+bi}{c+di}$,

1) Multiply the numerator & the denominator by the conjugate of the deno.

2) Simplify.

$$\frac{a+bi}{c+di} = \left(\frac{a+bi}{c+di}\right) \cdot \left(\frac{c-di}{c-di}\right) = \frac{(ac+bd) + (bc-ad)i}{c^2+d^2}$$

Exp . Write the quotient is standard form $a+bi$.

a) $\dfrac{3+5i}{1-2i}$ b) $\dfrac{7+3i}{4i}$ d) $\dfrac{3}{i}$

a) $\dfrac{3+5i}{1-2i}\cdot\dfrac{1+2i}{1+2i}=\dfrac{3+6i+5i-10}{1^2+2^2}=\dfrac{-7+11i}{5}$

$=\boxed{\left(-\frac{7}{5}\right)+\left(\frac{11}{5}\right)i}$

b) $\dfrac{7+3i}{4i}\cdot\dfrac{(-i)}{(-i)}=\dfrac{-7i-3i^2}{-4i^2}=\dfrac{-7i+3}{-4(-1)}=\boxed{\dfrac{3-7i}{4}}$

c) $\dfrac{3}{i}\cdot\dfrac{-i}{-i}=\dfrac{-3i}{1}=\boxed{-3i}$

Powers of i.

$i^1=i$, $i^2=-1$ $i^3=-i$ $i^4=1$

$(i^2 i=(-1)i)$ $(i^2\cdot i^2=(-1)(-1))$

Property

If $n\in\mathbb{N}$ & r is the remainder of divisim of n by 4 . then

$$\boxed{i^n=i^r}$$

Exp. Simplify

a) i^{175} , b) i^{-15}

a) $i^{173} = i^{75} = i^{15} = i^{3} = -i$

$i^{-15} = \frac{1}{i^{15}} = \frac{1}{-i} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{1} = \boxed{i}$

Exp. Simplify

1) $\frac{-8 + \sqrt{-128}}{4}$ 2) $\sqrt{-6} \cdot \sqrt{-10}$ 3) $\frac{\sqrt{-20}}{\sqrt{-2}}$

4) $\frac{\sqrt{-4}\left(\sqrt[3]{-27} - \sqrt{-16}\right)}{(1+i)^2}$ 5) $\frac{7-3i}{1+i} - i^{51}$