9.7 Algebra of Matrices

Objectives.

Definition of a matrix, its size, equality

Addition, subtraction & scalar multiplication.

Matrix multiplication, properties

A matrix is a rectangular array of numbers.

Eg. $B = \begin{pmatrix} 1 & -2 & 3 \\ \sqrt{2} & 0 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 0 \\ 7 & 8 \end{pmatrix}$

In general,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{i=1 \\ j=1}}^{m,n}$$

The size of A is number of rows × number of columns $= m \times n$

Equality $A = (a_{ij})$, $B = (b_{ij})$ are equal if they have the same size &

$\boxed{a_{ij} = b_{ij}}$ for any i, j

Exp $\begin{pmatrix} 1 & 2 & p \\ 3 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & x & -2 \\ 3 & y & 7 \end{pmatrix}$ iff $x = 2$, $p = -2$ & $y = 0$

Basic Operations

Addition: To be added 2 matrices need to be of the same size

$A = (a_{ij})$ & $B = (b_{ij})$ 2 $m \times n$-matrices

$$A + B = (a_{ij} + b_{ij})_{i=1, j=1}^{m, n}$$

Subtraction: Same condition as for addition

$$A - B = (a_{ij} - b_{ij})_{i=1, j=1}^{m, n}$$

Scalar Multiplication. $k \in \mathbb{R}$, $A = (a_{ij})_{1,1}^{m,n}$

$$kA = (k\, a_{ij})_{i=1, j=1}^{m, n}$$

Zero Matrix

$$O = O_{m \times n} = \underbrace{\begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}}_{n \text{ cols}} \Big\} m \text{ rows}$$

Exp

$$A = \begin{pmatrix} 4 & 3 \\ \frac{1}{2} & 1 \\ 0 & -3 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & -2 \\ 0 & 6 \\ -4 & 2 \end{pmatrix}$$

Compute $2A$, $2A + B$, $A - 3B$

Properties :

1) The matrix addition is associative, commutative.

$$(A+B)+C = A+(B+C)$$

$$A+B = B+A$$

2) $(c+d)A = cA + dA$ $(cd)A = c(dA)$

3 $c(A+B) = cA + cB$ $c(A)d = cd(A)$

Matrix Multiplication.

A $m\times n$-matrix, B $p\times q$-matrix. The matrix multiplication of A by B is defined only if $\boxed{n=p}$

ie $\boxed{\text{number of Col of A = number of row of B}}$

If A is $m\times n$, B is $n\times q$ $\Rightarrow$ $C = A\times B$ is $m\times q$

C takes the # rows of 1^{st} matrix & # col of 2^{nd} matrix

If $C = AB$ & $C = (c_{ij})_{i=1,\, j=1}^{m,\; q}$

$$c_{ij} = (i^{th}\text{ row of } A) \cdot \begin{pmatrix} j^{th} \\ \text{col} \\ \text{of} \\ B \end{pmatrix} = (a_{i1}\ a_{i2}\ \cdots\ a_{in}) \cdot \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

Exp. $A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$ $\quad B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & 5 \end{pmatrix}$

$C = \begin{pmatrix} 1 & 2 \\ 4 & 7 \\ -1 & -2 \end{pmatrix}$

$\underset{2\times 2}{A} \cdot \underset{2\times 3}{B}$ = defined

$\underset{2\times 3}{B} \times \underset{2\times 2}{A}$ undefined.

$$B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & 5 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} B = \begin{pmatrix} 2+2 & -1+6 & 0+10 \\ -6+4 & 3+12 & 0+20 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 10 \\ -2 & 15 & 20 \end{pmatrix}$$

$A \cdot C =$ $\qquad$ $C \cdot A =$

$B\,C =$ $\qquad$ $C \cdot B$

Remarks: If A or B are rectangular $(m \neq n)$, we can have AB defined & $B \cdot A$ undefined or AB & BA of different size.

Square Matrices

A square matrix is a matrix with #(rows) = # (columns)

If A is $n\times n$-matrix

$A^2 = A.A$ is defined

$A^3 = A^2.A$ --- $A^n = A^{n-1}.A$

If A, B are $n\times n$-matrices $\Rightarrow$ AB & BA are both defined & of same size $n\times n$.

But in general

$$AB \neq BA$$

Exp $A = \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}$ $B = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$

$$AB = \begin{pmatrix} 2 & x \\ x & x \end{pmatrix} \neq BA = \begin{pmatrix} -5 & x \\ x & x \end{pmatrix}$$

Consequences The identities for real nbrs are no more true.

1) $(A+B)^2 = A^2 + AB + BA + B^2$

$(A+B)^2 \neq A^2 + 2AB + B^2$

2) $(A-B)(A+B) \neq A^2 - B^2$

Ex. (Linear Matrix Equation)

Find a matrix X such that

$$A + 2X = 3B - X$$

where $A = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 5 & 6 \end{pmatrix}$ $B = \begin{pmatrix} -2 & 2 & 4 \\ -5 & 7 & 1 \end{pmatrix}$

Ex. Find the element in the 3rd row & 2nd col of BC if $B = \begin{pmatrix} 1 & -2 \\ 3 & 4 \\ 1 & -1 \end{pmatrix}$ $C = \begin{pmatrix} 1 & -1 & 0 \\ 4 & 2 & -2 \end{pmatrix}$
