Example. Calculation of a determinant by putting it in a triangular form.

$$A = \begin{pmatrix} 2 & 1 & -1 & 3 \\ 2 & 2 & 0 & 1 \\ 4 & 5 & 4 & -3 \\ 2 & 2 & 7 & -3 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & -1 & 3 \\ 2 & 2 & 0 & 1 \\ 4 & 5 & 4 & -3 \\ 2 & 2 & 7 & -3 \end{vmatrix} \underset{\substack{-2R_1+R_3 \\ -R_1+R_4}}{\overset{-R_1+R_2}{=}} \begin{vmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 3 & 6 & -9 \\ 0 & 1 & 8 & -6 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 1 & 8 & -6 \end{vmatrix} \underset{-R_2+R_4}{\overset{-R_2+R_3}{=}} 3 \begin{vmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 7 & -4 \end{vmatrix}$$

$$\overset{-7R_3+R_4}{=} 3 \begin{vmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 3\left(2(1)(1)\,3\right) = \boxed{18}$$

Exp. Find the cofactor of a_{23}.

$$C_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & -3 \\ 2 & 2 & -3 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 2 & 1 & 1 \\ 4 & 5 & -1 \\ 2 & 2 & 1 \end{vmatrix} \underset{-R_1+R_3}{\overset{R_1+R_2}{=}} -3 \begin{vmatrix} 2 & 1 & 1 \\ 6 & 6 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= -3(6) \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = -18\left(1 \cdot C_{13} + 0\, C_{23} + 0\, C_{33}\right)$$

$$= -18\, M_{13} = -18 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -18(1-0) = \boxed{-18}$$