Chapter

5.2

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Alright everyone, welcome to this segment of our Phys 608 Laser Spectroscopy course. Today, we're diving into a critical component of any laser system: the laser resonator. This is covered in Chapter 5, section 2, of our material. These notes have been prepared by Distinguished Professor Doctor M A Gondal for our course here at KFUPM, Term 251.

Laser resonators are truly at the heart of what makes a laser a laser – they provide the optical feedback necessary for sustained oscillation and play a crucial role in determining the laser's output characteristics, such as its mode structure, beam quality, and frequency spectrum. So, let's begin our exploration.

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To guide our discussion on laser resonators, let's first outline our overall orientation and learning goals for this topic. First, Goal 1: We aim to situate laser resonator theory within the broader context of basic electromagnetic cavity physics. Many of you will have encountered E&M cavities in other courses, perhaps in the context of microwaves. We'll build upon those foundational concepts and see how they adapt and specialize when we move to optical frequencies and the specific requirements of lasers. Understanding this lineage is key to grasping why resonators behave the way they do.

Second, Goal 2: We will strive to build a rigorous, symbol-by-symbol derivation of every key formula. This includes expressions for losses within the cavity, the quality factor (or Q factor, which is a measure of the resonator's efficiency), the Fresnel number (a crucial dimensionless parameter characterizing diffraction), and the mode spectra (the set of frequencies the resonator can support). A deep understanding here requires us to unpack the mathematics and the physics behind each term.

Third, Goal 3: We will contrast different types of resonator geometries. We'll start with idealized closed cavities, then move to the more practical open stable resonators which are common in many lasers. We'll also explore deliberately unstable resonators, which are used in high-gain systems, and ring geometries, which offer unique advantages like unidirectional operation. We'll examine these in step-wise detail, highlighting their respective pros and cons.

And finally, Goal 4, which is very practical: We want to provide design rules that a practicing experimentalist can immediately apply. This includes how to estimate and control beam spot sizes, how diffraction loss scales with resonator parameters, how to read and interpret stability charts for resonator design, and other crucial rules-of-thumb. The aim here is to bridge theory with practical application, enabling you to design or analyze real-world laser systems. These four goals will form the backbone of our journey through laser resonators.

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Let's begin our exploration with the simplest, most idealized starting point: Closed, Perfectly Conducting Cavities. Our first bullet point defines a "closed cavity." Imagine an enclosure, like a metallic box, whose walls are perfectly reflecting. This means that any electromagnetic wave encountering a wall is reflected with 100% efficiency, no energy is absorbed by the walls, and no energy is transmitted through them. Furthermore, there are no apertures, no holes or openings, so no radiation can escape. This is, of course, an idealization, but it provides a fundamental basis for understanding how electromagnetic fields can be confined and can form standing wave patterns.

The second point concerns dimensions. We'll often refer to a characteristic linear size, denoted by the symbol L. For example, if our cavity is a cube, L could be the length of an edge. If it's a more complex shape, L might represent the largest internal distance. This dimension is crucial because it sets the scale for the wavelengths of the electromagnetic modes that can exist within the cavity.

The third point is particularly relevant for our interest in lasers, which operate at optical frequencies. For optical frequencies, the wavelength, λ, is typically very, very small – on the order of hundreds of nanometers. The characteristic dimension of a macroscopic cavity, L, might be on the order of centimeters or meters. Therefore, we almost always have the condition that the wavelength λ is much, much less than the cavity dimension L ($\lambda \ll L$). This has profound implications. It means that the cavity is electromagnetically very large, and it can support an enormous number of resonant modes. Think of it like a violin string: a longer string can support more harmonics. A large cavity is similar, but in three dimensions. This high density of modes is something we'll come back to, as it presents both challenges and opportunities for laser design.

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Continuing with our discussion of closed, perfectly conducting cavities, we now consider the electromagnetic fields within them.

The first bullet point introduces the concept of electromagnetic normal modes. These are specific spatial patterns of the electromagnetic field that can exist as standing waves within the cavity. For a wave to be a "standing wave" or a "normal mode," it must satisfy the boundary conditions imposed by the perfectly conducting walls on all six faces of our conceptual box cavity. For a perfect conductor, the tangential component of the electric field must be zero at the walls, and the normal component of the magnetic field must also be zero. These conditions restrict the possible wavelengths and, therefore, the frequencies of the electromagnetic fields that can stably exist inside the cavity.

Now, if these cavity walls are not at absolute zero temperature, but are at some finite wall temperature, capital T, the cavity will be filled with thermal radiation. The system will reach thermal equilibrium. In this situation, the spectral energy density of the radiation field inside the cavity is described by Planck's distribution, a cornerstone of quantum mechanics.

The formula for Planck's distribution of spectral energy density, $\rho(\nu,T)$ (often written as $\rho_\nu(T)$), is given as:

$$\rho(\nu,T) = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}} - 1}$$

Let's break this down carefully.

On the left, $\rho(\nu,T)$, is the spectral energy density. Its units, as shown in the final bullet point, are Joules per cubic meter per Hertz $\left(\mathrm{J\ m^{-3}\ Hz^{-1}}\right)$. This means it represents the energy per unit volume, per unit frequency interval, at a specific frequency ν and temperature T.

Now for the terms on the right: h is Planck's constant, approximately 6.626×10^{-34} Joule-seconds. It's the fundamental constant of quantum mechanics, linking energy to frequency. ν (Greek letter nu) is the frequency of the electromagnetic radiation in Hertz. c is the speed of light in vacuum, approximately 3×10^8 meters per second. So, the term $\frac{8\pi h\nu^3}{c^3}$ represents the energy density of modes if each mode had exactly $h\nu$ energy, multiplied by the density of modes (which goes as ν^2, as we'll see).

The second part of the expression, $\frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}}-1}$, is the crucial Bose-Einstein factor. This term gives the average number of photons, or energy quanta, per mode at frequency ν and temperature T. k_B is Boltzmann's constant, approximately 1.38×10^{-23} Joules per Kelvin. It connects temperature to energy. T is the absolute temperature in Kelvin. The quantity $h\nu$ is the energy of a single photon of frequency ν. The ratio $\frac{h\nu}{k_\mathrm{B}T}$ compares the photon energy to the characteristic thermal energy.

This entire Planck distribution formula was revolutionary because it correctly described the black-body spectrum, avoiding the ultraviolet catastrophe predicted by classical physics. It arose from Planck's hypothesis that energy is quantized and from the statistical mechanics of photons as bosons.

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Let's continue by defining some of the constants we just encountered and then look more closely at the concept of mode density. First, a quick recap of the constants: 'h' is Planck's constant. 'k sub B' is the Boltzmann constant. 'c' is the vacuum speed of light. These are fundamental constants that appear throughout physics, especially in quantum mechanics and thermodynamics.

Now, a crucial component embedded within Planck's law is the mode density per unit volume. This tells us how many distinct electromagnetic modes, or resonant states, are available within a certain frequency interval, $d\nu$, per unit volume of the cavity.

The expression for this is given as:

$$n(\nu)\, d\nu = \frac{8\pi\nu^2}{c^3}\, d\nu$$

Here, 'n of nu' represents the number of modes per unit volume per unit frequency interval. So, $n(\nu)\, d\nu$ is the number of modes per unit volume in the small frequency range from ν to $\nu + d\nu$.

Notice the ν^2 dependence: the density of available states increases rapidly with frequency. The c^3 in the denominator means that for a given frequency, if c were smaller, there would be more modes. The factor of 8π comes from considering a 3D space and two polarization states for light.

The final bullet point gives us a vital piece of insight: "Result obtained by counting integer lattice points in k-space octant." This is a beautiful piece of physics. To derive the mode density, one typically considers a rectangular cavity (a box). The boundary conditions (that the tangential E-field is zero on the walls) force the wave solutions to be standing waves, meaning the wave vector components (k_x, k_y, k_z) can only take on discrete values, proportional to integers divided by the cavity dimensions (e.g., $k_x = \frac{n_x \pi}{L_x}$).

These allowed (k_x, k_y, k_z) values form a grid, or lattice, in "k-space" (wave-vector space). Each point on this lattice represents a possible mode. For a large cavity (where L is much greater than λ), these points are very closely packed. We can then approximate the number of modes by finding the volume in k-space corresponding to a certain frequency range and dividing by the "volume per mode" in k-space.

The "octant" refers to the fact that we only consider positive values for the integers n_x, n_y, n_z for unique standing wave solutions. Additionally, for each k-vector (each mode (n_x, n_y, n_z)), there are two independent polarization states for electromagnetic waves.

Combining these factors and converting from k (wavenumber, related to ν by $k = \frac{2\pi\nu}{c}$) to frequency ν leads to the $\frac{8\pi\nu^2}{c^3}$ result for mode density per unit volume per unit frequency. This derivation is a classic piece of statistical mechanics and E&M theory.

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Now, let's look at a direct consequence of Planck's law and the Bose-Einstein statistics for photons, specifically focusing on the mean photon occupation number per mode, especially in the optical range.

The first point presents the formula for the mean photon number per mode, which we'll denote as $\bar{n}(\nu, T)$. This is precisely the Bose-Einstein distribution factor that we saw in Planck's law:

$$\bar{n}(\nu, T) = \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}$$

This $\bar{n}$ represents the average number of photons occupying a single electromagnetic mode of frequency ν when the system is in thermal equilibrium at temperature T. It's a fundamental result from quantum statistical mechanics for bosons, which photons are.

Now, let's consider the implications for typical conditions encountered in laser spectroscopy. The second bullet point is crucial: "For visible/near-IR frequencies at room temperature, $h\nu$ is much, much greater than $k_B T$."

Let's quantify this. For visible light, say green light around 550 nanometers, the photon energy $h\nu$ is about 2.25 electron volts (eV). At room temperature, say 300 Kelvin, the thermal energy $k_B T$ is about 0.025 electron volts, or about 1/40th of an eV.

So, $h\nu$ is indeed much larger than $k_B T$ – roughly by a factor of 100 in this example.

What does this mean for $\bar{n}$? If $h\nu$ is much greater than $k_B T$, then the term $\frac{h\nu}{k_B T}$ in the exponential is a large positive number. Let's say it's X, where $X \gg 1$. Then $\exp(X)$ is a very, very large number. Subtracting 1 from a very, very large number still leaves a very, very large number. So, $\bar{n}$ becomes 1 divided by (a very, very large number), which means $\bar{n}$ is a very, very small number.

The slide indicates that $\bar{n}$ is approximately 10^{-19} to 10^{-12}.

Think about how incredibly small these numbers are! For instance, 10^{-12} means there's, on average, only one photon in a trillion modes of that frequency, or a mode is occupied by a photon for only one trillionth of the time. For 10^{-19}, it's even more sparse.

This is a profoundly important result: at room temperature, optical modes are essentially empty of thermal photons. The thermal background provides virtually no photons in the visible or near-infrared range.

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Given the extremely low thermal photon occupation numbers in the optical range that we just discussed, let's consider the consequences for emission processes within such a closed cavity.

The first bullet point states: "Hence stimulated emission (which is proportional to $\bar{n}$) is utterly negligible compared to spontaneous emission inside such a cavity."

Recall Einstein's A and B coefficients. The rate of stimulated emission is proportional to $B\,\rho$, where ρ is the energy density, or equivalently, proportional to $B\,\bar{n}$, the number of photons already in the mode. If $\bar{n}$ is vanishingly small (like 10^{-12} or less), then the rate of stimulated emission induced by these thermal photons will also be vanishingly small.

Spontaneous emission, on the other hand, occurs even if $\bar{n}$ is zero; its rate is given by the A coefficient. So, in a thermal cavity at room temperature, spontaneous emission will overwhelmingly dominate any stimulated emission that might be seeded by the thermal photon background.

The second bullet point draws a critical conclusion from this: "Therefore a large closed cavity with L much, much greater than λ is a poor laser resonator – the gain medium cannot build one mode far above the background."

Why is this so? Laser action relies on achieving a condition where stimulated emission dominates spontaneous emission *into the desired lasing mode*. If you place a gain medium (an مجموعه of excited atoms or molecules) inside this thermally cold cavity, the excited species will primarily decay via spontaneous emission, radiating photons into any of the myriad available modes more or less randomly.

Because $\bar{n}$ for the thermal field is so low, there's no pre-existing "strong" mode of thermal photons to preferentially stimulate emission into a single, coherent mode. The gain medium itself has to provide the photons to build up the field. While it does this, it's competing with spontaneous emission into all other modes. For a *closed* cavity with many, many modes (since $L \gg \lambda$), it's very difficult for the gain medium to funnel enough stimulated photons into *one specific mode* to make it stand out significantly from the "background" of spontaneously emitted light, which is diffuse and spread over many modes.

The phrase "cannot build one mode far above the background" means that achieving a high-intensity, coherent, single-mode laser output is extremely challenging under these conditions. This is precisely why practical lasers use *open* resonators, which have very different properties, as we will see.

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Now, we transition from the idealized closed cavity to concepts more directly applicable to real laser resonators. We'll start by introducing the idea of a loss factor, β_{k}, and the stored energy, W_{k}. The first bullet point asks us to "Consider the k-th cavity eigenmode (frequency ν_{k})." In any resonant structure, there will be a set of discrete frequencies, ν_{k} (where k is an index like 1, 2, 3, etc.), at which the cavity can resonate. Each of these corresponds to a specific spatial field pattern, an eigenmode.

Next, we define $W_{\mathrm{k}}(t)$ – that's capital W, subscript k, as a function of time t. This represents the instantaneous electromagnetic energy stored in that specific k-th mode. The units for this energy will be Joules, denoted by capital J.

Now, in any real cavity, energy is not stored indefinitely. There are always loss mechanisms. The third bullet point is key: "Lump all energy-removal processes (...) into a single first-order loss constant β_{k}." These energy-removal processes are numerous:

- Mirror transmission: If a mirror is not 100% reflective (e.g., an output coupler), some energy is lost through transmission. - Scattering: Imperfections on mirror surfaces or within intracavity components can scatter light out of the mode. - Diffraction: Due to the wave nature of light and finite mirror sizes, some light will spread out and miss the mirrors, especially in open cavities. - Absorption: Mirror coatings or other intracavity elements (even the gain medium itself if not pumped, or if there are parasitic absorptions) can absorb energy.

Instead of tracking each of these individually for now, we group their combined effect for the k-th mode into a single parameter: β_k. This β_k is called a "first-order loss constant," and its units are inverse seconds (s^{-1}). "First-order" means that the rate of energy loss from the mode is directly proportional to the energy currently stored in that mode. A higher β_k means higher losses, or a faster rate of energy decay.

This leads us to the fundamental differential equation shown at the bottom:

$$\frac{dW_k}{dt} = -\beta_k W_k$$

This equation states that the rate of change of energy stored in the k-th mode ($\frac{dW_k}{dt}$) is negative (indicating loss) and proportional to the energy W_k currently in that mode, with β_k being the constant of proportionality. This is the mathematical statement of a first-order decay process, identical in form to radioactive decay, for example.

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Given the first-order differential equation for energy decay in a mode, $\frac{dW_k}{dt} = -\beta_k W_k$, we can find its formal solution.

The first bullet point presents this solution:

$$W_k(t) = W_k(0)\exp(-\beta_k t)$$

Here, $W_{\mathrm{k}}(t)$ is the energy in the k-th mode at any time t. $W_{\mathrm{k}}(0)$ is the initial energy stored in that mode at time $t = 0$. The term $\exp(-\beta_{\mathrm{k}} t)$ shows that the energy decays exponentially with time, at a rate determined by the loss constant β_{k}.

The parenthetical note "(energy “ring-down” measurement!)" is very important. This exponential decay is directly observable. If you can somehow excite a cavity mode (inject some energy into it) and then abruptly stop the excitation, the light intensity (and thus energy) within the cavity will "ring down" or decay exponentially. By measuring the rate of this decay, you can experimentally determine β_{k}. This is the basis of cavity ring-down spectroscopy (CRDS), a very sensitive absorption measurement technique.

From the loss constant β_{k}, we can define a very intuitive quantity: the photon lifetime in that mode. This is presented in the second bullet point:

$$\tau_{\mathrm{k}} = \frac{1}{\beta_{\mathrm{k}}}$$

Here, τ_{k} (Greek letter tau, subscript k) is the photon lifetime for the k-th mode. It represents the characteristic time constant of the exponential decay. Specifically, it's the time it takes for the energy in the mode to decay to $\frac{1}{e}$ (about 37%) of its initial value.

A mode with high losses (large β_{k}) will have a short photon lifetime (small τ_{k}). Conversely, a low-loss mode (small β_{k}) will have a long photon lifetime – photons "survive" longer in such a mode. This concept of photon lifetime is central to understanding laser resonators. For lasing to occur, the gain must be able to overcome losses within the photon lifetime.

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Now we introduce another extremely important figure of merit for any resonant system, including laser cavities: the Quality Factor, usually denoted by capital Q. We'll look at the Q factor for the k-th mode, Q_k. This slide gives a precise definition and interpretation.

First, let's consider the duration of one optical cycle for the k-th mode: Capital $T = \frac{1}{\nu_\mathrm{k}}$, where ν_k is the frequency of the mode. This is just the period of the oscillation.

The slide then provides a definition for the Quality factor: $Q_\mathrm{k} = 2\pi \times \frac{\text{(energy stored)}}{\text{(energy lost per radian)}}$. Let's unpack this "energy lost per radian" idea, as it connects directly to the standard definition of Q.

The standard definition of Q for an oscillator is $Q = \omega_0 \frac{\text{(Average Energy Stored)}}{\text{(Average Power Loss)}}$, where $\omega_0 = 2\pi\nu_0$ is the resonant angular frequency.

In our case, Energy Stored is W_k. The Power Loss (rate of energy loss) is $-\frac{dW_\mathrm{k}}{dt}$, which from the previous page is $\beta_\mathrm{k} W_\mathrm{k}$.

So, $Q_\mathrm{k} = \frac{2\pi\nu_\mathrm{k} W_\mathrm{k}}{\beta_\mathrm{k} W_\mathrm{k}}$.

The W_k terms cancel, leading to: $Q_\mathrm{k} = \frac{2\pi\nu_\mathrm{k}}{\beta_\mathrm{k}}$. This is the equation shown on the slide derived from the first definition.

Now, let's see how the "energy lost per radian" definition connects.

Power loss $P_\mathrm{L} = \beta_\mathrm{k} W_\mathrm{k}$.

In one optical cycle (duration $T = \frac{1}{\nu_\mathrm{k}}$), the field oscillates through 2π radians of phase.

The energy lost in one full cycle (2π radians) is $\Delta W_\mathrm{cycle} = P_\mathrm{L} \times T = \frac{\beta_\mathrm{k} W_\mathrm{k}}{\nu_\mathrm{k}}$.

So, the energy lost per radian of optical oscillation is $\frac{\Delta W_\mathrm{cycle}}{2\pi} = \frac{\beta_\mathrm{k} W_\mathrm{k}}{2\pi\nu_\mathrm{k}}$.

If we use the slide's phrasing: $Q_{\mathrm{k}} = 2\pi \times \frac{\text{(Energy Stored)}}{\text{(Energy Lost per Cycle)}}$. No, the slide says: $Q_{\mathrm{k}} = \frac{\text{(Energy Stored)}}{\text{(Energy Lost per Radian of Oscillation)}}$. Using our expression for energy lost per radian: $Q_{\mathrm{k}} = \frac{W_{\mathrm{k}}}{\frac{\beta_{\mathrm{k}} W_{\mathrm{k}}}{2\pi\nu_{\mathrm{k}}}} = \frac{2\pi\nu_{\mathrm{k}}}{\beta_{\mathrm{k}}}$.

This is indeed consistent! So, Q is (Energy Stored) divided by (Energy Lost per Radian of Oscillation phase). It's a clever way to define it. The factor of 2π in the common $Q = 2\pi \frac{\text{(Energy Stored)}}{\text{(Energy Lost per Cycle)}}$ definition is absorbed into the "per radian" denominator here.

The Q factor is dimensionless. A high Q value means that the resonator stores energy efficiently, with relatively low losses per optical cycle.

The next bullet point emphasizes this: "High-Q implies photon executes many oscillations before amplitude drops appreciably." Since $Q_{\mathrm{k}} = \frac{2\pi\nu_{\mathrm{k}}}{\beta_{\mathrm{k}}}$, and the photon lifetime $\tau_{\mathrm{k}} = \frac{1}{\beta_{\mathrm{k}}}$, we can write $Q_{\mathrm{k}} = 2\pi\nu_{\mathrm{k}}\tau_{\mathrm{k}}$.

As $\nu_{\mathrm{k}}\tau_{\mathrm{k}}$ is the number of oscillations during one lifetime τ_{k} (since ν_{k} is cycles per second and τ_{k} is seconds), Q_{k} is 2π times the number of oscillations the energy effectively persists in the cavity. So a high Q means many oscillations.

Finally, "Rule-of-thumb conversions" refers to the relationships between Q, β_{k}, and other parameters like resonance width, which we'll see on the next slide.

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This page continues our discussion of the Quality Factor Q_{k} and its relationship to other important resonator parameters.

The first bullet point gives us a direct relationship between the photon lifetime τ_{k} and the Q-factor Q_{k}: $\tau_{\mathrm{k}} = \frac{Q_{\mathrm{k}}}{2\pi\nu_{\mathrm{k}}}$. This can be easily seen from the definition of Q_{k} we just established: $Q_{\mathrm{k}} = \frac{2\pi\nu_{\mathrm{k}}}{\beta_{\mathrm{k}}}$. Since $\tau_{\mathrm{k}} = \frac{1}{\beta_{\mathrm{k}}}$, we can substitute $\beta_{\mathrm{k}} = \frac{1}{\tau_{\mathrm{k}}}$ into the Q_{k} expression to get $Q_{\mathrm{k}} = 2\pi\nu_{\mathrm{k}}\tau_{\mathrm{k}}$. Rearranging this for τ_{k} gives the formula on the slide. This equation tells us that for a given frequency ν_{k}, a higher Q-factor directly implies a longer photon lifetime.

The second bullet point introduces the concept of resonance width: "Resonance half-width (HWHM) $\Delta\nu = \frac{\beta_{\mathrm{k}}}{2\pi}$, which also equals $\frac{\nu_{\mathrm{k}}}{Q_{\mathrm{k}}}$." Let's be precise about the width. When a resonator is driven by an external source, its response (e.g., the energy stored) will be sharply peaked around its resonant frequencies ν_{k}. This peak has a characteristic width.

If β_{k} is the energy decay rate (units s^{-1}), then the Full Width at Half Maximum (FWHM) of the power resonance curve, when plotted as a function of frequency (ν in Hertz), is given by $\Delta\nu_{\mathrm{FWHM}} = \frac{\beta_{\mathrm{k}}}{2\pi}$. The slide denotes $\Delta\nu$ as the "Resonance half-width (HWHM)". If $\Delta\nu$ is indeed HWHM, then $\mathrm{HWHM} = \frac{\mathrm{FWHM}}{2} = \frac{\beta_{\mathrm{k}}}{4\pi}$. However, the subsequent equality, $\Delta\nu = \frac{\nu_{\mathrm{k}}}{Q_{\mathrm{k}}}$, is standard if $\Delta\nu$ represents the FWHM. That is, $Q_{\mathrm{k}} = \frac{\nu_{\mathrm{k}}}{\Delta\nu_{\mathrm{FWHM}}}$.

Let's check the consistency: If $\Delta\nu = \frac{\beta_{\mathrm{k}}}{2\pi}$ (i.e., FWHM), then substituting $\beta_{\mathrm{k}} = \frac{2\pi\nu_{\mathrm{k}}}{Q_{\mathrm{k}}}$ gives

$$\Delta\nu = \frac{\frac{2\pi\nu_{\mathrm{k}}}{Q_{\mathrm{k}}}}{2\pi} = \frac{\nu_{\mathrm{k}}}{Q_{\mathrm{k}}}.$$

This is perfectly consistent. So, it appears that $\Delta\nu$ on this slide, despite being labeled HWHM, is being used in the formula as if it were the FWHM. Or, perhaps β_{k} is defined as an amplitude decay rate in some conventions that would lead to this. Given $\beta_{\mathrm{k}} = \frac{1}{\tau_{\mathrm{k}}}$ where τ_{k} is photon (energy) lifetime, β_{k} is an energy decay rate.

So, I will interpret $\Delta\nu$ here as the FWHM for consistency of the formulas: The Full Width at Half Maximum of the resonance, $\Delta\nu$, is given by $\frac{\beta_\text{k}}{2\pi}$. And this $\Delta\nu$ is also equal to $\frac{\nu_\text{k}}{Q_\text{k}}$.

This last relationship, $Q_\text{k} = \frac{\nu_\text{k}}{\Delta\nu}$ (FWHM), is a very common and useful definition of Q: it's the resonant frequency divided by the bandwidth of the resonance. A high Q resonator, therefore, not only has low losses and long photon lifetime but also a very sharp, narrow frequency response. This is critical for achieving frequency selectivity in lasers.

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Now we address a very practical aspect of laser design: Selecting a Single Oscillating Mode via Loss Engineering. Lasers often need to operate on a single mode for coherence and specific applications, but a resonator can typically support many modes. How do we choose just one?

The first bullet point sets the stage: "Assume different modes possess very different β_k." Recall that β_k is the loss constant for the k-th mode. If we can design the resonator such that one desired mode (say, the fundamental TEM_{00} mode) has a significantly lower β_k (lower loss) than all other modes, then that low-loss mode will be the first to reach the lasing threshold and will dominate, suppressing other modes. This is the principle of "loss engineering."

The second bullet point gives an example: "fabricate extra intracavity aperture or slightly mis-align mirrors such that only one Gaussian-like mode has small losses." An intracavity aperture (a small opening) will preferentially introduce losses for higher-order transverse modes because these modes typically have a larger spatial extent (they are "wider") than the fundamental Gaussian mode. So, they get clipped more by the aperture. Slightly misaligning the mirrors can also disproportionately increase the losses for higher-order modes or make the resonator unstable for them, while the fundamental mode might still be sustainable. The goal is to make β_k very small for the desired mode and much larger for all unwanted modes.

The slide then introduces the concept of gain. "Unsaturated gain per round-trip (small-signal) in active medium: $G(\nu) = \alpha(\nu)L$ where..." Here, $G(\nu)$ is the gain at frequency ν. $\alpha(\nu)$ is the small-signal gain coefficient, and L is the length of the gain medium. It's important to clarify what $G(\nu)$ represents. If $\alpha(\nu)$ is the small-signal intensity gain coefficient (units of per meter, e.g., m^{-1}, as indicated in the next bullet), then $\alpha(\nu)L$ is the total gain exponent for a single pass through a gain medium of length L. In this case, the intensity amplification factor would be $\exp(\alpha(\nu)L)$. If $\alpha(\nu)L$ is small, then $\exp(\alpha(\nu)L)$ is approximately $1 + \alpha(\nu)L$, and $G(\nu) = \alpha(\nu)L$ would represent the fractional intensity gain per pass. For lasing threshold conditions, we usually consider the round-trip gain. So, if L is the length of the gain medium and it's traversed twice per round trip, the round-trip gain exponent might be $2\alpha(\nu)L$. Let's assume for now that $G(\nu)$ as defined here refers to the gain exponent experienced over one effective pass or one round trip, and we'll see how it's used in the threshold condition on the next slide.

The final bullet point clarifies α: "$\alpha =$ small-signal gain coefficient [m^{-1}]." This is the gain per unit length provided by the active medium when it is not saturated (i.e., when the stimulating intensity is low). Its units are inverse meters.

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Continuing with the theme of selecting a single oscillating mode, this page refines the gain and loss considerations.

First, "capital L equals length of gain medium." This confirms the L used in the gain expression $G(\nu) = \alpha(\nu)L$ on the previous slide refers to the physical length of the active material providing amplification.

Now, the crucial "Net gain" condition for mode k is presented: $G(\nu_{\text{k}}) > \gamma_{\text{k}}$, where $\gamma_{\text{k}} = \beta_{\text{k}} \cdot \frac{2\,d}{c}$. And there's an extra "L" after this on the slide, which appears to be a typo; I will omit it as it's not standard in this context.

Let's interpret this carefully.

$G(\nu_{\text{k}})$ is the gain for mode k. Based on the previous slide, if $G(\nu_{\text{k}}) = \alpha(\nu_{\text{k}})L_{\text{gain medium}}$, this is the single-pass gain exponent. For a standing wave cavity, the light usually passes through the gain medium twice per round trip. So, the round-trip gain exponent would typically be $2\alpha(\nu_{\text{k}})L_{\text{gain medium}}$. Let's assume $G(\nu_{\text{k}})$ here implicitly refers to the *total round-trip gain exponent*.

On the right side, γ_{k} is the round-trip loss exponent. β_{k} is the loss rate in units of per second (s^{-1}). The term "2d over c" is the time it takes for light to make one round trip in a resonator of geometric length d (where d is the separation between mirrors, and c is the speed of light). Let's call this T_{rt} (round-trip time).

So, $\gamma_{\text{k}} = \beta_{\text{k}}\,T_{rt}$ represents the total dimensionless loss exponent experienced by the mode k in one round trip. For example, if energy decays as $\exp(-\beta_{\text{k}}t)$, then in one round trip time T_{rt}, the energy is reduced by a factor

$$\exp(-\beta_{\text{k}}T_{rt}) = \exp(-\gamma_{\text{k}}).$$

Therefore, the condition

$$G_{\text{round-trip exponent}}(\nu_{\text{k}}) > \gamma_{\text{round-trip loss exponent}}$$

means that the gain experienced in one round trip must be greater than the loss experienced in one round trip for the mode amplitude to grow and for lasing to occur. This is the fundamental laser threshold condition.

The next bullet clarifies:

$$\frac{2\,d}{c} = T_{rt} \quad \text{(round-trip time in resonator of geometric length } d\text{).}$$

Finally, the "Design principle" is reiterated: "make γ_k tiny for one target mode, large for all others." By engineering the losses (β_k, and thus γ_k) to be very different for different modes, and ensuring that the gain $G(\nu_\text{k})$ is sufficient to overcome the losses only for the desired mode (or a small number of modes), we can achieve single-mode, or few-mode, laser operation. The mode with the highest $G(\nu_\text{k}) - \gamma_\text{k}$ value will "win" the competition for the available gain.

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We've been discussing closed cavities and the general concepts of loss and gain. Now, let's focus on why practical lasers almost universally use "Open" Optical Resonators and understand their core motivation.

The first bullet point defines an "open" resonator: "Open = mirrors do not form a closed metallic box - free space exists outside." Unlike the idealized sealed cavities we discussed earlier, open resonators typically consist of two (or more) mirrors facing each other, but the space around and beyond the mirrors is open. This means radiation is not perfectly confined; it can escape, particularly through diffraction if the mirrors are finite, or intentionally through a partially transmissive mirror.

The slide then lists "Advantages relative to closed cavity." These are crucial for understanding why lasers are built this way.

1. "Dramatically smaller number of supportable transverse modes implies easier single-mode selection." In a closed cavity where the dimensions L are much larger than the wavelength λ, we found an extremely high density of modes. This makes it very hard to select just one. Open resonators, especially *stable* open resonators (which use curved mirrors, as we'll see), are designed to preferentially support a limited set of well-behaved transverse modes, often Gaussian-like beams (TEM_{mn} modes). Higher-

order transverse modes in these open resonators tend to be larger spatially or have different divergence properties, making them more susceptible to losses from finite mirror sizes or strategically placed apertures. This greatly simplifies the task of achieving single transverse mode operation, which is often desired for good beam quality.

2. "Light retraces same path (on-axis) many times, amplifying along gain medium each pass." While light also makes many passes in a closed cavity, open resonators are specifically engineered to ensure that a well-defined beam path exists between the mirrors. The gain medium is placed along this path. Each time the light reflects and passes through the gain medium, it is amplified. This cumulative amplification over many passes is what allows the laser intensity to build up from noise to a powerful, coherent beam. The "on-axis" part is key for stable resonators supporting a fundamental mode.

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Continuing with the advantages and characteristics of "Open" Optical Resonators:

3. "Coupling out usable laser power is trivial (make one mirror partially transmitting)." This is a huge practical advantage. If you had a perfect closed cavity with 100% reflecting walls, you couldn't get any light out! In an open resonator, one of the mirrors is intentionally designed to be partially transmitting. This mirror is called the output coupler. A certain fraction of the intracavity light "leaks" through this mirror on each pass, forming the usable output laser beam. The reflectivity of this output coupler is a critical design parameter, balancing the need for sufficient feedback (high reflectivity) with efficient power extraction (not too high reflectivity).

However, openness comes with a "Penalty," as the slide notes: "new loss channels appear." Because the system is no longer a perfectly sealed box, there are additional ways for energy to be lost from the desired resonating mode. These include:

"Walk-off or “geometrical” losses of off-axis rays."

If a light ray inside the resonator is not perfectly aligned with the optical axis defined by the mirrors, or if the mirrors themselves are slightly tilted, the ray can "walk off" the edge of one of the mirrors after a certain number of bounces. This is a geometrical loss because it depends on the ray's trajectory and the finite size of the mirrors.

"Diffraction losses at finite-sized mirrors."

Light is a wave, and when it is constrained by an aperture (like the edge of a mirror), it diffracts, meaning it spreads out. Even if a beam is perfectly aligned, some of its energy will spread beyond the edges of the mirrors on subsequent passes due to this diffraction. This is an unavoidable fundamental loss mechanism in any resonator with finite-sized components. The magnitude of diffraction loss depends on the wavelength, mirror sizes, and mirror separation, often characterized by the Fresnel number.

"Mirror coating transmission/absorption/scattering."

Real mirrors are not perfect reflectors. - Transmission: As mentioned, the output coupler is designed to transmit. Even mirrors intended to be "high reflectors" might have some residual transmission. - Absorption: The coatings on the mirrors (and the mirror substrate itself) can absorb a small fraction of the incident light, converting it to heat. - Scattering: Imperfections in the mirror surface (roughness, dust, coating defects) can scatter light out of the well-defined beam path. All these contribute to the overall loss that the gain in the laser medium must overcome.

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Now let's quantify one specific type of loss in an open resonator: Round-Trip Reflection Losses, and we'll look at an exact formula for this.

The first bullet says: "Let mirrors M_1, M_2 have intensity reflectivities R_1, R_2 (dimensionless, passive cavity)." So, we have a cavity formed by two mirrors, Mirror 1 and Mirror 2. R_1 is the fraction of intensity reflected by Mirror 1, and R_2 is the fraction reflected by Mirror 2. These are dimensionless numbers between 0 and 1 (e.g., $R = 0.99$ means 99% reflectivity). We are considering a "passive cavity," meaning no gain medium is present for now; we're just looking at how light bounces and gets attenuated by imperfect reflections.

The next line presents an equation:

$$I_{\mathrm{after}} = R_1 R_2 I_0 \exp(-\gamma_{\mathrm{R}})$$

This equation needs careful interpretation. If I_0 is the intensity just before starting a round trip, and the *only* losses are due to reflection from mirrors with reflectivities R_1 and R_2, then after one round trip (hitting M_1 then M_2, or M_2 then M_1, and returning to the starting plane in the same direction), the intensity I_{after} should simply be $I_0 \times R_1 \times R_2$.

The term $\exp(-\gamma_{\mathrm{R}})$ in the slide's equation $I_{\mathrm{after}} = R_1 R_2 I_0 \exp(-\gamma_{\mathrm{R}})$ seems to imply that γ_{R} accounts for *additional* losses beyond R_1 and R_2, or there's a slight redundancy if γ_{R} itself is defined *from* R_1 and R_2. Let's look at the next definition.

Define reflection loss exponent

$$\gamma_{\mathrm{R}} = -\ln(R_1 R_2) = |\ln R_1| + |\ln R_2|$$

This definition is standard for the round-trip loss *exponent* due to reflectivities R_1 and R_2. If $\exp(-\gamma_{\mathrm{R}})$ represents the fraction of intensity remaining after one round trip due to reflections only, then

$$\exp(-\gamma_{\mathrm{R}}) = R_1 \times R_2.$$

Taking the natural logarithm of both sides gives

$$-\gamma_{\mathrm{R}} = \ln(R_1 R_2),$$

so

$$\gamma_{\text{R}} = -\ln(R_1 R_2).$$

Since R_1 and R_2 are less than or equal to 1, $R_1 \times R_2$ is also less than or equal to 1. The natural logarithm of a number less than 1 is negative. So, $-\ln(R_1 R_2)$ is a positive quantity, which is what we expect for a loss exponent. For example, if $R_1 = 0.9$ and $R_2 = 0.9$, then

$$R_1 R_2 = 0.81.$$

$\ln(0.81)$ is approx -0.21. So $\gamma_{\text{R}} \approx 0.21$.

The equality

$$\gamma_{\text{R}} = -\ln(R_1 R_2) = -(\ln R_1 + \ln R_2) = -\ln R_1 - \ln R_2$$

is also correct. And since $\ln R_1$ and $\ln R_2$ are negative, $-\ln R_1$ is $|\ln R_1|$ and $-\ln R_2$ is $|\ln R_2|$. Thus,

$$\gamma_{\text{R}} = |\ln R_1| + |\ln R_2|.$$

This is a useful form, adding the individual logarithmic losses.

Given this definition of γ_{R}, the initial equation should perhaps be

$$I_{\text{after}} = I_0 \exp(-\gamma_{\text{R}})$$

if γ_{R} encompasses *all* round-trip losses. If γ_{R} is *only* due to reflection, then

$$\frac{I_{\text{after}}}{I_0} = R_1 \times R_2 = \exp(-\gamma_{\text{R}}).$$

I will proceed with the interpretation that

$$\gamma_{\text{R}} = -\ln(R_1 R_2)$$

is the round-trip intensity attenuation *exponent* due to mirror reflectivities.

The final bullet points to the next concept: "Time for one round-trip (two traversals of length d)." This time, T_{rt} will be

$$T_{rt} = \frac{2\,d}{c},$$

where d is the cavity length and c is the speed of light.

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This page continues from the definition of round-trip reflection losses.

First, we have the round- trip time: $T_{rt} = \frac{2\,d}{c}$. This is the time it takes light to travel from one mirror to the other and back again, a total distance of $2\,d$, at speed c.

Next, "Resulting decay constant associated *only* with mirror reflectances: $\beta_\mathrm{R} = \frac{\gamma_\mathrm{R}\,c}{2\,d}$." Let's understand this. β_R is a loss rate (units of s^{-1}), similar to the β_k we saw earlier, but here it's specifically the loss rate due to mirror reflectivities only. γ_R is the dimensionless loss *exponent per round trip*. This amount of loss occurs over a time $T_{rt} = \frac{2\,d}{c}$. So, the average rate of loss is indeed $\beta_\mathrm{R} = \frac{\gamma_\mathrm{R}}{T_{rt}} = \frac{\gamma_\mathrm{R}}{\frac{2\,d}{c}} = \frac{\gamma_\mathrm{R}\,c}{2\,d}$. This is correct.

The final bullet point defines the "Photon lifetime due solely to reflection," denoted as τ_R: $\tau_\mathrm{R} = \frac{1}{\beta_\mathrm{R}}$. This is the standard definition of lifetime from a decay rate. Substituting our expression for β_R, we get: $\tau_\mathrm{R} = \frac{2\,d}{c\,\gamma_\mathrm{R}}$.

Now, if we substitute $\gamma_\mathrm{R} = -\ln(R_1\,R_2)$ (which is a positive number if $R_1, R_2 < 1$), the expression becomes: $\tau_\mathrm{R} = \frac{2\,d}{c\,[-\ln(R_1\,R_2)]}$. The slide's final form is $\tau_\mathrm{R} = \frac{2\,d}{c\ln(R_1\,R_2)}$. This is problematic because $\ln(R_1\,R_2)$ is a negative quantity for $R_1, R_2 < 1$, which would make τ_R negative, and a lifetime must be positive. There's likely a minus sign missing in the denominator of the slide's final printed expression, or an implicit understanding that $\ln(R_1\,R_2)$ should be taken as its absolute magnitude in this context. To be correct and unambiguous, it should be: $\tau_\mathrm{R} = \frac{2\,d}{c\,[-\ln(R_1\,R_2)]}$ or $\tau_\mathrm{R} = \frac{2\,d}{c\,|\ln(R_1\,R_2)|}$.

Let's consider an example: if $R_1 = R_2 = R$ (symmetric cavity), then $\gamma_{\text{R}} = -2\ln R$. If R is very close to 1, let $R = 1 - \varepsilon$, where ε is small and positive (e.g., $\varepsilon = 1 - R$ is the loss per mirror). Then $\ln R = \ln(1 - \varepsilon)$ is approximately $-\varepsilon$. So, γ_{R} is approximately 2ε. Then τ_{R} is approximately $\frac{2\,d}{c\,2\varepsilon} = \frac{d}{c\,\varepsilon} = \frac{d}{c\,(1-R)}$. This makes sense: lifetime is cavity length divided by speed of light times loss per mirror (if only one mirror is lossy, or average loss if both are). This form is often seen for high reflectivity mirrors. The key is that γ_{R} or the term involving $\ln(R_1\,R_2)$ in the denominator must represent a positive loss quantity.

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This slide, titled "Geometric Walk-Off & Reflection Losses – Visual Aid," provides a helpful diagram to illustrate one of the loss mechanisms in open resonators.

The diagram depicts a simple two-mirror resonator. We see two parallel, flat mirrors, labeled M1 on the left and M2 on the right. They are separated by a distance 'd', which is indicated as $d = 450$ (the units, perhaps millimeters or arbitrary units, are not specified but should be consistent). The mirrors also have a finite vertical size, indicated by $D = 200$.

Two types of rays are shown:

1. An "On-axis ray (confined)" is shown in red. This ray starts perpendicular to M1, travels to M2, reflects, travels back to M1, reflects, and continues this pattern, remaining confined between the mirrors. This represents the ideal behavior for a perfectly aligned ray in a plane-parallel resonator.

2. An "Inclined ray (walk-off loss)" is shown in blue. This ray starts from M1 but is slightly tilted upwards. It travels to M2, reflects, then travels back to M1. After reflecting from M1, its angle is such that when it travels towards M2 again, it completely misses the edge of mirror M2 and "Escapes" the cavity.

This is precisely what "geometric walk-off loss" means. If rays are not perfectly aligned or if the mirrors are not perfectly parallel, or if rays originate off-axis, they can, after a number of reflections, simply miss one of the mirrors due to their trajectory and the finite extent of the mirrors. This constitutes a loss of energy from the resonator. This type of loss is particularly important for misaligned resonators or can be a dominant loss mechanism in certain types of "unstable" resonators, where it's sometimes used to extract the laser beam. It's distinct from diffraction loss, which is due to the wave nature of light spreading out. Walk-off is a ray optics concept.

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This slide, "Fabry-Perot Interferometer vs. Laser Resonator Geometry," draws a comparison between a familiar optical instrument and the resonators typically used in lasers, highlighting key geometrical differences.

First, it considers a "Standard FPI in spectroscopy" (FPI stands for Fabry-Perot Interferometer).

- A key characteristic of a typical FPI is that the "Mirror separation d is of order few millimetres." So, d is quite small.

- In contrast, the "Mirror diameter $2\,a$ is many times larger than d." Here, a would be the radius of the mirrors. This means that the mirrors are very broad compared to their separation ($a \gg d$).

Next, it contrasts this with a "Typical gas-laser resonator."

- For these, the mirror separation d is approximately 20 centimeters to 2 meters.

The mirror diameter for these laser resonators will be discussed on the next slide, and that will complete the comparison of aspect ratios. The key takeaway starting to emerge is that laser resonators are often "long and skinny" compared to FPIs being "short and broad."

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Continuing the comparison between Fabry-Perot Interferometer and Laser Resonator geometries:

For the typical gas-laser resonator, the slide now specifies: * "Mirror diameter $2\,a$ is approximately 1 centimeter." This means the mirror radius a is about 0.5 centimeters.

Now, let's combine this with the mirror separation for lasers ($d \sim 20$ cm to 2 m $= 200$ cm from the previous slide). If $a = 0.5$ cm and d ranges from 20 cm to 200 cm, it's clear that d is much larger than a. For example, if $d = 20$ cm, then $\frac{d}{a} = \frac{20}{0.5} = 40$. If $d = 200$ cm, $\frac{d}{a} = \frac{200}{0.5} = 400$. This leads to the crucial point: "Thus reversed aspect ratio $a \ll d$." In FPIs, we had $a \gg d$ (mirror radius much larger than separation). In many laser resonators, we have $a \ll d$ (mirror radius much smaller than separation, or at least not vastly larger).

The "Consequence" of this reversed aspect ratio is profound: * "Diffraction effects cannot be ignored in lasers; they dominate the round-trip loss budget if not mitigated." When the mirrors are not extremely large compared to their separation (and in relation to the wavelength), the natural spreading of light due to diffraction becomes a major loss mechanism. Light diffracting at the edge of one mirror might spread out too much to be fully intercepted by the other mirror. Mitigating these diffraction losses, often through the use of curved mirrors, is a central theme in laser resonator design.

The final point offers a bridge: "Many FPI analytic results (Airy spectrum, finesse) still apply, provided diffraction is handled explicitly." Concepts like the Airy function, which describes the transmission of a Fabry-Perot etalon as a function of frequency, and finesse, which measures the sharpness of the interference fringes, are still fundamentally relevant to understanding the spectral properties of laser resonators. However, for a complete picture of laser resonators, these must be augmented by a careful treatment of diffraction and the spatial mode structure of the beam, which are less critical in the idealized plane-wave analysis of FPIs that assume negligible diffraction.

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This slide, "Diffraction at a Finite Mirror – Aperture Analogy," delves into the nature of diffraction losses.

The first bullet point draws an important analogy: "Plane wave impinging on mirror of radius a = plane wave passing a circular aperture of same size." When a light wave reflects off a mirror of finite size (say, a circular mirror of radius 'a'), the reflected wave behaves as if it has passed through an aperture of the same dimensions. The edges of the mirror effectively "cut off" parts of the wavefront, leading to diffraction effects, just like an aperture would.

The second point describes a key feature of this diffraction: "First angular minima of Fraunhofer diffraction pattern at $\theta_1 \approx \pm\frac{1.22\lambda}{2\,a}$." This formula describes the angular position of the first dark ring in the Fraunhofer diffraction pattern (the Airy pattern) produced by a circular aperture of radius 'a' (so diameter $D = 2\,a$). - θ_1 is the angle, measured from the central axis, to this first minimum. - λ is the wavelength of light. - $2\,a$ is the diameter of the circular mirror/aperture.

The factor 1.22 is a well-known constant for circular apertures.

The third point clarifies the origin of this constant: "(1.22 originates from first zero of J_1)." The intensity distribution in the Airy pattern is described by a function involving the first-order Bessel function of the first kind, J_1. Specifically,

$$\text{Intensity}(\theta) \propto \left[\frac{2J_1(x)}{x}\right]^2,$$

where $x = \frac{\pi D}{\lambda}\sin(\theta)$. The first zero of the $J_1(x)$ function (not $\frac{J_1(x)}{x}$) occurs when its argument is approximately 3.8317. This value, when related back to the angle θ, gives rise to the 1.22 factor in the formula

$$\sin(\theta_1) = \frac{1.22\lambda}{D}.$$

For small angles, $\sin(\theta_1) \approx \theta_1$.

The final bullet point highlights the consequence of this diffraction: "Integrated intensity outside central lobe (the Airy disk) is approximately 16% of total implies that portion misses second mirror which implies it's lost." The central bright spot of the Airy pattern is called the Airy disk. It contains about 84% of the total energy of the diffracted wave. The remaining 16% is distributed in the outer rings. If the light, after diffracting from the first mirror, travels to a second mirror, and if that second mirror is not large enough to capture these outer rings (or even a significant portion of the central disk if the spreading is severe), then that 16% (or more) of the energy is lost from the resonator on that pass. This illustrates how diffraction contributes to resonator losses.

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Continuing from the discussion of diffraction at a finite mirror, this slide lists the key parameters that control the fraction of light lost due to this diffraction. If a portion of the diffracted light misses the second mirror and is lost, what determines how large this lost fraction is?

The slide identifies three key parameters:

1. "Wavelength λ." The extent of diffraction is directly proportional to the wavelength. Longer wavelengths diffract more for a given aperture size. So, θ_1 (the angle of the first minimum) is proportional to λ. If λ is larger, the beam spreads more, potentially leading to greater losses.

2. "Mirror radius a." The mirror radius a (or diameter $2\,a$) appears in the denominator of the diffraction angle formula ($\theta_1 \sim \frac{\lambda}{2\,a}$). This means that diffraction is inversely proportional to the mirror size. Smaller mirrors cause more angular spread. If a is decreased, θ_1 increases, leading to more spreading and potentially higher losses.

3. "Mirror separation d (determines solid angle accepted by opposite mirror)." After light diffracts from the first mirror with a certain angular spread (e.g., characterized by θ_1), it travels a distance d to the second mirror. Over this distance, the linear extent of the spread will be approximately d times θ_1.

The second mirror, of radius a, subtends a certain solid angle as viewed from the first mirror. If the diffracted beam spreads beyond this acceptance angle, energy is lost. A larger d means the beam has more distance over which to spread, so for a fixed angular diffraction and fixed second mirror size, a larger d will generally lead to more of the diffracted light missing the second mirror. Conversely, if d is very small, the beam might not spread much before hitting the second mirror, even if the angular diffraction is present.

These three parameters – λ, a, and d – are collectively captured in a single dimensionless quantity called the Fresnel Number, which we will define next. Understanding how these individual parameters influence loss helps in appreciating the significance of the Fresnel number.

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This slide introduces a very important dimensionless parameter in resonator theory: the Fresnel Number, denoted as N_{F} (or sometimes just N). It provides a quantitative measure of how significant diffraction effects are for a given resonator geometry.

The first bullet gives the "Formal definition for a two-mirror resonator":

• N_F equals a^2, divided by the product of λ and d. So, $N_\mathrm{F} = \frac{a^2}{\lambda d}$.

Here: – a is the radius of the mirrors (assuming circular mirrors of equal radius for simplicity here, or a characteristic transverse dimension if they are not circular or are unequal). – λ is the wavelength of light. – d is the separation distance between the mirrors.

The Fresnel number is dimensionless because a^2 has units of length-squared, and λd (wavelength times distance) also has units of length-squared.

Next, the slide offers "Interpretations (choose whichever aids intuition)":

1. "Zone counting – number of half-period Fresnel zones across mirror as seen from the opposite one."

This is a very insightful way to understand N_F. Imagine you are at the center of one mirror (say, $M1$) looking towards the other mirror ($M2$). You can divide the surface of $M2$ into concentric zones, called Fresnel zones, such that the path length from your observation point on $M1$ to successive zones on $M2$ differs by half a wavelength $\left(\frac{\lambda}{2}\right)$.

The radius of the m-th Fresnel zone on $M2$ (as viewed from the center of $M1$, distance d away) is approximately

$$r_\mathrm{m} = \sqrt{m\,\lambda\,d}\,.$$

If the actual radius of mirror $M2$ is a, and this radius encompasses N_F Fresnel zones, then we can set

$$a = r_{N_\mathrm{F}} = \sqrt{N_\mathrm{F}\,\lambda\,d}\,.$$

Squaring both sides gives

$$a^2 = N_\mathrm{F}\,\lambda\,d\,,$$

which rearranges to

$$N_\mathrm{F} = \frac{a^2}{\lambda\, d}.$$

So, N_F literally counts how many half-period Fresnel zones, as defined by the geometry and wavelength, fit into the radius of the mirror.

A large N_F (many zones) implies that the mirror is large relative to the scale of diffraction effects. This usually means diffraction losses are small, and the resonator behaves more like a geometrical optics system.

A small N_F (e.g., $N_\mathrm{F} \sim 1$ or less) means the mirror only covers one or a few Fresnel zones. This indicates that diffraction effects are very strong and will be a dominant factor in determining the resonator's properties and losses.

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We continue with interpretations of the Fresnel Number, N_F.

2. "Area ratio – mirror area divided by area of diffraction-limited spot after propagation d."

This interpretation needs to be handled with a bit of care in its phrasing. The Fresnel number $N_\mathrm{F} = \frac{a^2}{\lambda d}$ can be thought of as the ratio of the mirror's characteristic area (taken as a^2, not πa^2 for this definition) to a characteristic area λd.

The term λd represents the area of the first Fresnel zone (if π is absorbed into definition of "area"). More intuitively, λd is a measure of the transverse area over which light from a point source would spread due to diffraction after traveling a distance d.

So, N_F essentially tells you how many of these "diffraction cells" of area λd can fit into the mirror area a^2.

If N_F is large, the mirror is large enough to intercept many such diffraction-spread regions, implying diffraction doesn't cause the beam to "miss" the mirror significantly.

If N_{F} is small, the mirror area is comparable to or smaller than this characteristic diffraction area, so diffraction will cause a significant portion of the beam to spread beyond the mirror edges.

3. "Mode parameter – gauge of expected diffraction loss; larger N_{F} implies lower loss."

This is the most practical consequence. The Fresnel number is a direct indicator of the severity of diffraction losses.

- If $N_{\mathrm{F}} \gg 1$ (e.g., $N_{\mathrm{F}} > 10\text{-}100$): Diffraction losses are generally small. The resonator is said to be in the "geometrical optics limit" or "high Fresnel number regime."

- If N_{F} is around 1 (e.g., $0.5 < N_{\mathrm{F}} < 5$): Diffraction losses are significant and play a crucial role in shaping the modes and determining their stability.

- If $N_{\mathrm{F}} \ll 1$: Diffraction losses are extremely high, and it's very difficult to establish a stable resonating mode.

The slide also includes a "Simple condition for negligible diffraction after n passes of a plane wave: $N_{\mathrm{F}} > n$."

This rule of thumb often arises in the context of numerical modeling of resonators (like the Fox-Li method). It suggests that for iterative calculations to converge stably to a mode profile over n iterations (representing n round trips or passes), the Fresnel number should be larger than n. Physically, it implies that if a beam makes many passes, it will progressively spread due to diffraction. For the beam to remain well-confined within the mirrors after n such passes, the initial Fresnel number (characterizing a single pass) must be sufficiently large. If N_{F} is too small, the beam diffracts too rapidly, and a stable mode may not form or might be very lossy.

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This slide, "Worked Examples – Quantifying N_{F}," provides concrete calculations of the Fresnel number to build our intuition.

* "Example 1 – high-resolution planar Fabry-Perot Interferometer (FPI)" The parameters given are: - Mirror separation $d = 1\,\text{cm} = 0.01\,\text{m}$. - Mirror radius $a = 3\,\text{cm} = 0.03\,\text{m}$. (Note: For FPIs, often the diameter is much larger than d, here radius a is 3x the separation d, so $a \gg d$ holds). - Wavelength $\lambda = 500\,\text{nm} = 500 \times 10^{-9}\,\text{m} = 5 \times 10^{-7}\,\text{m}$. Now, we calculate N_{F} using the formula

$$N_{\mathsf{F}} = \frac{a^2}{\lambda d}.$$

$$N_{\mathsf{F}} = \frac{(0.03\,\text{m})^2}{(5 \times 10^{-7}\,\text{m})\,(0.01\,\text{m})}$$

$$N_{\mathsf{F}} = \frac{9 \times 10^{-4}\,\text{m}^2}{5 \times 10^{-9}\,\text{m}^2}$$

$$N_{\mathsf{F}} = \frac{9}{5} \times 10^{-4-(-9)} = 1.8 \times 10^5.$$

The slide calculation shows 1.8×10^5, which is correct. The implication is stated: "Immense N_{F} implies diffraction utterly negligible; FPI finesse limited by coating only." A Fresnel number of 180,000 is indeed very large. This confirms our earlier understanding that for typical FPIs with $a \gg d$, diffraction effects are minimal. The performance of such an FPI (like its ability to resolve closely spaced spectral lines, quantified by its finesse) will be primarily determined by the reflectivity of its mirror coatings and their flatness, not by light diffracting past the edges of the mirrors.

* "Example 2 – gas-laser resonator with plane mirrors" The parameters for this case are: - Mirror separation $d = 50\,\text{cm} = 0.5\,\text{m}$. - Mirror radius $a = 0.1\,\text{cm} = 0.001\,\text{m}$. - Wavelength $\lambda = 500\,\text{nm} = 5 \times 10^{-7}\,\text{m}$. The calculation for this example's N_{F} will be on the next slide. We can already anticipate

that since 'a' is much smaller than 'd' here (0.1 cm vs 50 cm), N_F is likely to be much smaller than in the FPI example.

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This page continues with the second worked example for quantifying the Fresnel Number, N_F, for a gas-laser resonator with plane mirrors.

The parameters were: $d = 50\,\mathrm{cm}$ (0.5 m), $a = 0.1\,\mathrm{cm}$ (0.001 m), and $\lambda = 500\,\mathrm{nm}$ (5×10^{-7} m).

The calculation of N_F is shown:

$$N_\mathrm{F} = \frac{a^2}{\lambda\, d}$$

$$N_\mathrm{F} = \frac{(0.001\,\mathrm{m})^2}{(5 \times 10^{-7}\,\mathrm{m})(0.5\,\mathrm{m})}$$

$$N_\mathrm{F} = \frac{1 \times 10^{-6}\,\mathrm{m}^2}{2.5 \times 10^{-7}\,\mathrm{m}^2}$$

$$N_\mathrm{F} = \frac{1}{0.25} = 4$$

So, for this gas-laser resonator configuration, the Fresnel number N_F is 4.

Now, let's consider the implication, as stated in the second bullet point:

"If $n \sim 50$ passes needed for gain to reach threshold, condition $N_\mathrm{F} > n$ not met implies diffraction dominates losses."

Here, n represents the number of passes (or round trips) light needs to make through the gain medium for the laser intensity to build up to the threshold for oscillation. If, for example, this laser requires about 50 passes ($n = 50$), we then compare N_F with n.

We found $N_\mathrm{F} = 4$. The condition for low diffraction effects over n passes was $N_\mathrm{F} > n$.

In this case, 4 is NOT greater than 50. The condition is not met.

The consequence is that "diffraction dominates losses." A Fresnel number of 4 for a plane-parallel resonator already indicates significant diffraction loss per pass.

Over 50 passes, the cumulative effect of this diffraction would be extremely severe. Such a laser configuration (long cavity, relatively small plane mirrors) would likely be very inefficient or might not lase at all, because the gain might not be able to overcome these substantial diffraction losses.

This highlights why simple plane-parallel mirror resonators can be problematic for typical laser dimensions unless N_{F} is kept reasonably high, or other mechanisms (like curved mirrors) are used to control diffraction.

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Now that we understand the Fresnel number N_{F}, let's look at how it relates to an "Approximate Diffraction Loss Per Transit."

The first bullet point states: "For slowly varying plane-wave assumption, empirical scaling: γ_{D} is approximately $\frac{1}{N_{\text{F}}}$." Here, γ_{D} (gamma sub D) represents the fractional power loss due to diffraction *per transit* (i.e., for a single pass from one mirror to the other). This is an empirical scaling, meaning it's a rule of thumb derived from observations and more complex calculations, particularly for plane-parallel resonators when the field can be approximated as a slowly varying plane wave. The condition for this approximation is "valid for $N_{\text{F}} \geq 1$." So, if $N_{\text{F}} = 4$ (as in our previous laser example), then γ_{D} would be approximately $\frac{1}{4}$, or 25%. This is a very high loss per single pass. If the light makes a round trip (two transits), the total diffraction loss would be even more substantial if this formula is applied naively per transit. Often, $\gamma_{\text{D}} \sim \frac{1}{N_{\text{F}}}$ is quoted as the round-trip diffraction loss for the lowest-order mode of a plane-parallel resonator. Let's assume for a moment that γ_{D} here refers to the round-trip diffraction loss exponent. If N_{F}

is large, say $N_{\mathrm{F}} = 100$, then $\gamma_{\mathrm{D}} \sim 0.01$, or 1% round-trip diffraction loss, which is more manageable.

The second bullet point considers the total loss: "Combined reflection + diffraction loss exponent: $\gamma_{\mathrm{tot}} = \gamma_{\mathrm{R}} + \gamma_{\mathrm{D}}$." Here, γ_{tot} is the total round-trip loss exponent. γ_{R} is the round-trip loss exponent due to mirror reflectivities, which we defined earlier as $\gamma_{\mathrm{R}} = -\ln(R_1 R_2)$. γ_{D} is the round-trip loss exponent due to diffraction. So, this equation simply states that the total loss exponent is the sum of the loss exponent from reflections and the loss exponent from diffraction. This assumes these are the only two loss mechanisms present, or the dominant ones. For a laser to operate, the round-trip gain must exceed this total round-trip loss exponent, γ_{tot}.

Page 28:

This page presents a "Design takeaway" related to controlling diffraction losses (γ_{D}) and reflection losses (γ_{R}).

The bullet point states: "Design takeaway – doubling mirror radius a halves γ_{D} (quadratic in a), whereas improving reflectivity requires exponential improvement (R approaches 1)."

Let's analyze the first part: "doubling mirror radius a halves γ_{D} (quadratic in a)." We just learned that γ_{D} is approximately proportional to $\frac{1}{N_{\mathrm{F}}}$ for round-trip losses (or per transit loss). And $N_{\mathrm{F}} = \frac{a^2}{\lambda d}$. So, γ_{D} is approximately proportional to $\frac{\lambda d}{a^2}$. This means γ_{D} is proportional to $\frac{1}{a^2}$. If you double the mirror radius a (i.e., a becomes $2\,a$), then a^2 becomes $(2\,a)^2 = 4\,a^2$. Therefore, γ_{D} should become $\frac{1}{4}$ of its original value, not $\frac{1}{2}$. The statement on the slide, "doubling mirror radius a halves γ_{D}," implies that γ_{D} is proportional to $\frac{1}{a}$, which is not consistent with $\gamma_{\mathrm{D}} \sim \frac{1}{N_{\mathrm{F}}}$. The parenthetical remark "(quadratic in a)" is also confusing in this context. If γ_{D} is halved when a is doubled, that's an inverse linear relationship. If it were quadratic in a in the denominator, $\gamma_\mathrm{D} \sim \frac{1}{a^2}$, then doubling a would make γ_D four times smaller. There seems to be an inconsistency here. I will proceed by emphasizing the $\gamma_\mathrm{D} \sim \frac{1}{a^2}$ relationship from the Fresnel number, which means diffraction loss decreases rapidly with increasing mirror radius.

Now for the second part: "whereas improving reflectivity requires exponential improvement (R approaches 1)." $\gamma_\mathrm{R} = -\ln(R_1 R_2)$. For simplicity, let $R_1 = R_2 = R$, so $\gamma_\mathrm{R} = -2\ln R$. If R is close to 1, let $R = 1 - \epsilon$, where $\epsilon = (1 - R)$ is the small fractional loss per mirror reflection. Then $\ln R = \ln(1 - \epsilon)$ is approximately $-\epsilon$ (for small ϵ). So, γ_R is approximately $2\epsilon = 2(1 - R)$. This means the round-trip reflection loss exponent is linearly proportional to $(1 - R)$, the imperfection in reflectivity. To halve γ_R, you need to halve $(1 - R)$. For example, if R goes from 0.98 (loss 0.02) to 0.99 (loss 0.01), γ_R is halved. The phrase "exponential improvement" when R approaches 1 might refer to the practical difficulty or cost of manufacturing mirrors with extremely high reflectivities. For example, to get from $R = 0.99$ to $R = 0.999$, and then to $R = 0.9999$, each step requires significantly more effort, more dielectric coating layers, and higher precision. So, while the mathematical relationship between γ_R and $(1 - R)$ is linear for high R, achieving those incremental improvements in R itself can be technologically challenging, perhaps in a way that feels "exponentially" harder. I will interpret it in terms of this practical challenge.

Page 29:

This slide marks an important conceptual shift: "Why Curved Mirrors Help – Intuitive Picture." We've seen that plane-parallel mirrors can suffer from high diffraction losses, especially if the Fresnel number is not very large. Curved mirrors offer a solution.

The first bullet explains the fundamental action: "Curved (concave) mirrors act like lenses implies they re-focus diffracted field back toward axis. Net effect." This is the key insight. A concave mirror has a focusing effect. As a beam propagates in a resonator and naturally spreads due to diffraction, a concave mirror can counteract this spreading by refocusing the light. If the mirror curvatures and separation are chosen correctly, the beam can be made to repeatedly pass between the mirrors, with its diffractive spreading continually compensated by the focusing action of the mirrors. This leads to a stable, confined beam path.

The second bullet point elaborates: "On-axis wavefront curvature matched to mirror curvature implies it stabilises Gaussian-like modes." For a stable mode to exist in a resonator with curved mirrors, the wavefront of the beam must match the curvature of the mirror *at the surface of that mirror*. If this condition is met at both mirrors, the beam effectively "fits" into the resonator, and its shape reproduces itself after each round trip. These self-reproducing modes in stable resonators with curved mirrors are often (and for the fundamental mode, always) Gaussian beams, or higher-order Hermite-Gaussian or Laguerre-Gaussian modes, which have a characteristic Gaussian intensity profile in their transverse dimension.

The third bullet addresses losses: "Outer parts of beam still lost, but dramatically less than in flat-flat cavity of same N_{F}." Even with curved mirrors, if the mirrors are of finite size, some diffraction loss will occur because the Gaussian beam has tails that extend indefinitely. However, because the beam is actively refocused and confined, these losses are typically much, much lower than for a plane-parallel (flat-flat) resonator with a comparable Fresnel number (calculated using mirror radius a and separation d).

Finally, a specific and very important configuration is mentioned: "Confocal configuration (to be detailed next) is..." (continued on next slide). The confocal resonator is a special type of stable resonator with curved mirrors that has particularly elegant properties and often serves as a benchmark.

Page 30:

This page continues the sentence about the confocal configuration from the previous slide. "...the optimal compromise of minimum spot size & minimum diffraction loss for a given d."

Let's expand on why the confocal resonator is often considered such an "optimal compromise." A confocal resonator, as we'll define formally soon, consists of two identical spherical mirrors of radius of curvature R, separated by a distance d equal to R.

This configuration offers several advantages:

1. **Low Diffraction Loss:** Confocal resonators are known for their very low diffraction losses compared to many other stable configurations, especially for a given mirror size. The refocusing is very effective.

2. **Well-Defined Mode Structure:** They robustly support Hermite-Gaussian (or Laguerre-Gaussian) modes.

3. **Alignment Stability:** Confocal resonators are relatively insensitive to small misalignments (tilts) of the mirrors compared to, for example, plane-parallel or concentric resonators. This makes them easier to work with in practice.

4. **Minimum Spot Size Consideration:** The phrase "minimum spot size" here should be understood in context. While a concentric resonator ($d = 2R$) can produce a smaller spot size at the mirrors, it is very sensitive to alignment. The confocal resonator provides a reasonably small beam waist at its center, and the spot sizes on the mirrors are $w_{\mathrm{s}} = \sqrt{2}\, w_0$, which are well-behaved. It strikes a good balance: the spot sizes are not excessively large (which would require very large mirrors) nor excessively small (which might lead to very high intensity and potential damage to optics in high-power lasers).

So, "optimal compromise" means it balances low loss, ease of alignment, and manageable mode sizes effectively for a given mirror separation d

(which equals R in this case). It's a very practical and widely analyzed resonator type.

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This slide presents a very effective visual comparison titled "Optical Resonator: Diffraction Spreading vs. Refocusing." It consists of two diagrams, (a) and (b).

Diagram (a) is labeled "Diffraction Spreading (Plane Mirrors)." It shows two flat, parallel mirrors, Mirror 1 on the left and Mirror 2 on the right, separated by a distance d. A beam of light, with an initial radius a at Mirror 1, is depicted propagating towards Mirror 2. As it propagates, the beam is shown spreading outwards due to diffraction. The shaded region in light blue represents the main part of the beam. A pink shaded area at the top and bottom, beyond the edge of Mirror 2, is labeled "Diffraction Loss." This visually illustrates how, in a plane-mirror resonator, the natural tendency of light to diffract causes part of the beam to expand beyond the dimensions of the second mirror, leading to energy loss.

Diagram (b) is labeled "Refocusing Effect (Concave Mirrors)." This diagram also shows two mirrors, Mirror 1 and Mirror 2, separated by distance d. However, these mirrors are now concave (curved inwards). A beam of light is shown propagating between them. This beam is labeled "Stable Gaussian Mode."

Crucially, the beam does not continuously spread out and miss the mirrors. Instead, it is shown to be confined. It has a narrower "waist" in the center of the resonator and expands as it approaches each mirror. The curvature of the beam's wavefront at each mirror is drawn to match the curvature of that mirror. This refocusing action of the concave mirrors counteracts the diffractive spreading.

A caption below this diagram states: "Refocusing minimizes diffraction loss."

This pair of diagrams powerfully illustrates the fundamental advantage of using curved mirrors in laser resonators: they can guide and confine the light, leading to the formation of stable modes with significantly lower diffraction losses compared to simple plane-mirror configurations of similar dimensions. The concept of the beam "fitting" the mirrors is central to stable resonator theory.

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This slide, "Transition to Field-Distribution Analysis – Beyond Plane Waves," signals a move towards a more rigorous, wave- optics based understanding of resonator modes.

The first bullet point makes a crucial statement: "Open resonator modes are not plane waves; amplitude $A(x, y)$ must adapt until self-reproducing." While plane waves are a useful concept for basic Fabry-Perot theory, the actual modes that exist in open resonators (especially those with curved mirrors or significant diffraction) have complex transverse amplitude and phase distributions, $A(x, y)$, where x and y are coordinates transverse to the direction of propagation. The defining characteristic of a resonator mode is that its field distribution must be "self-reproducing." This means that after one complete round trip within the resonator (e.g., from mirror 1 to mirror 2 and back to mirror 1), the electric field's complex amplitude distribution must be identical to its starting distribution, apart from a constant multiplicative factor. This factor accounts for the loss and phase shift incurred during the round trip.

The slide then introduces an "Iterative modelling technique (Fox-Li)" which was historically very important for first calculating these mode structures. It's a numerical approach:

1. "Start with arbitrary trial field on mirror." One begins by assuming some initial field distribution on one of the resonator mirrors. This could be a simple plane wave, a Gaussian, or even random noise. Let's call this $A_0(x', y')$.

2. "Propagate to other mirror using Kirchhoff-Fresnel integral." Using the principles of diffraction theory, specifically the Kirchhoff-Fresnel (or similar, like Rayleigh-Sommerfeld) diffraction integral, one calculates how this initial field $A_0(x', y')$ on the first mirror propagates through space to the second mirror. This will result in a new field distribution, say $A_{1\_\text{intermediate}}(x, y)$, on the surface of the second mirror.

3. "Multiply by mirror aperture." The second mirror has a finite size and a certain reflectivity. So, the field $A_{1\_\text{intermediate}}(x, y)$ is then "acted upon" by the second mirror. This typically means multiplying it by the mirror's reflectivity function (which might be complex if there are phase shifts on reflection) and an aperture function that is 1 inside the mirror's clear aperture and 0 outside. This gives the field $A_{1\_\text{reflected}}(x, y)$ just after reflection from the second mirror.

The process would then continue by propagating this field back to the first mirror, applying its aperture/reflectivity, and so on.

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This page continues the description of the Fox-Li iterative modeling technique.

4. "Repeat until convergence factor C stabilises (eigenvalue problem)."

After the field is propagated from mirror 1 to mirror 2 and reflected (as described on the previous slide), it is then propagated back from mirror 2 to mirror 1 and reflected from mirror 1. This completes one full round trip.

Let $A_\text{n}(x, y)$ be the field distribution on mirror 1 after 'n' such round trips. The Fox-Li method involves repeatedly applying this round-trip propagation operator. If a stable mode exists, this iterative process will eventually converge. This means that the shape of the field distribution $A_\text{n}(x, y)$ will stop changing significantly from one iteration to the next. Instead, it will reproduce itself, merely multiplied by a complex constant C after each round trip: $A_{\mathrm{n}}(x,y)$ will become $C \cdot A_{n-1}(x,y)$. This complex constant C is the eigenvalue of the round-trip propagation integral operator, and the converged field distribution $A(x,y)$ is the corresponding eigenfunction (the mode).

The magnitude of C, $|C|$, gives the attenuation of the field amplitude per round trip. So, $|C|^2$ is the intensity reduction factor per round trip. The loss per round trip is $1-|C|^2$. The phase of C, $\arg(C)$, gives the phase shift experienced by the mode during one round trip. This phase shift is crucial for determining the resonant frequencies of the cavity.

The slide further notes: "Integral kernel reflects scalar diffraction theory; solutions give both loss γ_{D} and phase shift ϕ." The "integral kernel" is the mathematical function (derived from Kirchhoff-Fresnel theory) that describes how each point on one mirror contributes to the field at each point on the other mirror. This kernel embodies the physics of scalar diffraction.

The solutions to this eigenvalue problem (the eigenfunctions $A(x,y)$ and eigenvalues C) provide: - The loss, γ_{D} (where γ_{D} might be related to $1-|C|^2$, representing diffraction loss primarily if mirror reflectivities are handled separately or assumed perfect in the propagation integral itself). - The phase shift per round trip, $\phi = \arg(C)$.

This iterative approach, pioneered by Fox and Li in the early 1960s, was instrumental in understanding the modes of open optical resonators for the first time, revealing the existence of Hermite-Gaussian modes.

Page 34:

This slide, "Kirchhoff-Fresnel Propagation Integral – Explicit Form," presents the mathematical tool used in the Fox-Li method for propagating the field.

The equation shown is:

$$A_{\text{n}}(x,y) = -\frac{i}{\lambda}\iint_{\text{mirror 1}} A_{n-1}(x',y')\exp[-ikp\cos(\theta)]\frac{dx'\,dy'}{p}$$

Let's break this down:

- $A_{\text{n}}(x,y)$ is the complex amplitude of the electric field at a point (x,y) on mirror 2 (the observation mirror). - $A_{n-1}(x',y')$ is the complex amplitude of the field at a point (x',y') on mirror 1 (the source mirror from the $(n-1)$th iteration or previous transit). - The integral sums the contributions from all such source points (x',y') on mirror 1 to the field at (x,y) on mirror 2. - λ is the wavelength of light. - i is the imaginary unit, $\sqrt{-1}$. So $-i/\lambda$ is a complex prefactor. - k is the wavenumber, $k = \frac{2\pi}{\lambda}$. - p is the distance between the source point (x',y') on mirror 1 and the observation point (x,y) on mirror 2. This is defined below. - The term $\frac{1}{p}$ represents the $1/r$ decrease in amplitude for a spherical wave emanating from (x',y'). - The term $\exp[-ikp\cos(\theta)]$ is the phase factor. Usually, a spherical wave from (x',y') arriving at (x,y) would have a phase $\exp(-ikp)$ or $\exp(ikp)$ depending on convention. The $\cos(\theta)$ term inside the exponent is unusual for the phase path length. More commonly, $\cos(\theta)$ (an obliquity factor, where θ is the angle p makes with the normal to the source surface) would appear as a multiplicative factor, often approximated as 1 for paraxial rays. If $p\cos(\theta)$ is meant to be the optical path length along the z-axis (d), then this is a strong approximation. Given the definition of p on the slide, it's most likely that the phase term should be $\exp(-ikp)$, and $\cos(\theta)$ is an obliquity factor that might be approximated differently in some forms of the Kirchhoff integral. I will assume standard Huygens-Fresnel form where $\exp(-ikp)$ is the phase.

The slide then provides "Coordinate definitions (see Fig.)," referring to a figure which is on page 36.

* $p = \sqrt{d^2 + (x-x')^2 + (y-y')^2}$ This is the correct Euclidean distance between a point (x',y') on mirror 1 (assumed to be at $z = 0$, for instance)

and a point (x, y) on mirror 2 (at $z = d$), where d is the axial separation of the mirrors.

* θ equals angle between p and optical axis. (Optical axis is the z-axis). This θ is typically the obliquity angle. $\cos(\theta) = \frac{d}{p}$.

* "Stationarity condition." This refers to the fact that when the iterative process converges, the field becomes "stationary" or self-reproducing under the action of this integral propagation for a round trip.

Page 35:

This page continues the discussion of the iterative propagation and convergence to an eigenmode.

The first line states: "$A_\mathrm{n}(x, y) = C \cdot A_{n-1}(x, y)$". This equation, in the context of the Fox-Li iteration, describes the condition when convergence to an eigenmode is achieved. If $A(x, y)$ is an eigenmode of the resonator, then after one complete round-trip propagation (which involves two applications of an integral like the one on the previous slide, one for each transit between mirrors, plus reflections), the resulting field distribution is simply the original field distribution $A(x, y)$ multiplied by a complex constant C. So, if $A_{n-1}(x, y)$ represents the field of a mode on a mirror at the start of a round trip, then $A_\mathrm{n}(x, y)$ is the field on the same mirror after that round trip, and it's equal to C times $A_{n-1}(x, y)$.

The complex constant C, the eigenvalue, is given as: "$C = \sqrt{1 - \gamma_\mathrm{D}} \exp(i\phi)$". Let's analyze this:

- The magnitude of C is $|C| = \sqrt{1 - \gamma_\mathrm{D}}$. Since C is the amplitude factor per round trip, $|C|^2$ is the intensity factor per round trip. So, $|C|^2 = 1 - \gamma_\mathrm{D}$. This means γ_D is the fractional intensity loss per round trip due to diffraction (assuming mirror reflectivities are handled separately or are perfect). This is consistent.

- The phase of C is ϕ. So, $\exp(i\phi)$ is the phase factor per round trip. ϕ represents the total phase shift the mode accumulates in one round trip, beyond the basic $2\pi q$ for an ideal plane wave in a simple cavity. This phase shift is crucial for determining the exact resonant frequencies of the modes.

The final bullet point summarizes the mathematical nature of the problem: "Ultimately leads to homogeneous Fredholm integral equation for unknown eigenfields $A(x, y)$." The condition $A(x, y) = C \cdot (\text{Integral Operator acting on } A(x', y'))$ is a homogeneous Fredholm integral equation of the second kind. It's an eigenvalue equation where $A(x, y)$ are the eigenfunctions (the transverse mode patterns) and C are the corresponding eigenvalues (giving loss and phase shift). "Homogeneous" means there is no external source term driving the system; we are looking for the natural resonant modes. Solving this equation yields the set of modes that the resonator can support.

Page 36:

This page contains the image that was referenced on page 34 for defining the coordinates in the Kirchhoff-Fresnel propagation integral. The caption requests: "[IMAGE REQUIRED: Geometry of two parallel square apertures separated by d; annotate x, x', y, y', p, θ.]"

The diagram shown depicts:

- Two parallel planes, representing Mirror 1 (M1) on the left and Mirror 2 (M2) on the right. They appear as square apertures.

- The mirrors are separated by an axial distance d, indicated along the horizontal optical axis (which we can consider the z-axis).

- A point on M1 is labeled with coordinates (x', y'). For clarity, this point is shown with a small red circle. The x' and y' axes are shown local to M1.

- A point on M2 is labeled with coordinates (x, y), also with a small red circle and local x, y axes.

- A blue line segment labeled p connects the point (x', y') on M1 to the point (x, y) on M2. This visually represents the distance p used in the integral.

- The angle θ (often denoted by Greek letter theta, though ϑ is used on the slide image if one looks closely at the symbol near point (x', y') on M1, representing the angle between p and the normal to M1) is the angle that the line p makes with the optical axis (the z-axis, normal to the mirrors if they are parallel to the x-y plane). From the geometry, if the mirrors are in planes $z = 0$ and $z = d$, then $\cos(\theta) = \frac{d}{p}$.

This diagram perfectly illustrates the geometry for the integral: we are calculating the field at (x, y) on M2 by summing contributions from all source points (x', y') on M1, with each contribution involving the distance p and potentially the obliquity angle θ.

Page 37:

This slide, "Paraxial Approximation & Series Expansion of p," introduces a crucial simplification for solving the propagation integral analytically in many cases.

The first bullet states: "For $|x|, |x'|, |y|, |y'| \ll d$ we expand p." This is the paraxial approximation. It assumes that the transverse dimensions of the beam (x, y on one mirror, x', y' on the other) are much smaller than the separation d between the mirrors. This means the light rays are propagating at small angles with respect to the optical axis (the z-axis).

Under this condition, the distance p can be approximated. Recall

$$p = \sqrt{d^2 + (x - x')^2 + (y - y')^2}.$$

We can write this as

$$p = d\sqrt{1 + \frac{(x - x')^2 + (y - y')^2}{d^2}}.$$

Using the binomial expansion $\sqrt{1+u} \approx 1 + \frac{u}{2}$ for small u (where $u = \frac{(x-x\prime)^2+(y-y\prime)^2}{d^2}$), we get:

$$p \approx d\left(1 + \frac{(x - x')^2 + (y - y')^2}{2\,d^2}\right)$$

$$p \approx d + \frac{(x - x')^2 + (y - y')^2}{2\,d}.$$

This is the Fresnel approximation for the path length, and it's shown correctly on the slide.

The second bullet point describes the consequences of using this approximation in the Kirchhoff-Fresnel integral: "Substitute in exponential, neglect slow prefactor variation implies integral separates into product of two 1-D Fresnel integrals."

- When this approximate p is substituted into the phase term $\exp(-ikp)$ of the integral, it becomes:

$$\exp(-ikd) \cdot \exp\left(-ik\ \frac{(x - x')^2 + (y - y')^2}{2\,d}\right).$$

The $\exp(-ikd)$ is a constant phase factor related to propagation along the axis. The second exponential contains a quadratic dependence on $(x - x')$ and $(y - y')$.

- "Neglect slow prefactor variation": The $1/p$ term in the integral is approximated as $1/d$ (since p is close to d). The obliquity factor $\cos(\theta)$ is approximated as 1. These are slowly varying compared to the rapid oscillations of the exponential phase term.

- "Integral separates into product of two 1-D Fresnel integrals": If the mirror aperture is also separable (e.g., a square mirror, so the integration limits for x' and y' are independent), then the integral involving $\exp\left(-ik\,\frac{(x-x\prime)^2}{2\,d}\right)dx'$ and the integral involving $\exp\left(-ik\,\frac{(y-y\prime)^2}{2\,d}\right)dy'$ can be separated. This reduces the 2D diffraction problem to two independent 1D problems.

The final bullet point highlights the "Key outcome": "eigenmodes in many practical cavities factorise: $A(x,y) = X(x)Y(y)$." This separation of variables means that the transverse mode solutions $A(x,y)$ can often be written as a product of a function X that depends only on x, and a function Y that depends only on y. For cavities with rectangular symmetry (like square mirrors), this leads to Hermite-Gaussian modes. For cavities with circular symmetry (circular mirrors), a similar analysis in polar coordinates leads to Laguerre-Gaussian modes. This factorisation greatly simplifies finding and understanding the mode structures.

Page 38:

This slide introduces a particularly important and analytically solvable resonator configuration: "Confocal Resonator – Geometry & Separation Condition."

The first bullet provides the "Definition": "both mirrors spherical, equal radius R, spacing $d = R$."

So, a confocal resonator consists of: 1. Two spherical mirrors. 2. They have the same radius of curvature, capital R. 3. The distance 'd' separating the mirrors is exactly equal to this radius of curvature R.

A key geometrical property is also stated: "Foci of each mirror coincide at cavity centre." A spherical mirror with radius of curvature R has a focal length $f = \frac{R}{2}$. Mirror 1 has its focal point at a distance $\frac{R}{2}$ from its surface. Mirror 2 has its focal point at a distance $\frac{R}{2}$ from its surface. Since the mirrors are separated by $d = R$, the center of the cavity is at a distance $\frac{R}{2}$ from Mirror 1 and also $\frac{R}{2}$ from Mirror 2. Therefore, the focal point of Mirror 1 (which is $\frac{R}{2}$ from M1, towards M2) lies exactly at the center of the cavity. Similarly, the focal point of Mirror 2 ($\frac{R}{2}$ from M2, towards M1) also lies at the center of the cavity. Thus, their two focal points coincide at the geometric center of the resonator. This is a defining feature.

The next point is crucial for its analysis: "Under above paraxial approximation the Fredholm equation yields analytic* solutions." When the paraxial approximation (small angles, $x, y \ll d$) is applied to the Kirchhoff-Fresnel integral equation for this specific confocal geometry ($d = R$), the integral equation can be solved analytically. This is a rare and fortunate case in resonator theory. Most other geometries require numerical solutions. The asterisk on "analytic*" solutions might hint that while analytic forms (like Hermite-Gaussians) emerge, some parameters within them might still need to be found by solving related equations.

The "Resulting eigenfields" are the Hermite-Gaussian modes, which will be detailed on the next slide. This analytical tractability makes the confocal resonator a cornerstone for understanding stable optical resonators.

Page 39:

This page shows the mathematical form of the resulting eigenfields for a confocal resonator, which are the Hermite-Gaussian modes.

The expression given is for $A_{mn}(x, y, z)$, representing the complex amplitude of the (m, n)-th transverse mode as a function of transverse coordinates x, y and axial coordinate z (where $z = 0$ is usually taken at the center of the confocal cavity, the common focal point).

$$A_{mn}(x, y, z) = C^* H_{\mathrm{m}}\left(\frac{\sqrt{2}\, x}{w(z)}\right) H_{\mathrm{n}}\left(\frac{\sqrt{2}\, y}{w(z)}\right) \exp\left(-\frac{r^2}{w^2(z)}\right) \exp\big(-i\, \phi_{mn}(r, z)\big)$$

Let's break this down term by term:

- C star (C^*) is a normalization constant, ensuring the mode has a certain power or energy.

- $H_\mathrm{m}(\dots)$ and $H_\mathrm{n}(\dots)$ are Hermite polynomials of order m and n respectively. These polynomials describe the spatial variation of the mode in the x and y directions. m is the mode index for x, and n is for y. They dictate the number of nodes (zeros) in the transverse field pattern.

- The arguments of the Hermite polynomials are $\frac{\sqrt{2}\,x}{w(z)}$ and $\frac{\sqrt{2}\,y}{w(z)}$. Here, $w(z)$ is a characteristic transverse scaling factor called the "spot size" or beam radius, which varies with position z along the resonator axis.

- $\exp\left(-\frac{r^2}{w^2(z)}\right)$ is the Gaussian envelope term. Here, $r^2 = x^2 + y^2$ is the square of the radial distance from the optical axis. This term gives the characteristic bell-shaped (Gaussian) fall-off in intensity away from the axis. The spot size $w(z)$ defines the radius at which the field amplitude drops to $1/e$ of its axial value.

- $\exp\big(-i\,\phi_{mn}(r,z)\big)$ is the complex phase factor. $\phi_{mn}(r,z)$ is the total phase, which includes: * A term like $k_0\,z$ for basic propagation (where k_0 is free-space wave number). * A term related to the curvature of the wavefronts. * The Gouy phase shift, which is an additional phase advance that focused beams acquire as they pass through a focus. This Gouy phase depends on the mode indices m and n.

The final bullet point says: "Components to be defined on next slide." This refers to $w(z)$, $\phi_{mn}(r,z)$, and the Hermite polynomials themselves. This expression is the general form for a Hermite-Gaussian beam, which are the natural modes of a stable resonator with rectangular symmetry, like a confocal resonator with spherical mirrors under paraxial approximation.

Page 40:

This slide, "Hermite-Gaussian Mode Parameters – Symbol-by-Symbol," provides definitions for the components of the eigenfield expression we saw on the previous page.

* "H_m, H_n – Hermite polynomials of order m, n." These are a standard set of orthogonal polynomials. $H_0(u) = 1,\ H_1(u) = 2\,u,\ H_2(u) = 4\,u^2 - 2$, and so on. They are solutions to Hermite's differential equation. "m" and "n" are non-negative integers $(0,1,2,\dots)$ that specify the mode order.

* "Orthogonal over (-infinity, infinity) with Gaussian weight." The Hermite polynomials $H_{\mathrm{m}}(u)$ are orthogonal with respect to a Gaussian weighting function, $\exp(-u^2)$, over the interval from $-\infty$ to ∞. That is, the integral of $H_{\mathrm{m}}(u) H_{\mathrm{p}}(u) \exp(-u^2)\, du$ from $-\infty$ to ∞ is zero if $m \neq p$. This orthogonality is important for decomposing an arbitrary beam into a sum of Hermite-Gaussian modes.

* "r equals square root of (x squared plus y squared) [in meters] radial coordinate." $r = \sqrt{x^2 + y^2}$ [in meters] radial coordinate.

* "Beam radius w(z)." This is a crucial parameter, defining the characteristic size of the Gaussian part of the mode at an axial position z. The formula given is specific to a confocal resonator of length "d" (where $d = R$, the mirror radius of curvature), with $z = 0$ defined at the center of the cavity:

$$w(z) = \sqrt{\frac{\lambda d}{2\pi}\left[1 + \left(\frac{2\,z}{d}\right)^2\right]}.$$

Let's analyze this $w(z)$ formula: - λ is the wavelength. - "d" here is the mirror separation, which for a confocal resonator is equal to R, the radius of curvature of the mirrors. - The term $\frac{\lambda d}{2\pi}$ will be related to the square of the beam waist radius, w_0^2 (as we'll see). - The term $[1 + (2\,z/d)^2]$ shows how the spot size varies with z. It's a hyperbolic dependence. The spot size

$w(z)$ is the radius at which the field amplitude drops to $1/e$ of its value on the axis, or where the intensity drops to $1/e^2$ of its axial value.

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* "Beam waist (minimum w): occurs at $z = 0$."

From the formula for $w(z)$ on the previous slide, $w(z) = \sqrt{\frac{\lambda d}{2\pi}\left[1 + \left(\frac{2z}{d}\right)^2\right]}$, we can see that $w(z)$ is minimized when the term $\left(\frac{2z}{d}\right)^2$ is zero, which occurs at $z = 0$. This $z = 0$ point is the center of the confocal resonator.

The minimum value of $w(z)$ is called the beam waist radius, denoted as w_0 (w-naught).

So, $w_0 = w(z = 0) = \sqrt{\frac{\lambda d}{2\pi}[1 + 0]} = \sqrt{\frac{\lambda d}{2\pi}}$.

The slide writes this as $w_0 = w(0) = \sqrt{\frac{\lambda R}{2\pi}}$. This is correct, as for a confocal resonator, the separation d is equal to the mirror radius of curvature R. This w_0 is a fundamental parameter characterizing the size of the beam at its narrowest point.

* "Phase factor ϕ_{mn} encodes Gouy phase + curvature; full expression given later."

The phase $\phi_{mn}(r, z)$ in the mode expression $\exp[-i\phi_{mn}(r, z)]$ is complex. It generally includes:

1. A longitudinal phase term, $-k_0 z$, representing the rapid phase advance of a wave propagating along z.

2. The Gouy phase shift, $\zeta_{mn}(z) = -(m + n + 1)\arctan\left(\frac{z}{z_{\mathrm{R}}}\right)$, where z_{R} is the Rayleigh range ($z_{\mathrm{R}} = \frac{\pi w_0^2}{\lambda}$). This is an extra phase shift that focused

beams experience as they pass through the focal region, compared to a plane wave or spherical wave. It depends on the mode order $(m + n + 1)$.

3. A term related to the radius of curvature of the wavefronts, $-\frac{k_0 r^2}{2R_{wf}(z)}$, where $R_{wf}(z) = z\left[1 + \left(\frac{z_R}{z}\right)^2\right]$ is the wavefront radius of curvature at position z.

The "full expression given later" indicates we might see this decomposition explicitly.

* "Mode naming: TEM$_{mn}$ where T Transverse, E Electric, M Magnetic; first index node count in x, second in y."

TEM stands for Transverse ElectroMagnetic, indicating that for these modes, the electric and magnetic field vectors are predominantly perpendicular (transverse) to the direction of propagation (the z-axis). This is a good approximation for paraxial beams.

The indices 'm' and 'n' in TEM$_{mn}$ correspond to the order of the Hermite polynomials H_m and H_n used in the mode description. Physically, 'm' gives the number of nulls (zeroes) in the transverse intensity pattern along the x-direction, and 'n' gives the number of nulls along the y-direction.

The TEM$_{00}$ mode (m=0, n=0) is the fundamental mode, with no nodes, and has a simple Gaussian intensity profile.

Page 42:

This slide, "Visual Library of Low-Order Confocal Modes," presents a gallery of images showing the transverse intensity patterns for various Hermite-Gaussian TEM$_{mn}$ modes. The caption indicates it's a 3×3 array of color maps. Axes are typically $\frac{x}{w_0}$ and $\frac{y}{w_0}$, or scaled by $w(z)$.

Let's describe what these would look like:

* TEM_{00} (Top-Left): This is the fundamental Gaussian mode $(m = 0, n = 0)$. It appears as a single, bright, circular (if $w_x = w_y$) spot with intensity peaking at the center and smoothly decaying outwards following a Gaussian profile. It has no nodes.

* TEM_{10} (Top-Middle): $(m = 1, n = 0)$. This mode has one node along the y-axis (a vertical dark line at $x = 0$). It appears as two bright lobes aligned horizontally, separated by the vertical nodal line.

* TEM_{01} (Top-Right): $(m = 0, n = 1)$. This mode has one node along the x-axis (a horizontal dark line at $y = 0$). It appears as two bright lobes aligned vertically, separated by the horizontal nodal line.

* TEM_{11} (Middle-Left): $(m = 1, n = 1)$. This mode has one node along the x-axis and one node along the y-axis. It appears as a four-lobe pattern, like a four-leaf clover, with dark lines (nodes) along both $x = 0$ and $y = 0$.

* TEM_{20} (Middle-Center): $(m = 2, n = 0)$. Two vertical nodal lines. This will show three bright lobes arranged horizontally.

* TEM_{02} (Middle-Right): $(m = 0, n = 2)$. Two horizontal nodal lines. This will show three bright lobes arranged vertically.

* TEM_{22} (Bottom-Left): $(m = 2, n = 2)$. Two vertical nodes and two horizontal nodes, forming a 3×3 grid of 9 lobes (some might be less distinct depending on scaling).

* TEM_{30} (Bottom-Middle): $(m = 3, n = 0)$. Three vertical nodal lines, resulting in four bright lobes arranged horizontally.

* TEM_{03} (Bottom-Right): $(m = 0, n = 3)$. Three horizontal nodal lines, resulting in four bright lobes arranged vertically.

These images provide an excellent visual understanding of how the mode indices m and n relate to the spatial complexity and nodal structure of the transverse intensity patterns. Higher-order modes occupy a larger overall area than the fundamental TEM_{00} mode.

Page 43:

This slide focuses on the "Fundamental Gaussian Mode Characteristics," which is the $TEM_{0\ 0}$ mode.

"For $m = n = 0$, Hermite polynomials reduce to unity implies circular symmetry."

The Hermite polynomial of zeroth order, $H_0(u)$, is simply 1. So, if $m = 0$ and $n = 0$, both H_m and H_n in the general mode expression become 1. The part of the mode involving Hermite polynomials simplifies to $1 \times 1 = 1$. The remaining spatial dependence is the Gaussian term $\exp\left(-r^2/w^2(z)\right)$, which is circularly symmetric (depends only on $r = \sqrt{x^2 + y^2}$, not on the angle).

"Intensity distribution $I_{00}(r, z) = I_0(z) \exp\left[-\frac{2\,r^2}{w^2(z)}\right]$."

This is the intensity profile of the $TEM_{0\ 0}$ mode. – $I_{00}(r, z)$ is the intensity at radial distance r from the axis and axial position z. – $I_0(z)$ is the peak intensity on the axis ($r = 0$) at position z. It varies with z because the spot size $w(z)$ changes, and for constant power, intensity changes as $\frac{1}{w^2(z)}$. – The exponent is $-\frac{2\,r^2}{w^2(z)}$. The factor of 2 is because intensity is proportional to the square of the electric field amplitude, and the field amplitude has an $\exp\left[-\frac{r^2}{w^2(z)}\right]$ dependence. So, $\left(\exp\left[-\frac{r^2}{w^2(z)}\right]\right)^2 = \exp\left[-\frac{2\,r^2}{w^2(z)}\right]$.

"Key metric – radius where intensity drops to $\frac{1}{e^2}$: by definition $r = w(z)$."

The spot size $w(z)$ is defined as the radial distance r at which the intensity $I_{00}(r, z)$ drops to $\frac{1}{e^2}$ (which is about 13.5%) of its on-axis value

$I_0(z)$. Let's check: if $r = w(z)$, then the exponent becomes $-\frac{2\,w^2(z)}{w^2(z)} = -2$. So, $I_{00}(w(z), z) = I_0(z)\exp(-2)$. This is correct. (Alternatively, $w(z)$ is sometimes defined as where the *field amplitude* drops to $\frac{1}{e}$ of its axial value, which is equivalent).

"Spot size on mirrors ($z = \pm d/2$): $w_\mathrm{s} = w(d/2) = \sqrt{2}\,w_0$."

This is specific to a confocal resonator where the length $d = R$, and $z = 0$ is at the center. The mirrors are located at $z = +d/2$ and $z = -d/2$ (or $\pm R/2$). We use the formula $w(z) = w_0\sqrt{1 + \left(\frac{2\,z}{d}\right)^2}$. Substitute $z = d/2$: $w(d/2) = w_0\sqrt{1 + \left(\frac{2(d/2)}{d}\right)^2} = w_0\sqrt{1 + \left(\frac{d}{d}\right)^2} = w_0\sqrt{1 + 1^2} = w_0\sqrt{2}$. So, the spot size on each mirror, w_s, is $\sqrt{2}$ times the waist spot size w_0. This means the beam is wider on the mirrors than it is at its waist in the center of the cavity. This is a characteristic feature of Gaussian beam propagation.

Page 44:

This page continues with characteristics of the fundamental Gaussian mode.

The single bullet point states:

"Independence from physical mirror diameter – but losses depend on how many w_s fit inside aperture."

Let's unpack this important distinction:

The mathematical solution for the Hermite-Gaussian modes, including the expressions for the beam waist w_0 and the spot size $w(z)$ (and thus w_s on the mirrors), are derived assuming the mirrors are infinitely large. So, the *shape and size parameters of the ideal mode itself* (like w_0 and w_s) do not depend on the actual physical diameter (say, $2\,a_\mathrm{mirror}$) of the real mirrors used in a laser. These mode parameters are determined by the wavelength (λ), and the resonator geometry (mirror curvatures and separation, e.g., d or R for confocal).

However, the "losses" for this mode *do* critically depend on the physical mirror diameter (or, more generally, the aperture size). The Gaussian intensity profile

$$\exp\left[-\frac{2\,r^2}{w_\mathrm{s}^2}\right]$$

extends theoretically to $r = \infty$, although it drops off very rapidly. If the physical radius of the mirror, let's call it a_m, is not significantly larger than the spot size w_s on that mirror, then the "tails" of the Gaussian beam will be clipped by the edge of the mirror. This clipped energy constitutes diffraction loss. The phrase "how many w_s fit inside aperture" is a colloquial way of saying what the ratio $\frac{a_\mathrm{m}}{w_\mathrm{s}}$ is.

If $\frac{a_\mathrm{m}}{w_\mathrm{s}}$ is small (e.g., 1 or less), losses will be very high because a large fraction of the beam energy is cut off.

If $\frac{a_\mathrm{m}}{w_\mathrm{s}}$ is around 1.5 to 2, losses become moderate.

If $\frac{a_\mathrm{m}}{w_\mathrm{s}}$ is 2.5 to 3 or more, diffraction losses for the fundamental TEM_{00} mode become very small (typically less than 1%, or much less).

So, while the mode *solution* is independent of mirror size, the *viability and efficiency* of that mode in a real resonator depend heavily on the mirror size relative to the mode's spot size on the mirror. This is a key consideration in practical resonator design to minimize unwanted losses.

Page 45:

This slide provides a "Numerical Example – He-Ne Confocal Resonator" to make the concept of beam waist more concrete.

The given "Parameters" are:

- Wavelength $\lambda = 633$ nm. This is the standard wavelength for a Helium-Neon laser, which is 633×10^{-9} m.
- Mirror radius of curvature R = mirror separation $d = 30$ cm. This confirms it's a confocal resonator. 30 cm $= 0.30$ m.

We need to "Compute" the beam waist radius, w_0.

The formula for the beam waist in a confocal resonator (where $z = 0$ is at the center) is:

$$w_0 = \sqrt{\frac{\lambda R}{2\pi}}$$

(using R for the confocal length $d = R$).

Let's plug in the values:

$$w_0 = \sqrt{\frac{633 \times 10^{-9}\,\text{m} \cdot 0.30\,\text{m}}{2\pi}}$$

$$w_0 = \sqrt{\frac{1.899 \times 10^{-7}\,\text{m}^2}{2 \cdot 3.14159265\,...}}$$

$$w_0 = \sqrt{\frac{1.899 \times 10^{-7}\,\text{m}^2}{6.2831853}}$$

$$w_0 = \sqrt{3.02240 \times 10^{-8}\,\text{m}^2}$$

$$w_0 = 1.7385 \times 10^{-4}\,\text{m}$$

Converting this to millimeters:

$$1.7385 \times 10^{-4}\,\text{m} = 0.17385\,\text{mm}$$

The slide shows the result as: $= 1.7 \times 10^{-4}$ m $= 0.17$ mm.

This calculation is consistent. The beam waist for a typical He-Ne confocal resonator with a 30 cm length is very small, about 0.17 mm in radius.

Page 46:

This page discusses the "Implication" of the numerical example for the He-Ne confocal resonator we just calculated.

We found that the beam waist radius w_0 is about 0.17 mm.

The implication stated is: "central mode is pencil-thin; thus a 1 mm mirror easily supports it with negligible diffraction."

Let's verify this.

The beam waist w_0 is 0.17 mm (radius).

The spot size on the mirrors, w_s, for a confocal resonator is

$$w_\mathrm{s} = \sqrt{2} \cdot w_0.$$

$w_\mathrm{s} = 1.4142 \times 0.17$ mm ≈ 0.240 mm (radius).

So, the spot *diameter* on the mirror is

$$2 \times w_\mathrm{s} \approx 0.48 \text{ mm}.$$

The slide considers a "1 mm mirror." This usually refers to the mirror diameter.

So, if the mirror diameter is $2 \cdot a_\mathrm{m} = 1$ mm, then the mirror radius is $a_\mathrm{m} = 0.5$ mm.

Now, we compare the mirror radius a_m to the beam spot radius w_s on the mirror:

Ratio

$$\frac{a_\mathrm{m}}{w_\mathrm{s}} = \frac{0.5\ \mathrm{mm}}{0.240\ \mathrm{mm}} \approx 2.08.$$

A common rule of thumb for low diffraction loss is that the mirror radius should be at least 2 to 3 times the spot radius $\left(\frac{a_\mathrm{m}}{w_\mathrm{s}} > 2 \text{ or } 3\right)$.

Here, $\frac{a_\mathrm{m}}{w_\mathrm{s}}$ is just over 2. This means that about 95% of the beam's power is contained within radius $r = 1.5 \times w_\mathrm{s}$ according to Gaussian integral tables (for power within e^{-2} radius).

The fraction of power transmitted past an aperture of radius a_m is given by

$$\exp\left(-2\left(\frac{a_\mathrm{m}}{w_\mathrm{s}}\right)^2\right).$$

For $\frac{a_\mathrm{m}}{w_\mathrm{s}} = 2.08$, the exponent is $-2 \times (2.08)^2 = -2 \times 4.3264 = -8.65$.

$\exp(-8.65)$ is approx 1.75×10^{-4}, or 0.0175%.

This is indeed a very small loss.

So, the statement that a 1 mm diameter mirror "easily supports it with negligible diffraction" is quite accurate. The term "pencil-thin" aptly describes a beam with a sub-millimeter diameter at the mirrors and an even smaller waist. This is characteristic of many gas lasers.

Page 47:

This slide, "Phase Fronts – Derivation of Radius of Curvature," delves into understanding the shape of the wavefronts for Hermite-Gaussian modes.

It begins: "Start from phase expression (Boyd-Kogelnik)." Boyd and Kogelnik published seminal work on Gaussian beams and resonators in the 1960s.

The phase expression $\phi(r,z)$ for a TEM$_{mn}$ mode (ignoring the overall kz propagation term for a moment, or considering the phase relative to a plane wave) is given as:

$$\phi(r,z) = \frac{2\pi}{\lambda}\left\{\frac{R}{2(1+\xi^2)} + \frac{r^2\,\xi}{R(1+\xi^2)}\right\} - (m+n+1)\frac{\pi}{2} - \arctan\left(\frac{1-\xi}{1+\xi}\right).$$

This expression looks like a specific form of the phase, possibly already simplified or for a specific plane. Let's check common forms.

The Gouy phase is $(m+n+1)\arctan(\xi)$, where $\xi = \frac{z}{z_R}$ (z_R is Rayleigh range).

Wavefront curvature term is $k\,\frac{r^2}{2R_{wf}(z)}$, where $R_{wf}(z) = z\left(1+\frac{1}{\xi^2}\right)$.

The expression on the slide looks like it might be related to the phase difference across the beam or a phase variation.

The term $\frac{2\pi}{\lambda}\{\cdots\}$ looks like $k\{\cdots\}$.

The term $(m+n+1)\frac{\pi}{2}$ is unusual. Gouy phase is usually $(m+n+1)\arctan\left(\frac{z}{z_R}\right)$. Perhaps at $z=\infty$, $\arctan(\infty) = \frac{\pi}{2}$.

The term $\arctan\left(\frac{1-\xi}{1+\xi}\right)$ also looks specific. Let's assume this $\phi(r,z)$ is a given phase function from Boyd-Kogelnik.

The slide seems to be working with a definition of $\xi = \frac{2z}{R}$ (or $\xi = \frac{2z}{d}$ for confocal, if $d = R$). This ξ is a normalized axial distance specific to confocal geometry perhaps, where $z_R = \frac{d}{2} = \frac{R}{2}$. So $\xi = \frac{z}{(R/2)} = \frac{2z}{R}$. This is consistent with common notation for confocal resonators.

with $\xi = \frac{2z}{R}$. For $m=n=0$ and small r, set constant-phase surface; after algebra, obtain spherical wavefront equation.

Page 48:

This page continues from the derivation of the wavefront shape. After the algebra mentioned on the previous slide (setting the phase constant for $m = n = 0$ and small r), one obtains a spherical wavefront equation:

$$x^2 + y^2 + (z - z_0)^2 = R_{\text{prime}}^2.$$

This is the standard equation of a sphere centered at $(0,0,z_0)$ with radius R_{prime}. This shows that for the $\text{TEM}_{0\ 0}$ Gaussian beam, the surfaces of constant phase are spherical, at least near the axis (paraxial approximation).

The next bullet point defines the "Effective radius of curvature along axis," $R_{\text{prime}}(z_0)$. This R_{prime} is the radius of curvature of the wavefront that passes through the axial point z_0. $R_{\text{prime}}(z_0)$ equals R times $(1 + \xi_0^2)$ divided by $(2\,\xi_0)$. Here, $\xi_0 = \frac{2\,z_0}{R}$ is the normalized axial position where the wavefront is being considered. And R is the mirror radius of curvature (and cavity length for confocal). This formula is a standard result for Gaussian beams if we relate ξ_0 to z_0/z_{R}.

For a Gaussian beam, the radius of curvature of the wavefront at axial position z is

$$R_{\text{wf}}(z) = z\left[1 + \left(\frac{z_{\text{R}}}{z}\right)^2\right],$$

where $z_{\text{R}} = \frac{\pi w_0^2}{\lambda}$ is the Rayleigh range.

For a confocal resonator, $R = d$, and $w_0^2 = \frac{\lambda d}{2\pi}$, so

$$z_{\text{R}} = \frac{\lambda d}{2\pi} \cdot \frac{\pi}{\lambda} = \frac{d}{2} = \frac{R}{2}.$$

So,

$$R_{\text{wf}}(z_0) = z_0\left[1+\left(\frac{R/2}{z_0}\right)^2\right] = z_0\left[1+\frac{R^2}{4\,z_0^2}\right] = z_0 + \frac{R^2}{4\,z_0}.$$

Let's check if this matches the slide's $R_{\text{prime}}(z_0) = \frac{R\,(1+\xi_0^2)}{2\,\xi_0}$ with $\xi_0 = \frac{2\,z_0}{R}$.

$$R_{\text{prime}}(z_0) = R\cdot\frac{1+\left(\frac{2\,z_0}{R}\right)^2}{2\cdot\left(\frac{2\,z_0}{R}\right)} = R\cdot\frac{1+\frac{4\,z_0^2}{R^2}}{\frac{4\,z_0}{R}} = R\cdot\frac{R^2+4\,z_0^2}{R^2}\cdot\frac{R}{4\,z_0} = \frac{R^2+4\,z_0^2}{4\,z_0}$$
$$= \frac{R^2}{4\,z_0} + z_0.$$

This is indeed identical to the standard formula $R_{\text{wf}}(z_0) = z_0 + \frac{R^2}{4\,z_0}$ when $z_{\text{R}} = \frac{R}{2}$. So the formula is correct.

"Special cases":

* "At mirrors $z_0 = \frac{d}{2}$ implies $\xi_0 = 1$ implies $R_{\text{prime}} = R$." For a confocal resonator, $d = R$. So at the mirrors, $z_0 = \pm\frac{R}{2}$. Then $\xi_0 = \frac{2(\pm R/2)}{R} = \pm 1$. So $\xi_0^2 = 1$. $R_{\text{prime}}\left(z_0 = \pm\frac{R}{2}\right) = R\cdot\frac{(1+1^2)}{2\cdot 1}$ (taking positive z_0 for $\xi_0 = 1$) $= R\cdot\frac{2}{2} = R$. This is a crucial self-consistency condition: the wavefront radius of curvature of the $\text{TEM}_{0\,0}$ mode at the mirrors is R, which exactly matches the physical radius of curvature of the confocal mirrors. This is why these modes are stable.

* "At beam waist $z_0 = 0$ implies $R_{\text{prime}} = \infty$ (plane wavefront)." At the beam waist (center of the confocal cavity), $z_0 = 0$. Then $\xi_0 = \frac{2\cdot 0}{R} = 0$. The formula $R_{\text{prime}}(z_0) = \frac{R^2}{4\,z_0} + z_0$ shows that as z_0 approaches 0, the $\frac{R^2}{4\,z_0}$ term dominates and goes to infinity. So, $R_{\text{prime}}(0) = \infty$. An infinite radius of curvature means the wavefront is planar at the beam waist. This is also a standard characteristic of Gaussian beams: they have a flat phase front at their narrowest point.

Page 49:

This slide discusses "General Spherical Resonators – Equivalence Criterion." We are moving beyond the specific confocal case $d = R,\ R_1 = R_2 = R$ to resonators with two spherical mirrors that might have different radii of curvature R_1, R_2 and arbitrary separation d.

The first bullet point states the principle: "Replace confocal mirrors by any pair whose local curvature equals that of Gaussian wavefront at same z." This is a powerful idea. If we have a Gaussian beam (like the TEM_00 mode of a confocal resonator), its wavefronts have a specific radius of curvature $R_{wf}(z)$ at each axial position z. We can, in principle, place mirrors at any two positions z_1 and z_2 along this beam, as long as the physical curvature of the mirror placed at z_1 matches $R_{wf}(z_1)$ and the curvature of the mirror at z_2 matches $R_{wf}(z_2)$. Such a resonator would also support that same Gaussian beam as a mode.

The slide then considers a specific case: "For symmetric resonator (\(R_1 = R_2 = R_\)) of length \(d_\), require..." So, we have two identical mirrors, each with radius of curvature \(R_\), separated by a distance \(d_\). For this resonator to support a Gaussian beam whose wavefront curvature matches the mirrors, the condition is given:

\[R_ = \frac{R^2 + d_^2}{2 d_*}\]

This equation looks like it's relating the parameters (\(R_, d_\)) of a general symmetric resonator to the R of some reference confocal resonator, or perhaps R is a parameter from the Gaussian beam solution itself (like $2\,z_{\mathrm{R}}$, the confocal length of the underlying beam). Let's interpret R in the numerator as $2\,z_{R,\mathrm{beam}}$, where $z_{R,\mathrm{beam}}$ is the Rayleigh range of the fundamental Gaussian beam that "fits" this resonator. For a beam defined by waist w_0,

$$z_{R,\mathrm{beam}} = \frac{\pi w_0^2}{\lambda}$$

The radius of curvature of its wavefront at position z (measured from the waist) is

$$R_{wf}(z) = z\left(1 + \left(\frac{z_{R,\text{beam}}}{z}\right)^2\right)$$

If the symmetric resonator of length \(d_\) has its mirrors at \(z = \pm \frac{d_}{2}\) from the waist, then the radius of curvature of the mirrors \(R_\) must be equal to \(R_{wf}\left(\frac{d_}{2}\right)\).

So,

\[R_ = \frac{d_}{2}\left[1 + \left(\frac{z_{R,\text{beam}}}{\frac{d_}{2}}\right)^2\right] = \frac{d_}{2} + \frac{z_{R,\text{beam}}^2}{\frac{d_*}{2}}\]

This means

\[R_ = \frac{\left(\frac{d_}{2}\right)^2 + z_{R,\text{beam}}^2}{\frac{d_}{2}} = \frac{d_^2 + 4\,z_{R,\text{beam}}^2}{2\,d_*}\]

Comparing this to the slide's

\[R_ = \frac{R^2 + d_^2}{2\,d_*}\]

it implies

$$R^2 = 4\,z_{R,\text{beam}}^2, \quad \text{or} \quad R = 2\,z_{R,\text{beam}}.$$

This R is the length of an equivalent confocal resonator that would produce the same beam waist w_0, since

$$z_\text{R} = \frac{R_\text{confocal}}{2}.$$

So, the equation relates the mirror curvature \(R_\) and separation \(d_\) of a general symmetric stable resonator to the parameter R of an equivalent confocal resonator that shares the same fundamental mode.

"which rearranges to..." (continued on next slide)

Page 50:

This page continues from the rearrangement of the equivalence criterion for general spherical resonators. The previous equation was

$$R_\star = \frac{R^2 + d_\star^2}{2d_\star}$$

We want to solve for $d_\star$.

$$2d_\star R_\star = R^2 + d_\star^2$$

$$d_\star^2 - 2R_\star d_\star + R^2 = 0$$

This is a quadratic equation for $d_\star$: $ax^2 + bx + c = 0$, where $x = d_\star$, $a = 1$, $b = -2\,R_\star$, $c = R^2$. The solution is

$$d_\star = \frac{-b \pm \sqrt{b^2 - 4\,ac}}{2\,a}$$

$$d_\star = \frac{2R_\star \pm \sqrt{(-2\,R_\star)^2 - 4 \cdot 1 \cdot R^2}}{2}$$

$$d_\star = \frac{2R_\star \pm \sqrt{4\,R_\star^2 - 4\,R^2}}{2}$$

$$d_\star = \frac{2R_\star \pm 2\sqrt{R_\star^2 - R^2}}{2}$$

$$d_\star = R_\star \pm \sqrt{R_\star^2 - R^2}$$

This is exactly what is shown on the slide:

$$d_\star = R_\star \pm \sqrt{R_\star^2 - R^2}$$

The interpretation is profound: "Hence an infinite family* of mirror curvatures yields identical internal field pattern to a reference confocal case."

Let's clarify. R is related to the beam waist w_0 by

$$w_0^2 = \frac{R\lambda}{2\pi}$$

(where R here is the length of the reference confocal resonator that gives this w_0). For a given desired beam (defined by w_0, or equivalently by

$$R = \frac{2\pi w_0^2}{\lambda}$$

), we can choose a mirror radius of curvature $R_\star$ (as long as

$$R_\star^2 \geq R^2$$

, which means

$$|R_\star| \geq |R|$$

). Then, there are generally two possible mirror separations $d_\star$ (the plus and minus solutions) that will result in a symmetric resonator (mirrors of curvature $R_\star$) that supports that *exact same internal Gaussian field pattern* (same w_0, same $w(z)$ profile). The asterisk on "infinite family*" might refer to the continuous choice of $R_\star$ (as long as

$$|R_\star| \geq |R|$$

). For each such $R_\star$, we find one or two $d_\star$ values. This means many different physical resonators (different $R_\star$ and $d_\star$ combinations) can support the same fundamental Gaussian mode. The "reference confocal case" is when

$$d_\star = R_\star = R$$

. In this case,

$$\sqrt{R_\star^2 - R^2} = 0$$

, so

$$d_\star = R_\star$$

, which is consistent.

Other examples include:

- Plane-parallel: $R_\star \to \infty$. This formula doesn't directly apply, but it's the limit where $d_\star$ can be anything, but R must also effectively be infinite (plane wave).

- Concentric: $d_\star = 2\,R_\star$. Here, if

$$d_\star = R_\star + \sqrt{R_\star^2 - R^2}$$

then

$$R_\star = \sqrt{R_\star^2 - R^2}$$

, which implies

$$R = 0$$

(infinite w_0). This needs care. If

$$d_\star = 2\,R_\star$$

, then from

$$R_\star = \frac{R^2 + d_\star^2}{2\,d_\star}$$

we get

$$R_\star = \frac{R^2 + 4\,R_\star^2}{4\,R_\star}$$

, so

$$4\,R_\star^2 = R^2 + 4\,R_\star^2$$

, which implies

$$R = 0$$

. This means w_0 would be zero (a perfect focus), which is an idealization.

The formula is valid for stable resonators where

$$R_\star^2 > R^2$$

.

Page 51:

This slide introduces "Beam Radius & g-Parameter Notation," which is essential for characterizing general stable resonators, not just confocal ones.

First, it defines the g-parameter:

$$g = 1 - \frac{d}{R'}$$

(The prime on R' is usually omitted, so $g = 1 - \frac{d}{R}$.) This "g" is a dimensionless parameter. For a resonator with two mirrors, M1 and M2, with radii of curvature R_1 and R_2, and separation "d", we define two g-parameters:

$$g_1 = 1 - \frac{d}{R_1}$$

$$g_2 = 1 - \frac{d}{R_2}$$

(By convention, R is positive for a concave mirror if the center of curvature is within the cavity, or more generally, R is positive if the mirror is concave as seen from inside the cavity). The slide uses R', perhaps to distinguish from the confocal R. I'll use R.

Then, formulas for the beam waist radius w_0 (squared) and the spot size on the mirrors w_1 (squared) and w_2 (squared) are given in terms of these g-parameters (assuming a symmetric resonator where $g_1 = g_2 = g$, or these are general formulas applicable to one of the mirrors).

The formula for w_0^2 looks like:

$$w_0^2 = \frac{d\lambda}{\pi}\sqrt{\frac{1+g}{4(1-g)}}$$

This needs checking. The standard formula for w_0^2 for a general stable resonator defined by g_1, g_2, d, λ is:

$$w_0^2 = \frac{\lambda d}{\pi}\sqrt{\frac{g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2\, g_1\, g_2)^2}}$$

This is quite complex. For a symmetric resonator ($R_1 = R_2 = R_{\text{mirror}}$, so $g_1 = g_2 = g$), where $g = 1 - \frac{d}{R_{\text{mirror}}}$, the waist w_0 is at the center ($z = 0$).

$$w_0^2 = \frac{\lambda d}{2\pi}\sqrt{\frac{1+g}{1-g}}$$

This is for a symmetric resonator. The slide has

$$w_0^2 = \frac{d\lambda}{\pi}\sqrt{\frac{1+g}{4(1-g)}}$$

which is equivalent to

$$w_0^2 = \frac{\lambda d}{2\pi}\sqrt{\frac{1+g}{1-g}}$$

if we take the 2 inside the square root as 4. Yes, this matches the standard for symmetric resonators.

The formula for w_1^2 (spot size on mirror 1) is given as:

$$w_1^2 = w_2^2 = \frac{d\lambda}{\pi}\frac{1}{\sqrt{1-g^2}}$$

This is also for a symmetric resonator ($g_1 = g_2 = g$). The standard formulas for spot sizes on mirrors M_1 and M_2 are:

$$w_1^2 = \frac{\lambda d}{\pi}\sqrt{\frac{g_2}{g_1(1-g_1\,g_2)}}$$

$$w_2^2 = \frac{\lambda d}{\pi}\sqrt{\frac{g_1}{g_2(1-g_1\,g_2)}}$$

If $g_1 = g_2 = g$, then

$$w_1^2 = w_2^2 = \frac{\lambda d}{\pi}\sqrt{\frac{g}{g(1-g^2)}} = \frac{\lambda d}{\pi}\frac{1}{\sqrt{1-g^2}}$$

This matches the slide's formula.

The final bullet point: "Minimum w_0 occurs at $g = 0$ implies precisely the confocal geometry." For a symmetric resonator,

$$g = 1 - \frac{d}{R_{\text{mirror}}}.$$

If $g = 0$, then

$$1 - \frac{d}{R_{\text{mirror}}} = 0,$$

which means

$$\frac{d}{R_{\text{mirror}}} = 1, \quad \text{so} \quad d = R_{\text{mirror}}.$$

This is exactly the condition for a confocal resonator. Let's check w_0^2 for $g = 0$:

$$w_0^2 = \frac{\lambda d}{2\pi}\sqrt{\frac{1+0}{1-0}} = \frac{\lambda d}{2\pi}.$$

This is indeed the formula for the waist in a confocal resonator of length "d".

Does this value of w_0 correspond to a minimum? We have w_0^2 proportional to

$$\sqrt{\frac{1+g}{1-g}}.$$

The stability condition for $g_1\, g_2$ is $0 < g_1\, g_2 < 1$. For symmetric, $0 < g^2 < 1$, so $-1 < g < 1$ (excluding $g = \pm 1$ which are marginally stable).

The function

$$f(g) = \sqrt{\frac{1+g}{1-g}}$$

over $(-1,1)$: As $g \to -1$ from above, $1 + g \to 0$, so $f(g) \to 0$. (This corresponds to d near $2\,R$, concentric). As $g \to 1$ from below, $1 - g \to 0$, so $f(g) \to \infty$. (This corresponds to d near 0, or R near infinity for plane-parallel). The derivative of $\frac{1+g}{1-g}$ is

$$\frac{(1-g)\cdot 1 - (1+g)\cdot(-1)}{(1-g)^2} = \frac{1-g+1+g}{(1-g)^2} = \frac{2}{(1-g)^2},$$

which is always positive. So, $\frac{1+g}{1-g}$ is an increasing function of g. Thus, $\sqrt{\frac{1+g}{1-g}}$ is also increasing.

Therefore, the minimum w_0 actually occurs as g approaches -1 (concentric limit), where w_0 approaches 0. The statement "Minimum w_0 occurs at $g = 0$" seems incorrect. At $g = 0$ (confocal), w_0 is finite and well-behaved. It's not the global minimum for w_0 within the stable range $-1 < g < 1$.

Perhaps it means minimum *practical* w_0 or minimum w_0 for a fixed d when varying R? If d is fixed, and R varies, then g varies.

$$w_0^2 = \frac{\lambda d}{2\pi}\sqrt{\frac{1+\left(1-\frac{d}{R}\right)}{1-\left(1-\frac{d}{R}\right)}} = \frac{\lambda d}{2\pi}\sqrt{\frac{2-\frac{d}{R}}{\frac{d}{R}}} = \frac{\lambda d}{2\pi}\sqrt{\frac{2\,R}{d}-1}.$$

For this to be real, $\frac{2\,R}{d} \geq 1$, or $R \geq \frac{d}{2}$. This expression for w_0^2 clearly decreases as R decreases towards $\frac{d}{2}$ ($g \to +1$). And as R increases ($g \to 1 - 0 = 1$ if $d/R \to 0$, or $g \to 1-$ (small positive) if d/R is small). The statement "Minimum w_0 occurs at $g = 0$" is problematic as stated. I will note this.

Page 52:

This slide discusses "Diffraction Loss in Confocal Cavities – Empirical Fit." We are back to the specific confocal case $d = R$, or $g_1 = g_2 = 0$.

The first bullet point provides a widely used approximation for the diffraction loss per round trip γ_D for the fundamental TEM_{00} mode in a confocal resonator with circular mirrors, when the Fresnel number N_F is greater than 1:

$$\gamma_\text{D} \approx 16\pi^2 N_\text{F} \exp(-4\pi N_\text{F})$$

Here, $N_\text{F} = \frac{a^2}{\lambda d} = \frac{a^2}{\lambda R}$ is the Fresnel number, where 'a' is the mirror radius. This formula shows that the diffraction loss depends on N_F. The term

$\exp(-4\pi N_\mathrm{F})$ is a very rapidly decreasing function of N_F because of the large factor 4π (approx 12.56) in the exponent.

The second bullet lists "Observations."

The third bullet elaborates: "Loss falls super-exponentially with N_F." This is due to the $\exp(-\text{constant} \cdot N_\mathrm{F})$ term. The prefactor $16\pi^2 N_\mathrm{F}$ actually increases with N_F, but the exponential decay is so strong that it dominates completely for $N_\mathrm{F} > 1$. So, as N_F increases (e.g., by using larger mirrors for a given λ and d), the diffraction loss plummets extremely quickly. This is why confocal resonators are known for their low diffraction losses if N_F is even moderately large.

Page 53:

This page continues with the implications of the diffraction loss formula for confocal cavities.

The first bullet states: "Becomes $< 10^{-6}$ for $N_\mathrm{F} > 5$." (Here 10^{-6} means one part in a million).

Let's check this. If $N_\mathrm{F} = 5$:

$$\gamma_\mathrm{D} \approx 16\pi^2 \times 5 \times \exp(-4\pi \times 5)$$

$$= 80\pi^2 \times \exp(-20\pi)$$

$$= 80 \times (9.87) \times \exp(-62.83)$$

$$= 789.6 \times \exp(-62.83)$$

$\exp(-62.83)$ is $\left(\exp(-10)\right)^{6.283}$. $\exp(-10)$ is 4.5×10^{-5}. This will be incredibly small.

$$\exp(-62.83) = \left(\frac{1}{e}\right)^{62.83}.$$

Using a calculator: $\exp(-62.83) \approx 2.0 \times 10^{-28}$.

So

$$\gamma_{\mathrm{D}} \approx 789.6 \times 2.0 \times 10^{-28} = 1.6 \times 10^{-25}.$$

This is vastly smaller than 10^{-6}.

Perhaps the formula was intended for something else, or there's a typo in the $N_{\mathrm{F}} > 5$ condition for 10^{-6}.

Let's try to find N_{F} for which $\gamma_{\mathrm{D}} \approx 10^{-6}$.

$$16\pi^2 N_{\mathrm{F}} \exp(-4\pi N_{\mathrm{F}}) = 10^{-6}.$$

Approximately,

$$158 N_{\mathrm{F}} \exp(-12.57 N_{\mathrm{F}}) = 10^{-6}.$$

If $N_{\mathrm{F}} = 1$,

$$\gamma_{\mathrm{D}} \approx 158 \times \exp(-12.57) \approx 158 \times 3.47 \times 10^{-6} \approx 5.5 \times 10^{-4}.$$

If $N_{\mathrm{F}} = 2$,

$$\gamma_{\mathrm{D}} \approx 158 \times 2 \times \exp(-25.14) \approx 316 \times 1.2 \times 10^{-11} \approx 3.8 \times 10^{-9}.$$

This is already much less than 10^{-6} for $N_{\mathrm{F}} = 2$.

So, the condition " $N_{\mathrm{F}} > 5$" for loss " $< 10^{-6}$" is extremely conservative if this formula is correct. The loss becomes negligible much sooner.

Siegman ("Lasers", Table 19.1) gives diffraction loss for confocal TEM_{00} as α_{00} (loss fraction) $\approx 8\pi\sigma\exp(-4\pi\sigma)$, where $\sigma = N_{\mathrm{F}}$. This is slightly different (factor of 2 in prefactor). For $N_{\mathrm{F}} = 1$, loss is 0.05%. For $N_{\mathrm{F}} = 1.5$, loss is 7×10^{-6}. For $N_{\mathrm{F}} = 2$, loss is 3×10^{-8}.

So, a loss less than 10^{-6} is achieved for N_{F} somewhere between 1.5 and 2.

The slide's claim " $N_{\mathrm{F}} > 5$" implies much higher N_{F} is needed than what the formula (or similar ones) suggests. I will proceed with the formula result, noting the slide's value.

The second part of the bullet point is a practical design implication:

* This formula underpins mirror-size selection: choose a so that computed $\gamma_\mathrm{D} <$ gain margin.

a is the mirror radius, which determines N_F (since $N_\mathrm{F} = \frac{a^2}{\lambda R}$).

You need to choose your mirror radius a large enough such that the resulting N_F gives a diffraction loss γ_D that is significantly smaller than the available net gain (round-trip gain minus all other losses like reflection, scattering, absorption). If γ_D is too large, the laser won't reach threshold. "Gain margin" is the excess gain available to overcome this specific loss.

So, you'd calculate the N_F required for your acceptable γ_D, and then from N_F, λ, and R, you'd find the minimum a.

Page 54:

This page shows a graph: "Diffraction Loss (γ_D) vs. Fresnel Number (N_F)." The vertical axis is Diffraction Loss (γ_D) on a logarithmic scale, ranging from 1 (or 10^0) down to 10^{-20}. The horizontal axis is Fresnel Number (N_F) on a linear scale, from 1 to 4.

There are two curves plotted:

1. A blue curve labeled "Confocal." This curve starts at $N_\mathrm{F} = 1$ with γ_D around 10^{-4}, and drops extremely steeply. By $N_\mathrm{F} = 2$, it's already below 10^{-8}. By $N_\mathrm{F} = 3$, it's around 10^{-14}. By $N_\mathrm{F} = 4$, it's below 10^{-19}. This illustrates the super-exponential decrease in loss with N_F for confocal resonators, consistent with the formula $\exp(-4\pi N_\mathrm{F})$.

2. An orange curve labeled "Plane-Parallel." This curve is much flatter. It starts near $\gamma_\mathrm{D} = 1$ (or slightly below, perhaps 0.5-0.8) at $N_\mathrm{F} = 1$ and very slowly decreases. By $N_\mathrm{F} = 4$, it might be around 0.1 or 0.2. This corresponds to the much weaker $\gamma_\mathrm{D} \sim \frac{1}{N_\mathrm{F}}$ dependence for plane-parallel resonators.

This graph visually drives home the immense advantage of confocal resonators over plane-parallel ones in terms of minimizing diffraction losses for a given Fresnel number (i.e., for given mirror size a, wavelength λ, and separation d). For $N_{\mathrm{F}} = 2$, the confocal loss is orders of magnitude (perhaps 7-8 orders) smaller than for plane-parallel. This is a very compelling illustration.

Page 55:

This slide introduces the "Stability Criterion – Derivation via Gaussian Beam Transport." This is a fundamental concept for designing any two-mirror resonator.

The first bullet says: "For general two-mirror system with radii R_1, R_2 and length d, define $g_1 = 1 - \frac{d}{R_1}$, and $g_2 = 1 - \frac{d}{R_2}$." These are the g-parameters we encountered earlier.

- R_1 is the radius of curvature of mirror 1. - R_2 is the radius of curvature of mirror 2. - d is the separation between the mirrors. - The sign convention for R_1 and R_2 is usually: $R > 0$ for a concave mirror (center of curvature lies towards the interior of the cavity), and $R < 0$ for a convex mirror. A plane mirror has $R = \infty$, so its g-parameter is 1.

The second bullet states: "ABCD-matrix analysis of Gaussian beam shows reproduction condition (finite spot size) implies $0 < g_1 g_2 < 1$." This is the famous resonator stability criterion.

- ABCD matrices are a tool used in paraxial optics to describe how a Gaussian beam's parameters (like spot size and wavefront curvature) transform as it passes through optical elements or free space. - For a Gaussian beam to be a stable mode of a resonator, it must reproduce its own spot size and wavefront curvature after one complete round trip. - When this condition is analyzed using ABCD matrices for a round trip in a two-mirror resonator, it leads to the requirement that the product of the g-parameters, $g_1 g_2$, must lie strictly between 0 and 1 for the resonator to be stable.

$$0 < g_1 g_2 < 1$$

- "Stable" here means that the beam remains confined; its spot size on the mirrors remains finite. If this condition is not met, the resonator is "unstable," and a simple Gaussian beam will either expand to infinite size (diverge) or focus down to a point within the cavity, effectively being lost.

This $g_1 g_2$ product is a compact and powerful way to determine if a given resonator geometry (defined by R_1, R_2, d) will support stable, confined modes.

Page 56:

This page elaborates on the consequences of the stability criterion $0 < g_{one} g_{two} < 1$.

The first bullet point explains: "If product outside interval implies beam either diverges to infinity or focuses before mirror." The "interval" referred to here is $(0,1)$, so "outside interval" means the product $g_{one} g_{two}$ is less than or equal to 0, or $g_{one} g_{two}$ is greater than or equal to 1.

As we discussed, the standard Gaussian beam formulas for finite spot sizes (w_{zero}, w_{one}, w_{two}) require the term $g_{one} g_{two}(1 - g_{one} g_{two})$ to be positive, which leads directly to the condition $0 < g_{one} g_{two} < 1$ for these simple Hermite-Gaussian mode solutions to be physically meaningful with real and finite spot sizes.

If this condition is not met: * If $g_{one} g_{two} \geq 1$ (e.g., for plane-parallel mirrors where $g_{one} = g_{two} = 1$, so the product is 1; or for two mirrors that are too weakly curved for their separation), the ABCD matrix analysis shows that a paraxial ray initially close to the axis will progressively move further away from the axis on successive round trips. The beam effectively "walks off" the mirrors or diverges so rapidly that it cannot be confined. The calculated spot sizes from the standard formulas would become infinite or imaginary. *If* $g_{one}g_{two} \leq 0$ (e.g., if one g is positive and the other is negative, or if one g is zero like in a confocal setup on the boundary), the situation is a bit more nuanced. While the simple Gaussian mode formulas might require careful interpretation or lead to imaginary spot sizes if they are naively applied*, some of these configurations can still be stable in a broader sense, or are marginally stable (like the confocal case where $g_{one}g_{two} = 0$). For instance, certain types of "unstable resonators" used in high-power lasers deliberately operate in regions where $g_{one}g_{two}$ is outside the $(0,1)$ interval, but they have different mode structures that are not simple confined Gaussians.

However, for the purpose of standard, low-loss, stable Hermite-Gaussian modes, the condition $0 < g_{one}g_{two} < 1$ is the key. When outside this, the beam either isn't confined (diverges) or its behavior is more complex than a simple stable Gaussian mode. The phrase "focuses before mirror" can also occur in unstable configurations where the beam comes to a sharp focus internally and then rapidly expands.

The second part of the bullet defines terminology: "* Term stable resonator reserved for geometries satisfying above; else unstable." So, within the context of seeking well-behaved, confined Hermite-Gaussian modes, a resonator is termed "stable" if its g-parameters satisfy $0 < g_{one}g_{two} < 1$. If this condition is not met, the resonator is considered "unstable" with respect to supporting these simple Gaussian modes. As experimentalists, this criterion is your first checkpoint when designing a cavity to support a well-behaved laser beam.

Page 57:

This slide, "Spot Sizes on Mirrors in Stable Case," provides algebraic results for the fundamental mode spot areas (or rather, radii squared, as w is radius) on the mirrors, attributed to Boyd & Gordon, for a stable resonator.

The first bullet presents the "Algebraic result for fundamental mode spot areas":

$$\pi w_{\text{one}}^2 = \lambda d \sqrt{\frac{g_{\text{two}}}{g_{\text{one}}(1 - g_{\text{one}} g_{\text{two}})}}$$

And,

$$\pi w_{\text{two}}^2 = \lambda d \sqrt{\frac{g_{\text{one}}}{g_{\text{two}}(1 - g_{\text{one}} g_{\text{two}})}}$$

Let's analyze these important formulas:

- w_{one} is the spot radius (1/e field amplitude radius) of the TEM_{00} mode on mirror 1.
- w_{two} is the spot radius on mirror 2.
- λ is the wavelength.
- d is the mirror separation.
- $g_{\text{one}} = 1 - \frac{d}{R_{\text{one}}}$ and $g_{\text{two}} = 1 - \frac{d}{R_{\text{two}}}$ are the g-parameters of the two mirrors.

These formulas are fundamental for designing stable resonators. They tell you how large the beam will be on each mirror, which is crucial for selecting appropriate mirror diameters to avoid excessive diffraction loss.

Notice the term $(1 - g_{\text{one}} g_{\text{two}})$ in the denominator under the square root. For w_{one} and w_{two} to be real and finite, we need $g_{\text{one}} g_{\text{two}} < 1$. Also, for the overall term under the square root to be positive, we need g_{one} and g_{two} to have the same sign if their product is positive, which implies $g_{\text{one}} g_{\text{two}} > 0$. Thus, these formulas are valid in the stable region

$$0 < g_{\text{one}} g_{\text{two}} < 1.$$

The second bullet considers "Special situations":

The third bullet gives an example: "g_one = g_two = g implies expression reduces to symmetric form earlier."

This refers to a symmetric resonator, where

$$R_{\text{one}} = R_{\text{two}} = R_{\text{mirror}},$$

so

$$g_{\text{one}} = g_{\text{two}} = g = 1 - \frac{d}{R_{\text{mirror}}}.$$

In this case, w_{one}^2 will equal w_{two}^2. Let's call it w_s^2 (spot size on either mirror).

$$\pi w_s^2 = \lambda d \sqrt{\frac{g}{g(1-g^2)}}$$

$$\pi w_s^2 = \lambda d \sqrt{\frac{1}{1-g^2}}$$

$$\pi w_s^2 = \frac{\lambda d}{\sqrt{1-g^2}}$$

So,

$$w_s^2 = \frac{\lambda d/\pi}{\sqrt{1-g^2}}.$$

This indeed matches the formula for $w_{\text{one}}^2 = w_{\text{two}}^2$ we saw on page 51 for a symmetric resonator. So, the general formulas correctly reduce to the symmetric case.

Page 58:

This page continues with special situations or limits related to the spot sizes in stable resonators.

The bullet point states: "$g_{one}g_{two}$ approaches 1 from below (1 minus) implies spot sizes blow up implies resonator on verge of instability."

Let's examine this. The formulas for w_{one}^2 and w_{two}^2 (from the previous page) both have a factor of $\sqrt{1 - g_{one}g_{two}}$ in the denominator.

As the product $g_{one}g_{two}$ approaches 1 (from values less than 1, ensuring the resonator is still in the stable region but moving towards the boundary of stability), the term $1 - g_{one}g_{two}$ approaches 0.

Therefore, the square root of $1 - g_{one}g_{two}$ also approaches 0.

Since this term is in the denominator of the expressions for w_{one}^2 and w_{two}^2, this means that w_{one}^2 and w_{two}^2 will approach infinity.

If the spot sizes on the mirrors "blow up" to become very large, the beam is no longer well-confined by mirrors of any practical finite size. This signifies that the resonator is becoming unstable.

The boundaries of the stability region, $g_{one}g_{two} = 1$, include:

1. Plane-parallel resonators: $R_{one} = \infty$ ($g_{one} = 1$), $R_{two} = \infty$ ($g_{two} = 1$). Product $g_{one}g_{two} = 1$. Here, the formulas would predict infinite spot size unless $d = 0$, which is not a resonator. Plane-parallel resonators are known to be very sensitive and their modes tend to fill the entire mirror aperture.

2. Concentric (or spherical) resonators: $d = R_{one} + R_{two}$. For a symmetric concentric resonator, $d = 2\,R_{mirror}$, so $g_{one} = g_{two} = 1 - \frac{d}{R_{mirror}} = 1 - \frac{2\,R_{mirror}}{R_{mirror}} = 1 - 2 = -1$. The product $g_{one}g_{two} = (-1) \cdot (-1) = 1$. Here again, the resonator is on the edge of stability, and the spot sizes on the mirrors are theoretically very large, while the waist at the center can be very small (approaching zero for ideal concentric).

So, this behavior – spot sizes becoming extremely large as $g_{one}g_{two}$ approaches 1 – is a key characteristic of resonators nearing the boundary of the stable operating regime. In practice, one usually designs resonators to be comfortably within the stable region $(0 < g_{one} g_{two} < 1)$ to ensure robustness against small perturbations in d or mirror curvatures.

Page 59:

This slide presents the "Stability Diagram – Visual Map," which is a graphical representation of the resonator stability criterion. It's often called the Boyd-Kogelnik stability diagram.

The diagram is a plot with g_{one} on the horizontal axis and g_{two} on the vertical axis. Both axes typically range from -1 to $+1$ or further, but the key features are usually within this range.

The stable regions are defined by the condition $0 \le g_{one}\, g_{two} \le 1$. The lines $g_{one}\, g_{two} = 0$ (which are the g_{one}-axis and the g_{two}-axis) and the hyperbolas $g_{one}\, g_{two} = 1$ form the boundaries of the stable regions.

The shaded area on the provided diagram represents these stable regions. It's the area *between* the g_{one} and g_{two} axes and the two branches of the hyperbola $g_{one}\, g_{two} = 1$.

Several specific resonator configurations are typically marked on this diagram:

* **Confocal (**$g_{one} = 0, g_{two} = 0$): This is the origin of the diagram. It lies on the boundary $g_{one}\, g_{two} = 0$, so it's marginally stable. It's shown on the slide.

* **Plane-Parallel (**$g_{one} = 1, g_{two} = 1$): This point is at the corner of one of the stable regions, on the boundary $g_{one}\, g_{two} = 1$. Also marginally stable. It's labeled "Plane-Plane" on the slide.

* **Concentric (symmetric:** $g_{one} = -1, g_{two} = -1$): This point is also on the boundary $g_{one}\, g_{two} = 1$. (Labeled ($g_{one} = -1, g_{two} = -1$) on the slide).

* **Semi-confocal (e.g., one plane mirror** $g_{one} = 1$, other mirror $R_{two} = d$, so $g_{two} = 0$): This would be at ($g_{one} = 1, g_{two} = 0$) or ($g_{one} = 0, g_{two} = 1$). These points are on the axes, hence $g_{one}\, g_{two} = 0$, marginally stable. (Labeled "Semiconfocal ($g_{one} = 0, g_{two} = 1$)" and another implied "Semiconfocal" at ($g_{one} = 1, g_{two} = 0$)).

The slide shows a light blue shaded region that seems to cover the square from $g_{one} = -1$ to 1 and $g_{two} = -1$ to 1, but with the regions *outside* the hyperbolas $g_{one}\, g_{two} = 1$ and *outside* the axes $g_{one} = 0, g_{two} = 0$ being implicitly unshaded or less emphasized. The primary stable regions are those enclosed by the axes and the hyperbolas.

Any point (g_{one}, g_{two}) that falls within the shaded stable regions represents a resonator configuration that will support confined Hermite-Gaussian modes. Points outside are unstable. This diagram is an invaluable tool for resonator designers. By choosing mirror curvatures (R_{one}, R_{two}) and separation (d), one calculates g_{one} and g_{two} and plots the point on this diagram to immediately see if the design is stable.

Page 60:

This slide provides a "Table of Common Resonators & Parameters (g_one, g_two, G)," where G seems to be the product $G = g_{\text{one}} \times g_{\text{two}}$. (Note: Sometimes capital G is used for other parameters like gain, so context is key).

Let's go through the listed common resonators:

"Confocal symmetric implies $g_{\text{one}} = g_{\text{two}} = 0$, $G = -1$."

For a symmetric confocal resonator, $R_{\text{one}} = R_{\text{two}} = d$. So, $g_{\text{one}} = 1 - \frac{d}{d} = 0$. And $g_{\text{two}} = 1 - \frac{d}{d} = 0$. The product $G = g_{\text{one}} \times g_{\text{two}} = 0 \times 0 = 0$. The slide states $G = -1$. This is inconsistent with G being the product $g_{\text{one}} \times g_{\text{two}}$. Let's assume G here is *not* $g_{\text{one}} \times g_{\text{two}}$ but another parameter (perhaps related to magnification in unstable resonators, or round trip matrix element). Given the context of listing g_1, g_2, G, it's most natural to assume $G = g_1 g_2$. If so, G should be 0 for confocal. I will proceed assuming G on this slide *should* be the product $g_1 g_2$ for consistency with stability discussions. If G is something else, the slide should define it. Assuming $G = g_1 g_2$: Confocal symmetric: $g_{\text{one}} = 0$, $g_{\text{two}} = 0$. Therefore, $G = 0$. This is on the boundary of stability.

"Plane-plane implies $g_{\text{one}} = g_{\text{two}} = 1$, $G = 1$ (marginally unstable)."

For plane-plane, $R_{\text{one}} = \infty$, $R_{\text{two}} = \infty$. So, $g_{\text{one}} = 1 - \frac{d}{\infty} = 1$. And $g_{\text{two}} = 1 - \frac{d}{\infty} = 1$. Product $G = g_{\text{one}} \times g_{\text{two}} = 1 \times 1 = 1$. This is on the boundary of stability ($0 \leq G \leq 1$). The term "marginally unstable" is often used, or "marginally stable." It's highly sensitive to mirror tilt.

"Concentric symmetric (mirror centres coincide) implies $g_{\text{one}} = g_{\text{two}} = -1$, $G = 1$ (unstable)."

For concentric symmetric, the cavity length $d = 2\,R_{\text{mirror}}$ (where R_{mirror} is the radius of curvature of each mirror, and the mirrors face each other, their centers of curvature coinciding at the midpoint of the cavity). $g_{\text{one}} = 1 - \frac{d}{R_{\text{mirror}}} = 1 - \frac{2\,R_{\text{mirror}}}{R_{\text{mirror}}} = 1 - 2 = -1$. $g_{\text{two}} = 1 - \frac{d}{R_{\text{mirror}}} = -1$. Product $G = g_{\text{one}} \times g_{\text{two}} = (-1) \times (-1) = 1$. This is also on the boundary of stability ($G = 1$). The term "unstable" is used here; it's extremely sensitive to alignment, often considered practically unstable or at best marginally stable.

"Semi-confocal (one flat, one $R = 2\,d$) implies $g_{\text{one}} = 1$, $g_{\text{two}} = \frac{1}{2}$, $G = 0$."

Let mirror 1 be flat: $R_{\text{one}} = \infty$, so $g_{\text{one}} = 1$. Let mirror 2 have radius of curvature $R_{\text{two}} = 2\,d$ (concave). So, $g_{\text{two}} = 1 - \frac{d}{R_{\text{two}}} = 1 - \frac{d}{2\,d} = 1 - \frac{1}{2} = \frac{1}{2}$. Product $G = g_{\text{one}} \times g_{\text{two}} = 1 \times \frac{1}{2} = \frac{1}{2}$. This value $G = \frac{1}{2}$ is within the stable region $0 \leq G \leq 1$ (in fact, $0 < G < 1$). So, a semi-confocal resonator is stable. The slide states $G = 0$. This would imply $g_{\text{one}} \times g_{\text{two}} = 0$. If $g_{\text{one}} = 1$,

$g_{\text{two}} = \frac{1}{2}$, then $g_{\text{one}} \times g_{\text{two}} = \frac{1}{2}$. The condition $R = 2\,d$ makes $g_{\text{two}} = \frac{1}{2}$. Perhaps the $G = 0$ on the slide for semi-confocal is an error, or G is not $g_1 g_2$. If G is $g_1 g_2$, then $G = \frac{1}{2}$ for this configuration.

Let me re-evaluate my assumption about G. If G is not $g_1 g_2$, what could it be? A common parameter in resonator theory is the overall magnification for unstable resonators or a parameter in mode frequency spacing. However, given g_1 and g_2 are listed, $g_1 g_2$ is the most direct "G" related to stability. Let's assume there's a typo in the G values on the slide and proceed with $G = g_1 g_2$. – Confocal: $g_1 = 0,\ g_2 = 0$. Product $g_1 g_2 = 0$. Stable (marginally). – Plane-plane: $g_1 = 1,\ g_2 = 1$. Product $g_1 g_2 = 1$. Stable (marginally, often considered difficult). – Concentric symmetric: $g_1 = -1,\ g_2 = -1$. Product $g_1 g_2 = 1$. Stable (marginally, very difficult). – Semi-confocal (one flat $R_1 = \infty$, other $R_2 = 2\,d$): $g_1 = 1$. $g_2 = 1 - \frac{d}{2\,d} = \frac{1}{2}$. Product $g_1 g_2 = \frac{1}{2}$. This is stable.

The slide's characterizations like "marginally unstable" or "unstable" for $G = 1$ points are reasonable. The $G = -1$ for confocal and $G = 0$ for semi-confocal on the slide are puzzling if $G = g_1 g_2$. Perhaps G refers to $(m + n + 1)$ part of the Gouy phase sum for mode frequencies? For confocal ($g_1 g_2 = 0$), mode frequencies have

$$\left(q + \frac{m + n + 1}{2}\right)\frac{c}{2\,d}$$

For general, it's

$$\left(q + \frac{m + n + 1}{\pi}\arccos\left(\sqrt{g_1 g_2}\right)\right)\frac{c}{2\,d}$$

If $\arccos\left(\sqrt{G_{\text{slide}}}\right)$ relates to this... $\arccos\left(\sqrt{-1}\right) \rightarrow$ not real for confocal $G = -1$. $\arccos\left(\sqrt{0}\right) \rightarrow \pi/2$ for semi-confocal $G = 0$, which is correct: $\frac{m+n+1}{2}$. This could be it! G_{slide} on the slide may represent the argument of the $\arccos$

in the frequency formula, or some transformation of it. Specifically, if the mode frequency is

$$\nu_{qmn} = \frac{c}{2\,d}\left[q + \frac{1}{\pi}(m+n+1)\arccos\left(\sqrt{g_1 g_2}\right)\right]$$

for some conventions, or $\arccos\left(\sqrt{g_1 g_2}\right)$ OR $\arccos\left(\text{sign} \times g_{\text{average}}\right)$ etc. A common form is

$$\nu = \frac{c}{2\,L}\left[q + \frac{m+n+1}{\pi}\arccos\left(\sqrt{g_1 g_2}\right)\right]$$

This doesn't match the G values on the slide directly.

Given the title "Table of Common Resonators & Parameters (g_1, g_2, G)", I will assume G IS $g_1 g_2$ and point out the discrepancies in the listed G values compared to calculated $g_1 g_2$ values, and use the calculated ones for assessing stability.

Page 61:

This slide presents a "Worked Design Problem – Ensure Gain Volume Filling." This is a very practical consideration.

The "Example requirement" is:

- Gain crystal diameter is 0.6 cm.
- We want the fundamental mode (TEM_{00}) to "fill it." This means the spot size of the laser mode inside the crystal should be comparable to the crystal's radius (which is 0.3 cm) to efficiently extract energy from the entire pumped volume.
- The problem states: "Choose $w_{\text{one}} = 0.3$ cm at mirror M_one." So, we are setting the spot radius on one of the mirrors to be 0.3 cm (3 mm). This mirror M_one might be adjacent to the crystal or the crystal might be near it.

- A "Fresnel number $N_{\mathrm{F}} = 3$" is also specified. This N_{F} is likely related to an aperture within the cavity or the crystal itself acting as an aperture, and $N_{\mathrm{F}} = 3$ suggests moderate diffraction conditions. The definition

$$N_{\mathrm{F}} = \frac{a^2}{\lambda\, d}$$

implies 'a' here would be the aperture radius for which this N_{F} is relevant, perhaps the crystal radius. If crystal radius $a_{\mathrm{cryst}} = 0.3\,\mathrm{cm}$, then

$$N_{F_{\mathrm{cryst}}} = \frac{a_{\mathrm{cryst}}^2}{\lambda\, L_{\mathrm{cryst}}}$$

if L_{cryst} is its length, or related to mirror spot size and cavity length. This $N_{\mathrm{F}} = 3$ specification seems a bit disconnected from $w_1 = 0.3\,\mathrm{cm}$ unless w_1 *is* the 'a' in N_{F} for the mirror M1 itself (i.e., $N_{F_{\mathrm{mirror1}}} = \frac{w_1^2}{\lambda\, d}$? This is not standard N_{F} definition).

Let's ignore the $N_{\mathrm{F}} = 3$ for a moment and focus on the core problem: "Invert spot size formula to find necessary g_{two}." We are given w_{one} (spot radius on mirror 1). We need to find g_{two} for mirror 2.

The formula for w_{one}^2 was (from page 57):

$$\pi\, w_{\mathrm{one}}^2 = \lambda\, d\, \sqrt{\frac{g_{\mathrm{two}}}{g_{\mathrm{one}}(1 - g_{\mathrm{one}}\, g_{\mathrm{two}})}}$$

This formula has both g_{one} and g_{two}. If we want to find g_{two}, we must know or choose g_{one}, λ, and d, in addition to the given w_{one}.

The problem on the slide then presents a formula for g_{two}:

$$g_{\mathrm{two}} = \frac{w_{\mathrm{one}}^2}{2\, N_{\mathrm{F}}\, d\, \lambda}.$$

This formula is very different from the one derived from the Boyd-Gordon spot size formula.

Let's try to understand where this

$$g_{\text{two}} = \frac{w_{\text{one}}^2}{2\, N_{\mathsf{F}}\, d\, \lambda}$$

might come from.

If N_{F} is defined as

$$N_{\mathsf{F}} = \frac{a^2}{\lambda\, d},$$

and if we assume the aperture a that defines this N_{F} is actually the spot size w_{one} on mirror 1 (so $a = w_{\text{one}}$), then

$$N_{\mathsf{F}} = \frac{w_{\text{one}}^2}{\lambda\, d}.$$

Substituting this into the slide's formula for g_{two}:

$$g_{\text{two}} = \frac{w_{\text{one}}^2}{2\left(\frac{w_{\text{one}}^2}{\lambda\, d}\right) d\, \lambda},$$

we get

$$g_{\text{two}} = \frac{w_{\text{one}}^2}{2\, w_{\text{one}}^2} = \frac{1}{2}.$$

So, this formula on the slide seems to imply that g_{two} must be $\frac{1}{2}$ if the N_{F} (related to w_{one} itself) is used in this way. This would mean the design is constrained to $g_{\text{two}} = \frac{1}{2}$ based on this particular definition/use of N_{F}.

This is a non-standard approach. A more typical design flow would be:

1. Choose desired w_{one} (e.g., to match crystal radius).

2. Choose cavity length d and wavelength λ.

3. Choose a value for g_{one} (e.g., make M1 flat, so $g_{\text{one}} = 1$, or choose some curvature).

4. Then use the Boyd-Gordon formula to solve for the required g_{two} that gives the desired w_{one}. This usually involves solving a more complex algebraic equation.

The slide's formula

$$g_{\text{two}} = \frac{w_{\text{one}}^2}{2\,N_{\mathsf{F}}\,d\,\lambda}$$

is very specific and seems to tie g_{two} directly to N_{F}, where N_{F} is defined based on w_{one} itself (i.e., $N_{\mathsf{F}} = \frac{w_{\text{one}}^2}{\lambda\,d}$ if the formula is to yield $g_2 = \frac{1}{2}$). If $N_{\mathsf{F}} = 3$ is an independent requirement (e.g., $N_{F_{\text{aperture}}} = \frac{a_{\text{ap}}^2}{\lambda\,d} = 3$), and w_1 is also specified, then this formula tries to link them.

Let's assume the slide's formula for g_{two} is what we must use for this problem.

$$g_{\text{two}} = \frac{w_{\text{one}}^2}{2\,N_{\mathsf{F}}\,d\,\lambda}.$$

We have $w_{\text{one}} = 0.3\,\text{cm} = 0.003\,\text{m}$.

$$N_{\mathsf{F}} = 3.$$

Parameters d and λ will be given on the next slide to complete the calculation.

The goal here is to find the g_{two} for the second mirror based on the parameters of the first mirror's spot size and some N_{F} constraint.

Page 62:

This page continues the worked design problem.

It says: "Insert $d = 0.5\,\text{m}$, $\lambda = 1\,\mu\text{m}$ ($10^{-6}\,\text{m}$)" into the formula for g_{two} from the previous slide:

$$g_{\text{two}} = \frac{w_{\text{one}}^2}{2 \cdot N_{\text{F}} \cdot d \cdot \lambda}.$$

We had $w_{\text{one}} = 0.003\,\text{m}$ and $N_{\text{F}} = 3$.

So,

$$g_{\text{two}} = \frac{(0.003\,\text{m})^2}{2 \cdot 3 \cdot (0.5\,\text{m}) \cdot (1 \cdot 10^{-6}\,\text{m})}.$$

$$g_{\text{two}} = \frac{9 \cdot 10^{-6}\,\text{m}^2}{6 \cdot (0.5 \cdot 10^{-6}\,\text{m}^{2)}}.$$

$$g_{\text{two}} = \frac{9 \cdot 10^{-6}\,\text{m}^2}{3 \cdot 10^{-6}\,\text{m}^2}.$$

$$g_{\text{two}} = 3.$$

The slide states: "implies $g_{\text{two}} = 3$ (implies concave $R_{\text{two}} = -0.25\,\text{m}$)."

Wait. If $g_{\text{two}} = 3$, and $g_{\text{two}} = 1 - \frac{d}{R_{\text{two}}}$, with $d = 0.5\,\text{m}$.

$$3 = 1 - \frac{0.5\,\text{m}}{R_{\text{two}}}$$

$$2 = -\frac{0.5\,\text{m}}{R_{\text{two}}}$$

$$R_{\text{two}} = -\frac{0.5\,\text{m}}{2} = -0.25\,\text{m}.$$

A negative radius of curvature $R_{\text{two}} = -0.25\,\text{m}$ means Mirror 2 is CONVEX (when viewed from inside the cavity). A concave mirror would have R_{two} positive in this convention (e.g., if its center of curvature is within d). So, $g_{\text{two}} = 3$ implies a convex mirror M2 with $R_2 = -0.25\,\text{m}$.

Now, the crucial check: "Stability satisfied? product $g_{\text{one}}\, g_{\text{two}} = 3 > 1$ implies actually unstable."

To check stability, we need g_{one}. The problem statement chose $w_{\text{one}} = 0.3\,\text{cm}$ at mirror M_{one}, but it didn't specify g_{one} for mirror M_{one}. Is there an implicit assumption about g_{one}? If we assume mirror M_{one} is flat (a common starting point for design), then $R_{\text{one}} = \infty$, so $g_{\text{one}} = 1$. If $g_{\text{one}} = 1$, then the product $g_{\text{one}}\, g_{\text{two}} = 1 \cdot 3 = 3$. Since $3 > 1$, this resonator configuration ($g_{\text{one}} = 1, g_{\text{two}} = 3$) is indeed UNSTABLE according to the criterion $0 < g_{\text{one}}\, g_{\text{two}} < 1$.

The conclusion is: "implies must adjust design (e.g. larger R_{two} or reduce w_{one})." If the design is unstable, it won't support the desired confined Gaussian mode. To make it stable, we need to change parameters such that $0 < g_{\text{one}}\, g_{\text{two}} < 1$.

- "larger R_{two}": If R_{two} (which was $-0.25\,\text{m}$, convex) is made "larger," it means making it less convex (e.g., $R_{\text{two}} = -0.5\,\text{m}$) or even flat ($R_{\text{two}} = \infty$, $g_{\text{two}} = 1$) or concave (R_{two} positive). If we keep M1 flat ($g_{\text{one}} = 1$), then we need $0 < 1 \cdot g_{\text{two}} < 1$, so $0 < g_{\text{two}} < 1$. This means we need $0 < \left(1 - \frac{d}{R_{\text{two}}}\right) < 1$. If $1 - \frac{d}{R_{\text{two}}} > 0$, then $1 > \frac{d}{R_{\text{two}}}$. If $R_{\text{two}} > 0$ (concave), then $R_{\text{two}} > d$. (This is a stable configuration). If $R_{\text{two}} < 0$ (convex), then $1 > -\frac{d}{|R_{\text{two}}|}$, which is always true if $d, |R_{\text{two}}|$ are positive. If $1 - \frac{d}{R_{\text{two}}} < 1$, then $-\frac{d}{R_{\text{two}}} < 0$, so $\frac{d}{R_{\text{two}}} > 0$. This requires R_{two} to be positive (concave). So, for $g_{\text{one}} = 1$, stability requires $R_{\text{two}} > d$ (concave mirror M2 with radius of curvature larger than cavity length). Our calculated $R_{\text{two}} = -0.25\,\text{m}$ (convex) clearly does not satisfy this. So, making R_{two} "larger" should mean choosing a concave $R_{\text{two}} > d$.

- "reduce w_{one}": If we reduce w_{one}, and if g_{two} is still calculated by that same formula

$$g_{\text{two}} = \frac{w_{\text{one}}^2}{2 \cdot N_{\text{F}} \cdot d \cdot \lambda},$$

then reducing w_{one} would reduce g_{two}. If g_{two} can be brought into the $(0,1)$ range (assuming $g_{\text{one}} = 1$), then it could become stable. For example, if we want $g_{\text{two}} = 0.5$ (stable with $g_{\text{one}} = 1$), then:

$$0.5 = \frac{w_{\text{one}}^2}{2 \cdot 3 \cdot 0.5\,\text{m} \cdot 10^{-6}\,\text{m}} = \frac{w_{\text{one}}^2}{3 \cdot 10^{-6}\,\text{m}^2}.$$

$$w_{\text{one}}^2 = 0.5 \cdot 3 \cdot 10^{-6}\,\text{m}^2 = 1.5 \cdot 10^{-6}\,\text{m}^2.$$

$$w_{\text{one}} = \sqrt{1.5} \cdot 10^{-3}\,\text{m} = 1.22 \cdot 10^{-3}\,\text{m} = 0.122\,\text{cm}.$$

So, if we aim for a smaller spot size $w_{\text{one}} = 0.122\,\text{cm}$ (instead of 0.3 cm), then g_{two} would be 0.5, and the resonator (with $g_{\text{one}} = 1$) would be stable.

This example, despite the unusual formula for g_{two}, illustrates the iterative nature of design: make an initial choice, check for stability, and adjust if necessary. The key takeaway is that simply achieving a desired spot size is not enough; the resulting resonator must also be stable.

Page 63:

This slide shifts focus to "Unstable Resonators – Motivation & Basic Geometry." We've primarily discussed stable resonators so far. Why would we want an unstable one?

The first bullet explains the motivation:

"High-gain bulk media (e.g. excimer, Nd-glass) waste inversion if confined to small Gaussian."

- Some laser media, like excimer gases or large Nd:glass slabs, can provide very high gain over a large volume (large transverse cross-section).

- If we use a stable resonator with these media, the fundamental TEM_{00} mode is typically a small, well-confined Gaussian beam. This means only a small fraction of the available excited atoms/molecules in the large gain medium actually interacts with the laser mode and contributes to stimulated emission. Much of the stored energy (population inversion) in the outer regions of the gain medium would be "wasted" (e.g., lost to fluorescence).

* Unstable cavity lets beam expand; more gain volume accessed; tolerates large intra-cavity power; natural output coupling via walk-off.

Unstable resonators are designed such that the mode is *not* confined in the same way as in a stable resonator. Instead, the mode typically expands on each round trip.

- "lets beam expand; more gain volume accessed": Because the mode expands to fill a larger transverse area, it can extract energy from a much larger portion of the high-gain bulk medium. This leads to higher output power and efficiency.

- "tolerates large intra-cavity power": Since the mode is large, the intensity (power per unit area) can be kept below the damage threshold of the optical components, even if the total intracavity power is very high.

- "natural output coupling via walk-off": In many unstable resonator designs, particularly "positive-branch" ones, the beam expands on each pass such that a portion of it "walks off" or spills around the edge of one of the mirrors (usually the output mirror). This spilled portion forms the output beam. This provides a way to extract a large fraction of the intracavity power, suitable for high-gain systems.

"* Geometry variants" refers to the different ways unstable resonators can be configured.

The second bullet point describes one common type: "Symmetric unstable (both R_i same sign, $|g_1|, |g_2| > 1$)."

- "both R_{i} same sign": For example, both mirrors are convex ($R_1 < 0$, $R_2 < 0$, as seen from inside), or both are concave but configured such that the g-parameters are large.

- "$|g_1|, |g_2| > 1$": Recall $g = 1 - \frac{d}{R}$. If R is convex, $R = -|R|$. So $g = 1 + \frac{d}{|R|}$, which is > 1.

- If both g_1 and g_2 are > 1, then their product $g_1 g_2$ will be > 1, which is the condition for instability.

- This type of resonator leads to a diverging geometric wave that expands on each pass. Output is often taken by diffraction around a smaller feedback mirror.

Page 64:

This page continues with geometry variants for unstable resonators.

The bullet point describes: "Asymmetric unstable (one mirror small curvature, one planar or opposite sign)." This refers to other ways to achieve $g_{one} g_{two} > 1$ or $g_{one} g_{two} < 0$, leading to instability.

- "one mirror small curvature, one planar": If mirror 1 is planar, $R_{one} = \infty$, so $g_{one} = 1$. If mirror 2 has a small curvature, it means $|R_{two}|$ is large. If M_2 is convex ($R_{two} = -|R_{two}|$), then

$$g_{two} = 1 - \frac{d}{-|R_{two}|} = 1 + \frac{d}{|R_{two}|}$$

Since $\frac{d}{|R_{two}|}$ is positive, $g_{two} > 1$. Then $g_{one} g_{two} = 1 \times g_{two} = g_{two} > 1$. This is unstable. This is a common setup: a flat mirror and a convex mirror. If M_2 is concave ($R_{two} > 0$) but with $R_{two} < d$ (i.e., cavity length is greater than mirror's radius of curvature), then

$$\frac{d}{R_{two}} > 1, \quad g_{two} = 1 - \frac{d}{R_{two}} < 0$$

Then $g_{one}g_{two} = 1 \times g_{two} < 0$. This is also in the unstable region (specifically, $g_{one}g_{two} < 0$ part). This can lead to a negative-branch unstable resonator.

- "one mirror small curvature, one ... opposite sign [of curvature]." This could mean one mirror is concave $(R_{one} > 0)$ and the other is convex $(R_{two} < 0)$. $g_{one} = 1 - \frac{d}{R_{one}}$. $g_{two} = 1 - \frac{d}{R_{two}} = 1 + \frac{d}{|R_{two}|}$. So, $g_{two} > 1$. If R_{one} is, for example, very large and positive (weakly concave) such that g_{one} is close to 1 (e.g., $g_{one} = 0.8$), then $g_{one}g_{two}$ could still be > 1 if g_{two} is large enough (e.g., $g_{two} = 1.5$ would give $g_{one}g_{two} = 1.2$, unstable). Or, if R_{one} is such that g_{one} is negative (e.g., R_{one} is concave, but $R_{one} < d$), and R_{two} is convex $(g_{two} > 1)$, then $g_{one}g_{two}$ will be negative, which is also unstable.

The key idea is that by choosing mirror curvatures and separation such that the product $g_{one}g_{two}$ falls outside the stable range $(0,1)$, various types of unstable resonators can be formed. These are categorized based on their geometry (e.g., positive-branch, negative-branch, confocal unstable, etc.) and have distinct mode properties, often involving geometrically expanding waves rather than confined Gaussian modes.

Page 65

This page has a title "Spherical Waves and Virtual Foci with Concave Mirrors" and a diagram. This diagram is likely meant to illustrate how waves behave in an unstable resonator, particularly a confocal unstable resonator.

The diagram shows: - An optical axis (horizontal line). - Two concave mirrors, M1 on the left and M2 on the right, facing each other. - A point F1 is marked to the left of M1. - A point F2 is marked to the right of M2. - A series of curved lines (wavefronts) are drawn between M1 and M2. - Blue wavefronts are shown expanding from left to right (originating from near F1, reflecting off M1, propagating towards M2). - Red wavefronts are shown expanding from right to left (originating from near F2, reflecting off M2, propagating towards M1).

This setup is characteristic of a confocal unstable resonator, specifically a positive-branch confocal unstable resonator.

In such a resonator: - Both mirrors are concave, say M1 has radius R_1 and M2 has radius R_2. - They are arranged such that the focal point of M1 (at $\frac{R_1}{2}$ from M1) coincides with the center of curvature of M2 (or a point related to R_2), and vice-versa, leading to a common virtual focus for waves expanding in opposite directions. - More precisely, for a positive-branch confocal unstable resonator, often $R_1 > d$ and $R_2 < 0$ (convex) or R_1 and R_2 are chosen such that $g_1 g_2 > 1$. The diagram shows two concave mirrors.

A common confocal unstable setup uses two concave mirrors, say M1 (radius R_1) and M2 (radius R_2, smaller than R_1). M2 is the smaller "feedback" mirror, and M1 is the larger "output" mirror. They share a common focal point. The light reflects off M2, expands towards M1, reflects off M1, and a portion of it is refocused back towards M2, while the outer part of the beam that misses M2 (or is scraped off) forms the output.

The points F1 and F2 in the diagram likely represent the common virtual centers of curvature for the expanding spherical waves that form the mode of this unstable resonator. The mode consists of a spherical wave appearing to emanate from F1, reflecting off M1, then appearing to emanate from F2 after reflecting off M2, and so on. The key is that the wavefront curvatures match the mirrors at incidence.

This diagram illustrates the geometric optic picture of modes in some unstable resonators, where the field is better described as expanding spherical waves rather than confined Gaussian beams.

Page 66:

This slide is titled "Magnification Factor & Loss per Round-Trip" for unstable resonators.

The first bullet states: "For generic unstable cavity, transverse diameter multiplies by M each pass."

In many unstable resonators, particularly positive-branch ones, the beam's transverse size (diameter or radius) increases by a geometric magnification factor, capital M, every time it makes a single pass through the resonator (or sometimes M is defined for a round trip). This M is typically greater than 1.

An expression for M is given:

$$M = \left(\frac{d + R_{\text{one}}}{R_{\text{one}}}\right)\left(\frac{d + R_{\text{two}}}{R_{\text{two}}}\right)$$

This seems incorrect based on standard forms. Let's look at the next part of the equality.

$$M = |G| \pm \sqrt{G^2 - 1}$$

And G is defined as

$$G = 2\, g_{\text{one}} g_{\text{two}} - 1$$

This $G = 2\, g1\, g2 - 1$ is a common parameter in unstable resonator theory. It's related to the eigenvalues of the round-trip ABCD matrix.

The magnification M is then one of the eigenvalues of the ray matrix analysis for transverse position and angle, often given as

$$M = \left|G + \sqrt{G^2 - 1}\right| \quad \text{or} \quad M = \left|G - \sqrt{G^2 - 1}\right|$$

where one is M and the other is $\frac{1}{M}$. For an expanding beam, $M > 1$.

So, $M = \left|G + \sqrt{G^2 - 1}\right|$ is typically the magnification if $|G| > 1$ (which is true for unstable resonators where $g1\, g2 > 1$ or $g1\, g2 < 0$, making $G^2 > 1$). The plus/minus on the slide could refer to the two eigenvalues, one being M and the other $\frac{1}{M}$ (or $-\frac{1}{M}$). The magnification is the one with magnitude > 1.

The second bullet addresses loss: "Fraction of power retained after reflection from small out-coupling mirror: $\frac{P_{\text{return}}}{P_{\text{zero}}} = \frac{1}{M^2}$."

This is a common result for the feedback coupling in unstable resonators where output is taken by geometric expansion.

- P_{zero} is the power incident on the feedback mirror (the smaller mirror). - P_{return} is the power reflected by this mirror that remains within the resonating mode. - If the beam diameter expands by M per pass, its area expands by M^2. - If the feedback mirror effectively re-captures only the central $\frac{1}{M^2}$ fraction of the incident beam's area to send it back for another pass, then the fraction of power retained is $\frac{1}{M^2}$.

The loss per pass (or round trip, depending on M's definition) due to this geometric out-coupling is

$$V = 1 - \frac{P_{\text{return}}}{P_{\text{zero}}} = 1 - \frac{1}{M^2}$$

The slide writes "V = 1 - 1/M^2". This V would be the fractional power *lost* (or coupled out) per round trip if M is the round-trip magnification.

Page 67:

This page continues the discussion of the loss per round-trip, V, in an unstable resonator.

"where $V =$ loss per round-trip." This confirms the interpretation of V from the previous slide: $V = 1 - \frac{1}{M^2}$ is the fraction of power lost (or coupled out as the useful beam) per round trip, if M is the round-trip magnification factor. This is a purely geometric loss due to the beam expanding past the edges of the feedback mirror.

* Typical V is approximately $0.9 - 0.99$ implies only 1–2 passes suffice; demands gain $>$ 5 times higher than stable designs.

This statement seems to have a misunderstanding of V or typical values. If V is the loss per round trip, then $V = 0.9$ means 90% loss per round trip. $V = 0.99$ means 99% loss per round trip. These are extremely high losses! If the loss is 90% per round trip, then the gain per round trip must be enormous to compensate (gain factor of 10). If loss is 99%, gain factor of 100 is needed.

"only 1–2 passes suffice": This implies that if the loss V is high, the laser doesn't need many passes to build up. But usually, for a laser to lase, the gain must overcome the loss. If V is this high, it means the output coupling is very large.

Perhaps V on the slide is actually meant to be the *reflectivity* of the effective output coupler $\left(R_{\text{eff}} = \frac{1}{M^2}\right)$, and the loss is $(1 - V)$. If $R_{\text{eff}} = \frac{1}{M^2}$ is typically 0.9–0.99, this would mean M^2 is $\frac{1}{0.9} \sim 1.11$ or $\frac{1}{0.99} \sim 1.01$. So M is very close to 1. This would be a very low magnification unstable resonator, with low output coupling (10% or 1%). This contradicts the idea of "natural output coupling via walk-off" for high gain systems usually needing large output coupling.

Let's re-read Siegman on unstable resonators. The geometric output coupling fraction is indeed

$$L = 1 - \frac{1}{M^2}$$

(where M is round-trip magnification). For high gain lasers, M is chosen to give substantial output coupling, e.g., $M = 2$ gives $L = 1 - \frac{1}{4} = 0.75$ (75% output coupling). In this case, $V = 0.75$.

The statement "Typical $V \sim 0.9 - 0.99$" if V is loss, implies M is very large:

If $V = 0.9$, then $\frac{1}{M^2} = 0.1$, $M^2 = 10$, $M \sim 3.16$.

If $V = 0.99$, then $\frac{1}{M^2} = 0.01$, $M^2 = 100$, $M = 10$.

These are reasonable magnifications for unstable resonators. So V is indeed loss. If V (loss) is 0.9 to 0.99, it means 90% to 99% of the energy is coupled out per round trip. This requires the single-pass gain to be very high. For example, if round-trip loss is 90%, then round-trip gain must be at least a factor of 10.

"only 1–2 passes suffice": This could mean that due to the high gain and high output coupling, the photons don't need to make many round trips before being extracted. The effective photon lifetime in such a cavity is very short.

Page 68:

This slide is titled "Near & Far-Field Patterns of Unstable Resonator." Part (a) shows the "Near-Field Intensity Profile (Unstable Resonator)." The caption says: "Output at coupling mirror plane, showing annular beam with dark central zone. Assumes magnification $M = 2$."

Two diagrams are shown:

1. "2D Beam Cross-Section": This is a circular view. It shows a bright outer ring (annulus), colored reddish-pink. The very center of this annulus is dark, labeled "Dark Zone." This doughnut shape is characteristic of the near-field output from many positive-branch unstable resonators. The dark central zone corresponds to the "shadow" of the smaller feedback mirror if the output is scraped around its edge, or it's the region that was blocked by the output coupler mirror itself if it's a hole-coupled unstable resonator (though scraping is more common for the $M = 2$ geometric case).

2. "1D Radial Intensity Profile": This shows a plot of intensity versus radial distance. It depicts a flat-topped profile for the bright annular region,

dropping sharply to zero intensity in the central "Dark Zone," and also dropping sharply to zero at the outer edge of the annulus. This idealized "top-hat" annular profile is what one might expect from a purely geometric optics model of the unstable resonator with uniform illumination.

The assumption $M = 2$ means the beam diameter magnifies by a factor of 2 on each pass. If the feedback mirror has diameter D_{fb}, the beam incident on it from the previous pass would have had diameter $M \times D_{fb} = 2 \times D_{fb}$. The feedback mirror reflects the central part of diameter D_{fb}, and the annulus from D_{fb} to $2 \times D_{fb}$ is the output. So the inner diameter of the output annulus is D_{fb} and the outer is $2 \times D_{fb}$. The central "dark zone" has diameter D_{fb}.

Page 69:

This page shows part (b): "Far-Field Intensity Profile (Unstable Resonator)." The caption says: "Comparison with Airy disk from uniformly illuminated circular aperture of same outer diameter (a)." (Here a likely means the outer radius of the annular near-field beam).

A "2D Far-Field Pattern (Unstable Resonator)" is shown on the left. It depicts a central bright spot, but it looks somewhat "speckled" or "structured" rather than perfectly smooth like an Airy disk. This is because the far-field pattern is the Fourier transform of the near-field annular aperture.

On the right, a "1D Radial Intensity Profile Comparison" graph is shown, titled "Far-Field Intensity ($M = 2, \quad \epsilon = 0.5$)." - The vertical axis is "Normalized Intensity." - The horizontal axis is "Normalized Angle ($u = k\, a \sin(\theta)$)", where a is the outer radius of the near-field annulus, $k = \frac{2\pi}{\lambda}$, and θ is the far-field angle. - $\epsilon = 0.5$ refers to the obscuration ratio: $\epsilon = \frac{\text{inner radius}}{\text{outer radius}}$. If $M = 2$, then outer diameter is twice the inner, so outer radius is twice the inner radius. Thus, $\epsilon = \frac{r_{\text{inner}}}{r_{\text{outer}}} = \frac{1}{M} = \frac{1}{2} = 0.5$. This is consistent.

Two curves are plotted: 1. "Unstable Resonator" (solid red line): This shows a central lobe that is narrower than the Airy disk's central lobe. However, its first side-lobe (and subsequent side-lobes) are significantly higher (stronger) than those of the Airy disk. The annotation points to "Narrower Central Lobe" and "Stronger Side-Wings." 2. "Airy Pattern (ref.)" (dashed blue line): This is the far-field diffraction pattern of a uniformly illuminated circular aperture of the same outer radius a (with no central obscuration). It has a wider central lobe but much lower side-lobes.

The key takeaway is that while the unstable resonator's far-field central spot can be narrower (implying better directivity for the very core of the beam), a significant amount of energy is thrown into the side-lobes due to the annular nature of the near-field. This can be an issue for applications requiring very clean beams with low side-lobe energy.

Page 70:

This slide introduces "Ring Resonators – Architecture & Directionality." So far, we've mostly considered standing-wave or linear resonators (light bounces back and forth along the same path). Ring resonators are different.

* "Consist of ≥ 3 reflectors forming closed polygonal optical path; no wave retraces exact opposite direction."

A ring resonator uses three or more mirrors to guide the light around a closed loop (e.g., a triangle for 3 mirrors, a rectangle for 4 mirrors). The key difference from a linear cavity is that light can, in principle, circulate around this loop in either the clockwise (CW) or counter-clockwise (CCW) direction, or both. Because the path is a closed loop, a wave traveling in one direction does not automatically retrace its path in the opposite direction unless there's a reflection that sends it backward, which is not the primary mode of operation.

* "Supports traveling waves only implies eliminates standing-wave spatial hole burning inside gain medium."

If the laser operates primarily with light traveling in only one direction around the ring (e.g., only CW), then the mode inside the gain medium is a traveling wave, not a standing wave. In a standing wave, there are fixed nodes (zero intensity) and antinodes (maximum intensity). "Spatial hole burning" occurs because the gain medium is depleted (atoms de-excite) more strongly at the antinodes, while inversion remains high at the nodes. This non-uniform gain saturation can allow other longitudinal modes (which have antinodes at different positions) to lase, leading to multi-mode operation. A traveling wave has uniform intensity (averaged over an optical cycle) along its path. This leads to more uniform gain saturation, which can help in achieving single longitudinal mode operation and more efficient energy extraction. This is a major advantage of ring lasers.

* "Enforce unidirectional operation using optical diode assembly:"

To get the benefit of a traveling wave, the laser needs to operate unidirectionally (either CW or CCW, but not both simultaneously). This is often achieved by introducing an "optical diode" or "optical isolator" element into the ring cavity. An optical diode allows light to pass with low loss in one direction but introduces high loss for light traveling in the opposite direction.

Components of a typical optical diode assembly for ring lasers:

* "Faraday rotator (non-reciprocal polarisation rotation by $\pm\alpha$)."

A Faraday rotator, when a magnetic field is applied along the direction of light propagation, rotates the plane of polarization of linearly polarized light. Crucially, this rotation is non-reciprocal: the direction of rotation (e.g., $+\alpha$ degrees) is the same regardless of whether the light passes forward or backward through the rotator. So, if light passes through, gets rotated by $+\alpha$, reflects, and passes back, it gets another $+\alpha$ rotation (total 2α relative to initial, or 0 relative to the rotated state if you consider single pass).

* "Birefringent reciprocal rotator ($+\alpha$)."

This is usually an optically active material like crystalline quartz, or a waveplate, that also rotates the plane of polarization, but this rotation IS reciprocal. If it rotates by $+\alpha$ in the forward direction, it rotates by $-\alpha$ (undoing the rotation) in the backward direction if it's a simple reciprocal rotator. The slide says "$+\alpha$" implying it adds to the rotation. Often, a half-wave plate is used whose axis is oriented to provide a certain rotation.

* "Polarisation-dependent Brewster windows provide differential loss."

By combining a non-reciprocal Faraday rotator with a reciprocal rotator, one can arrange it so that light traveling in the desired direction (e.g., CW) experiences a net polarization rotation that allows it to pass through polarizing elements (like Brewster windows on the gain medium, or intracavity polarizers) with low loss. Light traveling in the opposite direction (CCW) experiences a different net rotation, resulting in a polarization state that suffers high loss at these polarizing elements. This difference in loss favors lasing in only one direction.

Page 71:

This slide provides a diagram illustrating a "4-Mirror Ring Resonator with Optical Diode (CW Operation Favored)." This continues the discussion from the previous page about ring resonators and achieving unidirectional operation.

The diagram shows:

- Four mirrors (M1, M2, M3, M4) arranged in a roughly rectangular path.

- The optical path is shown with red lines and arrows indicating the Clockwise (CW) and Counter-Clockwise (CCW) directions.

- Between M1 and M2, an element labeled "FR ($-\alpha$ CW, $+\alpha$ CCW)" is shown. This is the Faraday Rotator. It rotates the polarization by $-\alpha$ for CW light and by $+\alpha$ for CCW light (or vice-versa, the signs indicate non-reciprocity).

- Between this FR and M2, another element "BR ($+\alpha$)" is shown. This is the Birefringent Reciprocal rotator. It provides a $+\alpha$ rotation for light passing in either direction through it.

- A "Net $+90°$" rotation is indicated for the CW path after FR and BR. This implies $-\alpha_{\text{FR, CW}} + \alpha_{\text{BR}}$ results in a specific desired rotation (e.g. to align with a polarizer).

- A "Net $0°$" rotation is indicated for the CCW path after BR and FR. This implies $+\alpha_{\text{FR, CCW}} + \alpha_{\text{BR, effective for CCW roundtrip}}$ results in a different net rotation for the CCW beam, perhaps one that is extinguished by a polarizer. The exact values here ($+45°, -45°$ are also annotated) show how the specific rotations are engineered.

- Near M4, "BW" (Brewster Window) is indicated, with P (parallel) and S (perpendicular) polarization states shown. Brewster windows transmit P-polarized light with very low loss but reflect S-polarized light. This acts as the polarizer.

The optical diode works as follows (simplified concept):

- Assume light starts P-polarized (passes BW with low loss).

- CW direction: The FR rotates it by $-\alpha_{\text{FR}}$. The BR rotates it by $+\alpha_{\text{BR}}$. The net rotation is designed such that the light arrives at the next polarizing element (e.g., another BW or the same one after a round trip) in a state that allows it to pass with low loss (e.g., still P-polarized or rotated back to P).

- CCW direction: The FR rotates by $+\alpha_{\text{FR}}$. The BR rotates by $+\alpha_{\text{BR}}$ (reciprocal). The net rotation for the CCW beam is different from the CW beam's net rotation. This difference is engineered such that the CCW beam arrives at the polarizing element in a state that experiences high loss (e.g., S-polarized or significantly mixed).

- The green arrows with polarization symbols (double arrow for P, dot for S) show how the polarization state evolves for the CW path, which is favored.

The specific angles ($-45°, +45°$,Net $+90°$,Net $0°$) shown are typical for an isolator setup that might rotate the polarization by $45°$ in the forward pass and effectively $0°$ or $90°$ (crossed) in the reverse to achieve isolation with polarizers. The diagram illustrates that CW operation is favored due to lower losses.

Page 72:

This slide outlines the "Benefits of Ring Geometry."

"Full inversion utilisation – no nodes implies potentially higher single-mode power."

This refers back to the elimination of spatial hole burning, which we discussed. In a standing-wave cavity, the gain medium has nodes and antinodes. At the nodes, the population inversion is not depleted by the lasing mode. This unused inversion can then provide gain for other longitudinal modes that happen to have antinodes at those locations, leading to multi-mode operation.

In a ring laser operating unidirectionally with a traveling wave, the intensity is (ideally) uniform along the gain medium (when averaged over time or many wavelengths). This leads to more uniform saturation of the gain. Because the entire gain medium interacts more homogeneously with the single traveling wave mode, there's less opportunity for other modes to reach threshold. This helps in achieving single longitudinal mode operation. And because the entire volume of the gain medium contributes efficiently to this single mode, potentially higher power can be extracted in that single mode compared to a standing-wave laser of similar size that might be multi-mode due to spatial hole burning.

"Natural frequency selection through travelling wave uni directionality; avoids Lamb dips caused by counter-propagating waves."

- "Natural frequency selection": Single longitudinal mode operation is easier to achieve.

- "avoids Lamb dips": The Lamb dip is a feature that appears in the power output versus frequency tuning curve of a standing-wave gas laser. It's a dip in power that occurs when the laser frequency is tuned exactly to the center of the atomic gain profile. It arises because at line center, both the forward and backward traveling components of the standing wave interact with the same group of atoms (those with zero axial velocity). This leads to a stronger saturation (double "hole burning" in the velocity distribution of atoms) at line center, reducing the gain and thus the output power.

In a unidirectional ring laser, there's only one traveling wave. So, there's no second counter-propagating wave to cause this specific type of saturation effect at line center. Therefore, ring lasers do not exhibit the Lamb dip, which can simplify frequency stabilization schemes if one wants to lock to the center of the gain profile.

Page 73:

This page continues with more benefits of the ring geometry.

"Convenient location of output coupler anywhere along path; can incorporate acousto-optic or electro-optic modulators with minimal perturbation."

- "Convenient location of output coupler": In a linear cavity, the output is usually taken from one of the end mirrors. In a ring cavity, since the light is circulating, an output coupling mirror (partially transmissive) can, in principle, be placed at any of the mirror locations in the ring, or a special output coupling element (like a frustrated total internal reflection coupler or a partially transmitting plate) could be inserted anywhere in the beam path within the ring. This offers more flexibility in the physical layout and design of the laser system.

- "can incorporate acousto-optic or electro-optic modulators with minimal perturbation": Ring cavities often provide more physical space and easier access for inserting intracavity components like modulators (AOMs for Q-switching or mode-locking, EOMs for mode-locking, frequency shifting, or stabilization). In a linear cavity, especially a short one, fitting these elements can be challenging.

"Minimal perturbation" might also refer to the fact that in a traveling wave, reflections from the surfaces of these intracavity components are less likely to cause coupled-cavity effects or feedback issues that can sometimes plague standing-wave lasers if the components are not perfectly anti-reflection coated and aligned. In a ring, a small reflection from a component surface would propagate in the "wrong" direction and ideally be suppressed by the optical diode or simply leave the main circulating path. This can lead to cleaner operation when intracavity elements are used.

Page 74:

This slide transitions to discussing the "Eigenfrequency Condition for Standing-Wave Cavities." We are returning to linear resonators to understand their resonant frequencies.

* "Generic requirement – field exactly reproduces after integer number of half-wavelengths along round-trip length $2\,d$."

For a standing wave to form in a cavity of round-trip length L_{rt} (which is $2\,d$ for a simple two-mirror linear cavity of length d), the total phase shift accumulated by the wave during one round trip must be an integer multiple of 2π.

This means L_{rt} must be equal to an integer number of full wavelengths within the medium, $L_{\text{rt}} = q \cdot \lambda_{\text{mode}}$.

Or, equivalently, if λ_{mode} is the wavelength of the mode in the cavity, then an integer number "q" of these wavelengths must fit into the round-trip path.

The slide phrases it as "integer number of half-wavelengths." This seems slightly off for the phase reproduction condition.

The condition for resonance is that the round trip phase is $q \cdot 2\pi$.

$$\text{Phase} = k \cdot L_{\text{rt}} = \left(\frac{2\pi}{\lambda_{\text{mode}}}\right) \cdot L_{\text{rt}} = q \cdot 2\pi.$$

So, $L_{\text{rt}} = q \cdot \lambda_{\text{mode}}$. The round trip length must be an integer number of *full* wavelengths.

Perhaps it's thinking of path length $d = q \cdot \lambda/2$ for a string fixed at both ends. For a resonator, it's the round-trip phase.

However, the statement can be salvaged if it refers to the *additional* phase shift beyond the basic $k \cdot L_{\text{rt}}$. The total phase accumulated in a round trip includes not only the phase from propagation ($k \cdot 2\,d$) but also phase shifts upon reflection from the mirrors (usually π per reflection for ideal metallic mirrors, or more complex for dielectric mirrors) and, crucially for Gaussian beams, the Gouy phase shift.

Let $\Phi_{\text{total round trip}}$ be this total phase. For resonance, $\Phi_{\text{total round trip}}$ must equal $q \cdot 2\pi$, where q is an integer (the longitudinal mode number).

" For confocal cavity $(R = d)$ and on-axis point, solving $\phi = q\pi$ yields..." (The equation for resonant frequency ν_{r} is given).

The condition given here, $\phi = q\pi$, also looks unusual if ϕ is the round trip phase. It should be $\phi_{\text{round trip}} = q \cdot 2\pi$.

Let's look at the frequency formula that results:

$$\nu_r = \frac{c}{2\,d}\left[q + \frac{1}{2}(m + n + 1)\right].$$

This is the well-known formula for the resonant frequencies of a confocal resonator $(R_1 = R_2 = d)$.

- q is the longitudinal mode index (a large integer). - m and n are the transverse mode indices $(0,1,2,...)$. - $\frac{c}{2\,d}$ is the fundamental longitudinal mode spacing (free spectral range for axial modes if m, n fixed and Gouy phase ignored).

The term $\frac{(m+n+1)}{2}$ arises from the Gouy phase shift. For a confocal resonator, the round-trip Gouy phase shift for a TEM$_{mn}$ mode is $(m+n+1)\pi$. (The Gouy phase shift from one mirror to the other, passing through the waist, is $(m+n+1)\frac{\pi}{2}$. So round trip is twice that).

The total phase for resonance is $k \cdot 2\,d - (m+n+1)\pi = q \cdot 2\pi$.

So, $\left(\frac{2\pi\nu}{c}\right) \cdot 2\,d = (q_{\text{effective}} + m + n + 1)\pi$.

$$\nu = \frac{c}{4\,d}(q_{\text{effective}} + m + n + 1).$$

This does not quite match the $\frac{c}{2\,d}$ prefactor with $\left(q + \frac{(m+n+1)}{2}\right)$.

Let's re-derive using $\phi_{\text{round trip}} = q_{\text{new}} \cdot 2\pi$.

The phase accumulated by a Gaussian beam in one round trip in a confocal resonator is:

$$k \cdot 2\,d - (m+n+1)\pi = q_{\text{new}} \cdot 2\pi \quad \text{(where } k = \frac{2\pi\nu}{c}\text{, and } q_{\text{new}} \text{ is integer mode number).}$$

$$\left(\frac{2\pi\nu}{c}\right) \cdot 2\,d = \left(q_{\text{new}} + \frac{(m+n+1)}{2}\right) \cdot 2\pi.$$

$$\nu = \frac{c}{2\,d}\left[q_{\text{new}} + \frac{(m+n+1)}{2}\right].$$

This matches the formula on the slide if we identify q on the slide with q_{new}.

So, the premise "solving $\phi = q\pi$" leading to this was perhaps a shorthand. The key is the resulting frequency formula is correct.

* "Axial modes $(m = n = 0)$: equally spaced by..." (continued on next slide).

If $m = 0$ and $n = 0$ (fundamental transverse mode), the frequencies are:

$$\nu_{q00} = \frac{c}{2\,d}\left[q + \frac{1}{2}\right].$$

The spacing between successive axial modes $(\Delta q = 1)$ would be $\frac{c}{2\,d}$.

Page 75:

This page continues from the previous one, giving the spacing for axial modes.

The formula for the frequency spacing, $\Delta\nu$, is:

$$\Delta\nu = \frac{c}{2\,d}$$

This is indeed the spacing between adjacent longitudinal modes (q and $q + 1$) if the transverse mode indices (m, n) are kept constant, and if the Gouy phase term $\frac{m+n+1}{2}$ does not change significantly with q or is the same for the modes being compared.

For purely axial modes $(m = 0, n = 0)$ in a confocal resonator, the frequencies are $\nu_{\mathrm{q}} = \frac{c}{2\,d}\left(q + \frac{1}{2}\right)$.

So, $\nu_{q+1} - \nu_{\mathrm{q}} = \frac{c}{2\,d}\left[\left(q + 1 + \frac{1}{2}\right) - \left(q + \frac{1}{2}\right)\right] = \frac{c}{2\,d}[1] = \frac{c}{2\,d}$.

This $\Delta\nu = \frac{c}{2\,d}$ is often called the Free Spectral Range (FSR) of the cavity for longitudinal modes. It's a fundamental quantity determined by the round-trip optical path length. For a cavity of length d, a pulse of light takes $\frac{2\,d}{c}$ to make a round trip. The inverse of this time is $\frac{c}{2\,d}$, which is the fundamental frequency spacing.

Page 76:

This slide discusses "Degeneracy & Free Spectral Range" specifically for the confocal case and then generalizes.

* "In confocal case, transverse indices with $m+n=2\,p-1$ fill halfway points between successive axial lines implies degeneracy." Let's look at the confocal frequency formula: $\nu=\frac{c}{2\,d}\left[q+\frac{(m+n+1)}{2}\right]$. Consider axial modes ($m=n=0$): $\nu_{q,0,0}=\frac{c}{2\,d}\left(q+\frac{1}{2}\right)$. These occur at $q+0.5$ units of $\frac{c}{2\,d}$. Now consider modes where $m+n=1$ (e.g., TEM_10 or TEM_01). Then $\frac{m+n+1}{2}=\frac{1+1}{2}=1$. So, $\nu_{q,m+n=1}=\frac{c}{2\,d}(q+1)$. These frequencies coincide with where the next axial mode $(q+1{,}0{,}0)$ would be if its Gouy term was $(q'+0.5)$. Let's be more careful. Frequencies are proportional to $2\,q+m+n+1$. An axial mode has frequency proportional to $2\,q+1$. (e.g., $q=10,\ m=n=0\rightarrow 21$) The next axial mode ($q+1, m=n=0$) has frequency proportional to $2(q+1)+1=2\,q+3$. A transverse mode with the same q, but $m+n=1$, has frequency prop. to $2\,q+1+1=2\,q+2$. This frequency $(2\,q+2)$ lies exactly halfway between $(2\,q+1)$ and $(2\,q+3)$. So, modes with $m+n$ odd (e.g., $m+n=2\,p-1$ for integer $p\geq 1$) have frequencies that are (integer + $1/2$)(c/d), while axial modes are (integer + $1/2$)(c/d). Let $K=m+n$. Freq ~ $q+\frac{K+1}{2}$. If K is even, $K=2\,p$, then $\frac{K+1}{2}=p+\frac{1}{2}$. Freq ~ $q+p+\frac{1}{2}$. (Same as axial modes but with shifted effective q). If K is odd, $K=2\,p-1$, then $\frac{K+1}{2}=p$. Freq ~ $q+p$. (Integer multiples of $c/2\,d$). So, in a confocal resonator, there are two sets of equally spaced resonant frequencies: Set 1 ($m+n$ is even): $\nu=\frac{c}{2\,d}\left(\text{Integer}_{\text{A}}+\frac{1}{2}\right)$ Set 2 ($m+n$ is odd): $\nu=\frac{c}{2\,d}\left(\text{Integer}_{\text{B}}\right)$ These two sets are interleaved. This means that modes with different (q,m,n) combinations can have the same resonant

frequency. This is called frequency degeneracy. For example, a $(q, m+n =$ even) mode can be degenerate with a $(q', m+n =$ odd) mode.

"*Therefore effective free spectral range (distance to next distinct* frequency) $\Delta\nu_{\text{confocal}} = \frac{c}{4\,d}$." Because of this interleaving, the spacing between *distinct* frequency groups is halved. The axial modes ($m+n =$ even, say $K=0$) are at $\left(q+\frac{1}{2}\right)\frac{c}{2\,d}$. The next group of modes ($m+n =$ odd, say $K=1$) are at $(q+1)\frac{c}{2\,d}$. The difference is $\frac{1}{2}\times\frac{c}{2\,d}=\frac{c}{4\,d}$. So, the effective FSR, considering all transverse modes, is $\frac{c}{4\,d}$ for a confocal cavity, which is half of the longitudinal FSR ($\frac{c}{2\,d}$).

* "Small perturbation $d \neq R$ lifts degeneracy; general symmetric cavity frequency formula..." If the cavity is not perfectly confocal (i.e., d is slightly different from R, but still symmetric $R_1 = R_2 = R_{\text{mirror}}$), the simple $\frac{m+n+1}{2}$ term for the Gouy phase changes. The degeneracy is lifted, meaning modes with different (m, n) but same $m+n$ sum will now have slightly different frequencies. The general symmetric cavity frequency formula is given on the next slide.

Page 77:

This page provides the "general symmetric cavity frequency formula" mentioned on the previous slide, for when d is not equal to R (but $R_1 = R_2 = R_{\text{mirror}}$ still).

The formula is:

$$\nu_{\text{r}} = \frac{c}{2\,d}\left\{q + \frac{1}{2}(m+n+1)\left[1 + \frac{4}{\pi}\arctan\left(\frac{d-R}{d+R}\right)\right]\right\}.$$

This formula needs careful checking against standard forms.

The Gouy phase shift for a symmetric resonator (mirrors R_{m} at $\pm\frac{d}{2}$ from waist) is $(m+n+1)\arccos(g)$, where $g=1-\frac{d}{R_{\mathrm{m}}}$. So the term in the frequency formula related to Gouy phase should be $(m+n+1)\frac{1}{\pi}\arccos(g)$. Let's see if

$$\frac{1}{2}\left[1+\frac{4}{\pi}\arctan\left(\frac{d-R}{d+R}\right)\right]$$

is equivalent to

$$\frac{1}{\pi}\arccos(g).$$

We have

$$g=1-\frac{d}{R_{\mathrm{m}}}.$$

Let's test some limits:

1. Confocal: $d=R_{\mathrm{m}}$. Then $g=0$ and $\arccos(0)=\frac{\pi}{2}$. So Gouy term becomes

$$(m+n+1)\frac{1}{\pi}\frac{\pi}{2}=\frac{m+n+1}{2}.$$

In the slide's formula, if $d=R_{\mathrm{m}}$, then

$$\frac{d-R_{\mathrm{m}}}{d+R_{\mathrm{m}}}=0 \quad \text{and} \quad \arctan(0)=0.$$

So term becomes

$$\frac{1}{2}(m+n+1)[1+0]=\frac{m+n+1}{2}.$$

This matches.

2. Plane-parallel: $R_\mathrm{m} \to \infty$. Then $g = 1$ and $\arccos(1) = 0$. Gouy term is 0. In the slide's formula, as $R_\mathrm{m} \to \infty$,

$$\frac{d - R_\mathrm{m}}{d + R_\mathrm{m}} \to \frac{-R_\mathrm{m}}{R_\mathrm{m}} = -1 \quad \text{and} \quad \arctan(-1) = -\frac{\pi}{4}.$$

Term becomes

$$\frac{1}{2}(m + n + 1)\left[1 + \frac{4}{\pi}\left(-\frac{\pi}{4}\right)\right] = \frac{1}{2}(m + n + 1)[1 - 1] = 0.$$

This also matches.

3. Concentric: $d = 2\,R_\mathrm{m}$. Then

$$g = 1 - \frac{2\,R_\mathrm{m}}{R_\mathrm{m}} = -1 \quad \text{and} \quad \arccos(-1) = \pi.$$

Gouy term is

$$(m + n + 1)\frac{1}{\pi}\pi = (m + n + 1).$$

In the slide's formula, with $d = 2\,R_\mathrm{m}$,

$$\frac{d - R_\mathrm{m}}{d + R_\mathrm{m}} = \frac{2\,R_\mathrm{m} - R_\mathrm{m}}{2\,R_\mathrm{m} + R_\mathrm{m}} = \frac{R_\mathrm{m}}{3\,R_\mathrm{m}} = \frac{1}{3},$$

so that

$$\arctan\left(\frac{1}{3}\right)$$

is obtained. The term then is

$$\frac{1}{2}(m + n + 1)\left[1 + \frac{4}{\pi}\arctan\left(\frac{1}{3}\right)\right].$$

This is not equal to $(m + n + 1)$. In fact,

$$\left[1 + \frac{4}{\pi}\arctan\left(\frac{1}{3}\right)\right] \approx 1 + \frac{4}{3.14159} \times 0.32175 \approx 1 + 0.4096 = 1.4096,$$

and we need this factor to be 2 for the $\frac{1}{2}$ to cancel and leave $(m+n+1)$. It's not.

There appears to be an issue with the slide's formula for the general symmetric case if it's meant to reduce to $\frac{1}{\pi}\arccos(g)$.

The standard form for resonant frequencies of a stable resonator with mirrors R_1, R_2, separation d is:

$$\nu_{qmn} = \frac{c}{2\,d}\left[q + \frac{(m+n+1)}{\pi}\arccos\left(\sqrt{g_1\,g_2}\right)\right] \quad \text{for } g_1\,g_2 \geq 0.$$

Or more generally, using

$$G_{\text{boyd}} = \frac{A+D}{2}$$

of the round trip matrix, where A and D are diagonal elements,

$$\nu_{qmn} = \frac{c}{2\,d}\left[q + \frac{(m+n+1)}{\pi}\arccos\left(G_{\text{boyd}}\right)\right].$$

For a symmetric resonator, $G_{\text{boyd}} = g$. So it should be

$$\nu_{qmn} = \frac{c}{2\,d}\left[q + \frac{(m+n+1)}{\pi}\arccos(g)\right].$$

The term

$$\frac{1}{2}\left[1 + \frac{4}{\pi}\arctan\left(\frac{d-R}{d+R}\right)\right]$$

must be equal to $\frac{1}{\pi}\arccos(g)$. This does not seem to be a general trigonometric identity.

The formula on the slide might be a specific approximation or a form from a particular textbook that uses different conventions or parameterizations. Given its correctness at $g = 0$ (confocal) and $g = 1$ (plane-parallel), it might be a valid alternative form, perhaps related by some trig identity involving $\arctan$ and $\arccos$ under specific variable transformations. I will proceed by stating the formula as given on the slide but noting that the $\arccos(g)$ form is more common.

Page 78:

This slide presents the "Most General Resonator Frequency Formula," applicable to resonators with unequal mirror radii R_1 and R_2.

The formula given is:

$$\nu_{\mathrm{r}} = \frac{c}{2\,d}\left[q + \frac{1}{\pi}(m+n+1)\arccos\left(\sqrt{g_1\,g_2}\right)\right]$$

This is a standard and widely accepted formula for the resonant frequencies of a stable two-mirror resonator, provided $g_1\,g_2$ is between 0 and 1 (inclusive of 0 for $\sqrt{\ }$).

- c is the speed of light. - d is the mirror separation. - q is the longitudinal mode index. - m and n are transverse mode indices. - $g_1 = 1 - \frac{d}{R_1}$, $g_2 = 1 - \frac{d}{R_2}$. - The term $\arccos\left(\sqrt{g_1\,g_2}\right)$ gives the Gouy phase contribution per half round trip, normalized by π. (Actually, $\arccos\left(\sqrt{g_1\,g_2}\right)$ itself is an angle; dividing by π and multiplying by $(m+n+1)$ gives the fractional shift in terms of mode number q.)

* "Works for unequal radii R_1, R_2." Yes, because g_1 and g_2 can be different.

" Parameter $\arccos\left(\sqrt{g_1\,g_2}\right)$ often called transverse mode spacing factor." This term determines how much the frequencies of modes with different transverse indices (m, n) are shifted relative to the purely axial modes (if one imagined them without Gouy phase). It dictates the spacing between different families of transverse modes.

* "Approaches $\pi/2$ in confocal limit; approaches 0 in plane-plane limit." Let's check these limits for the $\arccos\left(\sqrt{g_1\, g_2}\right)$ term: - Confocal limit: Can be $g_1 = 0$, $g_2 = 0$ (symmetric confocal). Then $\sqrt{g_1\, g_2} = 0$. $\arccos(0) = \pi/2$. This is correct. (If semi-confocal, $g_1 = 1$, $g_2 = 0$. $\sqrt{g_1\, g_2} = 0$. $\arccos(0) = \pi/2$. Correct.) - Plane-plane limit: $g_1 = 1$, $g_2 = 1$. Then $\sqrt{g_1\, g_2} = 1$. $\arccos(1) = 0$. This is correct.

These limits confirm the behavior of the Gouy phase factor in these important cases. The total Gouy phase per round trip is $2\arccos\left(\sqrt{g_1\, g_2}\right)$.

This formula is a cornerstone for understanding the mode spectrum of any stable resonator.

Page 79:

This slide is titled "Illustration of Mode Spectra Evolution." It shows a diagram with three panels, (a), (b), and (c), plotting mode frequencies. The horizontal axis is "Relative Frequency (units of $\frac{c}{2\,d}$)." The vertical lines represent the resonant frequencies of different modes.

(a) "Plane-Plane":

In a plane-parallel resonator, $g_1 = 1$, $g_2 = 1$, so $\arccos\left(\sqrt{g_1\, g_2}\right) = \arccos(1) = 0$. The frequency formula becomes $\nu = \frac{c}{2\,d} q$. This means all transverse modes (m, n) with the same longitudinal index 'q' are degenerate; they have the same frequency. The spectrum should show equally spaced lines at $q =$ integer multiples of $\frac{c}{2\,d}$. The diagram for (a) shows groups of lines. There's a tall black line (perhaps $q, m = n = 0$), and then shorter blue and orange lines very close to it, slightly offset. This implies some slight lifting of degeneracy or perhaps showing different $(m + n)$ families that are not perfectly degenerate in a real plane-parallel due to imperfections or finite mirror effects not captured by the simple formula. Ideally, they should all be at the same frequency for a given q. The slide may be anticipating that perfect degeneracy is an idealization.

(b) "Confocal":

In a confocal resonator, $\arccos\left(\sqrt{g_1 g_2}\right) = \arccos(0) = \frac{\pi}{2}$. The frequency formula is $\nu = \frac{c}{2d}\left[q + \frac{(m+n+1)}{2}\right]$. This means frequencies occur at (Integer + Fraction) multiples of $\frac{c}{2d}$. If $m+n$ is even (say $2p$), then $\frac{(m+n+1)}{2} = p + \frac{1}{2}$. Frequencies at $\left(q+p+\frac{1}{2}\right)\frac{c}{2d}$. If $m+n$ is odd (say $2p-1$), then $\frac{(m+n+1)}{2} = p$. Frequencies at $(q+p)\frac{c}{2d}$. The diagram for (b) shows equally spaced lines. A tall black line (axial mode, $m+n$ even) is shown, and then a blue line appears exactly halfway between two successive black lines. This blue line represents the $(m+n)$ odd family of modes. This is consistent with the effective FSR being $\frac{c}{4d}$. All modes with the same parity of $(m+n)$ are degenerate for a given effective q.

(c) "Near-Confocal":

Here, the degeneracy found in the perfect confocal case is lifted. The term $\arccos\left(\sqrt{g_1 g_2}\right)$ is no longer exactly $\frac{\pi}{2}$ or 0. So, $\nu = \frac{c}{2d}[q + (m+n+1) \cdot \text{constant}_{\text{A}}]$, where $\text{constant}_{\text{A}} = \frac{\arccos(\sqrt{g_1 g_2})}{\pi}$. Different $(m+n)$ values will lead to different frequency shifts. The diagram for (c) shows the black axial modes. The blue lines (e.g., from the $m+n=1$ family) are no longer exactly halfway but are shifted. The orange lines (e.g., from the $m+n=2$ family) are also shifted differently. The even spacing is broken, and the spectrum becomes more complex, with distinct frequencies for different (m,n) combinations.

This visual effectively shows how the resonator geometry (through g_1, g_2) influences the structure of the mode spectrum and the degeneracies.

Page 80:

This slide is titled "Resonance Width via Airy Finesse Approach." We are now looking at the sharpness of the resonance peaks.

- "Fabry-Perot transmission intensity $T(\nu) = \frac{1}{1+F_* \sin^2\left(\pi \frac{\nu}{\Delta\nu_{\text{fsr}}}\right)}$."

This is the standard Airy formula for the transmission of a Fabry-Perot interferometer. - $T(\nu)$ is the transmittance as a function of frequency ν. - $\Delta\nu_{\text{fsr}}$ is the Free Spectral Range, which is $\frac{c}{2\,d}$ for a simple cavity if we are considering only longitudinal modes. This is the spacing between transmission peaks. - The argument of $\sin^2$ is $\pi \frac{\nu}{\Delta\nu_{\text{fsr}}}$, which can also be written as $\frac{\phi}{2}$ where $\phi = \frac{2\pi\nu}{\Delta\nu_{\text{fsr}}}$ is the round-trip phase shift (modulo 2π). - \(F_\) *is the "coefficient of finesse," related to mirror reflectivity* R by \(F_ = \frac{4\,R}{(1-R)^2}\).

"where $F = \frac{4\,R}{(1-R)^2}$ for negligible diffraction."

The slide uses F for F_*. This F is indeed the coefficient of finesse. It is valid when losses are dominated by mirror transmission/reflection and diffraction is negligible (e.g., large Fresnel number).

- "delta_nu = c/(2d) free spectral range."

$\Delta\nu = \frac{c}{2\,d}$ free spectral range.

- "Half-width at half-maximum (HWHM):"

The Full Width at Half Maximum (FWHM) of the Airy transmission peaks is given by

$$\Delta\nu_{\text{FWHM}} = \frac{\Delta\nu_{\text{fsr}}}{F_\pi},$$

where F_π is the "finesse" of the cavity, $F_\pi = \frac{\pi\sqrt{F_*}}{2} = \frac{\pi\sqrt{R}}{1-R}$.

So, HWHM = FWHM / 2 = $\frac{\Delta\nu_{\text{fsr}}}{2\,F_\pi}$.

Let's see what formula the slide will give on the next page.

Page 81:

This page continues with the Half-Width at Half-Maximum (HWHM) of the resonance, denoted $\Delta\nu_{\text{r}}$.

The formula given is:

$$\Delta\nu_{\text{r}} = \frac{\delta\nu}{F_\star} \quad \text{(which is FSR } = \frac{c}{2\,d}\text{)}$$

and then equated to

$$\Delta\nu_{\text{r}} = \frac{c}{2\,d}\frac{(1-R)}{\pi\sqrt{R}}.$$

This formula seems to be for the FWHM, not HWHM, if $F_\star$ is the coefficient of finesse. Let's re-check: FWHM $= \frac{FSR}{\text{Finesse}}$. $F_\pi = \frac{\pi\sqrt{R}}{1-R}$. So FWHM $= \frac{c}{2\,d}\frac{(1-R)}{\pi\sqrt{R}}$. This matches the second part of the slide's expression exactly.

Therefore, $\Delta\nu_{\text{r}}$ on this slide represents the FWHM of the resonance peak, not the HWHM. The $F_\star$ in the first part, $\Delta\nu_{\text{r}} = \frac{\delta\nu}{F_\star}$, would then imply $F_\star = \frac{\pi\sqrt{R}}{1-R}$, which is the finesse F_π, not the coefficient of finesse $F_\star = \frac{4\,R}{(1-R)^2}$. There's a notation inconsistency here with $F_\star$. I will assume $\Delta\nu_{\text{r}}$ is FWHM and the formula $\frac{c}{2\,d}\frac{(1-R)}{\pi\sqrt{R}}$ is correct for FWHM. This FWHM is the spectral width of an individual cavity mode.

* "Mirrors with $R = 0.98$ and $d = 1\,m$ implies $\Delta\nu_{\text{r}} \approx 1\,$MHz (ideal)." Let's calculate this: $R = 0.98$. So $(1-R) = 0.02$. $\sqrt{R} = \sqrt{0.98} \approx 0.9899$. $d = 1\,$m. $c = 3 \times 10^8$ m/s. $\delta\nu_{\text{fsr}} = \frac{c}{2\,d} = \frac{3\times10^8\text{ m/s}}{2\times1\text{ m}} = 1.5 \times 10^8$ Hz $= 150\,$MHz. $F_\pi = \frac{\pi\sqrt{R}}{(1-R)} = \frac{\pi\times0.9899}{0.02} \approx \frac{3.14159\times0.9899}{0.02} \approx \frac{3.1098}{0.02} \approx 155.49$. $\Delta\nu_{\text{r}}$ (FWHM) $= \frac{\delta\nu_{\text{fsr}}}{F_\pi} =$

$\frac{150\,\text{MHz}}{155.49} \approx 0.9647\,\text{MHz}$. This is indeed approximately 1 MHz. So, for typical high-reflectivity mirrors (98%) in a 1-meter long cavity, the resonance linewidth is about 1 MHz. This is a very sharp resonance. "Ideal" means this calculation assumes only transmission losses from the mirrors define R, and no other losses (diffraction, scattering, absorption) are present.

Page 82:

This slide presents an "Alternate Derivation – Quality Factor Q" for the resonance width.

* "Photon lifetime for reflection-dominated losses: $T = \frac{d}{c \ln R}$." This formula for photon lifetime T (often denoted τ) needs a minus sign or absolute value for $\ln R$. Photon lifetime $\tau_{\text{ph}} = \frac{\text{(round trip time)}}{\text{(round trip fractional power loss)}}$. Round trip time = $\frac{2\,d}{c}$. Round trip loss fraction = $1 - R_{\text{eff\_round\_trip}}$. If only reflections from two mirrors R_1, R_2, $R_{\text{eff}} = R_1\,R_2$. If $R_1 = R_2 = R$ (symmetric), then loss = $1 - R^2$. For high R, $1 - R^2 = (1 - R)(1 + R) \approx 2(1 - R)$. So $\tau_{\text{ph}} \approx \frac{\frac{2\,d}{c}}{2(1-R)} = \frac{d}{c(1-R)}$. The slide uses $\ln R$. For high R, $\ln R = \ln\big(1 - (1 - R)\big) \approx -(1 - R)$. So, $\frac{d}{c(-\ln R)}$ or $\frac{d}{c|\ln R|}$ is the form that matches $\frac{d}{c(1-R)}$. The slide's " $T = \frac{d}{c \ln R}$" would be negative. It should be $T = -\frac{d}{c \ln R}$ or using absolute value for $\ln R$ to ensure T is positive. This " T" is the photon lifetime.

* "Lorentzian spectral density for exponentially decaying field implies HWHM: $\Delta\nu_{\text{r}} = \frac{1}{2\pi T}$." If a field amplitude decays exponentially with time constant T_{amp}, its power spectrum is a Lorentzian with FWHM = $\frac{1}{\pi T_{\text{amp}}}$. If energy decays with time constant T_{energy} (which is the photon lifetime T from the slide), then the field amplitude decays with $2\,T_{\text{energy}}$. So $T_{\text{amp}} = 2\,T_{\text{energy}}$. Then FWHM = $\frac{1}{\pi \cdot 2\,T_{\text{energy}}} = \frac{1}{2\pi T_{\text{energy}}}$. This means $\Delta\nu_{\text{r}}$ on the slide is

indeed the FWHM, if T is the energy lifetime. So, $\Delta\nu_{\mathrm{r}}$ (FWHM) $= \frac{1}{2 \pi T_{\text{photon\_\text{lifetime}}}}$. The slide then equates this to: $\frac{c|\ln R|}{2\pi d}$. This comes from substituting $T_{\text{photon\_\text{lifetime}}} = -\frac{d}{c \ln R}$ or $\frac{d}{c|\ln R|}$. $\Delta\nu_{\mathrm{r}} = \frac{1}{2\pi\left(\frac{d}{c|\ln R|}\right)} = \frac{c|\ln R|}{2\pi d}$. This is a correct expression for the FWHM linewidth based on photon lifetime due to reflection losses.

Page 83:

This page continues the discussion from the alternate derivation of resonance width.

* "For $|\ln R|$ is approximately $1 - R$ (when R is approximately 1), reproduces Airy result up to $\sqrt{R}$ factor." Let's check this.

From previous slide, using lifetime approach: $\Delta\nu_{\mathrm{r}}$ (FWHM) $= \frac{c\,|\ln R|}{2\pi d}$.

If $|\ln R| \approx (1 - R)$, then $\Delta\nu_{\mathrm{r}} \approx \frac{c(1-R)}{2\pi d}$.

From Airy finesse approach (page 81): $\Delta\nu_{\mathrm{r}}$ (FWHM) $= \frac{c}{2\,d}\frac{(1-R)}{\pi\sqrt{R}}$.

This can be written as:

$$\Delta\nu_{\mathrm{r}} = \frac{c(1-R)}{2\pi d\sqrt{R}}.$$

Comparing the two:

Lifetime: $\frac{c(1-R)}{2\pi d}$

Airy: $\frac{c(1-R)}{2\pi d\sqrt{R}}$

These two results differ by a factor of $\sqrt{R}$ in the denominator of the Airy result. So, the statement "reproduces Airy result up to $\sqrt{R}$ factor" is correct. The lifetime approach gives a slightly larger linewidth (by factor $\frac{1}{\sqrt{R}}$, which is

>1 if $R < 1$) than the Airy approach if R is not extremely close to 1. When R is very close to 1, $\sqrt{R}$ is also very close to 1, and the two results become nearly identical. This slight difference often arises from different approximations made in the two derivations (e.g., whether it's amplitude or intensity reflectivity used in defining finesse, or approximations for high R).

"Importance – shows directly how all loss mechanisms (not just mirror) broaden modes by shortening T." This is a very significant point. The lifetime approach ($\Delta\nu_{\text{r}} = \frac{1}{2\pi T_{\text{photon lifetime}}}$) is more general.

The photon lifetime T can be limited by *any* loss mechanism in the cavity, not just mirror transmission (which defines R).

If there are other losses (absorption, scattering, diffraction), they will contribute to reducing the overall photon lifetime.

Total loss rate

$$\frac{1}{T_{\text{total}}} = \frac{1}{T_{\text{mirror transmission}}} + \frac{1}{T_{\text{absorption}}} + \frac{1}{T_{\text{scattering}}} + \frac{1}{T_{\text{diffraction}}} + \cdots$$

A shorter T_{total} (due to any combination of these losses) will lead to a broader resonance width $\Delta\nu_{\text{r}}$.

The Airy formula, using only R, typically accounts only for the loss due to mirror reflectivity/transmissivity. To use it more generally, R would have to be an "effective" reflectivity that incorporates all losses.

The lifetime picture is more direct: find the total lifetime considering all loss sources, and that directly gives the linewidth. This is a very useful conceptual link.

Page 84:

This slide discusses "Real-World Finesse Limiting Factors." The ideal finesse calculated from mirror reflectivity R is often not achieved in practice.

* "Surface scattering from micro-roughness (proportional to λ^{-4})." Even the best mirrors are not perfectly smooth at the atomic level. Micro-roughness on the mirror surfaces causes light to scatter out of the specularly reflected beam. This scattering acts as a loss. The Rayleigh scattering criterion states that scattering intensity is often proportional to λ^{-4}, meaning it's much worse for shorter wavelengths (e.g., blue or UV light scatters more than red or IR). This can significantly reduce the effective reflectivity of the mirrors and thus limit the finesse.

″ Absorption in multi-layer dielectric stacks (materials, contamination)." Dielectric mirrors consist of many thin layers of alternating high and low refractive index materials. While these materials are chosen to be highly transparent at the laser wavelength, there's always some residual absorption in the layers. The amount of absorption depends on the materials used, their purity, and the quality of the deposition process. Contamination on the mirror surfaces (e.g., dust, organic films) can also absorb light. This absorption is another loss mechanism that reduces finesse.

″ Diffraction from finite mirror & aperture stops, esp. in power-built lasers with thermally distorted optics." We've discussed diffraction losses. Even if mirrors are large enough for low diffraction loss with an ideal beam, if the laser beam is distorted (e.g., due to thermal lensing in the gain medium in a high-power laser), it may no longer fit the mirrors well, leading to increased diffraction losses. Aperture stops (deliberate openings to control mode size or block stray light) also cause diffraction.

″ Misalignment (tilt implies TEM coupling), astigmatism (non-identical horizontal & vertical g values)." - Misalignment: If the mirrors are not perfectly parallel (for plane-parallel) or perfectly aligned along the optical axis, the losses can increase significantly. Tilt can also cause coupling of energy from the desired fundamental mode (TEM_{00}) to higher-order TEM modes, which might have different losses or be unwanted. - Astigmatism: If the mirrors have different radii of curvature in the horizontal (x) and vertical (y) planes (e.g., due to manufacturing imperfections or if cylindrical mirrors are used), then the g-parameters will be different for the x and y directions (g_x is not equal to g_y). This leads to astigmatic modes (e.g., elliptical spot shapes even for the fundamental mode) and can affect stability and losses.

* "Practical finesses: $50 \leq F \leq 100$ for typical research-grade cavities; up to 10^5 in ultra-high-reflection super-polished systems." - $F_\pi = \frac{\pi\sqrt{R}}{1-R}$. For $R = 0.98$, $F_\pi \sim 155$. For $R = 0.99$, $F_\pi \sim 312$. For $R = 0.97$, $F_\pi \sim 103$. So, a finesse of 50–100 corresponds to mirror reflectivities around 97% or slightly lower if other losses are dominant. This is typical for many lab lasers where good, but not "hero- experiment" level, mirrors are used, or where other intracavity losses are present. - A finesse of 10^5 (one hundred thousand) is extremely high. This requires extremely high reflectivity mirrors (e.g., $R > 0.99997$) and meticulous control of all other loss mechanisms (super-polished substrates to minimize scatter, ultra-pure coatings for low absorption, operation in vacuum to avoid contamination, etc.). Such high finesses are achieved in specialized applications like cavity ring-down spectroscopy, optical frequency standards, or gravitational wave detectors (though those are interferometers, the principle of high finesse is related).

Page 85:

This final slide summarizes "Key Design Take-Aways" for laser resonators. It's a condensed list of practical advice.

* "Size mirrors via Fresnel number to keep $\gamma_D \ll$ gain margin." Gamma_D is the diffraction loss. $N_F = \frac{a^2}{\lambda d}$. Choose mirror radius 'a' large enough to make N_F high enough so that γ_D is much smaller than the available gain (gain margin = gain - other losses). This ensures diffraction doesn't prevent lasing or dominate efficiency.

" Choose curvature to satisfy $0 < g_1 g_2 < 1$ Engineer losses (aperture, tilt) to select one desired mode; use ring or unidirectional schemes to eliminate standing-wave nodes." (The $0 seems to be a placeholder for the stability criterion). - Choose mirror curvatures \(R_1$, R_2 and separation d such that the resonator is stable: $0 < g_1 g_2 < 1$. - Loss engineering: Introduce elements like apertures, or carefully control alignment (tilt), to make losses higher for unwanted modes, thereby promoting oscillation in a single desired mode (often TEM00). - For single longitudinal mode operation or to avoid spatial hole burning effects, consider using a ring resonator with an optical diode to achieve unidirectional traveling-wave operation.

"Estimate bandwidths via either finesse or quality-factor method; ensure gain linewidth exceeds cavity mode width but is not too wide for single-mode." - The cavity mode width ($\Delta\nu_{\text{cavity}}$, e.g., FWHM) can be estimated using finesse ($\Delta\nu_{\text{cavity}} = \frac{\text{FSR}}{\text{Finesse}}$) or Q-factor ($\Delta\nu_{\text{cavity}} = \frac{\nu_{\text{laser}}}{Q}$). - The gain medium has its own gain bandwidth ($\Delta\nu_{\text{gain}}$), over which it can provide amplification. - For lasing to occur, the gain bandwidth must overlap with at least one cavity mode. So, $\Delta\nu_{\text{gain}}$ should be wide enough to cover a cavity mode. - However, if $\Delta\nu_{\text{gain}}$ is much wider than the cavity FSR ($\frac{c}{2\,d}$), then multiple longitudinal cavity modes might fall within the gain bandwidth and could lase simultaneously, leading to multi-longitudinal-mode operation. - For single longitudinal mode operation, one often tries to have the cavity FSR be comparable to or larger than $\Delta\nu_{\text{gain}}$ (hard to achieve for broad gain media), or ensure that only one cavity mode has net gain above threshold. The statement "not too* wide for single-mode" means that if the gain curve is very broad, you'll need additional mode selection techniques if only one longitudinal mode is desired.

" High-gain media may need unstable resonators; accept divergence, mitigate with external relay optics." - If the gain of the laser medium is very high (and often over a large volume), stable resonators might not be optimal for efficient power extraction or might lead to very high intracavity intensities. - Unstable resonators are often preferred for such systems because they allow the mode to fill a large volume of the gain medium and provide high output coupling. - The output beam from an unstable resonator is typically divergent (not collimated like an ideal Gaussian from a stable resonator) and may have a complex profile (e.g., annular with side-lobes in the far field). This divergence must be "accepted" as a characteristic. - "mitigate with external relay optics": The divergent output beam can often be reshaped, collimated, or focused using external lenses or mirror systems (relay optics) to make it suitable for specific applications.

This provides a good summary of the practical design philosophy for laser resonators, integrating many of the concepts we've discussed.

This concludes the lecture content from the provided slides.