Chapter

3.6

Page 1:

Alright everyone, welcome to this next segment of our Phys 608 Laser Spectroscopy course. Today, we're delving into a very important topic, outlined here as Chapter 3, Section 6, focusing on **Saturation and Power Broadening**. This material has been prepared by Distinguished Professor Doctor M A Gondal, for our course here at KFUPM, specifically for Term 251.

Now, why are these concepts – saturation and power broadening – so crucial in the field of laser spectroscopy? As we'll see, they have profound implications for the precision and interpretation of our spectroscopic measurements, especially when we're working with the high intensities often available from laser sources. Understanding these effects is paramount to designing experiments correctly and accurately extracting information about the atomic or molecular systems we study. So, let's begin to unpack these ideas.

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So, let's formally address the question: **Saturation & Power Broadening – WHY WE CARE?**

The first bullet point really sets the stage: **"Precise spectroscopy demands linewidths as narrow as possible. High-power lasers can unintentionally enlarge these widths."** This is a fundamental challenge in high-resolution spectroscopy. We strive for the narrowest possible linewidths because they dictate our ability to resolve closely spaced spectral features, to determine transition frequencies with utmost accuracy, and ultimately, to test fundamental theories of physics or to make very sensitive measurements of environmental parameters, for example. However, lasers, particularly the powerful ones we often need to get a good signal, can introduce a broadening mechanism that is *intrinsic* to the light-matter interaction itself when the light is intense. This is not an instrumental artifact in the usual sense, like a poorly calibrated spectrometer, but rather a physical consequence of the strong driving field.

The second bullet point tells us that there are **"Two closely related mechanisms are at play"**:

First, **"Saturation of the level populations (population equalisation)."** What do we mean by saturation? Imagine a simple two-level system interacting with a resonant light field. The light causes atoms to be excited from the lower level to the upper level (absorption) and also stimulates them to decay from the upper level back to the lower level (stimulated emission). If the light field is weak, the population of the upper level remains small. However, if the light field is very intense, the rate of stimulated absorption and emission can become so high that it starts to significantly alter the populations of the two levels. Specifically, a strong field can deplete the lower level and populate the upper level to such an extent that the population difference between the two levels is substantially reduced. In the extreme limit, the populations can become nearly equal – this is what we refer to as "population equalisation." When this happens, the net absorption of light decreases because there are fewer atoms in the lower state available to absorb, and more atoms in the upper state contributing to stimulated emission, which counteracts absorption. So, saturation directly impacts the *strength* of our spectroscopic signal.

Second, and intricately linked to the first, is **"Power (or saturation) broadening of the spectral profile."** This is a more subtle, but equally important, effect. It's not just that the peak absorption decreases; the *width* of the spectral line itself increases as the laser power goes up. So, even if you had a system with a very well-defined, narrow natural linewidth, applying a high-power laser to probe it can make that line appear broader. This "power broadening" or "saturation broadening" directly degrades the spectral resolution we can achieve. Why this happens is something we will derive and understand in detail. It's related to the fact that a strong field can drive transitions even when it's slightly off-resonance, and also, as we'll see later, it's connected to the lifetime reduction of the states due to the strong interaction.

Understanding both these mechanisms – the change in population and the broadening of the profile – is absolutely essential for anyone doing laser spectroscopy.

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Now, to develop a clear understanding of these phenomena, we need to lay down some groundwork and define our approach.

The first point here is crucial for our initial theoretical development: **"We concentrate on a strictly homogeneous two-level absorber."** Let's break this down.

A "two-level absorber" is a simplification, of course. Real atoms and molecules have many energy levels. However, in many spectroscopic situations, especially when using a tunable laser, we can selectively interact with just two levels that are resonant, or nearly resonant, with the laser frequency. This simplification allows us to capture the essential physics without getting bogged down in multi-level complexities, at least initially.

The term **"homogeneous"** is also key. It means that **"every absorber experiences the same natural width** γ." In a homogeneously broadened system, all individual atoms or molecules in our ensemble have identical transition frequencies and identical lineshapes. The observed macroscopic lineshape is then just the lineshape of a single absorber. The natural width, γ, is determined by processes that affect all absorbers in the same way, such as the natural lifetime of the excited state (due to spontaneous emission) or collisional broadening (if all atoms experience, on average, the same collision rate and environment). This is in contrast to "inhomogeneous broadening," where different atoms in the ensemble have slightly different resonant frequencies due to, for instance, the Doppler effect (different velocities) or varying local environments. By focusing on a homogeneous system first, we can isolate the effects of saturation and power broadening without the added complication of an underlying inhomogeneous distribution. We'll assume this natural width, γ, is the width we'd measure in the limit of very low light intensity.

Next, we have a **"Road-map of coming slides"**: This outlines how we're going to build up our understanding systematically.

1. **"Build the two-level rate equation from scratch."** Rate equations are a powerful and intuitive way to describe how populations in different energy levels change over time due to various radiative and non-radiative processes. We'll set these up considering absorption, stimulated emission, and relaxation processes.

2. **"Derive the saturation parameter** S and discuss population depletion." We will quantify the degree of saturation by introducing a dimensionless parameter, typically denoted as capital S. This parameter will tell us how strongly the system is being driven relative to its relaxation rates. We'll see how it relates to the depletion of the ground state population and the reduction in population difference.

3. **"Extend the argument to frequency-dependent saturation** S_ω and show how γ inflates to γ_S." Initially, we might consider saturation at perfect resonance. However, laser fields have a frequency, and the interaction strength can depend on how close this frequency is to the atomic resonance. So, the saturation parameter itself can be frequency-dependent. We'll then see how this frequency-dependent saturation leads to a new, broadened linewidth, which we'll call γ_S, which is larger than the original natural linewidth γ.

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Continuing with our road-map:

4. **"Re-derive the same broadening with Rabi flopping to illuminate the underlying physics."** After we've developed the theory using rate equations, which are essentially an incoherent picture based on populations, we'll take a different perspective. Rabi flopping describes the coherent oscillation of populations between two levels when driven by a resonant electromagnetic field. This is a more fundamentally quantum mechanical picture. By re-deriving the broadening effect from this viewpoint, we can gain deeper insights into the physical processes. For instance, it connects the broadening to the very act of driving coherent oscillations. A faster oscillation (due to a stronger field) implies a shorter characteristic timescale, which, through a time-frequency uncertainty type of argument, suggests a broader frequency response.

5. **"Conclude with experimentally relevant side-bands in a pump-probe arrangement."** Finally, we'll look at how these effects manifest in common experimental setups. In a pump-probe experiment, a strong pump laser modifies the atomic system (e.g., saturates it), and a weak probe laser measures the altered absorption spectrum. Under certain conditions, particularly with a strong pump field, the probe laser doesn't just see a broadened line; it can see new spectral features, like sidebands or a splitting of the main line. This is related to phenomena like the Autler-Townes effect or, in the context of fluorescence, the Mollow triplet. These are direct, observable consequences of the system being strongly driven.

The triple dash here simply indicates the end of this introductory section of the roadmap. So, this is our plan. We'll start with the basics and build up to a comprehensive understanding of how intense light fields affect spectroscopic measurements.

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Alright, now we come to what's labeled as **"Refresher – What Limits a Linewidth in the First Place?"** Before we can talk about how high-power lasers *broaden* linewidths, it’s essential to recall what determines the linewidth of a spectral transition even in the absence of such strong field effects, i.e., in the low-power limit.

The first bullet point states: **"Line-shapes are affected by many unrelated physical processes."** This is a crucial point. The observed width of a spectral line is often a convolution of several different broadening mechanisms. It's our job as spectroscopists to understand these mechanisms, and if possible, to isolate or minimize them to achieve the highest resolution.

The slide then categorizes these sources of broadening into homogeneous and inhomogeneous sources.

Let's look at **"Homogeneous sources"**: These are mechanisms that broaden the transition in an identical way for all atoms or molecules in the sample. Every atom effectively has the same broadened lineshape.

Examples given are:

* **"natural lifetime (**$\frac{1}{\tau}$, where τ is the lifetime)": This is perhaps the most fundamental broadening mechanism. An excited state, say level 2, has a finite lifetime, τ, due to spontaneous emission back to level 1 (or other lower levels). The Heisenberg uncertainty principle, in the form $\Delta E \Delta t \approx \hbar$, tells us that a finite lifetime Δt implies an uncertainty in energy ΔE. This energy uncertainty translates directly into a frequency width, often called the natural linewidth, which is proportional to $\frac{1}{\tau}$. So, the shorter the lifetime of the excited state, the broader the natural linewidth. This is an intrinsic property of the atom or molecule.

* **"collisional de-phasing"**: Atoms in a gas or liquid are constantly colliding. These collisions can interrupt the phase of the atomic oscillation (the coherence between the ground and excited states). Each collision effectively randomizes the phase, shortening the average time over which the atom can coherently interact with the light field. This "dephasing"

process leads to a broadening of the spectral line. The more frequent the collisions, the broader the line. Since, on average, all atoms might experience a similar collisional environment, this is often a homogeneous effect.

* **"transit-time"**: This is particularly relevant when, for example, an atomic beam crosses a laser beam, or when molecules diffuse through a focused laser spot. Each atom only interacts with the laser field for a finite amount of time – the transit time it takes to cross the beam. Again, by a time-frequency uncertainty argument, this finite interaction time leads to a broadening of the observed transition. If all atoms have roughly the same velocity and traverse the same beam geometry, this can be considered a homogeneous effect for that specific group of atoms.

* **"laser bandwidth, etc."**: We must also remember that our probe itself, the laser, is not perfectly monochromatic. It has its own finite bandwidth or linewidth. If the laser linewidth is broader than the true atomic linewidth, then the measured linewidth will be dominated by the laser. This is an instrumental effect, but it contributes to the observed homogeneous width if the laser output is stable.

Now for **"Inhomogeneous sources"**: These mechanisms cause different atoms or molecules in the ensemble to have slightly different resonant frequencies. The observed line profile is then a superposition of many narrower individual lines, each shifted by a certain amount, resulting in an overall broader line.

Examples include:

* **"Doppler shifts"**: In a gas, atoms are moving with a distribution of velocities (e.g., Maxwell-Boltzmann distribution). Atoms moving towards the laser see the light blue-shifted, and atoms moving away see it red-shifted. This range of Doppler shifts leads to a significant broadening of the spectral line, especially at higher temperatures and for lighter atoms. This is a

classic example of inhomogeneous broadening because each velocity group has a different resonant frequency in the lab frame.

* **"spatial field gradients"**: If the atoms are in an environment where there are spatial gradients in electric or magnetic fields (e.g., due to imperfect shielding or applied fields for trapping), and if the atomic energy levels are sensitive to these fields (Stark or Zeeman effects), then atoms at different positions will experience different shifts in their transition frequencies. This also leads to inhomogeneous broadening.

* **"unresolved hyperfine splittings, etc."**: Many atomic energy levels have hyperfine structure due to the interaction of the nuclear spin with the electrons. If these splittings are smaller than other broadening mechanisms or the resolution of our spectrometer, they might not be individually resolved and will instead contribute to an overall broadening of the observed spectral feature. This is inhomogeneous in the sense that we are averaging over distinct, but closely spaced, real transitions.

So, the "natural width" we often talk about is typically the homogeneous width due to lifetime effects, potentially modified by collisions, in the absence of strong fields. Understanding this baseline is key before we add power broadening into the mix.

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Now, having refreshed our memory on the various sources of linewidth, let's simplify our focus for the current discussion on saturation.

The first point states: **"For the moment we suppress every inhomogeneous mechanism."** This is a strategic simplification. We want to understand the intrinsic effects of an intense light field on a transition. Inhomogeneous broadening, like Doppler broadening, can often mask these more subtle effects or complicate the analysis considerably. So, we imagine a scenario where either there are no inhomogeneous effects, or we have somehow managed to eliminate them experimentally (e.g., by using a collimated atomic beam observed transversely, or through techniques like saturation spectroscopy itself, which we'll come to much later, that can select a zero-velocity group).

The asterisk clarifies this mathematically: **"Mathematically we keep a single Lorentzian function with half-width at half-maximum** $\frac{\gamma}{2}$ (Greek letter γ divided by 2)." A purely homogeneously broadened line, due to lifetime or collisional effects, typically has a Lorentzian lineshape. The full width at half maximum (FWHM) of this Lorentzian is γ. Therefore, the half-width at half-maximum (HWHM) is $\frac{\gamma}{2}$. So, our starting point is a collection of identical absorbers, each with this intrinsic Lorentzian profile.

Now, the crucial part: **"Intense radiation alters a homogeneous line by:"** How does a strong light field change this idealized, single Lorentzian line? There are two main ways listed here, which echo what we introduced earlier:

1. **"Depleting the lower level more strongly at resonance than in the wings."** When we tune our intense laser across the spectral line, the interaction is strongest when the laser is exactly on resonance (i.e., its frequency matches the peak of the Lorentzian). At this point, the rate of absorption (and stimulated emission) is highest. This means that the depletion of the lower state population and the buildup of the upper state population – in other words, saturation – will be most pronounced at the line center. As we detune the laser into the "wings" of the Lorentzian (i.e., further away from the exact resonance frequency), the interaction strength falls off, and so does the degree of saturation. This differential saturation – strong at the center, weaker in the wings – is key to understanding how the lineshape changes. It's not a uniform reduction in absorption across the entire profile.

2. **"Driving coherent Rabi oscillations that smear out the response."** This refers to the more quantum mechanical picture we alluded to earlier.

An intense, near-resonant field doesn't just cause random absorptions and emissions. It can drive coherent oscillations of the atomic system between the ground and excited states. These are Rabi oscillations, or Rabi flopping. The stronger the field, the faster these oscillations. If these oscillations are occurring on a timescale comparable to or shorter than other relaxation times, they fundamentally alter the way the atom responds to the light. This rapid driving can effectively "smear out" the spectral response, leading to a broadening. This perspective is especially important when the field is very strong and coherence effects dominate.

These two points are really two facets of the same underlying physics of strong light-matter interaction. The rate equation approach we'll start with primarily captures the first point (population effects), while a full quantum treatment (like the density matrix formalism or dressed atom picture, which are beyond this immediate discussion but underpin the Rabi oscillation idea) more directly addresses the second. However, we'll see that even our rate equation approach will predict the broadening.

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Following from the ways intense radiation alters a homogeneous line, we now consider the **"Outcome."**

The slide succinctly states: **"an apparently wider Lorentzian with a reduced peak absorption."**

This is the hallmark of saturation and power broadening for an initially homogeneous line. Let's elaborate on these two key changes:

First, **"reduced peak absorption."** As we discussed, saturation means that the population difference between the lower and upper states decreases. Since net absorption is proportional to this population difference $N_1 - N_2$, a reduction in this difference directly leads to a reduction in the absorption coefficient at the line center, where saturation is strongest. So, if you were to scan your laser across the transition, the peak height of your absorption signal would be lower with an intense laser than with a weak laser.

Second, **"an apparently wider Lorentzian."** This is the power broadening effect. Because saturation is most effective at the line center, it "eats away" at the peak more than it affects the wings of the line. To maintain (approximately, as we'll see) the total integrated area under the curve (which is related to the oscillator strength, a fundamental atomic property), if the peak height goes down, the width must go up. The line effectively "flattens and broadens." The important thing is that the *shape* often remains approximately Lorentzian, but it's a Lorentzian corresponding to a larger width parameter than the original γ. We will derive how this new, power-broadened width depends on the laser intensity and the intrinsic properties of the transition.

The triple dash again indicates a slight pause or transition. So, these are the macroscopic observables we expect to see: the absorption signal gets weaker at its peak, and the spectral feature itself becomes broader, even if it started as a nice, narrow homogeneous line. This has immediate practical consequences for resolution and sensitivity in our spectroscopic measurements.

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Now, let's start building our model. The title here is **"Prototype Two-Level System."** This is the foundational model we'll use to understand saturation and power broadening. As mentioned, while real systems are more complex, the two-level approximation is remarkably powerful for understanding resonant light-matter interactions.

The first bullet point defines our system: **"Two discrete, non-degenerate energy levels:"**

"Discrete" means the energy levels are well-defined and separated, not part of a continuum.

"Non-degenerate" means that each energy level corresponds to a unique quantum state. If there were degeneracy (multiple states with the same energy), we'd have to sum over them, which can complicate the bookkeeping, though the fundamental ideas of saturation would still apply. For simplicity, we assume $g_1 = g_2 = 1$, where g is the degeneracy.

Next, we label these levels:

"Lower level $|1\rangle$, energy E_1." We use the ket notation $|1\rangle$ (ket one) to represent the quantum state of the lower level, and its energy is E_1.

"Upper level $|2\rangle$, energy E_2 (with $E_2 > E_1$)." Similarly, $|2\rangle$ (ket two) is the upper level, and its energy E_2 is, of course, greater than E_1.

The interaction with light involves transitions between these two levels. The **"Transition angular frequency"** is defined by the Bohr frequency condition:

The equation shown is:

$$\omega_{21} = \frac{E_2 - E_1}{\hbar}$$

Let's break this down:

ω_{21} (omega sub 2 1) is the resonant angular frequency for the transition from state 1 to state 2 (or vice versa). It's an angular frequency, so its units would be radians per second. Sometimes you'll see ν_{21} (nu sub 2 1) for the frequency in Hertz, where $\omega = 2\pi\nu$ (omega equals two pi nu).

$E_2 - E_1$ (E sub 2 minus E sub 1) is the energy difference between the upper and lower states. This must be a positive quantity.

$\hbar$ is the reduced Planck constant (Planck's constant h divided by 2π). It has units of Joule-seconds or electronvolt-seconds, for instance.

This equation is fundamental: for resonant absorption or emission of a single photon, the photon's energy, $\hbar\omega$ (h-bar omega, where omega is the photon's angular frequency), must precisely match the energy difference $E_2 - E_1$ between the two levels. So, ω_{21} is the natural oscillation frequency of the two-level system.

Finally, we need to describe the state of our ensemble of two-level systems. We do this using **"Population variables (number of atoms per cm³)"**:

These will be N_1 (N sub 1) and N_2 (N sub 2), representing the number density of atoms in the lower level $|1\rangle$ and upper level $|2\rangle$, respectively, typically in units of atoms per cubic centimeter or per cubic meter. These populations, N_1 and N_2, will change under the influence of the light field and relaxation processes, and it's their dynamics that we want to model with rate equations.

This simple two-level system, characterized by its energies E_1 and E_2, its transition frequency ω_{21}, and the populations N_1 and N_2, is the workhorse for much of our discussion in laser spectroscopy.

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Continuing with our prototype two-level system, we now define the population variables more explicitly and list the **"Allowed processes"** that can change these populations.

At the top, we see $N_1(t), N_2(t)$ (N sub 1 of t, N sub 2 of t). This emphasizes that the populations of the lower level (N_1) and the upper level (N_2) are, in general, functions of time, t. Our goal with rate equations will be to write down $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$.

Now, let's consider the processes that cause atoms to move between levels $|1\rangle$ and $|2\rangle$:

* **"Stimulated absorption** $|1\rangle \rightarrow |2\rangle$." This is the process where an atom in the lower level $|1\rangle$ absorbs a photon from the incident light field and makes a transition to the upper level $|2\rangle$. The rate of this process is proportional to

the population of the lower level, N_1, and also to the energy density of the light field at the transition frequency. This is one of the key processes Einstein identified.

* **"Stimulated emission** $|2\rangle \rightarrow |1\rangle$." An atom in the upper level $|2\rangle$ can be stimulated by the incident light field to emit a photon that is identical in frequency, phase, direction, and polarization to the stimulating photon, and in doing so, the atom transitions back to the lower level $|1\rangle$. The rate of this process is proportional to the population of the upper level, N_2, and also to the energy density of the light field. This is the process that makes lasers possible – light amplification by *stimulated emission* of radiation.

* **"Spontaneous relaxation** $|2\rangle \rightarrow |1\rangle$ (rate R_2)." An atom in the upper level $|2\rangle$ can also decay to the lower level $|1\rangle$ by spontaneously emitting a photon, even in the absence of an external light field. This photon is emitted in a random direction with a random phase. The rate of this process is typically characterized by a constant R_2 (capital R sub 2), which is related to the Einstein A coefficient (A_{21}). R_2 would have units of per second (s^{-1}). For a simple two-level system where spontaneous decay only occurs from $|2\rangle$ to $|1\rangle$, R_2 is simply A_{21}. If there are other decay paths from level 2, R_2 would be the sum of all such spontaneous decay rates out of level 2 that ultimately feed level 1, or simply the total decay rate out of level 2 if we are only concerned with its depopulation. For our two-level system, we'll assume it means decay from $|2\rangle$ to $|1\rangle$.

* **"Non-radiative depopulation of** $|1\rangle$ (rate R_1)." This is a bit of a catch-all term. In a true, isolated two-level system, the ground state $|1\rangle$ might be stable. However, in more realistic scenarios, there might be processes that remove atoms from state $|1\rangle$ without involving radiation, or perhaps state $|1\rangle$ is not the true ground state of a larger multi-level system and can itself decay or be transferred to other states. The slide denotes a rate R_1 (capital R sub 1) for such processes. More commonly in a strict two-level system, R_1 might represent processes that *repopulate* level 1 from some reservoir,

or R_1 and R_2 might represent overall relaxation rates for levels 1 and 2 *back towards some equilibrium distribution* if they are perturbed. For our purpose here, let's interpret R_1 as a general relaxation rate affecting state 1, and R_2 as a relaxation rate affecting state 2 (primarily spontaneous emission to state 1). We will need to be careful how these are precisely defined when we write the rate equations.

* **"No other levels are involved; therefore** $N = N_1 + N_2$ remains constant." Here, N (capital N) represents the total number density of atoms participating in the two-level system. If we assume that atoms only shuttle between levels $|1\rangle$ and $|2\rangle$, and are not lost to or gained from other levels outside this pair, then the sum of the populations N_1 plus N_2 must be a constant, equal to the total population N. This conservation condition is very useful in solving the rate equations, as it means $\frac{dN_1}{dt}$ must equal $-\frac{dN_2}{dt}$ if there are no external feeds or losses from the two-level system as a whole.

These processes form the basis for writing our differential equations describing the population dynamics.

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This page provides a visual representation of the **"Prototype Two-Level System"** we've been discussing, specifically illustrating the processes outlined on the previous page. This diagram is crucial for understanding the terms in our upcoming rate equations.

At the top of the diagram, we see the chapter and section title: "Chap. 3.6 Saturation & Power Broadening," and "Prototype Two-Level System."

The diagram itself shows two horizontal lines representing the energy levels:

- The upper line is labeled $|2\rangle$ (ket two) on the left and E_2 (E sub 2) on the right, indicating the upper energy level.

- The lower line is labeled $|1\rangle$ (ket one) on the left and E_1 (E sub 1) on the right, indicating the lower energy level.

The energy difference between these levels is implicitly $E_2 - E_1$, which corresponds to the transition energy $\hbar\omega_{21}$ (h-bar omega sub 2 1), as labeled vertically on the far right of the energy gap.

Now let's look at the arrows, which represent the transitions:

- There's an **upward arrow** from level $|1\rangle$ to level $|2\rangle$. This represents **stimulated absorption**. It is labeled $B_{12}\rho(\omega)$ (B sub 1 2 rho of omega) or sometimes $B_{12}\rho_p(\omega)$ (B sub 1 2 rho sub p of omega, where p denotes 'pump'). This indicates that the rate of absorption is proportional to the Einstein B_{12} coefficient (for absorption) and the spectral energy density $\rho(\omega)$ (rho of omega) of the incident light field at angular frequency ω.

- There's a **downward arrow** from level $|2\rangle$ to level $|1\rangle$, parallel to the absorption arrow. This represents **stimulated emission**. It is labeled $B_{21}\rho(\omega)$ (B sub 2 1 rho of omega). The rate is proportional to the Einstein B_{21} coefficient (for stimulated emission) and $\rho(\omega)$. For non-degenerate levels, or levels with degeneracies g_1 and g_2, we know that $g_1 B_{12} = g_2 B_{21}$. If we assume $g_1 = g_2$, then $B_{12} = B_{21}$.

- There's a **wavy downward arrow** from level $|2\rangle$ to level $|1\rangle$. This represents **spontaneous emission**. It is labeled R_2 (capital R sub 2). This is the rate of spontaneous decay from the upper level to the lower level. In the simplest case, $R_2 = A_{21}$, the Einstein A_{21} coefficient for spontaneous emission.

- Finally, there's a **dashed downward arrow originating from level** $|1\rangle$ and pointing further down, away from the two-level system. This is labeled R_1 (capital R sub 1). This represents a non-radiative depopulation pathway or relaxation *out of* level 1 to some other states not explicitly part of our $|1\rangle$-$|2\rangle$ system, or perhaps relaxation of the entire system towards some thermal equilibrium not defined solely by $|1\rangle$ and $|2\rangle$. This R_1 term is sometimes omitted in the very simplest two-level models if $|1\rangle$ is considered absolutely stable and there's no mechanism to leave it other than absorption to $|2\rangle$. However, its inclusion allows for more general scenarios. Alternatively, if $|1\rangle$ is being repopulated from a reservoir, R_1 might represent that inflow, but the arrow direction here suggests an outflow. We need to be consistent with the rate equations. Given the equations we'll see, R_1 will act as a decay rate *from* level 1, and R_2 as a decay rate *from* level 2.

This diagram beautifully encapsulates the key processes: optical pumping upwards (absorption), stimulated de-excitation downwards, and spontaneous decay downwards. The R_1 term represents other relaxation processes that might be present. The energy $\hbar\omega_{21}$ is the characteristic energy of the transition. This visual aid will be very helpful as we now proceed to build the rate equations.

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Now we're ready to start **"Building the Rate Equation – Step by Step,"** as indicated by the title of Slide 4. This is where we translate the processes from the diagram into mathematical expressions.

The first bullet point defines the **"Stimulated absorption probability per atom per second:"**

The equation given is:

$$P_{12} = B_{12}\rho_p(\omega)$$

Let's dissect this:

* P_{12} (capital P sub 1 2) represents the rate of transition from level 1 to level 2 due to stimulated absorption, for a single atom. So, its units are per second (s^{-1}). This is essentially a transition rate constant.

* B_{12} (capital B sub 1 2) is the **Einstein B coefficient for absorption**. The slide provides its units: $s^{-1}\, m^3\, J^{-1}$ (per second, cubic meters, per Joule), or more standardly, $m^3\, J^{-1}\, s^{-2}$ (cubic meters per Joule per second squared) if

ρ is energy density per unit volume per unit angular frequency, or sometimes different units if ρ is defined per unit frequency (Hz). Let's assume the units provided here are consistent with the definition of $\rho_p(\omega)$. The B coefficient is a fundamental property of the atomic transition; it quantifies how strongly that particular transition couples to the radiation field.

$\rho_p(\omega)$ (rho sub p of omega) is the **"spectral energy density of the pump field at frequency ω** (omega)". The 'p' subscript emphasizes that this is the radiation field causing* the transitions, often called the pump field. "Spectral energy density" means it's the energy per unit volume per unit frequency interval. The units given are $J\,m^{-3}\,\text{Hz}^{-1}$ (Joules per cubic meter per Hertz). If ω is angular frequency, then typically $\rho(\omega)$ would be Joules per cubic meter per (radians per second), which is Joules-second per cubic meter. There's a common point of confusion here: $\rho(\nu)$ is energy per unit volume per unit frequency (Hz), while $\rho(\omega)$ is energy per unit volume per unit angular frequency (rad/s). They are related by $\rho(\omega) = \rho(\nu)/(2\pi)$. The Einstein B coefficients are defined differently depending on whether you use $\rho(\nu)$ or $\rho(\omega)$. Given the units of B_{12} ($s^{-1}\,m^3\,J^{-1}$) and $\rho_p(\omega)$ ($J\,m^{-3}\,\text{Hz}^{-1}$), if P_{12} is to be in s^{-1}, then B_{12} must be defined for $\rho_p(\omega)$ in $J\,m^{-3}\,(\text{rad/s})^{-1}$. Let's assume B_{12} here is the coefficient appropriate for $\rho_p(\omega)$ expressed in terms of energy density per unit angular frequency interval. Or, if $\rho_p(\omega)$ is indeed per Hertz, then $P_{12} = B_{12}^{(\nu)}\rho_p^{(\nu)}$. We need to be careful here. A common definition for B_{12} when $\rho(\omega)$ is spectral energy density per unit angular frequency (units: $J\,s\,m^{-3}$) is that B_{12} has units of $m^3\,J^{-1}\,s^{-2}$.

Let's assume for consistency with many texts that $\rho_p(\omega)$ is the energy density per unit angular frequency interval ($J\,s\,m^{-3}$). Then B_{12} would be $m^3\,J^{-1}\,s^{-2}$. If P_{12} is $B_{12}\rho_p(\omega)$, then the units would be $(m^3\,J^{-1}\,s^{-2}) \times (J\,s\,m^{-3}) = s^{-1}$, which is correct for a rate.

Alternatively, if $P_{12} = B_{12}\, I(\omega)/c$, where $I(\omega)$ is intensity and c is speed of light, then units also work out. The definition of B coefficients can sometimes vary. Let's proceed with the understanding that P_{12} is a rate (s^{-1}).

The slide then says: **"Define shorthand"**. This is very common to simplify the notation in the rate equations.

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Continuing from defining the shorthand, the slide introduces it:

$P = P_{12}$ (convenience symbol for pump rate).

So, capital P will now represent this stimulated transition rate, $B_{12}\rho_p(\omega)$. It's the probability per unit time that an atom in level 1 will absorb a photon and jump to level 2, and also the probability per unit time that an atom in level 2 will be stimulated to emit a photon and jump to level 1 (assuming $B_{12} = B_{21}$ for simplicity if degeneracies are equal, or that P implicitly includes the correct B coefficient for stimulated emission as well). Usually, P is taken to be the same for stimulated absorption and emission when degeneracies are equal.

Now, the core of this page: **"Full differential balance (population gains counted positive):"** These are the rate equations for the populations N_1 and N_2.

First, for the population of the lower level, N_1:

$$\frac{dN_1}{dt} = -PN_1 - R_1N_1 + PN_2 + R_2N_2$$

Let's interpret each term for $\frac{dN_1}{dt}$ (dee N sub 1 by dee t):

- $-PN_1$: This term represents the loss of population from level 1 due to stimulated absorption. P is the rate constant, N_1 is the population in level 1.

The minus sign indicates it's a loss from level 1. (Atoms leaving level 1 via absorption).

- $-R_1N_1$: This term represents the loss of population from level 1 due to other relaxation processes characterized by rate R_1, as discussed with the diagram. The minus sign indicates loss.

- $+PN_2$: This term represents the gain of population in level 1 due to stimulated emission from level 2. P is the rate constant (assuming $B_{12} = B_{21}$ or appropriate scaling), N_2 is the population in level 2. The plus sign indicates a gain for level 1.

- $+R_2N_2$: This term represents the gain of population in level 1 due to spontaneous emission (and possibly other relaxations included in R_2) from level 2. R_2 is the rate for this process, N_2 is the population in level 2. The plus sign indicates a gain for level 1.

Next, for the population of the upper level, N_2:

$$\frac{dN_2}{dt} = +PN_1 + R_1N_1 - PN_2 - R_2N_2$$

(Wait, there seems to be a typo here. R_1N_1 should not be a gain term for N_2 if R_1 is a loss from N_1. Let's re-examine the typical form.)

Let's reconsider the R_1 term. If R_1 is a decay *from* level 1 (as the arrow in the diagram on page 10 suggested, pointing away from the 1-2 system), then it shouldn't feed level 2. If R_1 were, for example, a pumping *into* level 1 from some external reservoir, that would be different. Assuming the diagram on page 10 is accurate where R_1 is a loss from level 1 to *outside* the two-level system, and R_2 is a loss from level 2 (primarily to level 1, but could include other channels if R_2 is a total decay rate from level 2), then the equations should be:

$$\frac{dN_1}{dt}$$

$= -PN_1 - R_1N_1 + PN_2 + A_{21}N_2$
$+$ (contributions to N_1 from R_2 if
R_2 includes decay to other levels that then feed N_1)

$$\frac{dN_2}{dt}$$

$= +PN_1 - PN_2 - R_2N_2$
$-$ (loss from N_2 to R_1 if R_1 is some common bath, which is unlikely)

Let's look at the common simplified two-level system rate equations where R_1 is often taken as a relaxation rate *of* level 1 (perhaps to a true ground state if 1 is metastable) and R_2 is the total relaxation rate *of* level 2 (A_{21} plus non- radiative decay). The terms for $\frac{dN_2}{dt}$ usually mirror $\frac{dN_1}{dt}$ but with opposite signs for processes linking the two levels, and with their own decay terms. If $N = N_1 + N_2$ is constant, then $\frac{dN_2}{dt} = -\frac{dN_1}{dt}$ *only if* R_1N_1 and R_2N_2 are solely internal exchanges or if $R_1 = R_2 = 0$. The slide states: **"Each term is individually interpretable"**: And then it lists for $\frac{dN_1}{dt}$:

- **"-P N_1 → atoms leaving level 1 via absorption."** This matches our interpretation.

Let's assume for the moment the equations are written as presented and try to make sense of them, or acknowledge a potential common simplification/typo. If $N = N_1 + N_2$ is constant, then

$$\frac{dN_1}{dt} + \frac{dN_2}{dt} = 0.$$

Adding the two equations given on the slide:

$$\frac{dN_1}{dt} + \frac{dN_2}{dt} = (-PN_1 - R_1N_1 + PN_2 + R_2N_2) + (+PN_1 + R_1N_1 - PN_2 - R_2N_2)$$

This sum is identically zero: $-PN_1$ cancels $+PN_1$, $-R_1N_1$ cancels $+R_1N_1$, $+PN_2$ cancels $-PN_2$, and $+R_2N_2$ cancels $-R_2N_2$. This means that with these specific equations, $\frac{d(N_1+N_2)}{dt} = 0$ is automatically satisfied. This implies that the terms R_1N_1 and R_2N_2 must be interpreted within the closed two-level system.

So, in this formulation:

- R_1N_1 must be a loss from level 1 that is exactly compensated by a gain for level 2 (the $+R_1N_1$ term in $\frac{dN_2}{dt}$). This would mean R_1 is actually an *incoherent pumping rate from level 1 to level 2*, which is unusual notation.
- Similarly, R_2N_2 is a gain for level 1 which must be compensated by a loss from level 2 (the $-R_2N_2$ term in $\frac{dN_2}{dt}$). This means R_2 is a relaxation rate from level 2 to level 1. This is standard (e.g., spontaneous emission A_{21} plus non- radiative decay from 2 to 1).

Let's stick to the slide's interpretation of terms for $\frac{dN_1}{dt}$, which is most standard. Page 13 will clarify the terms for R_1N_1 and R_2N_2.

Page 13:

This page continues the interpretation of the terms in the rate equation for $\frac{dN_1}{dt}$ (and implicitly for $\frac{dN_2}{dt}$, since $\frac{dN_2}{dt} = -\frac{dN_1}{dt}$ with the given forms).

* **"**$+PN_2$ → stimulated emission returns atoms to 1." This confirms our understanding of the $+PN_2$ term in the equation for $\frac{dN_1}{dt}$. It's a gain for level 1 due to atoms being stimulated to leave level 2 and arrive in level 1.

* **"**$-R_1N_1$ → non-radiative loss from 1." This is how the R_1 term is interpreted. It's a loss channel for population in level 1, occurring at a rate R_1, and it's described as "non-radiative." This implies that R_1 is a rate at which atoms in level 1 transition to some *other* state(s) not $|2\rangle$, or perhaps relax to a true ground state if $|1\rangle$ is an excited state itself.

Now, if this is a loss *from* the two-level system $|1\rangle - |2\rangle$, then the condition $\frac{d(N_1+N_2)}{dt} = 0$ (i.e., $N_1 + N_2 =$ constant) would only hold if there's a balancing inflow, or if R_1 is actually very small, or if the $R_1 N_1$ term in the $\frac{dN_2}{dt}$ equation on the previous page (which was $+R_1 N_1$) was a typo and should have been related to something else or absent.

Let's assume the standard rate equations for a two-level system where N is the total population within the two levels:

$$\frac{dN_1}{dt} = -P(N_1 - N_2) + A_{21}N_2 - (\text{other losses from } N_1) + (\text{other gains to } N_1)$$

$$\frac{dN_2}{dt} = +P(N_1 - N_2) - A_{21}N_2 - (\text{other losses from } N_2) + (\text{other gains to } N_2)$$

If we use the slide's equation from

Page 12:

$$\frac{dN_1}{dt} = -PN_1 - R_1\,N_1 + PN_2 + R_2\,N_2$$

$$\frac{dN_2}{dt} = +PN_1 + R_1\,N_1 - PN_2 - R_2\,N_2$$

The term $-R_1\,N_1$ in $\frac{dN_1}{dt}$ is a loss from 1. The term $+R_1\,N_1$ in $\frac{dN_2}{dt}$ means population lost from 1 via this R_1 process actually *appears* in level 2. This would make R_1 an incoherent pumping mechanism from level 1 to level 2 (e.g., collisional excitation). This is not "loss from 1" in the sense of leaving the system.

And similarly for R_2:

* **"**$+R_2\,N_2 \rightarrow$ spontaneous decay feeding 1."

This is standard. R_2 is the rate at which population from level 2 decays (spontaneously and perhaps non- radiatively) directly into level 1. This is a gain for level 1 and a loss for level 2. This matches the $R_2 N_2$ terms in both $\frac{dN_1}{dt}$ ($+R_2 N_2$) and $\frac{dN_2}{dt}$ ($-R_2 N_2$).

So, with the equations as written on page 12, which ensure $\frac{dN_1}{dt} = -\frac{dN_2}{dt}$:

* P represents the stimulated transition rate (absorption $1 \rightarrow 2$, emission $2 \rightarrow 1$). R_2 represents the relaxation rate from level 2 to* level 1. R_1 must represent an incoherent transfer rate from level 1 to* level 2.

This interpretation makes the system closed ($N_1 + N_2 =$ constant). The term "non- radiative loss from 1" for $-R_1 N_1$ (if R_1 is $1 \rightarrow 2$ transfer) is a bit misleading; it's a loss from state 1 *to state 2*. Perhaps "non- radiative transfer from 1" would be clearer if it goes to state 2.

Let's assume the equations on page 12 are the intended ones, implying a closed system where R_1 is a $1 \rightarrow 2$ incoherent rate and R_2 is a $2 \rightarrow 1$ relaxation rate. This is critical for the steady- state solutions that follow.

The triple dash suggests the end of this definitional section. The key is that whatever these R_1 and R_2 rates are, the structure of the equations $\frac{dN_1}{dt} = -\frac{dN_2}{dt}$ is what's used for finding the stationary solution.

Page 14

Slide 5: Stationary Solution of the Rate Equations.

In many spectroscopic experiments, we are interested in the steady-state behavior of the system, where the populations N_1 and N_2 are no longer changing in time, under continuous wave (CW) laser illumination.

The first bullet point defines the **Steady state condition**: This is mathematically expressed as:

$$\frac{dN_i}{dt} = 0 \quad (\text{for } i = 1,2)$$

This means that the rate of change of population in level 1 is zero, and the rate of change of population in level 2 is zero. The populations have reached a constant value. For this to happen, the total rate of processes filling a level must exactly balance the total rate of processes emptying it.

The second bullet point is **Enforce particle conservation**: This is the condition we discussed earlier, crucial for a closed two-level system:

$$N_1 + N_2 = N \quad \text{(total density, constant)}$$

Here, N (capital N) is the total number density of atoms in our two-level system. This equation allows us to express N_2 in terms of N_1 (i.e., $N_2 = N - N_1$) or vice versa, reducing the number of independent variables.

The third bullet point describes the algebraic procedure: **Insert P and solve algebraically for** N_1: We take one of the rate equations (say, for $\frac{dN_1}{dt}$), set it to zero, and use the conservation relation $N_2 = N - N_1$ to eliminate N_2. This will give us an algebraic equation that we can solve for N_1.

The specific equation shown is derived from setting $\frac{dN_1}{dt} = 0$: From page 12,

$$\frac{dN_1}{dt} = -PN_1 - R_1N_1 + PN_2 + R_2N_2.$$

Setting this to 0:

$$0 = -PN_1 - R_1N_1 + PN_2 + R_2N_2.$$

Substitute $N_2 = N - N_1$:

$$0 = -PN_1 - R_1N_1 + P\,(N - N_1) + R_2\,(N - N_1).$$

Expanding the terms gives:

$$0 = -PN_1 - R_1N_1 + PN - PN_1 + R_2N - R_2N_1.$$

Now, we want to group terms with N_1 and terms with N.

$$PN_1 + R_1N_1 + PN_1 + R_2N_1 = PN + R_2N$$

$$N_1(P + R_1 + P + R_2) = N(P + R_2)$$

$$N_1(2\,P + R_1 + R_2) = N(P + R_2).$$

The slide shows:

$$(P + R_1)N_1 = (P + R_2)(N - N_1).$$

Let's see how this form arises. It seems to be a rearrangement step or perhaps starting from a slightly different but equivalent form if some terms were grouped. Let's expand the slide's equation:

$$PN_1 + R_1N_1 = PN - PN_1 + R_2N - R_2N_1.$$

Collecting N_1 terms on the left:

$$PN_1 + R_1N_1 + PN_1 + R_2N_1 = PN + R_2N,$$

$$N_1(2\,P + R_1 + R_2) = (P + R_2)N.$$

This is the same result I got. So, the equation on the slide is an intermediate algebraic step that is correct. Specifically, if we have

$$0 = -(P + R_1)N_1 + (P + R_2)(N - N_1),$$

then

$$(P + R_1)N_1 = (P + R_2)(N - N_1).$$

This comes directly from:

$$0 = -PN_1 - R_1N_1 + P\,(N - N_1) + R_2\,(N - N_1),$$

$$0 = -(P + R_1)N_1 + (P + R_2)(N - N_1).$$

This makes sense. The left side, $(P + R_1)N_1$, represents all processes that empty N_1 *into* N_2 or other states characterized by R_1 (Here, my previous interpretation of R_1 as $1 \rightarrow 2$ transfer is consistent with this grouping). And

$(P + R_2)$ represents the rates for stimulated emission (P) and relaxation (R_2) from N_2, which has population $(N - N_1)$, that feed N_1.

No, this is not quite right. Let's restart the algebra from

$$0 = -PN_1 - R_1N_1 + PN_2 + R_2N_2.$$

This is:

$$(P + R_1)N_1 = PN_2 + R_2N_2 = (P + R_2)N_2.$$

Then, substituting $N_2 = N - N_1$:

$$(P + R_1)N_1 = (P + R_2)(N - N_1).$$

Yes, this is correct. The left side, $(P + R_1)N_1$, represents the total rate of atoms leaving level 1 *due to interaction* P and relaxation R_1. The right side, $(P + R_2)(N - N_1)$, represents the total rate of atoms arriving in level 1 *from level 2 (population* $N - N_1$) via stimulated emission P and relaxation R_2. In steady-state, these total rates of leaving and arriving must be equal. This is the physical meaning of this equation.

This algebraic equation can now be solved for N_1.

Page 15:

Following the algebraic setup, this page shows the results of the **"Rearrangement yields"**:

First, the steady-state population in the lower level, N_1:

$$N_1 = N \cdot \frac{P + R_2}{2\,P + R_1 + R_2}$$

Let's verify this from our previous equation: $N_1(2\,P + R_1 + R_2) = N(P + R_2)$.

Indeed, dividing by $(2\,P + R_1 + R_2)$ gives

$$N_1 = N \cdot \frac{P + R_2}{2\,P + R_1 + R_2}$$

This is correct.

Second, the steady-state population in the upper level, N_2:

$$N_2 = N \cdot \frac{P + R_1}{2\,P + R_1 + R_2}$$

We can obtain this by using $N_2 = N - N_1$.

$$N_2 = N - N \cdot \frac{P + R_2}{2\,P + R_1 + R_2}$$

$$N_2 = N \cdot \left(1 - \frac{P + R_2}{2\,P + R_1 + R_2}\right)$$

$$N_2 = N \cdot \frac{2\,P + R_1 + R_2 - P - R_2}{2\,P + R_1 + R_2}$$

$$N_2 = N \cdot \frac{P + R_1}{2\,P + R_1 + R_2}$$

This is also correct.

So these two equations give us the steady-state populations N_1 and N_2 as a function of the total population N, the pump rate P (which depends on laser intensity), and the relaxation rates R_1 (from level 1 to level 2, incoherently) and R_2 (from level 2 to level 1).

Now, we look at some **"Important limiting cases"**: These are crucial for building intuition about how these populations behave.

The first limiting case is: * $P \ll R_\text{i}$ (weak field) → returns thermal equilibrium populations.

Here, R_i presumably refers to both R_1 and R_2.

If the pump rate P is very small compared to the relaxation rates, it means the laser is not significantly perturbing the system from its natural equilibrium.

Let's see what happens if P approaches 0:

$$N_1 \to N \cdot \frac{0 + R_2}{0 + R_1 + R_2} = N \cdot \frac{R_2}{R_1 + R_2}$$

$$N_2 \to N \cdot \frac{0 + R_1}{0 + R_1 + R_2} = N \cdot \frac{R_1}{R_1 + R_2}$$

These are the populations determined solely by the balance of the relaxation rates R_1 and R_2.

If the system is in thermal equilibrium at some temperature T, then the ratio

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} \cdot \exp\left(-\frac{E_2 - E_1}{kT}\right)$$

would be given by $\frac{g_2}{g_1} \cdot \exp\left(-\frac{E_2 - E_1}{kT}\right)$, where k is Boltzmann's constant.

In our specific model where R_1 is an incoherent pump $1 \to 2$ and R_2 is relaxation $2 \to 1$, then in the absence of the coherent pump P, the ratio

$$\frac{N_2}{N_1} = \frac{R_1}{R_2}$$

So,

$$N_1 = N \cdot \frac{R_2}{R_1 + R_2} \quad \text{and} \quad N_2 = N \cdot \frac{R_1}{R_1 + R_2}.$$

This is exactly what we get.

The weak field limit is important: when the laser is very weak, it doesn't significantly change the populations from what they would be due to other relaxation/thermal processes.

Page 16:

Continuing with the important limiting cases:

The second case considered is: * $P \gg R_\text{i}$ (very strong field) → $N_1 = N_2 = \frac{N}{2}$. Medium becomes transparent because the populations equalise. (P much greater than R_i). Again, R_i refers to R_1 and R_2. This is the limit of a very intense laser field, where the stimulated rate P dominates over all relaxation rates.

Let's look at our expressions for N_1 and N_2 when P is very large:

$$N_1 = N \cdot \frac{P + R_2}{2\,P + R_1 + R_2}.$$

Divide numerator and denominator by P:

$$N_1 = N \cdot \frac{1 + \frac{R_2}{P}}{2 + \frac{R_1}{P} + \frac{R_2}{P}}.$$

As $P \to \infty$ (or $P \gg R_1$, R_2), the terms $\frac{R_2}{P}$, $\frac{R_1}{P}$ all go to zero.

So,

$$N_1 \to N \cdot \frac{1 + 0}{2 + 0 + 0} = N \cdot \frac{1}{2} = \frac{N}{2}.$$

Similarly for N_2:

$$N_2 = N \cdot \frac{P + R_1}{2\,P + R_1 + R_2}.$$

Divide by P:

$$N_2 = N \cdot \frac{1 + \frac{R_1}{P}}{2 + \frac{R_1}{P} + \frac{R_2}{P}}.$$

As $P \to \infty$,

$$N_2 \to N \cdot \frac{1 + 0}{2 + 0 + 0} = N \cdot \frac{1}{2} = \frac{N}{2}.$$

So, indeed, in the limit of a very strong driving field, $N_1 = N_2 = \frac{N}{2}$.

This is a profound result. It means that the populations in the lower and upper states become equal, each holding half of the total available population. This is the "population equalisation" we mentioned at the very beginning.

Why does this happen? When P is very large, stimulated absorption (emptying N_1, filling N_2) and stimulated emission (emptying N_2, filling N_1) are both occurring very rapidly. The system is driven so hard that it can't distinguish much between the up and down transitions; any atom that relaxes down is quickly pumped back up, and any atom pumped up can be quickly stimulated back down. The relaxation rates R_1 and R_2, which try to establish some other equilibrium, become negligible in comparison. The system reaches a state where the upward and downward stimulated rates nearly balance, which occurs when $N_1 \approx N_2$.

The consequence is that the "Medium becomes transparent." Why? Net absorption of light is proportional to the population difference $(N_1 - N_2)$ times the B_{12} coefficient, minus stimulated emission which is proportional to N_2 times B_{21}. If we assume $B_{12} = B_{21}$ (for equal degeneracies), then net absorption is proportional to $(N_1 - N_2)$. If $N_1 = N_2$, then $(N_1 - N_2) = 0$. This means there is no net absorption of light from the beam. The number of photons absorbed per unit time is exactly balanced by the number of photons added back into the beam by stimulated emission. The medium neither attenuates nor amplifies the beam; it becomes transparent to the strong resonant light. This phenomenon is called saturation. The transition is saturated.

The triple dash indicates the end of this thought. These two limits – very weak field and very strong field – provide crucial anchor points for understanding the behavior of the system. The transition from one regime to the other is governed by how P compares to R_1 and R_2.

Page 17

Now we arrive at **"Slide 6: Introducing the Saturation Parameter S."**

We've seen that the behavior of the system depends on how strong the pump rate P is compared to the relaxation rates. It's useful to define a dimensionless parameter that quantifies this.

First, let's define the **"Population difference (net absorbers)"**:

$$\Delta N = N_1 - N_2$$

(Delta N equals N sub 1 minus N sub 2).

This ΔN is crucial because, as we just discussed, the net rate of absorption of energy from the light field is proportional to this population difference (assuming equal degeneracies for simplicity, so $B_{12} = B_{21}$). If $\Delta N > 0$, there's net absorption. If $\Delta N < 0$ (population inversion, $N_2 > N_1$), there's net stimulated emission and thus optical gain. If $\Delta N = 0$, the medium is transparent at that frequency.

Next, we consider the **"Unsaturated difference (**$P = 0$)":

This is the population difference that would exist in the *absence* of the coherent pump field P. We denote this as ΔN_0 (Delta N sub zero).

Using our expressions for N_1 and N_2 from page 15 when $P = 0$:

$$N_{10} = N \cdot \left[\frac{R_2}{R_1 + R_2}\right]$$

$$N_{20} = N \cdot \left[\frac{R_1}{R_1 + R_2}\right]$$

So,

$$\Delta N_0 = N_{10} - N_{20} = N \cdot \left[\frac{(R_2 - R_1)}{(R_1 + R_2)}\right]$$

The slide shows this as:

$$\Delta N_0 = N \cdot \left[\frac{(R_2 - R_1)}{(R_1 + R_2)}\right]$$

This is correct. This ΔN_0 represents the "natural" population difference established by the incoherent relaxation and pumping processes R_1 (1→2) and R_2 (2→1). For net absorption to occur even with a weak probe, we typically need $\Delta N_0 > 0$, which means $R_2 > R_1$. If R_1 is thermal excitation and R_2 is thermal relaxation, R_2 will usually be greater than R_1 at reasonable temperatures for an optical transition where $E_2 - E_1 \gg kT$, so $\Delta N_0 \approx N$.

Now, the third bullet point: **"Substitute steady-state populations:"**

We want to find an expression for the actual population difference ΔN in the presence of the pump P, using the steady-state N_1 and N_2 we found on page 15:

$$N_1 = N \cdot \frac{P + R_2}{2\,P + R_1 + R_2}$$

$$N_2 = N \cdot \frac{P + R_1}{2\,P + R_1 + R_2}$$

So,

$$\Delta N = N_1 - N_2 = N \cdot \frac{(P + R_2) - (P + R_1)}{2\,P + R_1 + R_2}$$

$$\Delta N = N \cdot \frac{(R_2 - R_1)}{2\,P + R_1 + R_2}$$

This expression for ΔN will be used on the next page to define the saturation parameter S.

Notice that the numerator $N \cdot (R_2 - R_1)$ is exactly ΔN_0.

So, we can write:

$$\Delta N = \Delta N_0 \cdot \frac{R_1 + R_2}{2\,P + R_1 + R_2}$$

This relation is very important and will appear on the next page.

Page 18:

Continuing from the previous page, we have the expression for the population difference ΔN in the presence of the pump: The slide shows this relationship as:

$$\Delta N = \frac{\Delta N_0}{\left[1 + \left(\frac{2\,P}{R_1 + R_2}\right)\right]}$$

Let's verify this from what we derived:

$$\Delta N = \Delta N_0 \cdot \left[\frac{R_1 + R_2}{2\,P + R_1 + R_2}\right]$$

If we divide the numerator and denominator of the fraction inside the square brackets by $(R_1 + R_2)$, we get:

$$\Delta N = \Delta N_0 \cdot \left[\frac{1}{\left(\frac{2\,P}{R_1 + R_2}\right) + \left(\frac{R_1 + R_2}{R_1 + R_2}\right)}\right]$$

$$\Delta N = \Delta N_0 \cdot \left[\frac{1}{1 + \left(\frac{2\,P}{R_1 + R_2}\right)}\right]$$

This matches the slide perfectly. This is a very neat and common form. It shows that the actual population difference ΔN is the unsaturated population difference ΔN_0, reduced by a factor related to P and the relaxation rates R_1 and R_2.

Now, we **"Define"** the saturation parameter, S.

The denominator term $\left(\frac{2\,P}{R_1+R_2}\right)$ is defined as the saturation parameter S. So,

$$S = \frac{2\,P}{R_1 + R_2}$$

With this definition, the population difference becomes simply:

$$\mathit{\Delta} N = \frac{\mathit{\Delta} N_0}{1 + S}$$

This is a cornerstone equation in saturation physics! It elegantly shows how the population difference, and thus the absorption, is reduced from its unsaturated value $\mathit{\Delta} N_0$ by a factor of $(1 + S)$ due to the presence of the pump field P, which is embedded in S.

- If $S = 0$ (i.e., $P = 0$, no pump), then $\mathit{\Delta} N = \mathit{\Delta} N_0$ (unsaturated).
- If $S = 1$, then $\mathit{\Delta} N = \frac{\mathit{\Delta} N_0}{2}$ (population difference is halved). This occurs when $2\,P = R_1 + R_2$.
- If $S \gg 1$ (strong pump), then $\mathit{\Delta} N \approx \frac{\mathit{\Delta} N_0}{S}$, which tends to 0. (Populations equalize).

The slide also presents an alternative way to write S:

$$S = \frac{P}{\overline{R}}$$

where

$$\overline{R} = \frac{R_1 + R_2}{2}.$$

$\overline{R}$ can be thought of as a "mean relaxation rate" for the two-level system, averaging the rate R_1 (which we interpreted as $1 \rightarrow 2$ incoherent transfer) and R_2 ($2 \rightarrow 1$ relaxation). So, S is the ratio of the coherent pumping rate P

to this mean relaxation rate $\overline{R}$. This makes the physical meaning of S very clear.

Indeed, the next bullet point states:

- **"S is the ratio 'pumping rate per atom' ÷ 'mean relaxation rate'."** More precisely, P is the stimulated transition rate per atom (for one direction). The denominator $\frac{R_1+R_2}{2}$ is half the sum of the rates that tend to restore equilibrium or cause transitions between the levels in the absence of P (or in competition with P). When P becomes comparable to this mean relaxation rate (i.e., $S \approx 1$), saturation effects become significant.

Finally, a **"Special case: only spontaneous decay from upper level"**:

This is a very common scenario, especially for atoms in a vacuum where collisional relaxation is negligible.

- $R_1 = 0, R_2 = A_{21}$. If $R_1 = 0$, it means there's no incoherent process pumping atoms from level 1 to level 2. Level 1 only empties via stimulated absorption P. $R_2 = A_{21}$ means that the only way level 2 relaxes is by spontaneous emission to level 1, with rate A_{21} (the Einstein A coefficient).

In this very important special case:

- $\Delta N_0 = N \cdot \frac{(A_{21}-0)}{(0+A_{21})} = N$. (Assuming level 1 is the ground state, so initially all N atoms are in N_1, and $N_2 = 0$ without P.)

And the saturation parameter S becomes:

$$S = \frac{2\,P}{0 + A_{21}} = \frac{2\,P}{A_{21}}.$$

And $\overline{R} = \frac{A_{21}}{2}$. So $S = \frac{P}{A_{21}/2} = \frac{2\,P}{A_{21}}$. Consistent.

This specific form of S is frequently encountered.

The saturation parameter S is thus a central quantity that tells us how deeply we are into the saturation regime.

Page 19:

Now we connect the saturation parameter S to more experimentally accessible quantities like laser intensity and atomic cross- section.

The slide says: **"Using intensity** $I(\omega)$ and cross section $\sigma_{12}(\omega)$:"

Recall that our pump rate P was defined as $P = B_{12}\rho_p(\omega)$. The spectral energy density $\rho_p(\omega)$ can be related to the intensity $I(\omega)$ of the laser beam. Intensity $I(\omega)$ is power per unit area per unit angular frequency interval (e.g., W m^{-2}(rad/s)$^{-1}$). The relationship is $I(\omega) = c\, n\, \rho_p(\omega)$, where c is the speed of light in vacuum and n is the refractive index of the medium ($n \approx 1$ for gases). Or, if $\rho_p(\omega)$ is energy per unit volume per unit angular frequency, then $I(\omega) = \frac{c}{n}\, \rho_p(\omega)$. Let's assume $n = 1$ for simplicity, so $\rho_p(\omega) = \frac{I(\omega)}{c}$.

The stimulated rate P can also be expressed in terms of the absorption cross- section $\sigma_{12}(\omega)$ (sigma sub 1 2 of omega). The absorption cross- section is an effective area that an atom presents to incident photons for absorption. The rate of photon absorption by a single atom is given by the photon flux (photons per unit area per unit time) multiplied by the cross- section. Photon energy is $\hbar\omega$ (h- bar omega). Photon flux for a given intensity $I(\omega)$ (power per area) is $\frac{I(\omega)}{\hbar\omega}$ (photons per area per time, if $I(\omega)$ is just intensity, not spectral intensity).

If $I(\omega)$ is spectral intensity (power per area per frequency interval), then the rate of transitions P is related to $\sigma_{12}(\omega)$ by:

$$P = \frac{\sigma_{12}(\omega) I(\omega)}{\hbar\omega}$$

This equation is given on the slide. Let's check the units. $\sigma_{12}(\omega)$ has units of area (e.g., m^2). $I(\omega)$ is intensity (e.g., $\mathrm{W/m^2}$ or $\mathrm{J\ s^{-1}m^{-2}}$). $\hbar\omega$ is energy (J). So, P has units of $m^2 \times (\mathrm{J\ s^{-1}m^{-2}})/\mathrm{J} = s^{-1}$. This is correct; P is a rate.

Here, $I(\omega)$ is typically taken as the intensity of the laser at frequency ω, integrated over its narrow bandwidth if it's considered quasi- monochromatic, or it could be a spectral intensity if $\sigma_{12}(\omega)$ is appropriately defined. Usually, for a laser interacting with a narrow transition, $I(\omega)$ is simply the total intensity I of the laser if its frequency ω is near resonance, and $\sigma_{12}(\omega)$ is the cross- section at that frequency. So, $P = \frac{\sigma(\omega)I}{\hbar\omega}$.

Now, we can substitute this expression for P into our formula for S, particularly for the special case we just discussed where $R_1 = 0$ and $R_2 = A_{21}$ (spontaneous emission only). In that case, $S = \frac{2P}{A_{21}}$. Substituting P:

$$S = \frac{2\,\sigma_{12}(\omega)I(\omega)}{\hbar\omega A_{21}}$$

This equation is also given on the slide, and it's extremely useful. It directly relates the saturation parameter S to: • $\sigma_{12}(\omega)$: the absorption cross- section at the laser frequency ω (a property of the atom). • $I(\omega)$: the intensity of the laser (an experimental parameter we control). • $\hbar\omega$: the photon energy (defined by the laser frequency). • A_{21}: the spontaneous emission rate (a property of the atom).

This allows us to calculate S if we know the atomic parameters and the laser intensity. Often, we define a "saturation intensity" $I_{\text{sat}}(\omega)$ such that $S = \frac{I(\omega)}{I_{\text{sat}}(\omega)}$. Comparing with the formula for S, we can identify:

$$I_{\text{sat}}(\omega) = \frac{\hbar\omega A_{21}}{2\,\sigma_{12}(\omega)}$$

The saturation intensity I_{sat} is the intensity at which the saturation parameter S becomes equal to 1. It's a characteristic intensity for the transition. If your laser intensity I is much less than I_{sat}, you are in the weak field regime ($S \ll 1$). If I is much greater than I_{sat}, you are in the strong saturation regime ($S \gg 1$).

This connection to intensity and cross- section makes the concept of saturation very practical for experimentalists.

Page 20

This page presents a graph titled **"Saturation Behavior of a Two-Level System."** This graph visually illustrates the key equation $\Delta N = \frac{\Delta N_0}{1+S}$ that we derived.

Let's describe the graph:

- The **horizontal axis** is the **Saturation Parameter (**S), plotted on a logarithmic scale ranging from 10^{-2} (zero point zero one) to 10^{2} (one hundred). This wide range allows us to see the behavior from weak saturation to strong saturation.

- The **vertical axis** is the **Normalized Population Difference** $\left(\frac{\Delta N}{\Delta N_0}\right)$. This quantity ranges from 0 to 1. When $S = 0$, $\frac{\Delta N}{\Delta N_0} = 1$, meaning the population difference is at its unsaturated value. As S increases, $\frac{\Delta N}{\Delta N_0}$ decreases.

- A **blue curve** shows the relationship: $\frac{\Delta N}{\Delta N_0} = \frac{1}{1+S}$.

Let's trace the curve:

- At very small S (e.g., $S = 10^{-2}$), the value of $\frac{\Delta N}{\Delta N_0}$ is very close to 1. For example, if $S = 0.01$, then $\frac{\Delta N}{\Delta N_0} = \frac{1}{1.01} \approx 0.99$. This confirms that for weak pumping, the population difference is hardly affected.

- As S increases, $\frac{\Delta N}{\Delta N_0}$ starts to drop.

- A **key point** is highlighted on the graph with a red dot: $\left(S = 1, \frac{\Delta N}{\Delta N_0} = 0.5\right)$. This is exactly what the formula predicts: when $S = 1$, $\frac{\Delta N}{\Delta N_0} = \frac{1}{1+1} = \frac{1}{2} = 0.5$. So, when the saturation parameter is unity, the population difference is reduced to half of its unsaturated value. This is often taken as a practical definition of the onset of significant saturation.

- As S increases further beyond 1 (e.g., $S = 10$), $\frac{\Delta N}{\Delta N_0}$ drops significantly. For $S = 10$, $\frac{\Delta N}{\Delta N_0} = \frac{1}{1+10} = \frac{1}{11} \approx 0.09$.

- For $S = 100$, $\frac{\Delta N}{\Delta N_0} = \frac{1}{1+100} = \frac{1}{101} \approx 0.01$. As S approaches infinity, $\frac{\Delta N}{\Delta N_0}$ approaches zero, meaning complete saturation where $N_1 \approx N_2$.

This graph beautifully visualizes how the effectiveness of the medium as an absorber (which depends on ΔN) diminishes as the saturation parameter S (which depends on laser intensity) increases. It's a smooth transition from the unsaturated regime ($S \ll 1$) to the heavily saturated regime ($S \gg 1$), with a characteristic turning point around $S = 1$.

Understanding this curve is fundamental to interpreting experiments involving potentially saturating laser beams. If you measure an absorption signal, its strength is directly tied to this ΔN. If you increase your laser power, you are increasing S, and you will move along this curve to the right, observing a decrease in the normalized population difference and thus a decrease in the fractional absorption.

Page 21:

Now we connect the saturation of the population difference to the macroscopic absorption coefficient. This is **"Slide 7: Absorption Coefficient under Saturation."**

The first bullet point reminds us of the **"Microscopic absorption cross section** $\sigma_{12}(\omega)$ (units: m^2)." This, as we discussed, is an atomic property, representing the effective area for absorption at frequency ω.

The second bullet defines the **"Macroscopic coefficient"**, which is the linear absorption coefficient $\alpha(\omega)$ (alpha of omega). This is what typically appears in Beer-Lambert's law $\left(\frac{dI}{dz} = -\alpha I\right)$.

The relationship is given as:

$$\alpha(\omega) = \sigma_{12}(\omega)\Delta N$$

* $\alpha(\omega)$ is the absorption coefficient, usually in units of per meter (m^{-1}) or per centimeter (cm^{-1}). * $\sigma_{12}(\omega)$ is the cross-section (m^2). * ΔN is the population difference $(N_1 - N_2)$ in units of number per unit volume (m^{-3}).

So the units are m^2 × m^{-3} = m^{-1}, which is correct for $\alpha(\omega)$.

This equation states that the macroscopic absorption coefficient is simply the microscopic cross-section multiplied by the density of net absorbers $(N_1 - N_2)$. This makes intuitive sense.

Now, the third bullet point: **"Insert** ΔN expression:"

We know that in the presence of a saturating field,

$$\Delta N = \frac{\Delta N_0}{1 + S}$$

Substituting this into the expression for $\alpha(\omega)$:

$$\alpha(\omega) = \sigma_{12}(\omega)\left[\frac{\Delta N_0}{1 + S}\right]$$

We can group $\sigma_{12}(\omega)\Delta N_0$ together. What is this? It's the absorption coefficient we would have if the population difference were its unsaturated value ΔN_0. Let's call this $\alpha_0(\omega)$ (alpha sub zero of omega), the unsaturated absorption coefficient. So,

$$\alpha_0(\omega) = \sigma_{12}(\omega)\Delta N_0$$

With this definition, the saturated absorption coefficient $\alpha(\omega)$ becomes:

$$\alpha(\omega) = \frac{\alpha_0(\omega)}{1+S}$$

This fundamental equation is shown on the slide. It tells us that the absorption coefficient itself is reduced from its unsaturated value by the same factor $(1+S)$ that reduces the population difference.

Here, S is the on-resonance saturation parameter if $\alpha_0(\omega)$ is the peak unsaturated absorption coefficient. More generally, if we are considering off-resonance effects, S itself can become frequency dependent, $S(\omega)$, as we will see later. For now, if we are at a specific frequency ω (often the line center), S is the saturation parameter at that frequency.

The last bullet point confirms our definition: * **"**$\alpha_0(\omega) = \sigma_{12}(\omega)\Delta N_0$ is the unsaturated value."

This result is extremely important. It directly predicts how the measured absorption of a material will decrease as you increase the intensity of the probing laser beam (because S depends on intensity). This is the phenomenon of "saturable absorption," and materials exhibiting this are called "saturable absorbers," which have many applications, for example, in laser Q-switching and mode-locking. For spectroscopy, it means our signal strength (absorption) doesn't just scale with N, it's modified by this $(1+S)$ factor.

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This page discusses the **"Consequences"** of the formula

$$\alpha(\omega) = \frac{\alpha_0(\omega)}{1+S}$$

for the absorption coefficient under saturation.

The first consequence highlighted is: * **"Peak absorption diminishes as $\frac{1}{1+S}$."** This directly follows from the formula. If $\alpha_0(\omega)$ is the peak unsaturated absorption coefficient (i.e., at the line center), then the actual peak absorption $\alpha(\omega)$ is reduced by this factor $(1 + S)$. As S increases (due to higher laser intensity), the peak absorption gets smaller and smaller. For example, when $S = 1$, the peak absorption is halved. When S is very large, the peak absorption approaches zero.

This leads to the second point, which is the extreme limit: * **"In the limit** $S \to \infty$, $\alpha \to 0$: complete bleaching." (S approaches infinity, α approaches zero\). When the saturation parameter S becomes very, very large (e.g., due to extremely high laser intensity), the denominator $(1 + S)$ also becomes very large. Therefore, $\alpha(\omega) = \frac{\alpha_0(\omega)}{1+S}$ approaches zero. This means that the material effectively stops absorbing light at that frequency. The medium has become transparent, or "bleached." This is the ultimate effect of saturation: you can drive the population difference ΔN so close to zero that there's virtually no net absorption left. The term "bleaching" is very descriptive; it's like the color (absorption) has been removed from the material by the intense light. This is exploited in techniques like saturated absorption spectroscopy, where a strong pump beam bleaches a path for a probe beam, but it's also a caution: if you're trying to measure absorption, and your laser is too strong, you might be significantly underestimating the true (unsaturated) absorption coefficient because you're already in the saturation regime.

These consequences are direct and experimentally verifiable. The reduction in absorption is one of the primary signatures of saturation. The next step will be to consider how this affects the *lineshape*, not just the peak height.

Page 23:

This page provides another excellent visualization, titled **"Slide 7: Absorption Coefficient under Saturation"** (presumably a continuation or detailed view related to the previous Slide 7).

The graph shows the **"Absorption Coefficient (α_{S})** and Lower State Population ($\frac{N_1}{N_{\text{total}}}$) vs. Saturation Parameter (S)". (Here α_{S} is what we called $\alpha(\omega)$ or α).

Let's describe the graph:

* The **horizontal axis** is the **Saturation Parameter (**S), plotted on a linear scale this time, ranging from 0 to 15.

* The **vertical axis** is **Normalized Value**, ranging from 0.0 to 1.0.

* There are two curves plotted: 1. A **blue curve** representing the normalized absorption coefficient: $\frac{\alpha(S)}{\alpha_0} = \frac{1}{1+S}$. * At $S = 0$, $\frac{\alpha(S)}{\alpha_0} = 1$ (unsaturated absorption). * As S increases, this curve drops sharply at first, then more gradually. For example, at $S = 1$, it's 0.5. At $S = 3$, it's $\frac{1}{4} = 0.25$. At $S = 9$, it's $\frac{1}{10} = 0.1$. * An annotation points to the tail of this blue curve: **"**$\alpha/\alpha_0 \rightarrow 0$ (Full Transparency / Complete Bleaching)" as S gets large. This visually reinforces the concept of bleaching. 2. A **red curve** representing the normalized lower state population: $\frac{N_1(S)}{N_{\text{total}}} = \frac{S+2}{2(S+1)}$. * Let's verify this formula for $\frac{N_1}{N_{\text{total}}}$. We had

$$N_1 = N \cdot \frac{(P + R_2)}{(2\,P + R_1 + R_2)}.$$

And

$$S = \frac{2\,P}{R_1 + R_2}.$$

This means

$$P = \frac{S(R_1 + R_2)}{2}.$$

Let's consider the simple case where $R_1 = 0$ (no $1 \to 2$ incoherent pump) and R_2 is the relaxation rate from 2 to 1 (e.g., A_{21}). Then $\Delta N_0 = N_{\text{total}}$. * Our previous N_1 was

$$N \cdot \frac{(P + R_2)}{(2\,P + R_1 + R_2)}.$$

Let's assume the N_{total} here refers to N. * If $R_1 = 0$, then

$$\frac{N_1}{N} = \frac{(P + R_2)}{(2\,P + R_2)}.$$

And

$$S = \frac{2\,P}{R_2}.$$

So

$$P = \frac{SR_2}{2}.$$

Substituting P:

$$\frac{N_1}{N} = \frac{\frac{SR_2}{2} + R_2}{SR_2 + R_2} = \frac{R_2\left(\frac{S}{2} + 1\right)}{R_2(S + 1)} = \frac{\frac{S}{2} + 1}{S + 1} = \frac{S + 2}{2(S + 1)}.$$

This formula is correct for the case $R_1 = 0$. * Let's see how $\frac{N_1}{N}$ behaves: * At $S = 0$ (no pump P),

$$\frac{N_1}{N} = \frac{0 + 2}{2(0 + 1)} = \frac{2}{2} = 1.$$

This means all population is in the lower state, $N_1 = N_{\text{total}}$, which makes sense if $R_1 = 0$ (no way to get to N_2 without P) and we assume this is the initial state before P is applied. * As $S \to \infty$ (very strong pump),

$$\frac{N_1}{N} \to \frac{S}{2\,S} = \frac{1}{2}.$$

This means the lower state population approaches $\frac{N_{\text{total}}}{2}$, which is consistent with $N_1 = N_2 = \frac{N_{\text{total}}}{2}$ in the limit of strong saturation. * The red curve starts at 1 for $S = 0$ and decreases, approaching an asymptote at 0.5 for large S. A dashed horizontal line is drawn at a normalized value of 0.5, and an annotation for the red curve says "$\frac{N_1}{N_{\text{total}}} \to 0.5$".

What does this graph tell us?

The blue curve shows how the absorption coefficient plummets with increasing S. The red curve shows that the lower state population N_1 also decreases as S increases, but it doesn't go to zero. It goes to $\frac{N_{\text{total}}}{2}$. This is the source of the saturation: the "fuel" for absorption (atoms in N_1) is being depleted, but not entirely; it's being shared equally with N_2.

The fact that N_1 only drops to $0.5\,N_{\text{total}}$ while $\frac{\alpha(S)}{\alpha_0}$ drops all the way to 0 emphasizes that absorption depends on the *difference*

$$\Delta N = N_1 - N_2,$$

not just on N_1. When $N_1 = N_2 = \frac{N_{\text{total}}}{2}$, ΔN is zero, hence α is zero, even though N_1 is still substantial.

Page 24:

Now we make a crucial **"Conceptual Jump"** from just considering saturation of population at a single frequency to how this affects the entire spectral line shape. This is **"Slide 8: Saturation Broadening – Conceptual Jump."**

The first bullet point sets the context:

"So far we treated a single frequency at a time. Real lasers have finite spectral width."

Actually, the more important point here is that even if the laser is perfectly monochromatic, an atomic transition has a lineshape (e.g., a Lorentzian profile) meaning it interacts with light over a range of frequencies around the line center ω_0. When we talked about P, $\sigma(\omega)$, S, etc., these quantities can be frequency-dependent if the laser frequency ω is detuned from the line center ω_0. The saturation parameter S itself will be largest at resonance and smaller in the wings.

This leads to the second point:

"Saturation is strongest exactly at ω_0 (line centre) because the induced rate is highest there."

The interaction strength (and thus the absorption cross-section $\sigma(\omega)$ and the stimulated rate $P(\omega)$) is typically peaked at the resonant frequency ω_0. Therefore, the saturation parameter S will also be largest at ω_0. If we denote S_0 as the saturation parameter exactly on resonance, then $S(\omega)$ will be S_0 times some lineshape factor that decreases as $|\omega - \omega_0|$ increases.

The third point describes what happens in the wings:

"Wings of the Lorentzian are less saturated (smaller S_ω)."

(S sub omega, meaning S as a function of frequency ω). As we move away from the line center ω_0 into the wings of the Lorentzian profile, the atomic response (the cross-section) decreases. For a given laser intensity, this means the stimulated rate $P(\omega)$ will be smaller, and consequently, the frequency-dependent saturation parameter S_ω will be smaller than S_0. So, the center of the line is heavily saturated, while the wings are less saturated.

The final bullet point gives the **"Result"**:

"central part flattens while the area (integrated oscillator strength) stays conserved → line broadens outward."

This is the heart of power broadening. Because the center of the line is more saturated, its absorption is reduced more significantly than the absorption in the wings. This causes the peak of the spectral line to "flatten" or "dip." However, the "integrated oscillator strength" of the transition, which is related to the total area under the absorption curve $\int \alpha(\omega)\, d\omega$, is a fundamental atomic property and should remain (approximately) conserved, even under saturation. So, if the peak height of the absorption profile decreases due to saturation, but the total area under the curve must be conserved, then the width of the profile *must increase* to compensate. The line effectively "broadens outward." The energy that would have been absorbed at the peak (if it weren't saturated) is now effectively redistributed over a wider frequency range in the wings.

This conceptual jump is vital. Saturation doesn't just reduce the overall absorption; it preferentially reduces it at the line center, leading to a change in the lineshape – specifically, a broadening. We will now formalize this.

Page 25:

Now we're ready to look at the mathematics of this frequency-dependent saturation and the resulting broadening. This is **"Slide 9: Lorentzian Profile & Frequency-Dependent Saturation."**

First, let's define the **"Unsaturated homogeneous line (natural plus collisional)"**: We assume that in the absence of a strong saturating field (or in the limit of very weak probe light), our homogeneously broadened transition has a Lorentzian lineshape. This is given by:

$$L(\omega - \omega_0) = \frac{\left(\frac{\gamma}{2}\right)^2}{(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}.$$

Let's break this down:

- $L(\omega - \omega_0)$ is the Lorentzian lineshape function. It's a function of the detuning $(\omega - \omega_0)$ from the line center frequency ω_0.
- ω (omega) is the variable frequency (e.g., of our laser).
- ω_0 (omega sub zero) is the resonant frequency of the atomic transition (the line center).
- γ (gamma) is the **"full width at half maximum (FWHM)"** of this unsaturated Lorentzian profile. This γ is the homogeneous linewidth we discussed earlier, arising from natural lifetime and collisional broadening.
- The term $\frac{\gamma}{2}$ is therefore the half-width at half-maximum (HWHM).
- The Lorentzian is normalized such that $L(\omega = \omega_0) = 1$ (i.e., at the peak, $(\omega - \omega_0) = 0$, so $L = \frac{\left(\frac{\gamma}{2}\right)^2}{\left(\frac{\gamma}{2}\right)^2} = 1$). If this L is multiplying a peak absorption coefficient α_0, then $\alpha(\omega) = \alpha_0 \cdot L(\omega - \omega_0)$.

The second bullet point reiterates:

- **"**γ: full width at half maximum (FWHM)."

Now, we introduce the concept of a **"Local saturation parameter"**: Since the interaction strength depends on frequency (via the Lorentzian $L(\omega - \omega_0)$), the saturation parameter itself will be frequency-dependent. Let's call this S_ω (S sub omega). The absorption cross-section $\sigma(\omega)$ can be written as $\sigma_0 \cdot L(\omega - \omega_0)$, where σ_0 is the peak cross-section at ω_0. The pump rate $P(\omega)$ would then be $P_0 \cdot L(\omega - \omega_0)$, where P_0 is the pump rate at resonance.

Our original saturation parameter S (which we should now perhaps call S_0, the on-resonance saturation parameter) was defined as

$$S_0 = \frac{2\,P_0}{R_1 + R_2}.$$

The frequency-dependent saturation parameter S_ω will then be:

$$S_\omega = \frac{2\,P(\omega)}{R_1 + R_2} = \left[\frac{2\,P_0}{R_1 + R_2}\right] L(\omega - \omega_0).$$

So, $S_\omega = S_0 \cdot L(\omega - \omega_0)$.

This is what we'll see on the next page. The "local" saturation parameter S_ω depends on how far the current frequency ω is from resonance, through the Lorentzian factor.

Page 26:

This page continues the development of the frequency-dependent saturation parameter S_ω.

First, the equation for S_ω is given, based on our earlier definitions:

$$S_\omega = \frac{B_{12}\,\rho(\omega)}{\bar{R}}\,L(\omega - \omega_0)$$

Let's analyze this:

- S_ω (S sub omega) is the saturation parameter at a given frequency ω.
- $B_{12}\,\rho(\omega)$ is our stimulated rate $P(\omega)$ (assuming $\rho(\omega)$ is the spectral energy density of the pump at frequency ω, which is driving the saturation). More accurately,

$$P(\omega) = B_{12}\,\rho_{\text{pump}}\,L(\omega - \omega_0)$$

if ρ_{pump} is broadband, or if ρ_{pump} is monochromatic at ω, then

$$P(\omega) = B_{12}\,\rho_{\text{pump}}(\omega).$$

The slide seems to imply that $B_{12}\,\rho(\omega)$ here is the *on-resonance* stimulated rate P_0 if $L(\omega - \omega_0)$ is also present.

- $\bar{R}$ (R-bar) is the mean relaxation rate,

$$\bar{R} = \frac{R_1 + R_2}{2},$$

that we defined earlier such that the on-resonance saturation parameter $S_0 = \frac{P_0}{\bar{R}}$.

- $L(\omega - \omega_0)$ is the unsaturated Lorentzian lineshape function.

So, if

$$P_0 = B_{12}\,\rho(\omega_0)$$

is the stimulated rate exactly at resonance ($\omega = \omega_0$), then the term $\frac{B_{12}\,\rho(\omega)}{\bar{R}}$ should probably be $S_0 = \frac{P_0}{\bar{R}}$. The equation on the slide:

$$S_\omega = \frac{B_{12}\,\rho(\omega)}{\bar{R}}\,L(\omega - \omega_0)$$

This looks like it could be

$$S_\omega = S_0\,L(\omega - \omega_0)$$

if we define S_0 appropriately.

Let's assume $B_{12}\,\rho(\omega)$ refers to the on-resonance value of the pump interaction strength, P_0. So,

$$S_\omega = \frac{P_0}{\bar{R}}\,L(\omega - \omega_0).$$

Define peak value:

$S_0 = S_{\omega_0}$ (S sub zero equals S sub omega-sub-zero). This means S_0 is the saturation parameter evaluated exactly at the line center, $\omega = \omega_0$. Since

$$L(\omega_0 - \omega_0) = L(0) = 1,$$

then from our assumed formula

$$S_\omega = \frac{P_0}{\bar{R}}\, L(\omega - \omega_0),$$

we get:

$$S_0 = \frac{P_0}{\bar{R}}\, 1 = \frac{P_0}{\bar{R}}.$$

This is consistent. S_0 is the on-resonance saturation parameter we've been using, where P_0 is the stimulated transition rate at resonance.

Now, **"Frequency variation therefore reads"**:

Substituting $S_0 = \frac{P_0}{\bar{R}}$ back into

$$S_\omega = \frac{P_0}{\bar{R}}\, L(\omega - \omega_0),$$

we get:

$$S_\omega = S_0\, L(\omega - \omega_0).$$

And plugging in the formula for $L(\omega - \omega_0)$:

$$S_\omega = S_0 \left[\frac{\left(\frac{\gamma}{2}\right)^2}{(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}\right].$$

This equation is clearly shown on the slide and is exactly what we expected. It shows that the local saturation parameter S_ω is maximum (equal to S_0) at the line center ($\omega = \omega_0$) and falls off as a Lorentzian as we detune from resonance.

The final point is an important consequence:

- $S_\omega \to 0$ as $|\omega - \omega_0| \to \infty$. ($S_\omega$ approaches zero as the absolute value of $(\omega - \omega_0)$ approaches infinity). This means that far out in the wings of the line, where the detuning $|\omega - \omega_0|$ is very large, the Lorentzian factor $L(\omega - \omega_0)$ becomes very small. Consequently, S_ω also becomes very small, regardless of how large S_0 (the on-resonance saturation) is. This implies that far from resonance, the medium is essentially unsaturated. Saturation is a phenomenon that is primarily active near the line center. This differential saturation across the profile is what leads to power broadening.

Page 27:

Slide 10: Deriving the Broadened Linewidth γ_s

Now we reach **"Slide 10: Deriving the Broadened Linewidth γ_s**." This is where we see how the linewidth is affected.

The first bullet point considers the **"Power absorbed per unit volume per unit frequency"**: Let $\frac{dW_{12}(\omega)}{dt}$ represent this quantity. Each absorption event takes an energy $\hbar\omega$ from the field. The net rate of absorptions per unit volume is $P(\omega)\Delta N$, if $P(\omega)$ is $B_{12}\rho(\omega)$ where ΔN is $N_1 - N_2$. However, it's often written in terms of the population difference ΔN and the total relaxation rate of the population difference.

The slide gives the expression:

$$\frac{dW_{12}(\omega)}{dt} = \hbar\omega\,\overline{R}\left[\frac{\Delta N_0}{1+S_\omega}\right]$$

Let's try to understand this. We know

$$\Delta N(\omega) = \frac{\Delta N_0}{1+S_\omega},$$

which is the population difference at frequency ω. The rate of energy absorption is proportional to this population difference. Why $\hbar\omega\,\overline{R}$? $\overline{R}$ is defined as

$$\overline{R} = \frac{R_1 + R_2}{2}.$$

Let's consider the total power absorbed $P_{abs}(\omega)$. It should be $\hbar\omega$ times (rate of upward transitions minus rate of downward stimulated transitions). Rate of upward transitions from 1 to 2 = $P(\omega)N_1(\omega)$. Rate of downward stimulated transitions from 2 to 1 = $P(\omega)N_2(\omega)$ (assuming $B_{12} = B_{21}$ for equal degeneracies). Net rate of stimulated photon absorption per unit volume = $P(\omega)\big(N_1(\omega) - N_2(\omega)\big) = P(\omega)\Delta N(\omega)$. So, power absorbed per unit volume = $\hbar\omega\, P(\omega)\Delta N(\omega)$.

We know $P(\omega) = S_\omega\,\overline{R}$ (from $S_\omega = P(\omega)/\overline{R}$). So,

$$\frac{dW_{12}(\omega)}{dt} = \hbar\omega\,\left(S_\omega\,\overline{R}\right)\,\left[\frac{\Delta N_0}{1+S_\omega}\right].$$

This is

$$\frac{dW_{12}(\omega)}{dt} = \hbar\omega\,\overline{R}\,\Delta N_0\,\left[\frac{S_\omega}{1+S_\omega}\right].$$

The formula on the slide is missing an S_ω in the numerator compared to my derivation.

Let's re-evaluate. The power absorbed is proportional to $\alpha(\omega)$. And

$$\alpha(\omega) = \frac{\alpha_0(\omega)}{1+S_\omega(\omega)}$$

if S is defined locally, and

$$\alpha_0(\omega) = \sigma_0\,L(\omega-\omega_0)\,\Delta N_0.$$

Power absorbed is related to $\alpha(\omega)I$. Perhaps $\frac{dW_{12}(\omega)}{dt}$ refers to something slightly different, maybe related to the rate of cycling through the system.

Let's assume the slide's first equation is a given starting point, which might come from a more detailed derivation perhaps involving the total rate at which atoms are processed by the field and relaxation. The quantity

$$\frac{\Delta N_0}{1+S_\omega}$$

is indeed $\Delta N(\omega)$. So it's $\hbar\omega\,\overline{R}\,\Delta N(\omega)$. This term is proportional to the population difference.

Now, **"Substitute** S_ω expression": We have

$$S_\omega = S_0\,\frac{(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma/2)^2}.$$

Substituting this into the denominator of the slide's first equation:

$$1+S_\omega = 1+S_0\,\frac{(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma/2)^2},$$

$$1+S_\omega = \frac{(\omega-\omega_0)^2+(\gamma/2)^2+S_0(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma/2)^2},$$

$$1+S_\omega = \frac{(\omega-\omega_0)^2+(\gamma/2)^2(1+S_0)}{(\omega-\omega_0)^2+(\gamma/2)^2}.$$

So,

$$\frac{\Delta N_0}{1+S_\omega} = \Delta N_0\,\frac{(\omega-\omega_0)^2+(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma/2)^2(1+S_0)}.$$

If we use the slide's formula for

$$\frac{dW_{12}(\omega)}{dt} = \hbar\omega\,\overline{R}\,\frac{\Delta N_0}{1+S_\omega},$$

then:

$$\frac{dW_{12}(\omega)}{dt} = \hbar\omega\,\overline{R}\,\Delta N_0\,\frac{(\omega-\omega_0)^2+(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma/2)^2(1+S_0)}.$$

This does not look like the expression on the slide which is:

$$\frac{dW_{12}(\omega)}{dt} = \hbar\omega\,\overline{R}\,\Delta N_0\,\frac{S_0(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma/2)^2(1+S_0)}.$$

(A mistake in my transcription of slide? No, the slide has $S_0(\gamma/2)^2$ in the numerator).

Ah, I see the likely source of my confusion. The first equation on the slide must be for $\frac{dW_{12}(\omega)}{dt}$ integrated over the line, or it's a particular definition.

Let's assume the *second* equation on the slide is the correct expression for the spectral power absorption, possibly derived from $\hbar\omega\,P(\omega)\Delta N(\omega)$. $P(\omega) = P_0\,L(\omega-\omega_0) = P_0\,\frac{(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma/2)^2}$. $\Delta N(\omega) = \frac{\Delta N_0}{1+S_\omega} = \frac{\Delta N_0}{1+S_0\,L(\omega-\omega_0)}$. So,

$$\hbar\omega\,P(\omega)\Delta N(\omega) = \hbar\omega\,P_0\,L(\omega-\omega_0)\,\frac{\Delta N_0}{1+S_0\,L(\omega-\omega_0)}.$$

We know $P_0 = S_0\,\overline{R}$, so:

$$\hbar\omega\,P(\omega)\Delta N(\omega) = \hbar\omega\,S_0\,\overline{R}\,\Delta N_0\,\frac{L(\omega-\omega_0)}{1+S_0\,L(\omega-\omega_0)},$$

$$= \hbar\omega\,S_0\,\overline{R}\,\Delta N_0\left\{\frac{\frac{(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma/2)^2}}{1+\frac{S_0(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma/2)^2}}\right\},$$

$$= \hbar\omega\,S_0\,\overline{R}\,\Delta N_0\left[\frac{(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma/2)^2+S_0(\gamma/2)^2}\right],$$

$$= \hbar\omega\,S_0\,\overline{R}\,\Delta N_0\,\frac{(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma/2)^2(1+S_0)}.$$

This matches the second equation on the slide perfectly!

So, the first equation on the slide,

$$\frac{dW_{12}(\omega)}{dt} = \hbar\omega\,\overline{R}\,\frac{\Delta N_0}{1+S_\omega},$$

seems to be a conceptual intermediary, but the *actual spectral distribution* of power absorption is given by the second, more complex formula, which is correctly derived from $\hbar\omega\, P(\omega)\Delta N(\omega)$. This second formula is what we need to analyze for the lineshape.

Page 28:

Now we analyze the denominator of the expression for the power absorbed per unit volume per unit frequency, which we found to be:

$$(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2 (1 + S_0)$$

The first bullet point on this page states: * **"The denominator has exactly the structure of a Lorentzian with an enlarged half-width."** Let's recall the denominator of a standard Lorentzian: $(\omega - \omega_0)^2 +$ (HWHM)^2. In our case, the denominator is $(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2 (1 + S_0)$. We can rewrite $\left(\frac{\gamma}{2}\right)^2 (1 + S_0)$ as $\left[\frac{\gamma}{2}\sqrt{1 + S_0}\right]^2$. So, if we define a new, "saturated" half-width at half-maximum, let's call it $\frac{\gamma_\mathrm{s}}{2}$, as:

$$\frac{\gamma_\mathrm{s}}{2} = \frac{\gamma}{2}\sqrt{1 + S_0}$$

Then the denominator becomes $(\omega - \omega_0)^2 + \left(\frac{\gamma_\mathrm{s}}{2}\right)^2$. This is indeed the denominator of a new Lorentzian profile, but with a modified HWHM.

This leads directly to the definition of the broadened FWHM, γ_s:

$\gamma_\mathrm{s} = \gamma\sqrt{1 + S_0}$ (gamma sub s equals gamma times the square root of (one plus S sub zero)).

This is a very important result! It shows that the full width at half maximum of the absorption profile, γ_s, is increased from the original unsaturated width γ by a factor of $\sqrt{1 + S_0}$. This is **power broadening** (or saturation broadening). The stronger the on-resonance saturation S_0 (which depends on laser intensity), the larger the broadening factor $\sqrt{1+S_0}$, and thus the wider the observed spectral line.

Now for the **"Physical meaning"**:

* **"At** $S_0 = 1$ the width grows by $\sqrt{2}$." If $S_0 = 1$ (when the on-resonance pump rate P_0 equals the mean relaxation rate $\bar{R}$, or when the intensity I equals I_{sat}), then:

$$\gamma_{\text{s}} = \gamma\sqrt{1+1} = \gamma\sqrt{2}$$

So, when the saturation parameter is 1, the linewidth is increased by a factor of the square root of 2 (approximately 1.414). This is a significant broadening.

* **"Scaling is** $\propto \sqrt{P}$ for large P because $S_0 \propto P$." (Scaling is proportional to the square root of P, for large P, because S_0 is proportional to P). We know that $S_0 = \frac{P_0}{\bar{R}}$, and P_0 (the on-resonance stimulated rate) is proportional to the laser intensity I, or power P (if beam area is constant). Let's use I for intensity. So $S_0 \propto I$. For large S_0 (i.e., strong saturation, high intensity), the term $1 + S_0 \approx S_0$. So, $\gamma_{\text{s}} = \gamma\sqrt{1+S_0} \approx \gamma\sqrt{S_0}$. Since $S_0 \propto I$, then $\gamma_{\text{s}} \propto \sqrt{I}$. This means that in the highly saturated regime, the observed linewidth γ_{s} increases as the square root of the incident laser intensity. This is a key signature of power broadening. If you measure linewidth as a function of laser power and see it growing as $\sqrt{P}$, you are likely observing power broadening.

The triple dash indicates a pause. This formula $\gamma_{\text{s}} = \gamma\sqrt{1+S_0}$ is the quantitative expression for power broadening in a homogeneously broadened two-level system.

Page 29:

This slide, **"Slide 11: Absorption Profile in the Saturated Regime,"** summarizes the form of the absorption coefficient incorporating power broadening.

The first bullet gives the **"Saturated coefficient"** $\alpha_\mathrm{s}(\omega)$ (alpha sub s of omega):

$$\alpha_\mathrm{s}(\omega) = \alpha_0(\omega_0) \cdot \frac{\left(\frac{\gamma}{2}\right)^2}{(\omega - \omega_0)^2 + \left(\frac{\gamma_\mathrm{s}}{2}\right)^2}.$$

Let's analyze this expression:

$\alpha_\mathrm{s}(\omega)$ is the absorption coefficient at frequency ω in the saturated regime.

$\alpha_0(\omega_0)$ is the peak unsaturated absorption coefficient (i.e., at $\omega = \omega_0$, and $S_0 = 0$).

The term in the square brackets is a Lorentzian lineshape function, but notice the denominator:

- – $(\omega - \omega_0)^2$ is the squared detuning from resonance.
- – $\left(\frac{\gamma_\mathrm{s}}{2}\right)^2$ is the square of the *saturated* half-width at half-maximum.

  We know $\gamma_\mathrm{s} = \gamma\sqrt{1 + S_0}$. So $\left(\frac{\gamma_\mathrm{s}}{2}\right)^2 = \left(\frac{\gamma}{2}\right)^2 (1 + S_0)$.

This formula looks like a standard Lorentzian profile with a peak height that needs careful consideration and a width γ_s.

However, the peak height of this specific Lorentzian form would be $\alpha_0(\omega_0) \cdot \left[\frac{(\gamma/2)^2}{(\gamma_\mathrm{s}/2)^2}\right]$ if the numerator were $\left(\frac{\gamma_\mathrm{s}}{2}\right)^2$.

Peak height of $\alpha_\mathrm{s}(\omega)$ (at $\omega = \omega_0$) from this formula is

$$\alpha_0(\omega_0) \cdot \frac{\left(\frac{\gamma}{2}\right)^2}{\left(\frac{\gamma_s}{2}\right)^2}.$$

Substituting $\left(\frac{\gamma_s}{2}\right)^2 = \left(\frac{\gamma}{2}\right)^2 (1 + S_0)$:

$$\text{Peak height} = \frac{\alpha_0(\omega_0) \cdot \left(\frac{\gamma}{2}\right)^2}{\left(\frac{\gamma}{2}\right)^2 (1 + S_0)} = \frac{\alpha_0(\omega_0)}{1 + S_0}.$$

This is consistent with what we found earlier: the peak absorption is reduced by $(1 + S_0)$.

So, the saturated absorption profile is a Lorentzian, but it's broader (width γ_s) and its peak height is reduced (by a factor $1/(1 + S_0)$ compared to the *unsaturated peak* $\alpha_0(\omega_0)$).

The expression can be written as:

$$\alpha_s(\omega) = \frac{\alpha_0(\omega_0)}{1 + S_0} \cdot \frac{\left(\frac{\gamma_s}{2}\right)^2}{(\omega - \omega_0)^2 + \left(\frac{\gamma_s}{2}\right)^2} \cdot \frac{\left(\frac{\gamma}{2}\right)^2}{\left(\frac{\gamma_s}{2}\right)^2 \cdot (1 + S_0)}$$

? No.

Let me rewrite the slide's first equation using γ_s:

$$\alpha_s(\omega) = \alpha_0(\omega_0) \cdot \frac{\left(\frac{\gamma}{2}\right)^2}{(\omega - \omega_0)^2 + \left(\frac{\gamma_s}{2}\right)^2}.$$

This is a Lorentzian with HWHM $= \frac{\gamma}{2}\sqrt{1 + S_0} = \frac{\gamma_s}{2}$.

And its peak value (at $\omega = \omega_0$) is

$$\alpha_0(\omega_0) \cdot \frac{\left(\frac{\gamma}{2}\right)^2}{\left(\frac{\gamma_\mathrm{s}}{2}\right)^2}.$$

So,

$$\alpha_\mathrm{s}(\omega) = \frac{\alpha_0(\omega_0)}{1 + S_0} \cdot \frac{\left(\frac{\gamma_\mathrm{s}}{2}\right)^2}{(\omega - \omega_0)^2 + \left(\frac{\gamma_\mathrm{s}}{2}\right)^2}.$$

The first equation on the slide can be written as:

$$\alpha_\mathrm{s}(\omega) = \frac{\alpha_0(\omega_0)}{1 + S_0} \cdot \frac{\left(\frac{\gamma_\mathrm{s}}{2}\right)^2}{(\omega - \omega_0)^2 + \left(\frac{\gamma_\mathrm{s}}{2}\right)^2} \cdot \frac{\left(\frac{\gamma}{2}\right)^2}{\left(\frac{\gamma_\mathrm{s}}{2}\right)^2 \cdot (1 + S_0)}$$

? No.

Let me rewrite the slide's first equation using γ_s:

$$\alpha_\mathrm{s}(\omega) = \alpha_0(\omega_0) \cdot \frac{\left(\frac{\gamma}{2}\right)^2}{(\omega - \omega_0)^2 + \left(\frac{\gamma_\mathrm{s}}{2}\right)^2}$$

This is the peak unsaturated absorption coefficient $\alpha_0(\omega_0)$ multiplied by a frequency-dependent term that is NOT a simple normalized Lorentzian of width γ_s, because the numerator is $\left(\frac{\gamma}{2}\right)^2$ not $\left(\frac{\gamma_\mathrm{s}}{2}\right)^2$.

This form is equivalent to what we derived for the power absorbed, $\frac{dW_{12}(\omega)}{dt}$, on page 27 (if we divide by I and relate constants), because

$$\frac{dW_{12}(\omega)}{dt} = \alpha_\mathrm{s}(\omega) \cdot I(\omega)$$

(if $I(\omega)$ is spectral intensity).

The form was: (constant) $\cdot \frac{S_0\left(\frac{\gamma}{2}\right)^2}{(\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2(1+S_0)}$.

Our $\alpha_{\mathrm{s}}(\omega)$ is

$$\sigma(\omega)\Delta N(\omega) = \sigma_0\, L(\omega-\omega_0)\cdot\frac{\Delta N_0}{1+S_\omega}$$

$$= \sigma_0\Delta N_0\ \frac{L(\omega-\omega_0)}{1+S_0\, L(\omega-\omega_0)}$$

$$= \alpha_0(\omega_0)\frac{L(\omega-\omega_0)}{1+S_0\, L(\omega-\omega_0)}$$

$$= \alpha_0(\omega_0)\cdot\frac{\left(\frac{\gamma}{2}\right)^2/D_1}{1+S_0\left(\frac{\gamma}{2}\right)^2/D_1}$$

where $D_1 = (\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2$.

$$= \alpha_0(\omega_0)\cdot\frac{\left(\frac{\gamma}{2}\right)^2}{D_1+S_0\left(\frac{\gamma}{2}\right)^2}$$

$$= \alpha_0(\omega_0)\cdot\frac{\left(\frac{\gamma}{2}\right)^2}{(\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2+S_0\left(\frac{\gamma}{2}\right)^2}$$

$$= \alpha_0(\omega_0)\cdot\frac{\left(\frac{\gamma}{2}\right)^2}{(\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2(1+S_0)}.$$

This matches the first equation on the slide. This is the correct form.

It *is* a Lorentzian shape, with HWHM $=\frac{\gamma}{2}\sqrt{1+S_0}=\frac{\gamma_{\mathrm{s}}}{2}$.

And its peak value (at $\omega=\omega_0$) is

$$\alpha_0(\omega_0)\cdot\frac{\left(\frac{\gamma}{2}\right)^2}{\left(\frac{\gamma_\text{s}}{2}\right)^2}=\frac{\alpha_0(\omega_0)}{1+S_0}.$$

So,

$$\alpha_\text{s}(\omega)=\frac{\alpha_0(\omega_0)}{1+S_0}\cdot\frac{\left(\frac{\gamma_\text{s}}{2}\right)^2}{(\omega-\omega_0)^2+\left(\frac{\gamma_\text{s}}{2}\right)^2}.$$

Second bullet: **"Alternate but equivalent form"**:

$$\alpha_\text{s}(\omega)=\frac{\alpha_0(\omega)}{1+S_\omega}$$

Here, $\alpha_0(\omega)$ is the *unsaturated* absorption coefficient at frequency ω, so

$$\alpha_0(\omega)=\alpha_0(\omega_0)L(\omega-\omega_0)=\alpha_0(\omega_0)\cdot\frac{\left(\frac{\gamma}{2}\right)^2}{(\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2}.$$

And

$$S_\omega=S_0\,L(\omega-\omega_0)=S_0\cdot\frac{\left(\frac{\gamma}{2}\right)^2}{(\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2}.$$

So,

$$\alpha_\text{s}(\omega)=\frac{\alpha_0(\omega_0)L(\omega-\omega_0)}{1+S_0\,L(\omega-\omega_0)}.$$

This form is perhaps more intuitive. It says that the saturated absorption coefficient at any frequency ω is just the unsaturated coefficient at that same frequency, $\alpha_0(\omega)$, divided by one plus the *local* saturation parameter at that frequency, S_ω.

This is very elegant and was the basis of my derivation earlier.

It can be shown that these two forms are indeed equivalent.

$$\frac{\alpha_0(\omega_0)L(\omega-\omega_0)}{1+S_0\,L(\omega-\omega_0)}=\alpha_0(\omega_0)\cdot\frac{\left(\frac{\gamma}{2}\right)^2/D}{1+S_0\left(\frac{\gamma}{2}\right)^2/D}$$

where $D=(\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2$.

$$=\alpha_0(\omega_0)\cdot\frac{\left(\frac{\gamma}{2}\right)^2}{D+S_0\left(\frac{\gamma}{2}\right)^2}$$

$$=\alpha_0(\omega_0)\cdot\frac{\left(\frac{\gamma}{2}\right)^2}{(\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2+S_0\left(\frac{\gamma}{2}\right)^2}$$

$$=\alpha_0(\omega_0)\cdot\frac{\left(\frac{\gamma}{2}\right)^2}{(\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2(1+S_0)}.$$

This matches the first form.

So, both expressions are correct and useful.

Finally, **"Observations:"** These will be detailed on the next page.

Page 30:

This page continues with the **"Observations"** about the saturated absorption profile $\alpha_{\mathrm{s}}(\omega)$.

First observation: * **"Peak reduction factor $\frac{1}{1+S_0}$."** As we discussed, the peak of the saturated absorption profile occurs at $\omega=\omega_0$. At this frequency, $S_\omega=S_0$. So, $\alpha_{\mathrm{s}}(\omega_0)=\frac{\alpha_0(\omega_0)}{1+S_0}$. The peak height of the absorption line is reduced from its unsaturated value $\alpha_0(\omega_0)$ by the factor $\frac{1}{1+S_0}$. This is a direct consequence of saturation at the line center.

Second observation: * **"Area under $\alpha(\omega)$** remains (nearly) constant – oscillator strength conservation." Here, $\alpha(\omega)$ refers to $\alpha_\mathrm{s}(\omega)$. The integrated absorption $\int \alpha_\mathrm{s}(\omega)\,d\omega$ is proportional to the oscillator strength of the transition, which is a fundamental atomic property and should not change due to the intensity of the probing light (as long as the light is not so strong as to cause ionization or other non-linear processes that fundamentally alter the atomic structure). A Lorentzian profile $L(\omega) = H\frac{(\Gamma/2)^2}{(\omega-\omega_0)^2+(\Gamma/2)^2}$ has an integrated area of

$$\int L(\omega)\,d\omega = H\frac{(\Gamma/2)^2\,\pi}{\Gamma/2} = H\left(\frac{\Gamma}{2}\right)\pi.$$

The unsaturated profile is

$$\alpha_0(\omega) = \alpha_0(\omega_0)\frac{(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma/2)^2}.$$

Its area is $\alpha_0(\omega_0)\left(\frac{\gamma}{2}\right)\pi$. The saturated profile is

$$\alpha_\mathrm{s}(\omega) = \frac{\alpha_0(\omega_0)}{1+S_0}\frac{(\gamma_\mathrm{s}/2)^2}{(\omega-\omega_0)^2+(\gamma_\mathrm{s}/2)^2}$$

(using the form that clearly shows peak and width). Its area is

$$\frac{\alpha_0(\omega_0)}{1+S_0}\left(\frac{\gamma_\mathrm{s}}{2}\right)\pi.$$

Substitute $\gamma_\mathrm{s} = \gamma\sqrt{1+S_0}$:

$$\text{Area}_{\text{saturated}} = \frac{\alpha_0(\omega_0)}{1+S_0}\left(\frac{\gamma}{2}\sqrt{1+S_0}\right)\pi.$$

$$\text{Area}_{\text{saturated}} = \alpha_0(\omega_0)\left(\frac{\gamma}{2}\right)\pi\left[\frac{\sqrt{1+S_0}}{1+S_0}\right].$$

$$\text{Area}_{\text{saturated}} = \text{Area}_{\text{unsaturated}} \cdot \frac{1}{\sqrt{1+S_0}}.$$

So, the area actually *decreases* by a factor of $\frac{1}{\sqrt{1+S_0}}$. This means the statement "remains (nearly) constant" needs qualification. It's not strictly constant. However, often this effect is discussed in the context where the population difference ΔN is reduced, but if we only consider broadening due to lifetime effects (e.g. if strong field shortens lifetime), area might be conserved. The principle of oscillator strength conservation is fundamental. Perhaps the interpretation of "area under $\alpha(\omega)$" here refers to something subtly different, or the "nearly" is doing heavy lifting. In many treatments of power broadening, especially when derived from a dressed atom picture or when considering only the broadening effect without the reduction in ΔN, the area is stated to be conserved. However, with our rate equation approach, which explicitly includes the $\Delta N = \Delta N_0/(1+S)$ reduction, the area under $\alpha_{\text{s}}(\omega) = \sigma(\omega)\Delta N(\omega)$ is indeed reduced. The phrase "oscillator strength conservation" is more about the integral of the cross-section, $\int \sigma(\omega)\, d\omega$, which is related to A_{21} and fundamental constants. If the number of absorbers ΔN changes, then $\int \alpha(\omega)\, d\omega$ will change. Perhaps the statement implies that *if we could somehow maintain* ΔN_0 and only introduce broadening, the area would be conserved. Or, it refers to the fact that while the peak drops, the width increases, so the change in area is less dramatic than the change in peak height. For $S_0 = 1$, peak drops by 2, area drops by $\sqrt{2} \approx 1.41$. For $S_0 = 3$, peak drops by 4, area drops by $\sqrt{4} = 2$. It's a point that can sometimes cause confusion. Let's proceed with the formula derived.

Third observation: * **"Wings asymptotically merge with the unsaturated curve because $S_\omega \to 0$."** Far from resonance (i.e., $|\omega - \omega_0|$ is large), the local saturation parameter $S_\omega = S_0\, L(\omega - \omega_0)$ approaches zero. In this limit,

$$\alpha_\mathrm{s}(\omega) = \frac{\alpha_0(\omega)}{1 + S_\omega} \to \frac{\alpha_0(\omega)}{1 + 0} = \alpha_0(\omega).$$

This means that in the far wings of the spectral line, the saturated absorption profile $\alpha_\mathrm{s}(\omega)$ becomes identical to the unsaturated absorption profile $\alpha_0(\omega)$. This makes sense: saturation is negligible far from resonance, so the absorption there is unaffected. The power broadening effect "pulls" some of the absorption from the center out towards the wings, but very far out, the line looks like it would have without saturation.

The triple dash marks the end of these observations. The key takeaways are the reduced peak, the increased width, and the merging of the wings with the unsaturated profile.

Page 31:

Slide 12: Visualising Saturation Broadening

This page presents **"Slide 12: Visualising Saturation Broadening"** with a graph showing the **"Saturation Broadening of Lorentzian Profile."**

Let's describe the graph:

*The **horizontal axis** is the normalized detuning:* $\frac{\omega-\omega_0}{\gamma/2}$. This means 0 is the line center, +1 and -1 correspond to the HWHM points of the unsaturated* Lorentzian. The axis ranges from roughly -5 to +5.

* The **vertical axis** is the **Relative Absorption Coefficient**. It's normalized so that the peak of the unsaturated (S=0) line is 1.0. The axis ranges from 0.0 to 1.0.

* Several curves are plotted for different values of the on-resonance saturation parameter S_0 (which should be S_0): * **S = 0 (Blue curve)**: This is

the unsaturated Lorentzian. It peaks at 1.0 at the center. Its FWHM corresponds to a span of 2 units on this normalized x-axis (from -1 to +1). * **S = 0.5 (Orange curve)**: When $S_0 = 0.5$, the peak height is $\frac{1}{1+0.5} = \frac{1}{1.5} = \frac{2}{3} \approx 0.67$. The curve is visibly lower and slightly broader than the S=0 curve. * **S = 1.0 (Green curve)**: When $S_0 = 1.0$, the peak height is $\frac{1}{1+1} = 0.5$. The curve is further reduced in height and noticeably broader. The FWHM should be $\gamma\sqrt{2}$, so the HWHM is $(\gamma/2)\sqrt{2}$. On our normalized axis, this is $\sqrt{2} \approx 1.41$. * **S = 4.0 (Red curve)**: When $S_0 = 4.0$, the peak height is $\frac{1}{1+4} = \frac{1}{5} = 0.2$. This curve is much flatter and significantly broader. The FWHM should be $\gamma\sqrt{5}$, so HWHM is $(\gamma/2)\sqrt{5}$. On the normalized axis, this is $\sqrt{5} \approx 2.24$.

* **Dashed horizontal lines** are drawn to indicate the FWHM for each curve. For example, for the $S = 0$ blue curve, the dashed line is at height 0.5, and it intersects the curve at $x = -1$ and $x = +1$. For the $S = 1$ green curve, the peak is 0.5, so its FWHM is measured at height 0.25. The dashed line at 0.25 intersects the green curve at points wider than -1 and $+1$.

* An annotation on the right says: **"Dashed lines indicate FWHM."** And below it: **"Norm. FWHM =** $2\sqrt{1 + S_0}$" (using S_0 for S here). This refers to the FWHM in units of the original HWHM ($\gamma/2$). So,

$$\frac{\text{FWHM}}{\gamma/2} = \frac{\gamma\sqrt{1 + S_0}}{\gamma/2} = 2\sqrt{1 + S_0}.$$

This is correct. For $S_0 = 0$, Norm. FWHM = 2. For $S_0 = 1$, Norm. FWHM = $2\sqrt{2} \approx 2.82$. For $S_0 = 4$, Norm. FWHM = $2\sqrt{5} \approx 4.47$. The graph visually confirms these increasing widths.

* Another annotation: **"(γ = γ; width increases with S)"**. This should probably be γ_s (gamma sub s) for the broadened width, or it means γ is the original width. The statement "width increases with S" is the main point.

This graph provides an excellent visual summary of power broadening: 1. The peak absorption decreases as S_0 increases. 2. The FWHM of the line increases as S_0 increases (specifically as $\sqrt{1+S_0}$). 3. In the far wings, all curves tend to merge, as the effect of saturation diminishes away from resonance.

This clearly shows how an intense laser not only reduces the signal but also degrades spectral resolution by broadening the features.

Page 32:

"Slide 13: Power Broadening – An Alternative Microscopic View."

This approach will use the concept of Rabi oscillations, which arises from a more quantum mechanical treatment of the light-matter interaction, often using a semi-classical model (classical field, quantum atom).

First step: **"Treat the light classically: electric field"**

The laser field is represented as a classical monochromatic electromagnetic wave. We are interested in its electric field component $E(t)$ at the position of the atom:

$$E(t) = E_0 \cos(\omega t)$$

- $E(t)$ is the instantaneous electric field as a function of time t.

- E_0 (E sub zero) is the amplitude of the electric field. The intensity of the light is proportional to E_0^2.

- ω (omega) is the angular frequency of the light.

- $\cos(\omega t)$ describes the sinusoidal oscillation of the field.

Second step: **"Quantum mechanical two-level interaction Hamiltonian"**

The interaction between the two-level atom and this classical electric field is described by an interaction Hamiltonian, $H_{\text{int}}(t)$ (H sub int of t). For an electric dipole transition, this is given by the dipole interaction:

$$\widehat{H}_{\mathrm{int}}(t) = -\widehat{D} \cdot E(t)$$

(H-hat sub int of t equals minus D-hat dot E of t).

- $\widehat{H}_{\mathrm{int}}(t)$ is the interaction Hamiltonian operator. The hat (circumflex) denotes an operator.

- $\widehat{D}$ is the electric dipole moment operator of the atom.

- $E(t)$ is the classical electric field vector we defined above. The dot product indicates that the interaction depends on the relative orientation of the dipole moment and the electric field.

This interaction Hamiltonian is then used in the Schrödinger equation (or density matrix equations) to find how the atomic state evolves. When this is done for a two-level system, a characteristic frequency emerges, related to the strength of the interaction. This is the Rabi frequency.

The slide shows this leading to the Rabi frequency Ω_{R} (Omega sub R):

$$\Omega_{\mathrm{R}} = \frac{D_{ab} E_0}{\hbar}$$

Let's break this down:

- Ω_{R} is the **on-resonance Rabi frequency**. It has units of angular frequency (radians per second). It quantifies the rate at which the population coherently oscillates between the two levels when driven by a resonant field.

- D_{ab} is the **"dipole matrix element (C m)"** between the two states $|a\rangle$ and $|b\rangle$ (our $|1\rangle$ and $|2\rangle$). $D_{ab} = \langle a | \widehat{D}_{\mathrm{z}} | b \rangle$ if the field is polarized along z, for example. It represents the strength of the dipole transition. Its units are charge times distance, given here as Coulomb-meters (C m).

- E_0 is the amplitude of the classical electric field.

- $\hbar$ (h-bar) is the reduced Planck constant.

The Rabi frequency Ω_{R} is directly proportional to the field amplitude E_0 and the transition dipole moment D_{ab}. A stronger field or a stronger transition leads to a higher Rabi frequency, meaning faster oscillations. This Rabi frequency will be key to understanding power broadening from this coherent interaction perspective.

Page 33:

Continuing with the alternative microscopic view based on Rabi flopping:

The first bullet point reiterates the definition of "Ω_{R}: Rabi flopping frequency (rad s^{-1})." (Omega sub R, units of radians per second). This is the angular frequency of the coherent population oscillations between the two levels when driven by a resonant monochromatic field.

The second bullet describes what happens to the population: * **"Population of upper state** $|b\rangle$ without damping exhibits Rabi oscillations between 0 and 1." (Here $|b\rangle$ is our upper state $|2\rangle$). If we start with the atom in the lower state $|a\rangle$ (our $|1\rangle$) at $t = 0$, and turn on a resonant field $E_0 \cos(\omega_0 t)$, the probability of finding the atom in the upper state $|b\rangle$, $P_{\mathrm{b}}(t)$, will oscillate as:

$$P_{\mathrm{b}}(t) = \sin^2\left(\frac{\Omega_{\mathrm{R}} t}{2}\right)$$

This oscillates between 0 (when $\frac{\Omega_{\mathrm{R}} t}{2} = 0, \pi, 2\pi, ...$) and 1 (when $\frac{\Omega_{\mathrm{R}} t}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, ...$). The population of the lower state $|a\rangle$, $P_{\mathrm{a}}(t)$, will be $\cos^2\left(\frac{\Omega_{\mathrm{R}} t}{2}\right)$, so $P_{\mathrm{a}}(t) + P_{\mathrm{b}}(t) = 1$. This periodic transfer of population between the two levels is known as Rabi flopping or Rabi oscillations. The crucial part here is "without damping." If there are relaxation processes (like spontaneous emission or collisions), these oscillations will be damped, and the system will eventually reach a steady state, which is what our rate equations described. But the underlying tendency is to oscillate at Ω_{R}.

The third bullet point provides the **“Intuition”** connecting this to broadening: * **“stronger field** → faster flopping → broader spectral response (time-frequency uncertainty).” Let's unpack this intuitive argument: * **“Stronger field (larger** E_0) → faster flopping (larger Ω_R)”: This comes directly from

$$\Omega_\mathrm{R} = \frac{D_{ab}E_0}{\hbar}.$$

* **“Faster flopping** → broader spectral response”: If the population is oscillating rapidly, it means the system is changing its state on a short timescale (the period of Rabi oscillation is

$$T_\mathrm{R} = \frac{2\pi}{\Omega_\mathrm{R}}.$$

According to the time-frequency uncertainty principle ($\Delta E \Delta t \approx \hbar$, or $\Delta\omega\Delta t \approx 1$), a process that occurs on a timescale Δt is associated with a frequency spread $\Delta\omega \approx \frac{1}{\Delta t}$. If the Rabi oscillations are very fast (Ω_R is large, so T_R is small), the system doesn't have to be perfectly on resonance to be driven effectively. The strong field can "grab" the atom and force it to oscillate even if the laser frequency ω is somewhat detuned from the atomic resonance ω_0. The range of frequencies over which the atom responds effectively will be related to the Rabi frequency Ω_R itself. Essentially, the act of strongly driving the system at frequency Ω_R introduces a new characteristic frequency into the problem. The atom's energy levels can be thought of as being "dressed" by the strong field, leading to an effective splitting of the levels (Autler-Townes splitting), and the width of the response becomes related to Ω_R.

So, from this coherent interaction viewpoint, power broadening can be seen as a consequence of the rapid Rabi oscillations induced by the strong field. The spectral line broadens to roughly the Rabi frequency. We will see how this connects to the S_0 parameter from the rate equation approach.

The triple dash marks a pause. This provides a complementary physical picture to the population saturation model.

Page 34:

This slide, **"Slide 14: Exact Time-Dependent Excitation Probability,"** presents the mathematical result for the probability of finding the atom in the upper state when driven by a field that can be on or off resonance. This is a standard result from solving the time-dependent Schrödinger equation for a two-level system interacting with a classical field, usually within the Rotating Wave Approximation (RWA).

The first bullet introduces the **"Probability amplitude** $b(t)$ (rotating-wave approximation):" More precisely, what's shown is $|b(t)|^2$, which is the probability $P_\mathrm{b}(t)$ of being in the upper state $|b\rangle$ (our $|2\rangle$), assuming the system started in the lower state $|a\rangle$ (our $|1\rangle$) at $t = 0$.

The equation for $|b(t)|^2$ is:

$$|b(t)|^2 = \frac{D_{ab}^2\, E_0^2}{\hbar^2\left[(\omega_{ab}-\omega)^2 + \frac{D_{ab}^2\, E_0^2}{\hbar^2}\right]} \sin^2\left(\frac{1}{2}\sqrt{(\omega_{ab}-\omega)^2 + \varOmega_\mathrm{R}^2}\; t\right)$$

Let's break down this formidable equation:

$|b(t)|^2$: Probability of being in the upper state at time t.

D_{ab}: Transition dipole matrix element.

E_0: Amplitude of the applied electric field.

$\hbar$: Reduced Planck constant.

ω_{ab}: Resonant angular frequency of the transition $\frac{E_\mathrm{b}-E_\mathrm{a}}{\hbar}$. This is our ω_0.

ω: Angular frequency of the applied laser field.

$(\omega_{ab} - \omega)$: Detuning of the laser from resonance. Let's call this $\varDelta\omega = \omega_{ab} - \omega$.

$\Omega_\mathrm{R} = \frac{D_{ab}E_0}{\hbar}$: The on-resonance Rabi frequency, which we just defined.

Notice that $\frac{D_{ab}^2 E_0^2}{\hbar^2}$ in the denominator of the first fraction is simply Ω_R^2.

So, the equation can be rewritten more compactly using $\Delta\omega$ and Ω_R:

$$|b(t)|^2 = \frac{\Omega_\mathrm{R}^2}{(\Delta\omega)^2 + \Omega_\mathrm{R}^2} \sin^2\left(\frac{1}{2}\sqrt{(\Delta\omega)^2 + \Omega_\mathrm{R}^2}\, t\right)$$

Let's analyze this:

The term $\frac{\Omega_\mathrm{R}^2}{(\Delta\omega)^2+\Omega_\mathrm{R}^2}$ is the maximum amplitude of the oscillation.

If the laser is on resonance ($\Delta\omega = 0$), this amplitude is $\frac{\Omega_\mathrm{R}^2}{\Omega_\mathrm{R}^2} = 1$.

If the laser is far off resonance ($\Delta\omega \gg \Omega_\mathrm{R}$), this amplitude becomes $\left(\frac{\Omega_\mathrm{R}}{\Delta\omega}\right)^2$ which is very small. This means an off-resonant field is much less effective at exciting the atom.

The term $\sin^2\left(\frac{1}{2}\Omega'_\mathrm{R}\, t\right)$ describes the oscillation itself.

$\Omega'_\mathrm{R} = \sqrt{(\Delta\omega)^2 + \Omega_\mathrm{R}^2}$ is called the **generalized Rabi frequency** or off-resonance Rabi frequency. It's the frequency at which the population flops when there is a detuning $\Delta\omega$.

The $\sin^2$ term oscillates between 0 and 1. So, $P_\mathrm{b}(t)$ oscillates between 0 and the amplitude factor $\frac{\Omega_\mathrm{R}^2}{(\Delta\omega)^2+\Omega_\mathrm{R}^2}$.

The second bullet point on the slide considers a special case:

"At exact resonance $\omega = \omega_{ab}$ (so $\Delta\omega = 0$) the sine term oscillates with frequency Ω_{R}."

If $\Delta\omega = 0$, then the generalized Rabi frequency $\Omega'_{\mathrm{R}} = \sqrt{0^2 + \Omega_{\mathrm{R}}^2} = \Omega_{\mathrm{R}}$. The amplitude factor becomes $\frac{\Omega_{\mathrm{R}}^2}{\Omega_{\mathrm{R}}^2} = 1$. So, $|b(t)|^2 = 1 \cdot \sin^2\left(\frac{1}{2}\Omega_{\mathrm{R}}\, t\right)$. This is exactly the on-resonance Rabi flopping formula we mentioned earlier. The population in the upper state oscillates between 0 and 1 with a frequency component related to Ω_{R} (specifically, the argument of $\sin^2$ is $\frac{\Omega_{\mathrm{R}t}}{2}$, so the $P_{\mathrm{b}}(t)$ itself oscillates with angular frequency Ω_{R} because $\sin^2(x) = \frac{1-\cos(2\,x)}{2}$).

Page 35:

This page discusses a consequence of the time-dependent excitation probability in the absence of damping.

* **"Without damping the spectrum would develop many sharp sidebands (Fourier series of a square wave)."**

Let's elaborate on this. The probability $|b(t)|^2$ we saw on the previous page describes oscillations that, in the absence of any relaxation or damping mechanisms, would continue indefinitely. If you have a persistent oscillation in the time domain, its Fourier transform in the frequency domain will show features related to that oscillation frequency.

Consider the on-resonance case: $|b(t)|^2 = \sin^2\left(\frac{\Omega_{\mathrm{R}}t}{2}\right) = \frac{1-\cos(\Omega_{\mathrm{R}}t)}{2}$.

If you were to probe the system (e.g., by looking at its absorption spectrum with a weak second laser, or its fluorescence), the spectrum wouldn't just be a single line at ω_0. The strong driving field $E_0\cos(\omega_0 t)$ effectively "dresses" the atomic states. The energy levels are split by an amount related to $\hbar\Omega_{\mathrm{R}}$ (this is the Autler-Townes splitting). A probe laser would see transitions not just at ω_0, but also at $\omega_0 \pm \Omega_{\mathrm{R}}$.

If the driving field is very strong such that the Rabi oscillations are like a square wave (atom spends half its time in $|a\rangle$, half in $|b\rangle$, switching rapidly), the Fourier series of a square wave contains the fundamental frequency (related to Ω_{R}) and also its odd harmonics ($3\Omega_{\mathrm{R}}$, $5\Omega_{\mathrm{R}}$, etc.). This can lead to a complex spectrum with multiple sidebands. This is the regime of strong, coherent driving. In fluorescence, this can manifest as the Mollow triplet, where the central fluorescence peak at ω_0 is accompanied by two sidebands at $\omega_0 \pm \Omega_{\mathrm{R}}$, provided Ω_{R} is larger than the damping rates.

The key here is "without damping." In reality, there's always some damping (spontaneous emission, collisions, finite laser coherence time). Damping will broaden these sharp sidebands and can cause them to merge if the damping rate is comparable to or larger than Ω_{R}.

This sets the stage for the next slide, which will introduce relaxation. The undamped picture gives us the fundamental frequencies involved (like Ω_{R}), and damping will determine the observed linewidths of features related to these frequencies.

The triple dash indicates a short pause. This highlights that the purely coherent, undamped picture is an idealization, but a useful one for identifying the characteristic frequencies induced by the field.

Page 36:

Slide 15: Including Relaxation – Time Averaging.

Now we move to a more realistic scenario by **"Slide 15: Including Relaxation – Time Averaging."** The purely coherent Rabi oscillations are an idealization. In any real system, relaxation processes (damping) are present and will affect the outcome.

The first bullet addresses this:

- **"Exponential decay of coherence and population from both levels:"** Relaxation processes, such as spontaneous emission or collisions, cause the populations of the energy levels to decay towards some equilibrium. Crucially, they also cause the *coherence* between the levels to decay. Coherence refers to a definite phase relationship between the amplitudes of the lower and upper states in a superposition. Rabi oscillations are a coherent process. If this coherence is destroyed (dephasing), the oscillations will damp out. This decay is often modeled as exponential.

Second bullet:

- **"We introduce damping constant** γ (s^{-1})." (γ, with units of per second). This γ represents an overall relaxation rate. It could be the rate of decay of the coherence between the two levels (often denoted $\gamma_{\perp}$ or $1/T_2$), or an average population decay rate (related to $\gamma_{\parallel}$ or $1/T_1$). In many simple models, a single effective damping constant γ is used to represent the combined effect of all relaxation processes that limit the duration of coherent interaction. This γ will be related to the homogeneous linewidth of the transition in the absence of power broadening.

Third bullet: How do we get a steady-state spectral response from the time-dependent $|b(t)|^2$ when damping is present?

- **"Mean excitation probability obtained via weighted integral"**: If we want to find the average probability of being in the upper state, $P_{\mathrm{b}}(\omega)$, when the system is continuously driven by a laser of frequency ω and subject to relaxation rate γ, we can average the time-dependent probability $|b(t)|^2$ over time, but weighted by the probability that the coherent interaction has survived up to time t. This survival probability is often taken as $e^{-\gamma t}$ (e to the power of minus gamma t).

The formula given is for $P_{\mathrm{b}}(\omega)$, the mean excitation probability in the upper state:

$$P_{\mathrm{b}}(\omega) = \int_0^\infty \gamma\, e^{-\gamma t}\, |b(t,\omega)|^2\, dt$$

Let's understand this integral:

- $P_{\mathrm{b}}(\omega)$ is the time-averaged probability of finding the atom in the upper state $|b\rangle$ when driven by a field of frequency ω. This is what we would measure in a steady-state experiment.

- $|b(t,\omega)|^2$ is the time-dependent probability of being in state $|b\rangle$ if coherence were maintained (from page 34), explicitly showing its dependence on the driving frequency ω.

- $e^{-\gamma t}$ is the probability that the atom has "survived" without undergoing a randomizing relaxation event up to time t.

- γ in front of the integral is a normalization factor. The integral of $\gamma\, e^{-\gamma t}$ from 0 to ∞ is 1. This makes it a properly weighted average. It's essentially like saying interactions are constantly being re-initiated by relaxation events, and we are averaging over many such interaction segments, each lasting a random time with an average duration of $1/\gamma$.

Final bullet:

- **"Result after algebra:"** Performing this integral with the full expression for $|b(t,\omega)|^2$ (from page 34) is a non-trivial algebraic exercise. We will see the result on the next page. The outcome will be an expression for $P_{\mathrm{b}}(\omega)$ that looks like a Lorentzian profile, where the width depends on both the intrinsic damping γ and the Rabi frequency Ω_{R} (i.e., on the laser power).

This approach of time-averaging the coherent response in the presence of damping is a powerful way to bridge the gap between the purely coherent Rabi oscillation picture and the steady-state, broadened lineshapes observed in experiments.

Page 37:

This page presents the **"Result after algebra"** for the mean excitation probability $P_\mathrm{b}(\omega)$ that was set up on the previous slide.

The expression for $P_\mathrm{b}(\omega)$ is:

$$P_\mathrm{b}(\omega) = \frac{1}{2}\left[\frac{D_{ab}^2 E_0^2/\hbar^2}{(\omega_{ab}-\omega)^2 + \gamma^2(1+S)}\right]$$

Let's break this down and connect it to familiar terms:

* $P_\mathrm{b}(\omega)$: The time-averaged probability of the atom being in the upper state $|b\rangle$, as a function of the laser frequency ω. * $\frac{D_{ab}^2 E_0^2}{\hbar^2}$: This is simply Ω_R^2, the square of the on-resonance Rabi frequency. * $(\omega_{ab}-\omega)^2$: The square of the detuning from resonance $(\Delta\omega)^2$. * γ^2: The square of the damping constant (related to the homogeneous linewidth). * S: A saturation parameter. Now, how is this S defined in this context?

The slide provides the definition for S used in this formula:

$$S = \frac{D_{ab}^2 E_0^2}{\hbar^2 \gamma^2}$$

Let's analyze this S:

* Numerator: $\frac{D_{ab}^2 E_0^2}{\hbar^2} = \Omega_\mathrm{R}^2$. * Denominator: γ^2.

So, $S = \frac{\Omega_\mathrm{R}^2}{\gamma^2}$ or $S = \left(\frac{\Omega_\mathrm{R}}{\gamma}\right)^2$.

This definition of S is very important. It's the square of the ratio of the Rabi frequency to the damping rate.

* If $\Omega_\mathrm{R} \ll \gamma$ (weak field), then $S \ll 1$. The Rabi frequency is small compared to the damping rate. Coherent oscillations are quickly damped out. * If $\Omega_\mathrm{R} \gg \gamma$ (strong field), then $S \gg 1$. Rabi oscillations can occur many times before being damped. This is the strong coherent driving regime.

Now, let's substitute S back into the expression for $P_\text{b}(\omega)$:

$$P_\text{b}(\omega) = \frac{1}{2}\left[\frac{\Omega_\text{R}^2}{(\omega_{ab} - \omega)^2 + \gamma^2\left(1 + \frac{\Omega_\text{R}^2}{\gamma^2}\right)}\right]$$

$$P_\text{b}(\omega) = \frac{1}{2}\left[\frac{\Omega_\text{R}^2}{(\omega_{ab} - \omega)^2 + \gamma^2 + \Omega_\text{R}^2}\right]$$

This can also be written by multiplying the numerator and denominator inside the main fraction by $\frac{1}{\gamma^2}$:

$$P_\text{b}(\omega) = \frac{1}{2} \left[\frac{\Omega_\text{R}^{2/\gamma^2}}{\left[(\omega_{ab} - \omega)^2/\gamma^2\right] + 1 + \left(\Omega_\text{R}^{2/\gamma^2\right)}} \right]$$

$$P_\text{b}(\omega) = \frac{1}{2}\left[\frac{S}{[(\omega_{ab} - \omega)/\gamma]^2 + 1 + S}\right]$$

This $P_\text{b}(\omega)$ is proportional to the absorption lineshape. The denominator $(\omega_{ab} - \omega)^2 + \gamma^2(1 + S)$ has the form of a Lorentzian.

The HWHM of this Lorentzian is $\gamma\sqrt{1 + S}$.

The FWHM is $\gamma_\text{eff} = 2\gamma\sqrt{1 + S}$. Wait, this should be $\gamma\sqrt{1 + S}$.

Let $\text{HWHM}_\text{new} = \gamma\sqrt{1 + S}$. Then $\text{FWHM}_\text{new} = 2\gamma\sqrt{1 + S}$.

The original FWHM (due to damping γ) would be just γ (if γ is HWHM) or 2γ (if γ is FWHM).

Let's assume γ in this context is the HWHM of the underlying transition if $S = 0$. Then the lineshape is $1/(\Delta\omega^2 + \gamma^2)$.

The power broadened line has a HWHM of $\gamma_\text{s} = \gamma\sqrt{1 + S}$.

So the denominator is $(\omega_{ab} - \omega)^2 + \gamma_{\mathrm{s}}^2$.

The final bullet point makes a key observation about S:

* **"S again appears as the square of the field amplitude normalised to damping."** Since $S = \frac{\Omega_{\mathrm{R}}^2}{\gamma^2}$ and $\Omega_{\mathrm{R}} = \frac{D_{ab} E_0}{\hbar}$, then $S = \frac{D_{ab}^2 E_0^2}{\hbar^2 \gamma^2}$. E_0^2 is proportional to the laser intensity. So, S is proportional to intensity. S is dimensionless. It compares the strength of the coherent driving term (proportional to E_0^2) to the strength of the damping (proportional to γ^2). This S is very similar in spirit to the saturation parameter S we derived from rate equations ($S = P_0/\overline{R}$). We will see on the next slide how they are connected. If $P_0 \propto E_0^2$ and $\overline{R} \propto \gamma$ (or some function of relaxation rates), then the forms are analogous.

This result for $P_{\mathrm{b}}(\omega)$ gives us an absorption lineshape that is a Lorentzian, whose width depends on the laser intensity (through S). This is power broadening, derived from a coherent interaction picture with damping.

Page 38:

This slide, **"Slide 16: Identical Broadening Law from the Rabi Picture,"** connects the result we just obtained from the coherent Rabi picture (with damping) to the power broadening law we found earlier using rate equations.

"Because $P_{\mathrm{b}}(\omega)$ directly scales with the absorption line, we identify the same Lorentzian as before, but now with"

The mean excitation probability $P_{\mathrm{b}}(\omega)$ that we found:

$$P_{\mathrm{b}}(\omega) = \frac{1}{2} \frac{S}{\left(\frac{\omega_{ab} - \omega}{\gamma}\right)^2 + 1 + S}$$

This has a Lorentzian frequency dependence in the denominator: $\left(\frac{\omega_{ab}-\omega}{\gamma}\right)^2 + (1+S)$. Or, as $(\omega_{ab}-\omega)^2 + \gamma^2(1+S)$ if we use the form from the previous slide.

This denominator $(\omega_{ab}-\omega)^2 + \gamma^2(1+S)$ means the lineshape is a Lorentzian with a half-width at half-maximum (HWHM) of $\gamma\sqrt{1+S}$.

Therefore, the full width at half-maximum (FWHM) of this line, let's call it γ_s (gamma sub s), is:

$$\gamma_\text{s} = \gamma\sqrt{1+S}$$

(Here, γ would be the FWHM of the transition if $S=0$, from the damping. If γ on prev slide was HWHM, then $\gamma_{s,\text{FWHM}} = 2\gamma_\text{HWHM}\sqrt{1+S}$.)

Let's be consistent. If γ in $S = \frac{\Omega_\text{R}^2}{\gamma^2}$ is the HWHM from damping, then the FWHM of the unbroadened line is 2γ. The power-broadened FWHM will be $\gamma_\text{s} = 2\gamma\sqrt{1+S}$. If γ in $S = \frac{\Omega_\text{R}^2}{\gamma^2}$ is already the FWHM due to damping, then the power-broadened FWHM is $\gamma_\text{s} = \gamma\sqrt{1+S}$.

The slide uses $\gamma\sqrt{1+S}$, implying γ is the original FWHM. This is exactly the same power broadening law we found from the rate equation approach (page 28), where γ was the unsaturated FWHM and S_0 was the on-resonance saturation parameter from that model.

"Connection to the earlier definition:"

How does the saturation parameter $S = \frac{\Omega_\text{R}^2}{\gamma^2}$ (from the Rabi picture) relate to $S_0 = \frac{P_0}{\bar{R}}$ (from the rate equations)?

"Replace $\frac{D_{ab}^2 E_0^2}{\hbar^2}$ by $\frac{B_{12} I}{c}$ (classical ↔ Einstein coefficients)."

$\frac{D_{ab}^2 E_0^2}{\hbar^2}$ is Ω_{R}^2. The Einstein B_{12} coefficient is related to the dipole moment D_{ab} by

$$B_{12} = \frac{\pi D_{ab}^2}{3\epsilon_0 \hbar^2 g_1},$$

(depending on definition of B and averaging over orientations). And intensity I is $\frac{1}{2} c\epsilon_0 E_0^2$.

So, $\Omega_{\mathrm{R}}^2 = \frac{D_{ab}^2 E_0^2}{\hbar^2}$ can indeed be related to $\frac{B_{12} I}{c}$.

The pump rate P_0 in the rate equation model was

$$P_0 = B_{12}\rho(\omega_0) \quad \text{or} \quad P_0 = \frac{\sigma(\omega_0) I}{\hbar\omega_0}.$$

And the saturation parameter from rate equations was

$$S_0 = \frac{2\,P_0}{R_1 + R_2} = \frac{P_0}{\bar{R}}.$$

Let's assume $\bar{R}$, the mean relaxation rate, is proportional to γ (the damping rate or FWHM from the Rabi model). For example, in many cases, γ (FWHM) $= R_1 + R_2$. So $\bar{R} = \gamma/2$. Then $S_0 = \frac{P_0}{(\gamma/2)} = \frac{2\,P_0}{\gamma}$. And P_0 itself is proportional to $\frac{\Omega_{\mathrm{R}}^2}{\gamma}$ if one relates the coherent interaction strength to the incoherent pump rate. More directly: $P_0 = B_{12}\rho(\omega_0)$ (if B_{12} is defined for intensity) $= \frac{\sigma I}{\hbar\omega}$. We know P_0 drives transitions. Ω_{R}^2 is also proportional to I (since $E_0^2 \propto I$). It's known that for a two-level system, the on-resonance stimulated rate P_0 can be written as

$$P_0 = \frac{1}{2}\frac{\Omega_{\mathrm{R}}^2}{\gamma}$$

evaluated at $\Delta\omega = 0$, giving $P_0 = \frac{1}{2}\frac{\Omega_\mathrm{R}^2}{\gamma}$ (This requires γ to be HWHM related to T_2). If this is the case, then

$$S_0 = \frac{P_0}{\bar{R}} = \left(\frac{\Omega_\mathrm{R}^2}{2\gamma}\right)/\bar{R}.$$

If $\bar{R} \approx \frac{\gamma}{2}$, then $S_0 \approx \frac{\Omega_\mathrm{R}^2}{\gamma^2}$. The exact correspondence can be intricate depending on definitions of γ, R_1, R_2, B_{12}, etc.

"One recovers $S = \frac{B_{12}\rho(\omega_0)}{\bar{R}} = S_0$."

This means that the S from the Rabi picture, $S = \frac{\Omega_\mathrm{R}^2}{\gamma^2}$, must be numerically equivalent to the S_0 from the rate equation picture, $S_0 = \frac{B_{12}\rho(\omega_0)}{\bar{R}}$ (where $B_{12}\rho(\omega_0)$ is P_0). So, the assertion is:

$$\frac{\Omega_\mathrm{R}^2}{\gamma^2} = \frac{P_0}{\bar{R}}.$$

This equality holds if one carefully defines all terms. For example, often γ (the FWHM of the Lorentzian due to damping) is simply $R_1 + R_2$ if these are the only population relaxation channels. And P_0, the incoherent pumping rate, is related to the coherent driving Ω_R and the damping γ through expressions like

$$P_0 \approx \frac{\Omega_\mathrm{R}^2}{2\gamma}$$

(if γ is HWHM, T_2 type rate).

The key message is that the S parameter, whether defined as $\frac{\Omega_\mathrm{R}^2}{\gamma^2}$ or $\frac{P_0}{\bar{R}}$, plays the same role in the broadening formula $\gamma_\mathrm{s} = \gamma\sqrt{1+S}$, and these two definitions of S are physically equivalent measures of how strongly the system is driven relative to its relaxation.

This unification of the two pictures is a very satisfying result.

Page 39:

This page offers a **"Conceptual bonus"** from the Rabi picture approach.

The bullet point states: * **"Conceptual bonus: we see that the broadening stems from the Fourier transform of damped Rabi oscillations."**

Let's elaborate on this insight, which is a powerful way to think about power broadening.

In the purely coherent picture (no damping), Rabi oscillations of population occur at the generalized Rabi frequency $\Omega'_\mathrm{R} = \sqrt{(\omega - \omega_0)^2 + \Omega_\mathrm{R}^2}$. These oscillations would persist indefinitely. When we introduce damping (with rate γ), these oscillations no longer last forever. They decay exponentially, typically with a time constant related to $1/\gamma$ (e.g., T_2 for coherence decay, T_1 for population decay). So, what we have is a *damped oscillation* in the time domain.

We know from Fourier analysis that:

* An undamped sinusoid in time (e.g., $\cos(\Omega_\mathrm{R} t)$) transforms to a pair of delta functions in frequency (at $\pm\Omega_\mathrm{R}$). * An exponential decay in time (e.g., $e^{-\gamma t}$) transforms to a Lorentzian in frequency (with width γ). * A damped sinusoid (e.g., $e^{-\gamma t}\cos(\Omega_\mathrm{R} t)$) is the product of these two. The Fourier transform of a product in time is the convolution of their Fourier transforms in frequency. So, we would convolve the delta functions with the Lorentzian. This results in two Lorentzian peaks in the frequency domain, centered at $\pm\Omega_\mathrm{R}$, each with a width determined by γ. This is the Autler-Townes splitting.

The power broadening we've been discussing for a single absorption line can be thought of as a limiting case or a different manifestation. If we are looking at the absorption spectrum $P_\mathrm{b}(\omega)$, which came from integrating $\gamma e^{-\gamma t}|b(t)|^2 dt$, this averaging process effectively performs the Fourier transform. The term $|b(t)|^2$ contains $\sin^2(\Omega'_\mathrm{R} t/2)$ which is $\frac{1-\cos(\Omega'_\mathrm{R} t)}{2}$. So we are looking at the Fourier transform of (something related to) a constant term and a damped cosine term. The constant term gives a feature at zero frequency (in the rotating frame, so at ω_0 in lab frame). The damped cosine gives features related to $\pm\Omega'_\mathrm{R}$.

When Ω_R is not too large compared to γ, these features merge. The overall lineshape of $P_\mathrm{b}(\omega)$ that we found, $\frac{1}{2}\left[\frac{S}{\left(\frac{\omega_{ab}-\omega}{\gamma}\right)^2+1+S}\right]$, is a single Lorentzian centered at ω_{ab} (or ω_0), but its width $\gamma\sqrt{1+S}$ now incorporates both the original damping γ and the driving strength $S=\frac{\Omega_\mathrm{R}^2}{\gamma^2}$.

So, the "smearing" or broadening of the spectral line can be intuitively understood as arising because the Rabi oscillations are damped. The stronger the field (larger Ω_R), the faster the oscillations it tries to induce. Even though these are damped, the "attempt" to oscillate faster means the system responds over a broader range of frequencies, as dictated by the uncertainty principle or, more formally, by the Fourier transform properties.

This conceptual link to damped oscillations and Fourier transforms provides a deeper physical intuition for why power broadening occurs, beyond just the mathematical outcome of the rate equations or the formal solution of the damped Bloch equations (which is what underlies $P_\mathrm{b}(\omega)$). It connects the spectral domain (broadening) to the time domain (damped oscillations).

The triple dash signals the end of this point.

Page 40:

Slide 17: Pump-Probe Scenario & Autler-Townes-Type Sidebands.

Now we move to an experimentally very relevant scenario: **"Slide 17: Pump-Probe Scenario & Autler-Townes-Type Sidebands."** This is where the coherent effects of a strong field, like Rabi oscillations, can lead to observable new spectral features, not just broadening of a single line.

First bullet point:

* **"Strong 'pump' laser exactly resonant: large** Ω_R." We imagine an experiment with two lasers. The first is a strong "pump" laser, tuned exactly to the atomic resonance ω_0 (so $\Delta\omega = 0$ for the pump). "Strong" means that the Rabi frequency Ω_R induced by this pump laser is large. Specifically, it should be large compared to the relaxation rates/linewidth γ (i.e., $S = \frac{\Omega_\mathrm{R}^2}{\gamma^2} \gg 1$).

Second bullet point:

* **"Weak tunable 'probe' interrogates the modified absorption profile."** A second laser, called the "probe" laser, is used. This probe laser is weak, meaning it doesn't significantly perturb the system itself (it doesn't cause significant saturation or Rabi oscillations on its own). Its frequency ω_probe is tunable, so we can scan it across the region around ω_0 to measure the absorption spectrum of the atom *as modified by the strong pump laser*.

Third bullet point: What does the probe see?

* **"Populations and coherences are modulated at** Ω_R; probe perceives sidebands at"

$$\omega = \omega_0 \pm \Omega_\mathrm{R}$$

Because the strong resonant pump is driving Rabi oscillations at frequency Ω_R, the populations of the atomic levels are oscillating in time at this frequency. More subtly, the atomic coherence (the off-diagonal element of the density matrix) is also modulated.

The atom, when "dressed" by the strong pump field, effectively has its energy levels split. The original two levels $|a\rangle$ and $|b\rangle$, separated by $\hbar\omega_0$, are replaced by new dressed states. Transitions between these dressed states can occur when probed by the weak laser.

These new allowed transitions for the probe laser appear at frequencies shifted from the original resonance. Specifically, instead of a single absorption peak at ω_0, the probe laser will see features (absorption or gain, depending on details) at frequencies $\omega_0 + \mathit{\Omega}_\mathrm{R}$ and $\omega_0 - \mathit{\Omega}_\mathrm{R}$.

This splitting of the absorption line into a doublet (or sometimes a triplet in fluorescence, the Mollow triplet) centered around ω_0, with a separation of $2\mathit{\Omega}_\mathrm{R}$, is known as the **Autler-Townes effect** (or AC Stark splitting, dynamic Stark splitting).

These features at $\omega_0 \pm \mathit{\Omega}_\mathrm{R}$ are the "sidebands" referred to. They are a direct consequence of the coherent Rabi oscillations driven by the strong pump.

This pump-probe configuration is a standard method in laser spectroscopy to observe such strong-field effects. The pump creates the dressed states, and the probe measures their spectrum.

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Continuing with the pump-probe scenario and Autler-Townes sidebands:

First bullet point:

* **"Requirement for clearly resolved doublet"**

For the two sidebands at $\omega_0 \pm \mathit{\Omega}_\mathrm{R}$ to be seen as distinct peaks (a resolved doublet), the splitting between them (which is $2\mathit{\Omega}_\mathrm{R}$, or the shift of each from center is $\mathit{\Omega}_\mathrm{R}$) must be larger than their individual linewidths. The linewidth of these features will be related to the power-broadened homogeneous linewidth $\gamma_\mathrm{s} = \gamma\sqrt{1+S}$, where S is the saturation parameter related to the strong pump.

So, the condition is given as:

$\Omega_{\mathrm{R}} > \gamma_{\mathrm{s}}$ (Omega sub R must be greater than gamma sub s).

Actually, the peaks are at $\pm\Omega_{\mathrm{R}}$, and each has a width of approximately γ_{s} (or related to it). So for them to be resolved, Ω_{R} should be significantly larger than $\gamma_{\mathrm{s}}/2$ (their HWHM). Or, the separation $2\Omega_{\mathrm{R}}$ must be greater than γ_{s}. So, $\Omega_{\mathrm{R}} > \gamma_{\mathrm{s}}/2$ is a more accurate condition for resolvability. If Ω_{R} is not much larger than γ_{s}, the two sidebands will merge with each other and possibly with the central feature (if any), resulting in a single, very broad, and possibly strangely shaped line. When $\Omega_{\mathrm{R}} \gg \gamma_{\mathrm{s}}$, we see two distinct peaks.

Second bullet: **"Observed features:"**

What does the probe spectrum look like when Ω_{R} is large?

* **"Central dip (sometimes called 'hole burning' or 'Mollow triplet' in fluorescence context)."**

For absorption by a probe, if the pump is exactly resonant, the absorption at the exact line center ω_0 can be strongly suppressed due to saturation by the pump. This creates a "dip" or "hole" in the absorption profile at ω_0. This is related to "hole burning" in Doppler-broadened lines, but here it's in a homogeneous line due to the strong resonant pump. In fluorescence, the Mollow triplet consists of a central peak at ω_0 and two sidebands at $\omega_0 \pm \Omega_{\mathrm{R}}$. The relative heights depend on parameters. The "central dip" in absorption corresponds to this structure.

* **"Two Lorentzian satellites each of width** γ_{s}."

These are the Autler-Townes components. They appear as sidebands (satellites) to the original transition frequency. They are located at approximately $\omega_0 + \Omega_{\mathrm{R}}$ and $\omega_0 - \Omega_{\mathrm{R}}$. Each of these satellite peaks will have a lineshape that is approximately Lorentzian, and their width will be related to the saturated linewidth γ_{s} (which includes the original damping γ and the power broadening effect from the strong pump). More precisely, the widths are often given in terms of T_1 and T_2 relaxation times, and can be different for the central feature and the sidebands in more complex theories, but γ_s is a good first approximation.

So, a strong resonant pump doesn't just broaden the line; it can fundamentally alter the spectrum by splitting it into multiple components. This is a hallmark of the coherent interaction dominating over incoherent relaxation. The ability to resolve these components depends on the Rabi frequency being large enough compared to the broadened linewidth.

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This page presents a graph illustrating the **"Autler-Townes Doublet Absorption Profile."** This shows what the probe laser's absorption spectrum looks like under the conditions described on the previous pages (strong resonant pump).

Let's describe the graph:

* The **horizontal axis** is the detuning of the probe laser frequency ω from the atomic resonance ω_0, specifically labeled $(\omega - \omega_0)$. The center is 0. Points labeled $-\Omega_\mathrm{R}$ and $+\Omega_\mathrm{R}$ are marked, as well as $2\Omega_\mathrm{R}$ indicating the separation between these points.

* The **vertical axis** is **Absorption**, presumably the absorption of the weak probe laser. The scale goes from 0.0 to 1.0 (likely normalized).

* A **blue curve** shows the absorption profile. It has two distinct peaks, forming a doublet. * These two peaks are centered at $\omega - \omega_0 = -\Omega_\mathrm{R}$ and $\omega - \omega_0 = +\Omega_\mathrm{R}$. So, the absorption maxima are at $\omega = \omega_0 - \Omega_\mathrm{R}$ and $\omega = \omega_0 + \Omega_\mathrm{R}$. * There is a significant **dip in absorption at the line center** $(\omega - \omega_0 = 0)$. The absorption is very low, almost zero, at the exact original resonance frequency ω_0. This is the "hole" created by the strong resonant pump. * Each of the two peaks appears to be roughly Lorentzian in shape. * A vertical double arrow near the right peak indicates its width, labeled γ_s

(gamma sub s). This suggests that the FWHM of each of these Autler-Townes components is approximately the saturated linewidth γ_s.

What this graph beautifully illustrates is:

1. The splitting of the original single absorption line into two components due to the strong resonant pump. The separation between the peaks of these components is $2\Omega_\mathrm{R}$.

2. The suppression of absorption at the original line center ω_0.

3. The fact that these new spectral features (the sidebands or satellites) themselves have a finite width, γ_s, which is determined by the natural damping processes and power broadening by the strong pump.

For this doublet to be clearly resolved, as it is in the graph, the Rabi frequency Ω_R must be significantly larger than the width γ_s of each component (specifically, $\Omega_\mathrm{R} > \gamma_\mathrm{s}/2$). This Autler-Townes splitting is a direct spectroscopic signature of the system being in the strong, coherent driving regime. Measuring the splitting ($2\Omega_\mathrm{R}$) allows for a direct determination of the Rabi frequency, and thus, if D_{ab} is known, the electric field amplitude E_0 of the pump laser. It's a powerful diagnostic tool in laser spectroscopy.

Page 43:

This slide, **"Slide 18: Frequently Used Working Equations,"** serves as a useful summary of the key formulas we've derived and discussed related to saturation and power broadening. These are the equations one would often refer back to when analyzing experiments or solving problems.

First, the **"Saturation parameter"** S: Three equivalent forms are given for the on-resonance saturation parameter (which we called S_0, but here just S):

- $S = \frac{2P}{R_1+R_2}$: This was our definition from the rate equation model, where P is the stimulated transition rate (e.g., $B_{12}\rho(\omega_0)$), and R_1 and R_2 are the incoherent relaxation/transfer rates between levels 1 and 2 (R_1: 1→2, R_2: 2→1).

- $S = \frac{B_{12}\rho(\omega_0)}{\bar{R}}$: This is equivalent to the first, where $P = B_{12}\rho(\omega_0)$ (on-resonance pump rate) and $\bar{R} = \frac{R_1+R_2}{2}$ (mean relaxation rate).

- $S = \frac{D_{ab}^2 E_0^2}{\hbar^2\gamma^2}$: This was the definition of S from the coherent Rabi interaction model with damping. Here, $\frac{D_{ab}E_0}{\hbar}$ is the Rabi frequency Ω_{R}, and γ is the damping rate (related to FWHM of the undamped line). So, $S = \frac{\Omega_{\mathrm{R}}^2}{\gamma^2}$.

Second, the **"Population difference"** ΔN:

$$\Delta N = \frac{\Delta N_0}{1+S}$$

This shows how the actual population difference ΔN is reduced from its unsaturated value ΔN_0 by the factor $(1+S)$, where S is the on-resonance saturation parameter.

Third, the **"Saturated absorption coefficient (homogeneous line)"**: This will be continued on the next page. It refers to $\alpha_{\mathrm{s}}(\omega)$.

These first few equations encapsulate the core of saturation: how the driving field (via P, ρ, or E_0) competes with relaxation (R_1, R_2, γ) to determine the saturation parameter S, which in turn governs the reduction in the population difference ΔN.

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Continuing with **"Frequently Used Working Equations"**:

Third (continued from previous page), the **"Saturated absorption coefficient (homogeneous line)"** $\alpha_{\mathrm{s}}(\omega)$: Two forms are given:

$$\alpha_{\mathrm{s}}(\omega) = \frac{\alpha_0(\omega)}{1+S_\omega}$$

This is the very intuitive form: the saturated absorption at frequency ω is the unsaturated absorption at that frequency, $\alpha_0(\omega)$, divided by one plus the local (frequency-dependent) saturation parameter S_ω.

And the local saturation parameter S_ω is given by:

$$S_\omega = S_0 \left[\frac{(\gamma/2)^2}{(\omega - \omega_0)^2 + (\gamma/2)^2}\right]$$

Here, S_0 is the on-resonance saturation parameter (from page 43). γ is the FWHM of the unsaturated Lorentzian. This shows $S_\omega = S_0 \cdot L(\omega - \omega_0)$, where L is the normalized unsaturated Lorentzian.

Fourth, the **"Power-broadened half-width"** (this should be FWHM, or $\gamma_{\mathrm{s}}/2$ is the HWHM): The slide shows

$$\gamma_{\mathrm{s}} = \gamma\sqrt{1 + S_0}$$

Here: * γ_{s} is the FWHM of the power-broadened line. * γ is the FWHM of the unsaturated homogeneous line. * S_0 is the on-resonance saturation parameter. This is our key formula for power broadening.

Fifth, the **"Rabi frequency (dipole interaction)"** Ω_{R}:

$$\Omega_{\mathrm{R}} = \frac{D_{ab}E_0}{\hbar}$$

This defines the on-resonance Rabi frequency in terms of the transition dipole moment D_{ab}, the electric field amplitude E_0 of the laser, and $\hbar$. This frequency is crucial for understanding coherent effects like Autler-Townes splitting and for the definition of S in the coherent picture $\left(S = \frac{\Omega_{\mathrm{R}}^2}{\gamma^2}\right)$.

The triple dash indicates the end of this summary. These equations form a toolkit for analyzing experiments where saturation and power broadening are significant. They connect microscopic atomic properties (D_{ab}, γ, R_1, R_2) and laser parameters (E_0, ρ, ω) to observable macroscopic quantities (ΔN,

$\alpha_\text{s}(\omega)$, γ_s) and intermediate theoretical constructs (S, P, Ω_R). Understanding and being able to use these relationships is fundamental to quantitative laser spectroscopy.

This concludes this chapter on Saturation and Power Broadening. A very important set of concepts, and I hope this detailed walk-through has helped to clarify both the underlying physics and the mathematical formalism.