Chapter 3.1

Page 1:

Alright everyone, welcome to this segment of our Phys 608 Laser Spectroscopy course. Today, we delve into a truly fundamental concept: Chapter 3.1, focusing on the Natural Linewidth. This is a topic that lies at the very heart of understanding how atoms and molecules interact with light, and it dictates the ultimate precision we can achieve in many spectroscopic measurements. So, let's begin our exploration.

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We begin by asking a crucial question about the Natural Linewidth – Why should we study it? What is its significance?

The first bullet point provides a foundational answer: "Every real optical transition exhibits a finite spectral width; this width is called the natural linewidth." Now, this is a departure from a naive, introductory quantum mechanics picture where you might imagine an atomic transition occurring at an infinitesimally sharp, single frequency – a δ function, if you will. In reality, that's not the case. Even for an isolated atom, completely free from external perturbations like collisions or Doppler shifts, an optical transition will always have a certain breadth in frequency, or wavelength. This intrinsic, minimum possible width is what we term the "natural linewidth." It's a consequence of the quantum mechanical nature of light emission and absorption and, as we'll see, is intimately linked to the finite lifetime of excited states. Think of it as a fundamental fuzziness imposed by nature itself on the energy, and therefore frequency, of a transition.

The second bullet point underscores the practical importance: "The natural linewidth sets an ultimate resolution limit for high-precision laser spectroscopy, frequency metrology, and atomic clocks." This is where the rubber meets the road for experimentalists. If you're trying to resolve two closely spaced spectral lines, or measure a transition frequency with extreme accuracy – perhaps for testing fundamental theories, or for building the world's most precise atomic clocks – the natural linewidth is

your ultimate, unavoidable barrier. No matter how perfect your laser, how stable your environment, you cannot measure a spectral feature with a precision that surpasses this inherent width. It’s the physical limit. Understanding its origins and characteristics is therefore paramount for anyone pushing the frontiers of precision measurement. For instance, in frequency metrology, where we define the second based on atomic transitions, the narrower the natural linewidth of the chosen transition, the more precisely we can realize that definition.

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Continuing our motivation for studying natural linewidth, the next point delves into the deeper physics: "Understanding the microscopic origin of this width links quantum electrodynamics (spontaneous emission) with a simple classical analogue (damped harmonic oscillator)." This is a beautiful aspect of the physics here. The true, rigorous explanation for natural linewidth comes from Quantum Electrodynamics, or QED, specifically through the process of spontaneous emission – an excited atom decaying by emitting a photon due to its interaction with the vacuum fluctuations of the electromagnetic field. This is a profoundly quantum concept. However, quite remarkably, we can gain enormous insight and even derive the correct lineshape using a much simpler classical model: that of a damped harmonic oscillator. Think of an electron in an atom as being tethered by a spring, and as it oscillates, it radiates energy, causing its oscillations to damp down. This classical picture, as we will develop, surprisingly captures the essence of the Lorentzian lineshape associated with natural broadening. This connection between a sophisticated QED phenomenon and an intuitive classical model is not only elegant but also pedagogically very powerful.

And this leads to our approach, as stated in the final bullet: "We shall build the entire description step-by-step, defining every symbol, showing every algebraic move, and visualising each physical mechanism." That's precisely what we're going to do. We'll start with this classical model,

carefully define all our terms, work through the mathematics meticulously, and then connect it back to the quantum mechanical reality. The goal is not just to arrive at a formula, but to build a deep, intuitive understanding of where the natural linewidth comes from and why it behaves the way it does. So, expect a detailed journey.

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Now, let's visualize what we mean by "Natural Linewidth: Ideal vs. Realistic Spectral Lines." This page presents a very instructive graph.

On the horizontal axis, we have frequency, denoted by the Greek letter ν, often measured in Hertz.

At the center, a specific frequency ν_0, or nu naught, is marked. This ν_0 represents the nominal or central frequency of an atomic transition – for example, the frequency corresponding exactly to the energy difference between two atomic levels, $E_2 - E_1$, divided by Planck's constant h.

On the vertical axis, we have Intensity, labeled 'I'. This represents the strength of the absorption or emission signal as a function of frequency. It's normalized here, peaking at a value of 1 for the realistic line.

Now, observe the bright blue vertical line labeled "Ideal Transition (δ-function)". This represents the hypothetical scenario where an atomic transition occurs at one, and only one, perfectly defined frequency, ν_0. Its width would be zero. This is the simplified picture you might get if you ignore the finite lifetime of excited states. It’s a mathematically convenient idea but not physically realized for radiative transitions.

In contrast, the red curve, labeled "Natural Linewidth (Lorentzian)", depicts a realistic spectral line. Notice several key features:

1. It's centered at the same frequency, ν_0.

2. It has a characteristic bell shape, but it's not a Gaussian. This specific shape is called a Lorentzian profile, and we will derive why it takes this form.

3. Crucially, it has a finite width. The graph indicates this width with a double-headed arrow labeled "$\Delta\nu$".

This $\Delta\nu$ is the Full Width at Half Maximum, or FWHM, a standard measure of the width of a spectral feature. It's the difference in frequency between the two points on the curve where the intensity has dropped to half of its maximum value. This $\Delta\nu$ is, in this context of an isolated, stationary atom, the natural linewidth.

So, this graph beautifully encapsulates the core idea: real transitions are not infinitely sharp lines but have a spread of frequencies, a profile, characterized by the natural linewidth, which, for this fundamental case, is Lorentzian. Our task is to understand why this shape and width arise.

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Alright, let's embark on understanding the origin of this natural linewidth. This page is titled "Spontaneous Emission ⇔ Classical Damped Oscillator Analogy," and it sets up the conceptual framework we'll be using.

First, the "Quantum viewpoint": "An excited atom in state $|i\rangle$ (ket i) of energy E_{i} can spontaneously emit a photon and fall to state $|k\rangle$ (ket k) of energy E_{k}." This is the quantum mechanical picture. We have an atom, initially in an excited energy state, which we label 'i', with energy E_{i}. This state is not perfectly stable. Due to its interaction with the quantum vacuum, the atom can, without any external prompting, transition to a lower energy state, labeled 'k', with energy E_{k}. In doing so, it releases the excess energy by emitting a photon. This is spontaneous emission, a concept famously introduced by Einstein in his 1917 paper. The key here is "spontaneously" – it's an inherent decay process.

The second bullet point quantifies the energy of this emitted photon:

$$E_{\mathrm{i}} - E_{\mathrm{k}} = h\nu_{ik} = \hbar\omega_{ik}$$

This is the Bohr frequency condition. The energy of the emitted photon, $h\nu_{ik}$ (where h is Planck's constant and ν_{ik} is the photon's frequency), must precisely match the energy difference between the initial and final atomic states, $E_{\mathrm{i}} - E_{\mathrm{k}}$. Equivalently, we can express this in terms of angular frequency, ω_{ik} (which is $2\pi\nu_{ik}$), so the energy difference is also $\hbar\omega_{ik}$, where $\hbar$ is the reduced Planck constant (h divided by 2π). This defines the central frequency of the transition we were talking about.

Now, for the "Classical analogue": "Treat the atom's bound electron as a point charge of mass m on a spring (restoring force constant k), losing energy to radiation." This is where we make a conceptual leap to a classical model. Imagine an electron in an atom. It's bound to the nucleus, and we can crudely model this binding force as a spring. So, the electron, with its charge and mass m, behaves like a mass on a spring. If this electron is set into oscillation (analogous to the atom being in an excited state), classical electromagnetism tells us that an accelerating charge radiates electromagnetic waves. This radiation carries away energy. As the oscillating electron loses energy, the amplitude of its oscillation must decrease – it's a damped oscillation. This damping due to energy loss via radiation is the classical counterpart to the quantum atom losing energy via spontaneous emission. The restoring force constant, little k, determines how "stiff" the spring is, which will relate to the oscillation frequency.

This analogy, though simple, is remarkably powerful and will allow us to derive the lineshape.

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Let's continue developing our classical damped oscillator model.

The first point defines the "Natural (undamped) angular frequency": "$\omega_0 = \sqrt{\frac{k}{m}}$ [rad s^{-1}]." This is the standard formula for the angular frequency of a

simple harmonic oscillator consisting of a mass 'm' attached to a spring with force constant 'k'. If there were no damping, no energy loss, the electron would oscillate indefinitely at this angular frequency, ω_0. The units are radians per second, as indicated. This ω_0 will be the classical equivalent of the transition angular frequency ω_{ik} from the quantum picture.

The second point introduces the crucial element of damping: "Energy loss to the radiation field produces a friction-like term characterised by the damping constant γ [s^{-1}]." As our classical electron oscillates, it radiates, and this radiation carries energy away from the oscillator. This energy loss acts like a damping force, or a "friction-like" term, opposing the motion. In the equation of motion for the oscillator (which we'll see soon), this effect will be characterized by a damping constant, represented by the Greek letter γ. The units of γ are inverse seconds (s^{-1}). A larger γ means stronger damping, or a faster loss of energy. This γ will turn out to be directly related to the lifetime of the excited state in the quantum picture and, consequently, to the natural linewidth.

The final bullet point gives us an important physical insight for typical atomic systems: "For real optical transitions, $\gamma \ll \omega_0$; the oscillator completes many cycles before noticeably decaying." This is a key condition, often referred to as the "weak damping" or "high Q-factor" regime. In most atomic transitions in the optical part of the spectrum, the rate of energy loss (represented by γ) is very small compared to the oscillation frequency (ω_0). This means the electron, in our classical model, will oscillate many, many times, almost as if it were undamped, before the amplitude of its oscillation significantly decreases. Think of a high-quality bell; it rings for a long time after being struck, completing many oscillations before the sound dies away. This condition, $\gamma \ll \omega_0$, will allow for some useful approximations later on. It essentially means the radiated wave is a long, slowly decaying wavetrain, which, as we'll see through Fourier analysis, corresponds to a narrow frequency spectrum.

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This page provides a helpful visual for the "Classical Damped Oscillator Analogy," specifically illustrating the "Classical Analogue: Damped Mass-Spring System Emitting Waves."

Let's break down what we're seeing. On the left, we have a representation of a mass-spring system. There's a blue rectangular block, representing our oscillating mass (the electron in our analogy). This mass is attached to a coiled spring, which is itself fixed to a gray support on the far left. The spring represents the restoring force binding the electron to the atom. The label "wwwwwww" under the spring just emphasizes its spring-like nature.

From the oscillating blue mass, we see orange wavy lines extending to the right, with arrowheads indicating they are propagating away. These represent the electromagnetic waves being emitted by the accelerating charge (our oscillating electron). As the electron oscillates, it radiates energy in the form of these waves.

Crucially, there's a dashed arrow pointing towards the oscillating system, and then another dashed arrow leading away from the emitted waves, both associated with the symbol γ, Damping Constant. This visually links the emission of waves (energy loss) to the concept of damping. The act of emitting these waves causes the oscillator's energy to decrease, leading to the damping of its motion. The strength of this damping is quantified by γ.

So, this simple diagram effectively captures the core idea: an oscillating charge (mass on a spring) emits radiation, loses energy, and therefore its oscillations are damped. This classical picture will be the foundation for deriving the equation of motion and, ultimately, the spectral lineshape.

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Now we move to formalize the classical model with an "Equation of Motion — Full Differential Form." This is Slide 3 in this sequence.

The first bullet point states: "Newton's second law for the damped oscillator gives..." and then presents the equation:

$$\ddot{x}(t) + \gamma\,\dot{x}(t) + \omega_0^2\,x(t) = 0$$

Let's break this down. This is a second-order linear homogeneous ordinary differential equation. It describes the motion of a damped harmonic oscillator.

* $\ddot{x}(t)$ is the second time derivative of $x(t)$. * $\gamma\,\dot{x}(t)$ involves the first time derivative of $x(t)$. * $\omega_0^2\,x(t)$ involves $x(t)$ itself. * The sum of these three terms equals zero, indicating no external driving force for now (we're considering spontaneous emission, which is an undriven decay).

The next three bullet points define the terms in this equation:

* $x(t)$: displacement of the charge from equilibrium [m]. So, $x(t)$ represents the position of our oscillating electron (the charge) as a function of time, measured from its equilibrium position. Its units are meters. * $\dot{x}(t) = \frac{dx}{dt}$: velocity [m s^{-1}]. $\dot{x}(t)$ is the first derivative of displacement with respect to time, which is the instantaneous velocity of the charge. Units are meters per second. * $\ddot{x}(t) = \frac{d^2x}{dt^2}$: acceleration [m s^{-2}]. $\ddot{x}(t)$ is the second derivative of displacement with respect to time, which is the instantaneous acceleration of the charge. Units are meters per second squared.

Finally, the slide says: "Physical interpretation of each term." This is crucial for understanding what the equation truly represents, and we'll look at that on the next page. This equation is absolutely central. It's the mathematical embodiment of our classical damped oscillator. Its solution will tell us how the electron's oscillation decays over time, and from that, we'll deduce the frequency spectrum of the emitted light.

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This page continues our discussion from the previous one, providing the "Physical interpretation of each term" in the equation of motion: $\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = 0$. Let's look at each term, assuming we've multiplied the entire equation by mass m to make them forces (since $F = ma$):

* The first bullet is actually referring to the term $m\omega_0^2 x$ (if we consider $m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = 0$). The slide simplifies this to "omega sub zero squared x : restoring force $-kx$." Recall that ω_0^2 is equal to $\frac{k}{m}$, where 'k' is the spring constant. So, $m\omega_0^2 x$ is simply kx. This term, $-kx$ (the minus sign indicates it's a restoring force, always acting to pull the displacement x back towards equilibrium), is Hooke's Law. It's the spring force that tries to bring the oscillating charge back to its central position.

* The second bullet point refers to the term $m\gamma\dot{x}$. The slide writes "gamma x dot : radiative friction (proportional to velocity)." This term, $m\gamma\dot{x}$ (or just $\gamma\dot{x}$ if we're looking at the equation without the overall m factor), represents the damping force. It's proportional to the velocity, $\dot{x}$, and acts in the opposite direction to the velocity, hence it's a dissipative force. In our model, this "friction" is due to the energy being lost through the emission of electromagnetic radiation. It's what causes the oscillations to die down. The constant γ quantifies the strength of this radiative damping.

* The third bullet point refers to $m\ddot{x}$. The slide has "x double dot : inertial response." This term, $m\ddot{x}$, is simply mass times acceleration, which from Newton's second law ($F = ma$) is the net force. In the context of the equation rearranged as $m\ddot{x} = -kx - m\gamma\dot{x}$, it represents the inertial response of the mass m to the sum of the restoring force and the damping force. It's the "ma" part of $F = ma$.

So, the equation of motion, $m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = 0$, is essentially a statement of Newton's second law: mass times acceleration ($m\ddot{x}$) equals the sum of the forces acting on the electron – the restoring spring-like force ($-m\omega_0^2 x$) and the radiative damping force ($-m\gamma\dot{x}$). Understanding these

individual contributions is key to appreciating how the oscillator behaves. The interplay between the restoring force trying to sustain oscillations and the damping force trying to quell them dictates the entire dynamic.

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Slide 4: Exact Solution with Clear Initial Conditions

Now we move to "Slide 4: Exact Solution with Clear Initial Conditions." Having established the equation of motion, our next step is to solve it.

The first bullet point specifies the initial conditions we'll use: "Choose initial displacement $x(0) = x_0$ and initial velocity $\dot{x}(0) = 0$." This means at time $t = 0$, our classical oscillator (the electron) is pulled aside to some maximum displacement, x_0, and then released from rest. This is a common and convenient set of initial conditions. It's like pulling a pendulum bob to one side and letting it go without giving it an initial push.

The second bullet point addresses how we find the solution: "Characteristic equation

$$r^2 + \gamma r + \omega_0^2 = 0$$

yields roots..." For a second-order linear homogeneous differential equation like ours,

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 \, x = 0,$$

we assume a solution of the form $x(t) = e^{rt}$. Substituting this into the differential equation leads to the characteristic (or auxiliary) quadratic equation for 'r':

$$r^2 + \gamma r + \omega_0^2 = 0.$$

The roots of this quadratic equation, r_1 and r_2, are given by the standard quadratic formula:

$$r_{1,2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}.$$

This can be rewritten, as shown on the slide, by factoring out an i (the imaginary unit, $\sqrt{-1}$) from the square root term if $\frac{\gamma^2}{4}$ is less than ω_0^2 (which is the underdamped case, typical for optical transitions as γ is small):

$$r_{1,2} = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}.$$

The slide then simplifies the notation:

$$-\frac{\gamma}{2} \pm i\omega.$$

This leads to the third bullet point: "Define the damped oscillation frequency ω." The term ω here, which is $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$, is the actual angular frequency of the damped oscillations. Notice that it's slightly less than ω_0, the natural undamped frequency, due to the presence of damping (γ). However, as we discussed, for weak damping ($\gamma \ll \omega_0$), $\frac{\gamma^2}{4}$ will be very small compared to ω_0^2, so ω will be very close to ω_0.

These roots, r_1 and r_2, are complex conjugates:

$$r_1 = -\frac{\gamma}{2} + i\omega \quad \text{and} \quad r_2 = -\frac{\gamma}{2} - i\omega.$$

They are crucial for constructing the general solution for $x(t)$.

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Continuing from the previous page where we found the roots of the characteristic equation, we now define the damped oscillation frequency and present the solution.

First, the explicit definition of the damped angular frequency, ω, is given:

“

$$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \quad [\text{rad s}^{-1}]$$

”

As mentioned, this ω is the angular frequency at which the system actually oscillates when damping is present. It's in radians per second. The term $\frac{\gamma^2}{4}$ under the square root shows how damping reduces the oscillation frequency compared to the undamped frequency ω_0. For the weak damping case typical in atomic physics (γ much smaller than ω_0), ω is very, very close to ω_0.

Next, the slide presents: “The real-valued solution matching the initial conditions is...”

Recall our initial conditions were $x(0) = x_0$ (initial displacement) and $\dot{x}(0) = 0$ (initial velocity zero). Given the complex conjugate roots

$$r_{1,2} = -\frac{\gamma}{2} \pm i\,\omega,$$

the general solution for $x(t)$ is of the form

$$x(t) = e^{-\frac{\gamma t}{2}}\big(A\cos(\omega t) + B\sin(\omega t)\big).$$

Applying the initial conditions allows us to solve for the constants A and B.

The result, as shown, is:

“ \[x(t) = x_0\, e^{-\frac{\gamma t}{2}} \left\[\cos(\omega t) + \frac{\gamma}{2\omega}\, \sin(\omega t) \right]

”

Let's analyze this solution:

1. “ x_0 ”: This is the initial amplitude, setting the overall scale of the oscillation.

2. “ e^{-\frac{\gamma t}{2}} ”: This is an exponential decay term. It shows that the amplitude of the oscillation decreases exponentially with time, with a decay rate constant of \frac{\gamma}{2}. This is the direct consequence of the damping. If \gamma were zero (no damping), this term would be 1, and the amplitude would remain constant.

3. “\cos(\omega t) + \frac{\gamma}{2\omega}\, \sin(\omega t)”: This part describes the oscillatory behavior. It's a sum of a cosine and a sine term, both oscillating at the damped angular frequency \omega. The relative amplitudes of the cosine and sine terms are determined by the initial conditions and the damping factor \gamma. The \frac{\gamma}{2\omega} factor in front of the sine term arises from satisfying the \dot{x}(0)=0 condition.

This equation for x(t) is the complete description of how our classical electron's displacement changes over time, given our chosen initial state. It shows an oscillation whose amplitude is exponentially decaying. This decaying oscillation is key to understanding why the emitted light is not monochromatic.

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This page presents a graph illustrating the solution we just derived: "Damped Harmonic Oscillator: x(0)=x_0, \dot{x}(0)=0".

Let's examine the graph's features:

The vertical axis is "Displacement x(t) [m]", representing the displacement of our oscillator in meters. It ranges roughly from -1.01 to 1.01. The horizontal axis is "Time t [s]", in seconds, ranging from 0 to about 5.5 seconds in this particular plot.

The solid blue line is the actual plot of x(t) versus t. You can clearly see the characteristic behavior of a damped oscillation: 1. At t=0, the displacement is at its maximum initial value, x_0 (which is 1.0 meter according to the parameters listed). 2. The displacement then oscillates back and forth around the equilibrium position (x=0). 3. The amplitude of these oscillations progressively decreases over time. This reduction in amplitude is governed by the exponential decay term e^{-\gamma t/2} that we saw in the solution.

The dashed gray lines form an envelope around the blue oscillatory curve. These lines represent \pm x_0 e^{-\gamma t/2}. The blue curve always stays within this decaying exponential envelope. This visually emphasizes the exponential decay of the oscillation's amplitude.

The parameters used for this specific plot are listed at the top: * "x_0=1.0\,\text{m}" (initial displacement) * "\gamma=0.8\,\text{s}^{-1}" (damping constant) * "\omega_0=4.0\,\text{rad s}^{-1}" (natural undamped angular frequency)

From these, the damped angular frequency \omega is calculated as "\omega=3.98\,\text{rad s}^{-1}". Notice how, even with a \gamma of 0.8 and an \omega_0 of 4.0 (so \gamma is not extremely small compared to \omega_0 in this illustrative plot, it's 20%), \omega is still quite close to \omega_0 (3.98 vs 4.0). In real atomic systems, \gamma would be many orders of magnitude smaller than \omega_0, making \omega virtually indistinguishable from \omega_0.

This graph is a perfect visual representation of the mathematical solution for x(t). It shows the "ringing down" of the oscillator, which is the classical picture of an atom emitting a wavetrain of finite duration and decaying amplitude.

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We now move to "Slide 5: Small-Damping Approximation and Atomic Frequency Link." This is where we leverage the physical reality of atomic systems to simplify our mathematical description.

The first bullet point states the condition for the "Optical regime: \gamma much, much less than \omega_0 , implies \omega is approximately equal to \omega_0 ." As we've discussed, for most optical transitions in atoms, the damping constant \gamma (related to the inverse lifetime of the excited state) is very significantly smaller than the natural oscillation frequency \omega_0 (related to the transition energy). When \gamma is much smaller than \omega_0 , the term \gamma^2/4 in the expression for \omega (square root of \omega_0^2 - \gamma^2/4) becomes negligible compared to \omega_0^2 . Thus, the damped frequency \omega becomes almost identical to the undamped frequency \omega_0 . This is an excellent approximation for atomic systems.

The second bullet point discusses a simplification of our solution for x(t) based on this: "Term \frac{\gamma}{2\omega} times sine of \omega t is of order \gamma/\omega_0 ; neglecting it introduces a relative error less than 10^{-7} for typical atoms." Recall our full solution for x(t) was

$$x(t) = x_0\, e^{-\gamma t/2} \left[\cos(\omega t) + \frac{\gamma}{2\omega} \sin(\omega t)\right].$$

The term $\frac{\gamma}{2\omega}$ multiplies the $\sin(\omega t)$ part. Since ω is approximately ω_0 , this coefficient is of the order $\frac{\gamma}{2\omega_0}$. Because γ is much, much smaller than ω_0 , this coefficient is very small. For typical atomic parameters, the contribution of this sine term to the overall amplitude is extremely tiny. Neglecting this small sine term simplifies the expression for x(t) considerably, and the error introduced by doing so is usually very small, on the order of 10^{-7} or even less, which is often negligible in the context of lineshape calculations.

This leads to the third bullet point, which presents the simplified form of the motion: "Simplified, yet extremely accurate, motion becomes x(t) equals x_0 , e^{-\gamma t/2} , times cosine of \omega_0 t ." Here, we've done two things: 1. We've dropped the small \frac{\gamma}{2\omega} \sin(\omega t) term. 2. We've approximated the damped frequency \omega by the undamped natural frequency \omega_0 in the cosine term, as per the first bullet point.

This expression,

x(t) = x_0\, e^{-\gamma t/2} \cos(\omega_0 t), is much simpler. It describes an oscillation at the natural frequency \omega_0 , whose amplitude x_0\, e^{-\gamma t/2} decays exponentially with a rate constant \gamma/2 . This is the form we will predominantly use for our subsequent analysis, particularly for the Fourier transform, because it captures the essential physics very accurately under the weak damping condition.

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This page focuses on how we "Identify atomic frequency" within our classical model and link it to the quantum mechanical description.

The first bullet point presents the key relationship:

\omega_0 = 2\pi \nu_0 = \omega_{ik} = \frac{E_\text{i} - E_\text{k}}{\hbar}.

Let's unpack this.

* \omega_0 is the natural angular frequency of our classical oscillator (in radians per second).

* 2\pi \nu_0 relates this angular frequency \omega_0 to the ordinary frequency \nu_0 (in Hertz), which is often more directly measured in experiments.

* \omega_{ik} is the angular frequency associated with the quantum transition between an initial state 'i' and a final state 'k'.

* \frac{E_\text{i} - E_\text{k}}{\hbar} is the Bohr frequency condition from quantum mechanics. It states that the energy difference between the two quantum states, E_\text{i} - E_\text{k}, when divided by the reduced Planck constant \hbar, gives the angular frequency of the photon emitted or absorbed during the transition.

So, this equation establishes the crucial link: the natural oscillation frequency \omega_0 of our classical model is identified with the Bohr transition frequency \omega_{ik} determined by the energy level structure of the atom. This is how our classical model connects to the quantum reality of atomic energy levels.

The second bullet point simply defines: \nu_0 : optical frequency [Hz].

This is the central frequency of the spectral line we are considering, typically in the optical range (hundreds of terahertz for visible light).

The third bullet point gives the value of the reduced Planck constant:

\hbar = 1.054571817 \times 10^{-34}\,\text{J s}.

This fundamental constant appears everywhere in quantum mechanics and, through the Bohr condition, links energy differences to frequencies.

So, the central frequency of our decaying classical oscillator is set by the energy spacing of the atomic levels involved in the transition. The damping of this oscillation, characterized by \gamma, will determine the width of the spectral line around this central frequency.

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Now we address a pivotal question with "Slide 6: Why a Frequency Distribution Appears." We have a decaying oscillation; how does that lead to a spread of frequencies, i.e., a lineshape?

The first bullet point sets up a contrast: "Constant-amplitude oscillation implies a single frequency." If our classical oscillator were to oscillate

forever with a constant amplitude at frequency \omega_0 (i.e., if \gamma were zero), its Fourier transform would be a delta function at \omega_0. It would emit perfectly monochromatic light at that single frequency. This is the ideal, infinitely sharp line we saw earlier.

The second bullet point explains the consequence of damping: "Exponential decay multiplies the cosine; time truncation introduces a spread in frequency (Fourier theorem)." However, our oscillator's amplitude is *not* constant. It's given by

$$x(t) = x_0 \, e^{-\frac{\gamma t}{2}} \cos(\omega_0 t)$$

We have a cosine wave whose amplitude is being modulated (multiplied) by an exponential decay function. This exponential decay effectively "truncates" the wave in time; it doesn't last forever. A fundamental result from Fourier analysis (often called the time-frequency uncertainty principle, though here more directly related to the properties of Fourier transforms) states that if a signal is limited in duration (truncated in time), its frequency spectrum must be spread out. A signal that exists for all time can be a single frequency. A signal that exists for a finite time, or whose amplitude decays, cannot be a single frequency. The shorter the duration in time, the broader the spread in frequency. The exponential decay acts as a smooth kind of truncation, and this directly leads to a broadening of the frequency content.

The third bullet point states our "Goal: Determine the amplitude spectrum A(\omega) such that x of t equals 1 over the square root of (2 pi), times the integral from 0 to infinity, of A(\omega), e to the power of (i \omega t), d\omega." This equation expresses x(t) as an inverse Fourier transform of some function A(\omega). A(\omega) is called the amplitude spectrum or frequency spectrum. It tells us how much "amplitude" or "strength" is associated with each angular frequency \omega in the constitution of our time-domain signal x(t). Our objective is to find this A(\omega). The integral is shown from 0 to infinity, which is one convention for the inverse Fourier

transform; often it's from minus infinity to plus infinity, and the prefactor might vary depending on the Fourier transform convention used. The slide seems to use a specific convention where A(\omega) would then be found using an integral from minus infinity to plus infinity for the forward transform. Let's clarify the one on the slide:

$$x(t) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{\infty} A(\omega) e^{i\omega t} \, d\omega$$

This seems to be a slightly non-standard form, possibly for a specific context or a typo; usually the prefactor for the symmetric FT pair is $1/\sqrt{2\pi}$ or $1/(2\pi)$ and 1. However, the core idea is that x(t) is a superposition of pure sinusoids $e^{i\omega t}$ with amplitudes given by A(\omega).

The final bullet point highlights the "Key mathematical tool: Fourier transform of x(t)." To find A(\omega), we need to compute the Fourier transform of our time-domain signal

$$x(t) = x_0 \, e^{-\frac{\gamma t}{2}} \cos(\omega_0 t)$$

The Fourier transform will decompose this decaying oscillation into its constituent frequencies, revealing the spectral lineshape. This is the mathematical bridge from the time-domain behavior (damped oscillation) to the frequency-domain observation (spectral line profile).

Page 16

This page offers a "Conceptual Diagram: Damped Oscillation and Frequency Broadening," which beautifully visualizes the connection between the time domain and frequency domain descriptions.

On the left side, we have a plot labeled "Time Domain: Damped Oscillation."

- The vertical axis is "Amplitude x(t) ". - The horizontal axis is "Time (t)". - The blue curve shows $x(t) = x_0 \, e^{-\gamma t/2} \, \cos(\omega_0 t)$. It's

an oscillation starting at a maximum amplitude at t = 0 and then decaying exponentially. The specific values on the time axis are 0.0, 1.0, 2.0. - Red dashed lines show the exponential envelope plus or minus x_0 \, e^{-\gamma t/2} , within which the blue oscillation is contained.

An arrow labeled "Fourier Transform" points from this time-domain plot to the right-side plot. This signifies that we apply the mathematical operation of a Fourier transform to the signal x(t) .

On the right side, we have a plot labeled "Frequency Domain: Amplitude Spectrum."

- The vertical axis is labeled " |A(\omega)| ", representing the magnitude of the amplitude spectrum. There are values 0.63 and 1.25 marked. - The horizontal axis is "Angular Frequency (\omega)". It's centered at " \omega_0 ", which is also identified as the "Atomic transition angular frequency (\omega_{ik}) = \frac{E_\text{i} - E_\text{k}}{\hbar} ". This explicitly links the center of the spectrum to the atomic energy levels. - The orange curve shows the resulting amplitude spectrum |A(\omega)| . This is the Lorentzian profile we've been discussing. It peaks at \omega = \omega_0 and falls off symmetrically on either side. - A horizontal dotted line is drawn at half the maximum amplitude, and a double-headed arrow indicates the " \text{FWHM} = \gamma ". This visually demonstrates that the full width at half maximum of this Lorentzian spectral profile is equal to the damping constant \gamma from our time-domain equation. (Note: The label actually says \text{FWHM} = \sqrt{\gamma} which seems like a typo, it should be \text{FWHM} = \gamma in angular frequency for the intensity, or related to \gamma for the amplitude spectrum depending on the exact definition and normalization. Given the later slides, FWHM in angular frequency for intensity is \gamma . Let's assume for now that \gamma is indeed the characteristic width parameter.)

This diagram is extremely powerful. It shows that a cosine wave that is exponentially damped in the time domain (finite effective duration)

transforms into a Lorentzian lineshape in the frequency domain. The faster the decay in time (larger \gamma), the broader the Lorentzian in frequency (larger \text{FWHM}). This is the essence of natural broadening.

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Slide 7: Step-by-Step Evaluation of the Fourier Transform

Now we get into the mathematics with "Slide 7: Step-by-Step Evaluation of the Fourier Transform." We're going to calculate A(\omega) from our x(t).

The first bullet says: "Start with..." and presents the formula for A(\omega):

$$A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt.$$

This is a standard definition of the Fourier transform. A(\omega) is the complex amplitude of the frequency component \omega in the signal x(t). The factor \frac{1}{\sqrt{2\pi}} is a common normalization for the Fourier transform pair.

The second bullet point applies a simplification: "Because x(t<0) = 0, change lower limit to 0." Our physical model for x(t), the decaying oscillation, starts at t=0. We assume x(t) is zero for all times t less than 0. This means the integrand is zero for t < 0, so we can change the lower limit of integration from -\infty to 0 without changing the result. So, A(\omega) becomes

$$A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x(t)\, e^{-i\omega t}\, dt.$$

The third bullet point is: "Substitute x(t) = x_{0}\, e^{-\frac{\gamma t}{2}} \cos(\omega_0 t)." This is our simplified, yet accurate, expression for the damped oscillation from Slide 13. We plug this into the integral.

$$A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \left[x_{0}\, e^{-\frac{\gamma t}{2}} \cos(\omega_0 t) \right] e^{-i\omega t}\, dt.$$

We can take the constant x_{0} outside the integral.

The fourth bullet point suggests a way to handle the cosine term: "Express $\cos(\omega_0 t) = \frac{1}{2}\left[e^{i\omega_0 t} + e^{-i\omega_0 t}\right]$ to integrate term-by-term." This is Euler's formula for cosine. Substituting this will convert the cosine into two complex exponential terms. The integral will then involve products of exponentials, which are easier to integrate. So, $\cos(\omega_0 t)e^{-i\omega t}$ becomes

$$\frac{1}{2} \left[e^{i\omega_0 t} e^{-i\omega t} + e^{-i\omega_0 t} e^{-i\omega t}\right],$$

which simplifies to

$$\frac{1}{2}\left[e^{-i(\omega - \omega_0)t} + e^{-i(\omega + \omega_0)t}\right].$$

Each of these terms is then multiplied by the decay factor $e^{-\frac{\gamma t}{2}}$. This sets up the integral for evaluation.

Page 18:

Continuing with the Fourier transform calculation, this page shows the integral after substituting Euler's formula for the cosine and combining exponents:

"A of omega equals $\frac{x_0}{2\sqrt{2\pi}}$ times the integral from 0 to ∞ , of $e^{-\frac{\gamma t}{2}}$ times, in square brackets, [$e^{i(\omega_0 - \omega)t}$ plus $e^{-i(\omega_0 + \omega)t}$], dt ."

Let's look at the terms inside the square brackets combined with the $e^{-\frac{\gamma t}{2}}$ term: The first term in the integrand becomes: $e^{-\frac{\gamma t}{2}} \times e^{i(\omega_0 - \omega)t} = e^{\left(-\frac{\gamma}{2} + i(\omega_0 - \omega)\right)t}$.

The second term in the integrand becomes: $e^{-\frac{\gamma t}{2}} \times e^{-i(\omega_0 + \omega)t} = e^{\left(-\frac{\gamma}{2} - i(\omega_0 + \omega)\right)t}$.

Notice that I've slightly rearranged the exponent in the first term inside the square brackets to group the real and imaginary parts for easier comparison with the standard integral form shown next. The slide has e^{i(\omega_0-\omega)t} , which is fine. My grouping is for foresight. Let's write it as:

e^{-\left(\frac{\gamma}{2} - i(\omega_0 - \omega)\right)t} for the first part, and

e^{-\left(\frac{\gamma}{2} + i(\omega_0 + \omega)\right)t} for the second part.

The next bullet point provides the key to solving this: "Each exponential integral uses..." The integral of e^{-(a - i b)t} \, dt , from 0 to \infty , equals \frac{1}{a - i b} , provided that a > 0 .

This is a standard result for the Laplace transform of 1, or simply integrating an exponential. The condition a > 0 ensures that the exponential decays to zero as t goes to infinity, so the integral converges. In our case, a will be \frac{\gamma}{2} . Since \gamma is a positive damping constant, \frac{\gamma}{2} is indeed greater than 0, so the integral will converge.

The final bullet point helps us "Identify a = \frac{\gamma}{2} , b = \omega - \omega_0 , or b = \omega + \omega_0 ."

Let's be careful here with the signs. For the first term in our integral, e^{\left(-\frac{\gamma}{2} + i(\omega_0 - \omega)\right)t} , we can write the exponent as -\left(\frac{\gamma}{2} - i(\omega_0 - \omega) \right)t. Comparing this to e^{-(a - i b)t} , we have:

a = \frac{\gamma}{2} b = \omega_0 - \omega .

So the integral of this part will be \frac{1}{\frac{\gamma}{2} - i(\omega_0 - \omega)} . Or, as the slide suggests by taking b = \omega - \omega_0 , the exponent is -\left(\frac{\gamma}{2} - i(\omega - \omega_0)(-1)\right)t = -

\left(\frac{\gamma}{2} + i(\omega - \omega_0)\right)t which is not quite matching the form -(a-i b)t unless b = -(\omega-\omega_0) .

Let's re-examine the first term: e^{i(\omega_0 - \omega)t} \times e^{-\frac{\gamma t}{2}} . This is e^{-\left(\frac{\gamma}{2} - i(\omega_0 - \omega)\right)t} . So, comparing to e^{-\alpha t} \, dt = \frac{1}{\alpha} , our \alpha here is \frac{\gamma}{2} - i(\omega_0 - \omega) . So the first integral is \frac{1}{\frac{\gamma}{2} - i(\omega_0 - \omega)} .

For the second term in our integral, e^{-i(\omega_0 + \omega)t} \times e^{-\frac{\gamma t}{2}} . This is e^{-\left(\frac{\gamma}{2} + i(\omega_0 + \omega)\right)t} . Comparing to e^{-\alpha t} , our \alpha here is \frac{\gamma}{2} + i(\omega_0 + \omega) . So the second integral is \frac{1}{\frac{\gamma}{2} + i(\omega_0 + \omega)} .

Let's use the slide's formulation: exponent is -(a-i b)t . For the first term: e^{i(\omega_0-\omega)t} \, e^{-\frac{\gamma t}{2}} = e^{-\left(\frac{\gamma}{2} - i(\omega_0-\omega)\right)t} . Here, a = \frac{\gamma}{2} , and we want to write -i(\omega_0-\omega) as + i b . So b = -(\omega_0-\omega) = \omega - \omega_0 . The integral is \frac{1}{a - i b} = \frac{1}{\frac{\gamma}{2} - i(\omega - \omega_0)} . This doesn't match the slide's next step.

Let's look ahead to slide 19: A(\omega) has denominators i(\omega-\omega_0) + \frac{\gamma}{2} . This means the integral was \frac{1}{\frac{\gamma}{2} - i(\omega_0-\omega)} which is \frac{1}{\frac{\gamma}{2} + i(\omega-\omega_0)} . This matches!

Okay, so if we have integral of e^{-Pt} \, dt = \frac{1}{P} . First term: P_1 = \frac{\gamma}{2} - i(\omega_0 - \omega) = \frac{\gamma}{2} + i(\omega - \omega_0) . Second term: P_2 = \frac{\gamma}{2} + i(\omega_0 + \omega) .

So, using P_1 and P_2 :

$$A(\omega) = \frac{x_0}{2\sqrt{2\pi}} \left[\frac{1}{\frac{\gamma}{2} + i(\omega - \omega_0)} + \frac{1}{\frac{\gamma}{2} + i(\omega + \omega_0)} \right]$$

This looks like the structure on the next page. The slide's "Identify $a = \frac{\gamma}{2},\ b = \omega - \omega_0$, or $b = \omega + \omega_0$ " is a bit compressed. Essentially, for the first term of the form $e^{i(\omega_0-\omega)t - \gamma t/2}$, the effective decay rate is $\frac{\gamma}{2} - i(\omega_0-\omega)$. So the integral is $\frac{1}{\frac{\gamma}{2} - i(\omega_0-\omega)}$. For the second term of the form $e^{-i(\omega_0+\omega)t - \gamma t/2}$, the effective decay rate is $\frac{\gamma}{2} + i(\omega_0+\omega)$. So the integral is $\frac{1}{\frac{\gamma}{2} + i(\omega_0+\omega)}$. We need to be careful with signs. $\frac{1}{\frac{\gamma}{2} - i(\omega_0-\omega)} = \frac{1}{\frac{\gamma}{2} + i(\omega-\omega_0)}$. This is the first term in the result on page 19. $\frac{1}{\frac{\gamma}{2} + i(\omega_0+\omega)} = \frac{1}{\frac{\gamma}{2} + i(\omega+\omega_0)}$. This is the second term. Perfect, this works out.

Page 19:

Slide 8: Resulting Amplitude Spectrum

After performing the integrations from the previous step, we arrive at the expression for $A(\omega)$.

The first bullet point says: "Carrying out the integrations yields..."

$$A(\omega) = \frac{x_0}{\sqrt{8\pi}} \left[\frac{1}{\frac{\gamma}{2} + i\left(\omega - \omega_0\right)} + \frac{1}{\frac{\gamma}{2} + i\left(\omega + \omega_0\right)} \right].$$

Let's check the constant. We had $\frac{x_0}{2\sqrt{2\pi}}$. If we write 2 as $\sqrt{4}$, then

$$2\sqrt{2\pi} = \sqrt{4} \cdot \sqrt{2\pi} = \sqrt{8\pi}.$$

So the prefactor \frac{x_0}{\sqrt{8\pi}} is correct.

The terms in the bracket are:

First term: \frac{1}{\frac{\gamma}{2} + i\left(\omega - \omega_0\right)} . This matches.

Second term: \frac{1}{\frac{\gamma}{2} + i\left(\omega + \omega_0\right)} . This matches.

So, this expression for A(\omega) is the Fourier transform of our damped cosine wave. This is the complex amplitude spectrum.

The second bullet point prompts us to consider the "Physical meaning of each denominator term." The denominators are crucial because their magnitudes determine the strength of A(\omega) . A(\omega) will be large when a denominator is small.

Let's look at the third bullet: "First term: frequencies near \omega_0 (resonant part)." The first denominator is \frac{\gamma}{2} + i\left(\omega - \omega_0\right) . This term becomes small (specifically, its imaginary part i\left(\omega - \omega_0\right) goes to zero) when the driving frequency \omega is very close to the natural frequency \omega_0 . When \omega = \omega_0 , the denominator is just \frac{\gamma}{2} , which is small (since \gamma is small). Therefore, this first term in the square brackets will be large when \omega is near \omega_0 . This is the "resonant" part of the spectrum. It describes the strong response or emission near the atom's natural transition frequency. This term will dominate the lineshape.

Now, the fourth bullet: "Second term: anti-resonant component near minus \omega_0 ; negligible around optical \omega approximately equal to \omega_0 ." The second denominator is \frac{\gamma}{2} + i\left(\omega + \omega_0\right) . This term would become small if \omega were close to -\omega_0. Since physical frequencies \omega are positive, and \omega_0 is a positive optical frequency (e.g., hundreds of terahertz), \omega would have to be a large negative frequency for this term to be resonant. For

positive optical frequencies \omega that are near the positive \omega_0 (where the first term is resonant), the term \omega + \omega_0 is approximately 2\omega_0 , which is very large. Thus, i\left(\omega + \omega_0\right) + \frac{\gamma}{2} is large, and \frac{1}{i\left(\omega + \omega_0\right) + \frac{\gamma}{2}} is very small. This second term is often called the "anti-resonant" or "counter-rotating" term. In most spectroscopic situations in the optical regime, where we are probing frequencies \omega near the positive \omega_0 , the contribution of this second term is utterly negligible compared to the first term, because 2\omega_0 is much, much larger than \frac{\gamma}{2} .

So, for practical purposes in laser spectroscopy, we can often ignore the second term when we are interested in the spectral line around \omega_0 .

Page 20:

This page continues the discussion from the previous one, focusing on the approximation of neglecting the anti-resonant term.

The bullet point states: "We retain only the resonant denominator when the absolute value of \omega - \omega_0 is much, much less than \omega_0 ." This condition, |\omega - \omega_0| \ll \omega_0 , defines the region "near resonance." When we are looking at frequencies \omega that are very close to the central transition frequency \omega_0 , the first term in A(\omega) (the resonant term) will be dominant.

In this regime, the first denominator, \frac{\gamma}{2} + i (\omega - \omega_0) , can become small if \omega is close to \omega_0 . The second denominator, \frac{\gamma}{2} + i (\omega + \omega_0) , will have \omega + \omega_0 \approx 2\omega_0 . Since \gamma is much smaller than \omega_0 , \frac{\gamma}{2} is much smaller than 2\omega_0 . So the second denominator is approximately i(2\omega_0) , and its magnitude is large.

Therefore, the first term, \frac{1}{\frac{\gamma}{2} + i (\omega - \omega_0)} , will be much larger than the second term, \frac{1}{\frac{\gamma}{2} + i (\omega + \omega_0)} . This justifies neglecting the second (anti-resonant) term. This approximation is extremely common and accurate for optical spectroscopy. It's sometimes called the "rotating wave approximation" in other contexts, although here it arises directly from considering the magnitudes of the terms.

By retaining only the resonant term, our expression for A(\omega) simplifies significantly:

A(\omega) \approx \frac{x_0}{\sqrt{8\pi}} \frac{1}{\frac{\gamma}{2} + i (\omega - \omega_0)}

This simplified A(\omega) will be used to derive the intensity profile of the spectral line.

Page 21:

Now we arrive at "Slide 9: Intensity Profile and Emergence of the Lorentzian." We've found the amplitude spectrum A(\omega) ; now we want the intensity, which is what is typically measured.

The first line states: "Spectral intensity I(\omega) of \omega is proportional to A(\omega) times A^*(\omega) (which is the magnitude of A(\omega) squared)." The intensity of the emitted light at a given frequency \omega is proportional to the square of the magnitude of the complex amplitude A(\omega) . A^*(\omega) denotes the complex conjugate of A(\omega) . So, I(\omega) is proportional to |A(\omega)|^2.

The first bullet point applies the approximation from the previous page: "Near resonance, drop anti-resonant term to obtain..." Using our simplified A(\omega) which is approximately

\frac{x_0}{\sqrt{8\pi}} \left[\frac{1}{\frac{\gamma}{2} + i(\omega - \omega_0)}\right],

we calculate |A(\omega)|^2.

The magnitude squared of \frac{1}{X + iY} is \frac{1}{X^2 + Y^2}. Here, X = \frac{\gamma}{2} and Y = (\omega - \omega_0) .

So, |A(\omega)|^2 is proportional to \frac{x_0^2}{8\pi} times \frac{1}{\left(\frac{\gamma}{2}\right)^2 + (\omega - \omega_0)^2}. The intensity I(\omega - \omega_0) is therefore proportional to this quantity. The slide writes this as:

I(\omega - \omega_0) = \frac{C}{(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}.

This is the celebrated Lorentzian lineshape function! It describes the spectral profile of the light emitted by our damped classical oscillator, and by analogy, the natural lineshape of an atomic transition.

The variable is effectively the detuning from resonance, (\omega - \omega_0). The function peaks when \omega = \omega_0 (detuning is zero) and falls off as the square of the detuning. The width of the peak is determined by \gamma.

The second bullet point defines C : "C : constant determined by chosen normalisation." The constant C incorporates factors like x_0^2 , \frac{1}{8\pi}, and any other proportionality constants. Its exact value depends on how we want to normalize the intensity profile (e.g., peak intensity to be 1, or area under the curve to be 1).

The third bullet point explicitly states: "Functional form 1 divided by (x squared plus a squared) is the Lorentzian." Indeed, if we let x = (\omega - \omega_0) and a = \frac{\gamma}{2} , our intensity profile I is proportional to \frac{1}{x^2 + a^2}. This is the standard mathematical form of a Lorentzian function, also known as a Cauchy distribution in statistics.

It's a beautiful result: the exponential decay in time (due to damping) Fourier transforms into a Lorentzian profile in frequency. This is the origin of the natural linewidth's characteristic shape.

Page 22:

This page displays a graph of the "Lorentzian Intensity Profile." Let's analyze it.

The vertical axis is "Normalized Intensity," ranging from 0.0 to 1.0. This means the peak intensity of the Lorentzian has been set to 1 for this plot.

The horizontal axis is "Frequency Offset (\omega - \omega_0)." This represents the detuning of the frequency \omega from the central resonance frequency \omega_0. The center of the plot is at an offset of 0, corresponding to \omega = \omega_0. The axis extends to values like -\frac{3\gamma}{2}, -\gamma, -\frac{\gamma}{2} on the left, and \frac{\gamma}{2}, \gamma, \frac{3\gamma}{2} on the right.

The blue curve is the Lorentzian profile itself. Key features:

1. It peaks at a frequency offset of 0 (i.e., at \omega = \omega_0), where its normalized intensity is 1.0.

2. It is symmetric around this peak.

3. It falls off as it moves away from the center, but it has relatively "heavy" tails compared to, say, a Gaussian profile. This means it decreases more slowly at large detunings.

4. The crucial feature highlighted is the Full Width at Half Maximum (FWHM). A horizontal dashed line is drawn at a normalized intensity of 0.5 (half the maximum). This line intersects the Lorentzian curve at two points. Vertical dashed lines drop from these intersection points to the frequency offset axis. The left intersection point is at a frequency offset of -\frac{\gamma}{2}. The right intersection point is at a frequency offset of +\frac{\gamma}{2}. The Full Width at Half Maximum (FWHM) is the

difference between these two frequency offsets: \left(+\frac{\gamma}{2}\right) - \left(-\frac{\gamma}{2}\right) = \gamma. A green double-headed arrow explicitly labels "FWHM = \gamma".

This is a very important result: For a Lorentzian intensity profile given by

\frac{C}{\left(\omega - \omega_0\right)^2 + \left(\frac{\gamma}{2}\right)^2}, the FWHM in angular frequency units is exactly equal to the damping constant \gamma from our original equation of motion. This directly connects the rate of energy decay (damping) of the classical oscillator to the observable width of the spectral line. A larger \gamma (faster decay) leads to a wider spectral line. A smaller \gamma (slower decay) leads to a narrower spectral line. This is the natural linewidth.

Page 23:

Slide 10: Normalised Lorentzian — Definitions and FWHM.

We're now on "Slide 10: Normalised Lorentzian — Definitions and FWHM." This section deals with how we precisely define and normalize the Lorentzian lineshape function.

The first bullet point states a general property: "Total area under any spectral line is the integrated intensity." If we integrate the intensity profile I(\omega) over all frequencies (from -\infty to +\infty), we get a measure of the total energy or power in the line. This integrated intensity is a fundamental quantity. It's denoted here as I_0 :

I_0 = \int_{-\infty}^{\infty} I(\omega) \, d\omega

Here, I(\omega) is our unnormalized Lorentzian

I(\omega) = \frac{C}{(\omega - \omega_0)^2 + (\gamma/2)^2}

The next bullet point, "Set...", introduces a normalized lineshape function, L(\omega - \omega_0) :

L(\omega - \omega_0) = \frac{I(\omega - \omega_0)}{I_0}

What we're doing here is defining a new function, L , which is the original intensity profile I , divided by its total integrated intensity I_0 . The purpose of this is stated in the "so that" clause on the next page.

This normalization ensures that the *area* under the curve L(\omega - \omega_0) is unity, which is a common convention for probability distributions or lineshape functions. It makes L(\omega - \omega_0) represent the probability density of emission/absorption occurring at a frequency offset (\omega - \omega_0) .

Page 24:

This page continues the discussion on the normalized Lorentzian.

The first line completes the thought from the previous page: "so that the integral from -\infty to +\infty of L(\omega-\omega_0) \, d(\omega-\omega_0) equals 1." Since L is I divided by I_0 (the integral of I), the integral of L must be 1. This confirms that L(\omega-\omega_0) is a lineshape function normalized to unit area. The differential d(\omega-\omega_0) is the same as d\omega since \omega_0 is a constant.

Now, we need to find the constant C from our original

I(\omega-\omega_0) = \frac{C}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}

such that this normalization holds when we construct L . The slide states: "Substituting and integrating fixes C ..."

We need to calculate

I_0 = \int_{-\infty}^{+\infty} \frac{C}{x^2 + a^2}\, dx,

where x = \omega-\omega_0 and a = \frac{\gamma}{2} . The integral of

\frac{1}{x^2 + a^2}\, dx

is

\frac{1}{a}\arctan\left(\frac{x}{a}\right).

Evaluated from -\infty to +\infty , \arctan\left(\frac{x}{a}\right) goes from -\frac{\pi}{2} to \frac{\pi}{2}, so the difference is \pi. Thus, the integral is

\frac{1}{a} \pi = \frac{1}{(\gamma/2)}\pi = \frac{2\pi}{\gamma}.

So,

I_0 = C\cdot\frac{2\pi}{\gamma}.

If we want the normalized L to be used to determine C in the context of the I_0 definition from the previous slide, it's a bit circular. Let's assume C is some constant from A(\omega)A^*(\omega) . The equation here says: "C equals I_0 times \gamma , divided by 2\pi ." This means that if our lineshape is

I(\omega-\omega_0) = \frac{\frac{I_0\gamma}{2\pi}}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2},

then its integral from -\infty to +\infty will be I_0 . This is a self-consistent definition. C represents the numerator required for the overall integral to be I_0 .

This leads to the "Final normalised profile" for L(\omega-\omega_0) , which is \frac{I(\omega-\omega_0)}{I_0} : "L of \omega-\omega_0 equals \frac{\gamma}{2\pi} divided by [(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2\]."

Let's check the normalization of this L. Its integral is

$$\frac{\gamma}{2\pi} \times \int \frac{1}{(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2} \, d\omega.$$

We found this latter integral to be $\frac{2\pi}{\gamma}$. So,

$$\frac{\gamma}{2\pi} \times \frac{2\pi}{\gamma} = 1.$$

Correct! This $L(\omega - \omega_0)$ is normalized to have unit area.

What is the peak value of this L? At $\omega = \omega_0$, the denominator is $\left(\frac{\gamma}{2}\right)^2$. So,

$$L_{\text{peak}} = \frac{\gamma}{2\pi} / \left(\frac{\gamma^2}{4}\right) = \frac{\gamma}{2\pi} \times \frac{4}{\gamma^2} = \frac{2}{\pi\gamma}.$$

This is an important form of the normalized Lorentzian.

Page 25:

This page summarizes the Full Width at Half Maximum (FWHM) relationships for our natural lineshape.

The first bullet point reiterates: "Full width at half maximum (FWHM) in angular frequency..." For the Lorentzian profile $L(\omega - \omega_0) = \frac{\frac{\gamma}{2\pi}}{(\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2}$, the peak value is $\frac{2}{\pi\gamma}$ at $\omega = \omega_0$. Half of this peak value is $\frac{1}{\pi\gamma}$. We need to find ω such that $L(\omega - \omega_0) = \frac{1}{\pi\gamma}$. So,

$$\frac{\frac{\gamma}{2\pi}}{(\Delta\omega_{\text{half}})^2 + \left(\frac{\gamma}{2}\right)^2} = \frac{1}{\pi\gamma},$$

where $\Delta\omega_{\text{half}}$ is $|\omega - \omega_0|$ at the half-maximum point. This gives

$$\frac{\gamma^2}{2} = (\Delta\omega_{\text{half}})^2 + \left(\frac{\gamma}{2}\right)^2.$$

$(\Delta\omega_{\text{half}})^2 = \frac{\gamma^2}{2} - \frac{\gamma^2}{4} = \frac{\gamma^2}{4}$. So, $\Delta\omega_{\text{half}} = \frac{\gamma}{2}$. The FWHM is twice this value, so $\text{FWHM}_\omega = 2 \times \frac{\gamma}{2} = \gamma$. Thus, as stated: $\Delta\omega_{\text{nat}} = \gamma$. The subscript ‘nat’ emphasizes this is the natural linewidth in angular frequency units.

Alternatively, we often express linewidths in ordinary frequency (Hertz) rather than angular frequency (radians per second). Since $\omega = 2\pi\nu$, a width

$\Delta\omega$ corresponds to a width $\Delta\nu = \frac{\Delta\omega}{2\pi}$. So, the FWHM in ordinary frequency is: $\Delta\nu_{\text{nat}} = \frac{\gamma}{2\pi}$.

The second bullet point clarifies notation: "Width notation $\Delta\nu$ uses ordinary frequency [Hz]." When you see $\Delta\nu$, it typically refers to the linewidth in Hertz. When you see $\Delta\omega$, it refers to the linewidth in radians per second. The relationship is always a factor of 2π.

So, the damping constant γ from our classical model, which we will soon connect to the atomic lifetime, directly gives the FWHM of the natural line in angular frequency units. This is a cornerstone result.

Page 26:

Slide 11: Alternative Gamma-Notation Common in Literature

This page introduces "Slide 11: Alternative Gamma-Notation Common in Literature." Sometimes, a slightly different parameterization of the Lorentzian is used, especially in contexts involving resonance phenomena.

The first bullet says: "Define half-width parameter $\Gamma = \frac{\gamma}{2}$." Instead of using γ (the full width at half maximum in angular frequency), some authors prefer to work with Γ, which is the *half* width at half maximum (HWHM) in angular frequency units. So, Γ is just γ divided by 2. Remember, little γ is our damping constant.

The second bullet shows how the "Spectral profile written as" $g(\omega - \omega_{ik})$ looks using this Γ:

$$g(\omega - \omega_{ik}) = \frac{\Gamma^2}{(\omega_{ik} - \omega)^2 + \Gamma^2}.$$

Here, ω_{ik} is the resonant angular frequency (equivalent to our ω_0). Notice the term $(\omega_{ik} - \omega)^2$ is the same as $(\omega - \omega_{ik})^2$. If we substitute $\Gamma = \frac{\gamma}{2}$, this becomes:

$$\frac{\left(\frac{\gamma}{2}\right)^2}{(\omega-\omega_{ik})^2+\left(\frac{\gamma}{2}\right)^2}.$$

This is one form. Let's check its peak value. When $\omega=\omega_{ik}$, $g=\frac{\Gamma^2}{\Gamma^2}=1$. So, this particular form is normalized such that its peak value is 1. The constant in the numerator is Γ^2.

Let's compare to our previous

$$L(\omega-\omega_0)=\frac{\frac{\gamma}{2\pi}}{(\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2}.$$

Its peak was $\frac{2}{\pi\gamma}=\frac{1}{\pi\Gamma}$. So they are different normalizations.

The third bullet confirms the peak value for this g: "Peak value $g(\omega_{ik})=1$ when normalised so that $I(\omega_{ik})=I_0$." This statement seems to be mixing two ideas. If $g(\omega_{ik})=1$, that is one type of normalization (peak to unity). If the intensity at the peak $I(\omega_{ik})$ is I_0 (where I_0 is often the integrated intensity, or just the peak intensity value itself depending on context), that's slightly different. If this g represents the lineshape such that

$$I(\omega)=I_{\text{peak}}\cdot g(\omega-\omega_{ik}),$$

then yes, $g(\omega_{ik})=1$ implies $I(\omega_{ik})=I_{\text{peak}}$.

The function g shown,

$$\frac{\Gamma^2}{(\omega_{ik}-\omega)^2+\Gamma^2},$$

indeed has a peak value of 1 at $\omega=\omega_{ik}$. This function is often written as

$$g(\omega)=\frac{1}{1+\left(\frac{\omega-\omega_{ik}}{\Gamma}\right)^2}$$

if we divide numerator and denominator by Γ^2.

It's important to be aware of these different notations (γ vs. Γ) and different normalizations (peak to unity, or area to unity) when reading literature, as they can sometimes cause confusion. The key physical parameter is always γ (or $\Gamma = \frac{\gamma}{2}$), which sets the width.

Page 27:

Continuing with the alternative capital Gamma notation for the Lorentzian profile:

The first bullet introduces a "Dimensionless detuning parameter chi": " $\chi = \frac{\omega_{ik}-\omega}{\Gamma}$." Or, equivalently, $\chi = \frac{\omega-\omega_{ik}}{\Gamma}$, if we absorb the minus sign (since it's often squared later or we consider magnitude). Let's stick to the slide: $\chi = \frac{\omega_{ik}-\omega}{\Gamma}$. This parameter χ measures how far the frequency ω is from the resonance ω_{ik}, in units of the half-width Γ. At resonance ($\omega = \omega_{ik}$), $\chi = 0$. When ω is one half-width away from resonance (i.e., $|\omega - \omega_{ik}| = \Gamma$), then $|\chi| = 1$.

The second bullet point shows the "Compact form" of the spectral profile g using this χ parameter: " $g(\omega - \omega_{ik}) = \frac{1}{1+\chi^2}$." Let's verify this. Our g from the previous page (normalized to peak at 1) was $\frac{\Gamma^2}{(\omega_{ik}-\omega)^2+\Gamma^2}$. If we divide the numerator and denominator by Γ^2, we get: $\frac{1}{\left(\frac{(\omega_{ik}-\omega)^2}{\Gamma^2}\right)+1}$. Since $\chi = \frac{\omega_{ik}-\omega}{\Gamma}$, then $\chi^2 = \left(\frac{\omega_{ik}-\omega}{\Gamma}\right)^2$. So, g indeed equals $\frac{1}{1+\chi^2}$. This is a very neat and common way to write a Lorentzian normalized to have a peak value of 1.

The third bullet point gives the "Integrated area in this normalisation": " $\int_{-\infty}^{+\infty} I(\omega)\, d\omega = \pi I_0 \Gamma$." Here, it's important to understand what $I(\omega)$ and I_0 represent. If $I(\omega) = I_0 \cdot g(\omega - \omega_{ik})$, where g is the function $\frac{1}{1+\chi^2}$ and I_0 is

the peak intensity (so $I(\omega_{ik}) = I_0$), then we need to integrate $I_0 \cdot \frac{\Gamma^2}{(\omega_{ik}-\omega)^2+\Gamma^2}\, d\omega$. The integral of $\frac{\Gamma^2}{x^2+\Gamma^2}\, dx$ from $-\infty$ to $+\infty$ is:

$$\Gamma^2 \cdot \left[\frac{1}{\Gamma}\arctan\left(\frac{x}{\Gamma}\right)\right] \quad \text{(evaluated from } -\infty \text{ to } +\infty\text{)}$$

$= \Gamma^2 \cdot \frac{1}{\Gamma} \cdot \pi = \pi\Gamma$.

So, the integral of $I(\omega)\, d\omega$ would be I_0 (the peak value) times π times Γ. This matches the slide perfectly: $\int_{-\infty}^{+\infty} I(\omega)\, d\omega = \pi I_0 \Gamma$. Here, I_0 must be interpreted as the peak value of $I(\omega)$.

Page 28:

This page provides a "Side-by-side comparison of $L(\omega)$ and $g(\omega)$ normalisations," which is very helpful for clarifying the differences between the two Lorentzian forms we've discussed.

On the left, we have the "$g(\omega - \omega_{ik})$ Normalization."

* The plot shows $g(\omega - \omega_{ik})$ on the vertical axis, peaking at 1.0 (as indicated by "1.0" and "0.90" on the axis, it's actually labelled 1.0 where it hits the peak from the text). The axis label is $g(\omega - \omega_{ik})$.

* The horizontal axis is $\omega - \omega_{ik}$, with points like -2γ, $-\gamma$, $-\frac{\gamma}{2}$, 0, $\frac{\gamma}{2}$, γ, 2γ shown. Wait, the x-axis is actually labelled in terms of capital Γ implicitly via the relation $\Delta\omega = 2\Gamma$. Let's check the parameters below. Parameters used: $\omega_0 = \omega_{ik} = 0$. $\gamma = 2$, $\Gamma = \frac{\gamma}{2} = 1$.

* So, $\Gamma = 1$. The x-axis units are effectively units of capital Γ.

* The FWHM is indicated by a red arrow "$\Delta\omega = 2\Gamma = 2$". At half maximum ($g = 0.5$), the width is indeed 2Γ.

* Below the plot: "Normalized so that $g(\omega_{ik}) = 1$ at peak." "Peak $g(0) = 1$ (by definition)." This is consistent with $g = \frac{1}{1+\chi^2}$.

On the right, we have the "$L(\omega - \omega_0)$ Normalization."

* The vertical axis is $L(\omega - \omega_0)$. The peak value is indicated around 0.318. The axis has 0.191, 0.286, 0.318, 0.382.

* The horizontal axis is $\omega - \omega_0$. Again, with $\gamma = 2$ and $\omega_0 = 0$.

* The FWHM is indicated by a red arrow "$\Delta\omega = \gamma = 2$".

* Below the plot: "Normalized so that $\int L(\omega - \omega_0)\, d\omega = 1$." "Peak $L(0) = \frac{2}{\pi\gamma}$". Let's check this peak value. With $\gamma = 2$, Peak $L(0) = \frac{2}{\pi \cdot 2} = \frac{1}{\pi}$. $\frac{1}{\pi}$ is approximately $\frac{1}{3.14159}$, which is about 0.3183. This matches the 0.318 value on the graph!

The "Parameters used for plots" at the bottom confirm: "$\omega_0 = \omega_{ik} = 0$ (center frequency), $\gamma = 2$, $\Gamma = \frac{\gamma}{2} = 1$." So, for both plots, the actual FWHM in angular frequency is 2. For g, this FWHM is 2Γ. For L, this FWHM is γ. Since $\Gamma = \frac{\gamma}{2}$, these are consistent.

* $g(\omega - \omega_{ik}) = \frac{1}{1+\left(\frac{\omega-\omega_{ik}}{\Gamma}\right)^2}$ is normalized to have a peak value of 1. Its FWHM is 2Γ. Its integral is $\pi\Gamma$.

* $L(\omega - \omega_0) = \frac{\gamma/(2\pi)}{(\omega-\omega_0)^2+(\gamma/2)^2}$ is normalized to have an area of 1. Its FWHM is γ. Its peak value is $\frac{2}{\pi\gamma}$.

Students often get confused by these different forms. This slide makes it very clear that they are just different ways of scaling the same underlying Lorentzian shape, chosen for convenience depending on whether one wants to emphasize the peak value or the total area. The crucial physical width parameter (γ, or 2Γ) remains the same.

Page 29

Slide 12: Energy Loss Rate Derived Directly From the Equation of Motion

Now we shift gears slightly with "Slide 12: Energy Loss Rate Derived Directly From the Equation of Motion." We're going back to our original differential equation to look at the energy dissipation.

The first bullet says: "Multiply the motion equation by $m\dot{x}(t)$."

Our equation of motion (from page 8, after multiplying by m) is:

$$m\ddot{x}(t) + m\gamma\dot{x}(t) + m\omega_0^2 x(t) = 0.$$

Let's multiply every term by $\dot{x}(t)$ (not $m\dot{x}(t)$ as the slide says, because the m is already in the equation if we started from Newton's law with m, or if we take the original $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$ and multiply by $m\dot{x}$ now):

Let's use

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

and multiply by $m\dot{x}$.

This gives:

$$m\dot{x}\ddot{x} + m\gamma(\dot{x})^2 + m\omega_0^2 x\dot{x} = 0.$$

The slide says: " $m\dot{x}\ddot{x} + m\omega_0^2 x\dot{x} = -m\gamma(\dot{x})^2$."

This is equivalent to:

$$m\dot{x}\ddot{x} + m\omega_0^2 x\dot{x} = -m\gamma(\dot{x})^2.$$

Ah, the slide's equation is slightly different: " $m\ddot{x}\dot{x} + m\omega_0^2 x\dot{x} = -m\gamma(\dot{x})^2$." This matches what I wrote if I move the gamma term to the right.

The second bullet is key: "Recognise left side as the total time derivative of mechanical energy."

The left side is $m\dot{x}\ddot{x} + m\omega_0^2 x\dot{x}$.

Let's consider the mechanical energy $W(t)$ of the oscillator. It has two parts: kinetic energy and potential energy.

Kinetic Energy (KE) = $\frac{1}{2}m(\dot{x})^2$.

Potential Energy (PE) associated with the spring = $\frac{1}{2}kx^2$. Since $\omega_0^2 = \frac{k}{m}$, then $k = m\omega_0^2$. So, PE = $\frac{1}{2}m\omega_0^2x^2$.

The total mechanical energy $W(t)$ is:

$$W(t) = \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}m\omega_0^2x^2.$$

Now, let's find the time derivative of $W(t)$, $\frac{dW}{dt}$:

$$\frac{dW}{dt} = \frac{d}{dt}\left[\frac{1}{2}m(\dot{x})^2\right] + \frac{d}{dt}\left[\frac{1}{2}m\omega_0^2x^2\right].$$

Using the chain rule:

$$\frac{d}{dt}\left[\frac{1}{2}m(\dot{x})^2\right] = \frac{1}{2}m \cdot 2\dot{x}\ddot{x} = m\dot{x}\ddot{x},$$

and

$$\frac{d}{dt}\left[\frac{1}{2}m\omega_0^2x^2\right] = \frac{1}{2}m\omega_0^2 \cdot 2x\dot{x} = m\omega_0^2x\dot{x}.$$

So,

$$\frac{dW}{dt} = m\dot{x}\ddot{x} + m\omega_0^2x\dot{x}.$$

This is precisely the left-hand side of the equation we derived in the first bullet!

Therefore, the third bullet point states "Therefore..." and this implies we can substitute $\frac{dW}{dt}$ into our equation.

PAGE 30:

Continuing from the previous page, since the left side of our modified equation of motion is $\frac{dW}{dt}$ (the rate of change of mechanical energy), we can write:

$$\frac{dW}{dt} = -\gamma m \left(\dot{x}(t)\right)^2$$

This equation is profoundly important. It states that the rate at which the mechanical energy W of the oscillator changes is equal to $-\gamma m$ times the velocity squared. Since m (mass) and $(\dot{x})^2$ (velocity squared) are always non-negative, and γ (damping constant) is positive, $\frac{dW}{dt}$ is always less than or equal to zero. This means the mechanical energy of the oscillator is always decreasing (or constant if $\dot{x} = 0$), which makes perfect sense: the damping term is dissipating energy. This energy is being lost to radiation in our model. The rate of this energy loss is proportional to the damping constant γ and the square of the velocity.

The second bullet says: "Insert $x(t) = x_0 e^{-\gamma t/2}\cos(\omega_0 t)$ and discard γ^2 terms (small)." This is our approximate solution for $x(t)$ under weak damping. We first need $\dot{x}(t)$.

$$\dot{x}(t) = \frac{d}{dt}\left[x_0 e^{-\gamma t/2}\cos(\omega_0 t)\right]$$

Using the product rule:

$$\dot{x}(t) = x_0\left[\left(-\frac{\gamma}{2}\right)e^{-\gamma t/2}\cos(\omega_0 t) - \omega_0 e^{-\gamma t/2}\sin(\omega_0 t)\right]$$

$$\dot{x}(t) = -x_0 e^{-\gamma t/2}\left[\frac{\gamma}{2}\cos(\omega_0 t) + \omega_0\sin(\omega_0 t)\right]$$

Now we need $\left(\dot{x}(t)\right)^2$. This will be $x_0^2 e^{-\gamma t}$ times the square of the term in the bracket.

$$\left[\frac{\gamma}{2}\cos(\omega_0 t) + \omega_0 \sin(\omega_0 t)\right]^2$$

$$= \frac{\gamma^2}{4}\cos^2(\omega_0 t) + \omega_0^2 \sin^2(\omega_0 t) + \gamma\omega_0 \cos(\omega_0 t)\sin(\omega_0 t)$$

The instruction says to "discard γ^2 terms." So we neglect the $\frac{\gamma^2}{4}\cos^2(\omega_0 t)$ term. We also should probably neglect the $\gamma\omega_0$ term if ω_0 is much larger than γ, or if we are averaging.

Let's see what the slide simplifies this to. The slide gives the result for $\frac{dW}{dt}$ after this substitution:

$$\frac{dW}{dt} = -\gamma m x_0^2 e^{-\gamma t}\sin^2(\omega_0 t)$$

This expression is missing an ω_0^2. If $\dot{x} \approx -x_0\omega_0 e^{-\gamma t/2}\sin(\omega_0 t)$ (which comes from neglecting the $\gamma/2$ term in $\dot{x}$ as $\gamma \ll \omega_0$), then $(\dot{x})^2 \approx x_0^2\omega_0^2 e^{-\gamma t}\sin^2(\omega_0 t)$. Then

$$\frac{dW}{dt} = -\gamma\, m\, x_0^2\omega_0^2 e^{-\gamma t}\sin^2(\omega_0 t)$$

Yes, the slide seems to have a typo and is missing ω_0^2. Let's proceed with what the slide provides and see if it corrects later or if this is a particular approximation path.

The third bullet says: "Time-average over many optical cycles, using $\langle\sin^2\rangle = \frac{1}{2}$." The term $\sin^2(\omega_0 t)$ oscillates rapidly. Over many cycles of $\omega_0 t$, the average value of $\sin^2(\omega_0 t)$ is $\frac{1}{2}$. (Similarly, $\langle\cos^2\rangle = \frac{1}{2}$.)

So, the time-averaged power radiated, $\bar{P}$, which is the magnitude of $\langle\frac{dW}{dt}\rangle$, becomes: "P bar equals the magnitude of , equals $\gamma/2$, times m, times x_0^2, times ω_0^2, times $e^{-\gamma t}$." In symbolic form,

$$\bar{P}(t) = \frac{1}{2}\,\gamma\, m\, \omega_0^2\, x_0^2\, e^{-\gamma t}$$

Aha! The ω_0^2 has reappeared here in the $\bar{P}$ expression. This confirms my suspicion of a typo in the intermediate $\frac{dW}{dt}$ expression on the slide. The correct time-averaged power radiated is indeed proportional to ω_0^2. So,

$$\bar{P}(t) = \frac{1}{2}\,\gamma\, m\, \omega_0^2\, x_0^2\, e^{-\gamma t}$$

This shows that the average power radiated by the oscillator also decays exponentially, but with a rate constant γ, not $\gamma/2$. This is because power is related to amplitude squared, and if amplitude decays as $e^{-\gamma t/2}$, then amplitude squared (and hence power) decays as $\left(e^{-\gamma t/2}\right)^2 = e^{-\gamma t}$.

Page 31:

This page provides a concise conclusion from the energy loss rate calculation.

The single bullet point states: "Radiant power decays with the same time constant $\tau = \frac{1}{\gamma}$."

From the previous page, the time-averaged power radiated, $\bar{P}(t)$, was found to be proportional to $e^{-\gamma t}$.

A function that decays as $e^{-t/\tau}$ has a time constant τ.

Comparing $e^{-\gamma t}$ with $e^{-t/\tau}$, we see that γ corresponds to $\frac{1}{\tau}$, or equivalently, $\tau = \frac{1}{\gamma}$.

So, the characteristic time for the *power* to decay by a factor of $\frac{1}{e}$ is $\tau = \frac{1}{\gamma}$.

Recall that the *amplitude* $x(t)$ of the oscillation decayed as $e^{-\gamma t/2}$. The time constant for the amplitude decay is therefore $\frac{2}{\gamma}$.

It's important to distinguish between the decay of amplitude (time constant $\frac{2}{\gamma}$) and the decay of energy or power (time constant $\frac{1}{\gamma}$). Since energy is proportional to amplitude squared, its decay rate is twice as fast (or its time constant is half as long).

This time constant $\tau = \frac{1}{\gamma}$ is directly related to the lifetime of the excited state in the quantum mechanical picture, as we will see shortly. The damping constant γ, which determines the FWHM of the spectral line ($\Delta\omega_{\text{nat}} = \gamma$), is also the reciprocal of the power decay time constant. This interconnectedness is central to understanding natural linewidth.

Page 32:

Now we explicitly bridge our classical findings to the quantum world with "Slide 13: Connecting Damping Constant γ to Quantum Lifetime τ."

The first bullet point introduces a key quantum concept: "Quantum spontaneous-emission rate for state $|i\rangle$: Einstein A_{i} coefficient [s^{-1}]." In quantum mechanics, an atom in an excited state $|i\rangle$ can spontaneously decay to lower energy states. If we consider all possible decay channels from state $|i\rangle$, the total rate at which the population of state $|i\rangle$ depletes due to spontaneous emission is given by the Einstein A_{i} coefficient, often denoted A_{i} (or sometimes A_{ik} if referring to a specific transition to state k, but A_{i} usually implies the total decay rate out of state i). The units of A_{i} are inverse seconds (s^{-1}), representing a probability per unit time of decay.

The second bullet point defines "Mean lifetime":

$$\tau_{\text{i}} = \frac{1}{A_{\text{i}}}$$

The mean lifetime, τ_{i}, of an excited state $|i\rangle$ is simply the reciprocal of its total spontaneous emission rate A_{i}. If A_{i} is large (high probability of decay per second), then the lifetime τ_{i} is short. If A_{i} is small (low probability of

decay per second), the lifetime τ_i is long. This is analogous to radioactive decay.

The third, crucial bullet point makes the connection: "Identify $\gamma = A_i$."

This is the pivotal identification. The damping constant γ in our classical damped oscillator model, which governs the rate of energy loss and determines the spectral linewidth, is identified with the total spontaneous emission rate A_i from the excited state in the quantum mechanical picture. This means that the classical "friction" due to radiation is the macroscopic manifestation of the quantum process of spontaneous emission. Since $A_i = \frac{1}{\tau_i}$, this also means $\gamma = \frac{1}{\tau_i}$. This is a profound link. The lifetime of the excited atomic state directly determines the damping in our classical model and hence the natural linewidth. Longer lifetime means smaller A_i, smaller γ, and thus a narrower line. Shorter lifetime means larger A_i, larger γ, and a broader line.

Page 33:

Building on the connection $\gamma = A_i = \frac{1}{\tau_i}$, this page expresses the "Natural linewidth in frequency units."

The first bullet point presents the formula: " $\Delta\nu_{nat} = \frac{A_i}{2\pi} = \frac{1}{2\pi\tau_i}$ "

We established earlier (Page 25) that the FWHM natural linewidth in ordinary frequency units (Hertz) is $\Delta\nu_{nat} = \frac{\gamma}{2\pi}$.

Now, substituting $\gamma = A_i$, we get $\Delta\nu_{nat} = \frac{A_i}{2\pi}$.

And since $A_i = \frac{1}{\tau_i}$ (where τ_i is the lifetime of the upper state involved in the emission), we can also write:

$$\Delta\nu_{nat} = \frac{1}{2\pi\tau_i}$$

This is one of the most fundamental equations in spectroscopy concerning natural linewidths. It directly relates an observable spectral feature (the linewidth $\Delta\nu_{\text{nat}}$) to a fundamental property of the excited atomic state (its lifetime τ_{i}). This relationship has units of Hertz (s^{-1}) since τ_{i} is in seconds and 2π is dimensionless.

The second bullet point highlights the implication: "Any decrease in τ_{i} widens the line; longer-lived states generate ultra-narrow features."

* If the lifetime τ_{i} of the excited state is short (it decays quickly), then $\frac{1}{\tau_{\text{i}}}$ is large, leading to a large natural linewidth $\Delta\nu_{\text{nat}}$. The spectral line will be broad.

* Conversely, if the lifetime τ_{i} is long (the state is metastable or very slow to decay), then 1τi$\frac{1}{\tau_{\text{i}}}$ is small, resulting in a small natural linewidth $\Delta\nu_{\text{nat}}$. The spectral line will be very narrow.

This is why transitions from metastable states, which have very long lifetimes, are candidates for ultra-high resolution spectroscopy and atomic clocks – their natural linewidths are exceptionally small. For example, "forbidden" transitions often have very small A coefficients and thus very long lifetimes, leading to extremely narrow natural linewidths.

Page 34:

Now we approach the natural linewidth from a different but complementary angle: "Slide 14: Uncertainty Principle Perspective."

The first bullet point introduces the "Energy-time uncertainty relation":

$$\Delta E_{\text{i}}\,\Delta t \geq \frac{\hbar}{2}$$

This is one of Heisenberg's uncertainty principles. It states that there is a fundamental limit to the precision with which certain pairs of complementary physical properties of a particle can be known

simultaneously. For energy and time, it means that if a system exists in an energy state E_i for a limited duration Δt, then the energy of that state cannot be perfectly defined; it will have an inherent uncertainty or spread, ΔE_i. The shorter the duration Δt, the larger the uncertainty ΔE_i must be. $\hbar$ is the reduced Planck constant.

The second bullet point connects this to our system: "For exponential decay, $\Delta t \approx \tau_i$."

The third bullet point states: "Hence..." Substituting $\Delta t \approx \tau_i$ into the uncertainty relation (and often for order-of-magnitude estimates, the "greater than or equal to $\hbar/2$" is taken as "approximately $\hbar$"), we get:

$$\Delta E_i \approx \frac{\hbar}{\tau_i}$$

This means that due to its finite lifetime τ_i, an excited state $|i\rangle$ does not have an infinitely sharp energy. Instead, its energy is "smeared out" or uncertain by an amount ΔE_i, which is inversely proportional to its lifetime τ_i. This ΔE_i is often called the energy width of the state.

A very short-lived state (small τ_i) will have a large energy width (large ΔE_i). A long-lived state (large τ_i) will have a small energy width (small ΔE_i). This is a direct consequence of the wave nature of quantum particles and the Fourier relationship between time and frequency (or energy).

Page 35:

Continuing with the uncertainty principle perspective:

The first bullet point relates "Frequency uncertainty (linewidth)" to the energy uncertainty we just discussed. We have $\Delta E_i \approx \frac{\hbar}{\tau_i}$. Since energy E and angular frequency ω are related by $E = \hbar\omega$ (for photons, and by analogy for energy level differences), an uncertainty in energy ΔE_i

corresponds to an uncertainty in angular frequency $\Delta\omega$ given by $\Delta E_{\text{i}} = \hbar\Delta\omega$. So, $\hbar\Delta\omega \approx \frac{\hbar}{\tau_{\text{i}}}$. Dividing by $\hbar$, we get:

$$\Delta\omega = \frac{\Delta E_{\text{i}}}{\hbar} = \frac{1}{\tau_{\text{i}}}.$$

This $\Delta\omega$ represents the uncertainty in the angular frequency of the transition originating from state $|i\rangle$, which is essentially the natural linewidth in angular frequency units. Compare this to our earlier result: $\Delta\omega_{\text{nat}} = \gamma$. And we identified $\gamma = A_{\text{i}} = \frac{1}{\tau_{\text{i}}}$. So, $\Delta\omega_{\text{nat}} = \frac{1}{\tau_{\text{i}}}$. The uncertainty principle gives us the same result! This consistency is very satisfying and shows that the natural linewidth can be understood as a direct manifestation of the energy-time uncertainty principle applied to unstable excited states.

The second bullet point considers a more general case: "If both upper and lower states are unstable (lifetimes τ_{i}, τ_{k}), total uncertainty adds in quadrature." Often, an optical transition occurs between an excited state $|i\rangle$ (lifetime τ_{i}) and a lower state $|k\rangle$ (lifetime τ_{k}). If the lower state $|k\rangle$ is not the ground state, it too might be unstable and have a finite lifetime τ_{k}. In this case, both state $|i\rangle$ and state $|k\rangle$ have energy widths:

$$\Delta E_{\text{i}} \approx \frac{\hbar}{\tau_{\text{i}}}$$

$$\Delta E_{\text{k}} \approx \frac{\hbar}{\tau_{\text{k}}}$$

The width of the spectral line for the transition $i \to k$ will then depend on the energy widths of *both* states. Since these uncertainties are typically independent, they add in quadrature (like errors). The total energy width of the transition, $\Delta E_{\text{transition}}$, would be

$$\Delta E_{\text{transition}} = \sqrt{(\Delta E_{\text{i}})^2 + (\Delta E_{\text{k}})^2}.$$

The corresponding total linewidth in angular frequency, $\Delta\omega$, is given by

$$\hbar\Delta\omega = \sqrt{\left(\frac{\hbar}{\tau_{\mathrm{i}}}\right)^2 + \left(\frac{\hbar}{\tau_{\mathrm{k}}}\right)^2}.$$

Dividing by $\hbar$, we get:

$$\Delta\omega = \sqrt{\left(\frac{1}{\tau_{\mathrm{i}}}\right)^2 + \left(\frac{1}{\tau_{\mathrm{k}}}\right)^2}.$$

This is the general formula for the natural linewidth (in angular frequency) when both the initial and final states of the transition have finite lifetimes. If the lower state $|k\rangle$ is the ground state, it is perfectly stable, so τ_{k} is infinite, and $\frac{1}{\tau_{\mathrm{k}}}$ is zero. In this case, $\Delta\omega$ simplifies back to $\frac{1}{\tau_{\mathrm{i}}}$, which is the result we had before.

So, the linewidth is determined by the sum of the decay rates ($A_{\mathrm{i}} + A_{\mathrm{k}}$ if you like, since $A = \frac{1}{\tau}$), or more precisely,

$$\gamma_{\mathrm{total}} = \gamma_{\mathrm{i}} + \gamma_{\mathrm{k}} = A_{\mathrm{i}} + A_{\mathrm{k}}.$$

Then $\Delta\omega = \gamma_{\mathrm{total}}$. This means

$$\left(\frac{1}{\tau_{\mathrm{total}}}\right)^2 = \left(\frac{1}{\tau_{\mathrm{i}}}\right)^2 + \left(\frac{1}{\tau_{\mathrm{k}}}\right)^2$$

is not quite right. Actually, the total decay rate relevant for the linewidth is

$$\gamma_{\mathrm{total}} = \frac{1}{\tau_{\mathrm{i}}} + \frac{1}{\tau_{\mathrm{k}}}.$$

So, the total linewidth $\Delta\omega$ should be

$$\Delta\omega = \frac{1}{\tau_{\mathrm{i}}} + \frac{1}{\tau_{\mathrm{k}}}.$$

The quadrature addition is usually for independent, random errors. Lifetimes contribute to widths. Let's check standard texts. The linewidth

(FWHM in angular frequency) of a transition between two unstable states with decay rates γ_i and γ_k is $\gamma_i + \gamma_k$. So $\Delta\omega = \gamma_i + \gamma_k$. Perhaps the slide's formula

$$\Delta\omega = \sqrt{\left(\frac{1}{\tau_i}\right)^2 + \left(\frac{1}{\tau_k}\right)^2}$$

is related to a specific definition or context for "uncertainty." Let's assume it's presented this way for a reason within this course's framework, possibly relating to independent probability distributions for energy. The more common form for FWHM is additive for decay rates. I will proceed with the slide's formula but note this potential nuance.

Page 36:

This page provides a visual representation of "Energy Level Uncertainty and Transition Linewidth," corresponding to the discussion on the previous page.

On the left, we see two energy levels depicted:

* An upper energy level E_i, shown as a light blue, somewhat "fuzzy" or broadened horizontal band. An arrow next to it indicates its energy width, $\Delta E_i \approx \frac{\hbar}{\tau_i}$. This visually represents that the energy of state E_i is not perfectly sharp but has a spread due to its finite lifetime τ_i.

* A lower energy level E_k, shown as a light green, similarly fuzzy horizontal band. An arrow indicates its width, $\Delta E_k \approx \frac{\hbar}{\tau_k}$. This state also has an energy spread due to its lifetime τ_k.

On the right, we have equations and a diagram:

"Total Linewidth (Energy):"

$$\hbar\Delta\omega = \sqrt{(\Delta E_i)^2 + (\Delta E_k)^2}$$

This explicitly states that the total energy width of the transition ($\hbar\Delta\omega$) is found by adding the individual energy widths of the two states (ΔE_i and ΔE_k) in quadrature.

"Total Linewidth (Frequency):"

$$\Delta\omega = \sqrt{\frac{1}{\tau_\text{i}^2} + \frac{1}{\tau_\text{k}^2}}$$

This is the same formula for the angular frequency linewidth as on the previous slide, derived from the energy width formula by substituting $\Delta E_\text{i} = \frac{\hbar}{\tau_\text{i}}$ and $\Delta E_\text{k} = \frac{\hbar}{\tau_\text{k}}$.

Below these equations, there's a vector diagram illustrating the quadrature addition for energies.

* A vertical vector represents ΔE_i.

* A horizontal vector represents ΔE_k.

* The resultant vector, representing the total energy uncertainty $\hbar\Delta\omega$, is the hypotenuse of the right triangle formed by ΔE_i and ΔE_k. Its length is indeed $\sqrt{(\Delta E_\text{i})^2 + (\Delta E_\text{k})^2}$.

The caption summarizes: "The energy uncertainties ΔE_i and ΔE_k of the two states add in quadrature to give the total transition linewidth $\hbar\Delta\omega$."

This visual makes the concept of quadrature addition very clear. As noted before, this specific method of combining widths (quadrature) might differ from the simple sum of decay rates ($\gamma_\text{i} + \gamma_\text{k}$) found in some other contexts for convoluted Lorentzians. However, the slide is consistent with its own presentation. The key takeaway is that if both levels involved in a transition are unstable, both contribute to the overall natural linewidth of that transition.

Page 37:

We now shift to "Slide 15: Power Emitted by an Ensemble of Atoms." So far, we've mostly considered a single atom or oscillator. What happens when we have many?

The first bullet point defines: " N_i : number density [m^{-3}] of atoms in state $|i\rangle$." N_i represents the number of atoms per unit volume that are in the specific excited state $|i\rangle$ from which emission will occur. Its units are typically atoms per cubic meter.

The second bullet gives the "Radiant power for transition $i \to k$":

$$\frac{dW_{ik}}{dt} = N_i A_{ik} h\omega_{ik}$$

Let's interpret this: * A_{ik} is the Einstein A coefficient for the specific transition from state $|i\rangle$ to state $|k\rangle$. It's the probability per unit time that one atom in state $|i\rangle$ will decay to state $|k\rangle$ by emitting a photon. Units: s^{-1}. * $h\omega_{ik}$ (or $h\nu_{ik}$) is the energy of each photon emitted during this $i \to k$ transition. Units: Joules. $N_i A_{ik}$ gives the number of $i \to k$ transitions occurring per unit volume per unit time. (Number of atoms/volume transitions/atom/time = transitions/volume/time). * So, $N_i A_{ik} h\omega_{ik}$ is the total energy emitted per unit volume per unit time due to the $i \to k$ transition. This is the power density.

The slide uses $\frac{dW_{ik}}{dt}$, which usually signifies power. If N_i is number density, then this expression is power per unit volume. If N_i were total number of atoms in state i in some volume ΔV, then $N_i A_{ik} h\omega_{ik}$ would be total power from that volume. Let's assume N_i here is number density, so $\frac{dW_{ik}}{dt}$ is power density.

The third bullet point concerns detection: "Detector of area capital A at distance r subtends solid angle $d\Omega = \frac{A}{r^2}$." This is the standard definition of solid angle. If we have a detector with a surface area A, placed at a distance r from the light source, and if the detector surface is perpendicular

to the line of sight and r is much larger than the dimensions of A and the source, then the solid angle $d\Omega$ that the detector "sees" or collects light from is approximately $\frac{A}{r^2}$. Units are steradians.

The fourth bullet point sets an assumption: "Assuming isotropic emission, power reaching detector..." Isotropic emission means the atoms emit photons equally in all directions. The total solid angle around a point is 4π steradians. If the emission is isotropic, the power emitted per unit solid angle is the total power (or power density times volume) divided by 4π. The power reaching the detector will then be this power per unit solid angle multiplied by the solid angle $d\Omega$ subtended by the detector.

We'll see the formula on the next page.

Page 38:

This page continues from the previous one, giving the formula for the power reaching the detector.

The equation presented is:

$$P_{ik} = N_{\text{i}}\, A_{ik}\, h\, \omega_{ik}\, \Delta V \cdot \frac{A}{4\pi r^2}$$

Let's break this down:

* $N_{\text{i}}\, A_{ik}\, h\, \omega_{ik}$: As discussed, this is the power emitted per unit volume from the $i \to k$ transition (assuming N_{i} is number density).

* ΔV: This term, capital ΔV, must represent the volume of the ensemble of atoms that we are observing. So, $N_{\text{i}}\, A_{ik}\, h\, \omega_{ik}\, \Delta V$ is the total power emitted isotropically by the atoms in the volume ΔV.

* $\frac{A}{r^2}$: This is the solid angle $d\Omega$ subtended by the detector of area A at distance r, as defined on the previous page.

* $\frac{1}{4\pi}$: This factor accounts for isotropic emission. The total power is spread over 4π steradians. So, $\frac{\text{Total Power}}{4\pi}$ is the power per unit solid angle.

* Multiplying power per unit solid angle by the detector's solid angle $\frac{A}{r^2}$ gives the power P_{ik} that is intercepted by the detector.

So,

$$P_{ik} = [(N_i\, \Delta V)\, A_{ik}\, h\, \omega_{ik}] \cdot \left[\frac{d\Omega}{4\pi}\right]$$

$(N_i\, \Delta V)$ is the total number of atoms in state i in the observed volume. Let this be $N_{i,\text{total}}$.

Then

$$P_{ik} = N_{i,\text{total}}\, A_{ik}\, h\, \omega_{ik} \times (\text{fraction of solid angle covered by detector})$$

This formula makes sense. It tells us that the power received by the detector is proportional to:

1. The number of atoms in the upper state within the observed volume $(N_i\, \Delta V)$.

2. The spontaneous emission rate for the specific transition (A_{ik}).

3. The energy of each photon $(h\, \omega_{ik})$.

4. The fraction of the total solid angle that the detector covers $\left(\frac{A}{4\pi r^2}\right)$.

The bullet point below the equation states: "Measuring P_{ik} allows determination of N_i provided A_{ik} is known."

This is a key application. If we can measure the power P_{ik} arriving at our detector, and if we know the geometry (A, r, ΔV), the transition properties (A_{ik}, $h\, \omega_{ik}$), then we can rearrange the formula to solve for N_i, the number density of atoms in the excited state. This is a fundamental way to probe

populations in atomic and molecular systems, for example, in plasmas, flames, or interstellar gas clouds. Of course, calibrating such measurements (knowing A_{ik}, ΔV accurately, detector efficiency etc.) can be challenging.

Page 39:

This page provides a visual for "Slide 15: Power Emitted by an Ensemble of Atoms."

The diagram shows:

* A central blue sphere labeled ΔV. This represents the small volume containing the ensemble of emitting atoms.

* Numerous arrows radiate outwards from ΔV in all directions, signifying isotropic emission of photons. These photons are labeled with energy $h\omega_{ik}$.

* A dashed circle surrounds ΔV, suggesting a sphere over which the radiation is spreading.

* To the right, at a distance r from ΔV, there is a red curved segment labeled A. This represents the area of a detector.

* Orange lines define a cone from ΔV to the edges of the detector A, indicating the solid angle $\Delta\Omega$ (or $d\Omega$) that the detector subtends from the source.

* A green arrow labeled P_{ik} points from ΔV towards the detector A, representing the power from the i-k transition that reaches the detector.

This diagram nicely illustrates all the components of the equation from the previous page: the emitting volume ΔV, the isotropic emission of photons of energy $h\omega_{ik}$, the detector of area A at distance r, the solid angle $\Delta\Omega$ it covers, and the resulting power P_{ik} received. It helps to visualize how only a fraction of the total isotropically emitted power is captured by a detector of finite size.

Page 40:

Now we switch from emission to "Slide 16: Absorption Geometry and Beer's Law." This is fundamental for understanding how light intensity decreases as it passes through an absorbing medium.

The first bullet describes the setup: "Plane monochromatic wave enters sample of thickness z." Imagine a beam of light, which is a plane wave (wavefronts are planes) and monochromatic (single frequency or very narrow band of frequencies), incident on a sample of absorbing material. The sample has a certain thickness z along the direction of light propagation.

The second bullet describes the "Intensity decrement over dz": "dI equals minus $\alpha_{ik}(\omega)$ of omega, times I, times dz."

$$dI = -\alpha_{ik}(\omega)\, I\, dz$$

Let's break this down: * I is the intensity of the light at some point within the sample. * dz is an infinitesimally small thickness of the sample that the light is passing through. * dI is the change in intensity as the light traverses this small thickness dz. *The minus sign indicates that the intensity* decreases* due to absorption. * $\alpha_{ik}(\omega)$ is the "absorption coefficient." It's a property of the material for the specific transition $i \to k$ at the light's angular frequency ω. It quantifies how strongly the material absorbs light at that frequency. A larger $\alpha_{ik}(\omega)$ means stronger absorption. Its units will be inverse length (e.g., m^{-1}) so that $\alpha_{ik}(\omega)\, dz$ is dimensionless.

This equation states that the fractional decrease in intensity $\frac{dI}{I}$ is proportional to the thickness dz traversed:

$$\frac{dI}{I} = -\alpha_{ik}(\omega)\, dz$$

The third bullet says: "Integrate from 0 to z." We want to find the total change in intensity as the light passes through the entire sample of

thickness z. We assume the light enters the sample at position 0 with an initial intensity I_0.

Integrating $\frac{dI}{I} = -\alpha_{ik}(\omega)\, dz$ from $I = I_0$ to $I = I(z)$ and from position 0 to z:

Integral $\left(\frac{dI}{I}\right)$ from I_0 to $I(z)$ gives $\ln\left(\frac{I(z)}{I_0}\right)$. Integral $(-\alpha_{ik}(\omega)\, dz)$ from 0 to z gives $-\alpha_{ik}(\omega)\, z$ (assuming $\alpha_{ik}(\omega)$ is constant throughout the sample).

So,

$$\ln\left(\frac{I(z)}{I_0}\right) = -\alpha_{ik}(\omega)\, z$$

Exponentiating both sides gives:

$$\frac{I(z)}{I_0} = e^{-\alpha_{ik}(\omega)\, z}$$

Or, "I of z equals I_0, e to the power of (minus $\alpha_{ik}(\omega)$, times z)."

This is Beer's Law (or the Beer-Lambert Law). It describes the exponential attenuation of light intensity as it passes through an absorbing medium. The intensity decays exponentially with penetration depth z, and the rate of decay is determined by the absorption coefficient $\alpha_{ik}(\omega)$.

The fourth bullet defines: "$\alpha_{ik}(\omega)$: absorption coefficient [m^{-1}]." As mentioned, its units are inverse length, for example, per meter. If $\alpha_{ik}(\omega)$ is large, the light is absorbed very quickly over a short distance. If $\alpha_{ik}(\omega)$ is small, the light can penetrate much further before being significantly attenuated. This $\alpha_{ik}(\omega)$ will also have a frequency dependence, typically peaking at the resonance frequency ω_0, and having a lineshape (e.g., Lorentzian if dominated by natural broadening).

Page 41:

This page provides more detail on the absorption coefficient $\alpha_{ik}(\omega)$, expressing it in terms of microscopic quantities.

The first bullet states: "Expressed via cross-section $\sigma_{ik}(\omega)$ and populations..." "

$$\alpha_{ik}(\omega) = \sigma_{ik}(\omega) \times \left[N_{\mathrm{i}} - \left(\frac{g_{\mathrm{i}}}{g_{\mathrm{k}}}\right) N_{\mathrm{k}}\right]$$

"

Let's dissect this important formula:

* $\sigma_{ik}(\omega)$ is the "absorption cross-section" for the transition $i \to k$ at angular frequency ω. It has units of area (e.g., m^2). It represents the effective area that an atom in state |i> presents to an incoming photon of frequency ω for an absorption event to occur. Like $\alpha_{ik}(\omega)$, $\sigma_{ik}(\omega)$ also has a frequency dependence (a lineshape).

* N_{i} is the number density (population per unit volume) of atoms in the lower state |i> (the initial state for absorption).

* N_{k} is the number density of atoms in the upper state |k> (the final state for absorption).

* g_{i} and g_{k} are the statistical weights (degeneracies) of the lower state |i> and upper state |k>, respectively. These are integers (e.g., $2J + 1$ for an atomic level with total angular momentum J).

The term in the square brackets,

$$\left[N_{\mathrm{i}} - \left(\frac{g_{\mathrm{i}}}{g_{\mathrm{k}}}\right) N_{\mathrm{k}}\right],$$

is crucial.

* N_{i} represents the population available for absorption.

$\left(\frac{g_{\mathrm{i}}}{g_{\mathrm{k}}}\right) N_{\mathrm{k}}$ represents the population in the upper state that can contribute to stimulated emission*. Stimulated emission is a process where an incoming photon of the correct frequency can induce an atom in the upper state |k>

to emit a second, identical photon and transition down to state |i>. This process adds light rather than absorbing it. The factor $\frac{g_i}{g_k}$ accounts for the degeneracies.

* So, the net absorption is proportional to the difference between the effective population in the lower state that can absorb (N_i) and the effective population in the upper state that can be stimulated to emit ($\left(\frac{g_i}{g_k}\right) N_k$).

If $N_i > \left(\frac{g_i}{g_k}\right) N_k$, then α_{ik} is positive, and we have net absorption. This is the usual case in thermal equilibrium, where lower states are more populated.

If $N_i < \left(\frac{g_i}{g_k}\right) N_k$, then α_{ik} would be negative. This corresponds to net stimulated emission, or optical gain. This condition is known as "population inversion" and is the basis for laser operation.

If $N_i = \left(\frac{g_i}{g_k}\right) N_k$, then $\alpha_{ik} = 0$, and the medium is transparent to light at that frequency (absorption and stimulated emission balance out).

The second bullet considers a common scenario: "For N_k much, much less than N_i (ground state dominated): α_{ik} is approximately $\sigma_{ik} N_i$."

If the population of the upper state N_k is negligible compared to the population of the lower state N_i (e.g., if state |i> is the ground state or a very low-lying state, and the temperature is not extremely high), then the stimulated emission term $\left(\frac{g_i}{g_k}\right) N_k$ can be ignored.

In this case, the absorption coefficient simplifies to

$$\alpha_{ik}(\omega) \approx \sigma_{ik}(\omega) \times N_i.$$

The absorption is then directly proportional to the absorption cross-section and the number density of absorbers in the lower state. This is a very common approximation for absorption spectroscopy of dilute gases or weakly excited systems.

Page 42:

We now return to our classical damped oscillator model, but this time we consider what happens when it's driven by an external field. This leads to

Slide 17: Driven Damped Oscillator Yields Absorption Profile.

The first bullet introduces the "External electric field of light":

$E(t) = E_0 e^{i\omega t}$. This represents a monochromatic electromagnetic wave (light) with angular frequency ω and amplitude E_0, interacting with our atom/oscillator. We're using complex notation for the field; the actual electric field would be the real part, $E_0 \cos(\omega t)$, if E_0 is real.

The second bullet states: "Charge $q = e$ experiences force $qE(t)$."

Our classical oscillator is an electron with charge e (which is -1.602×10^{-19} Coulombs, though often e is used for the magnitude of the charge, and the sign handled by context). This charge, when placed in the external electric field $E(t)$, experiences a force $F_{\text{ext}} = eE(t)$. This force will drive the oscillator.

The equation of motion for the damped oscillator (from Page 8),

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = 0,$$

now needs to include this external driving force on the right-hand side:

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = eE_0 e^{i\omega t}.$$

Or, dividing by m:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{eE_0}{m} e^{i\omega t}.$$

This is the equation for a driven, damped harmonic oscillator.

The third bullet point connects this to previous work: "Linear response theory (from Sect. 2.6) gives frequency-dependent absorption coefficient near resonance."

Solving the driven oscillator equation (typically looking for the steady-state solution where the oscillator oscillates at the driving frequency ω) allows us to find the induced dipole moment of the atom, $p(t) = ex(t)$. From the induced dipole moment, we can find the complex polarizability of the atom, and from that, the macroscopic susceptibility of the medium, and finally the complex refractive index. The imaginary part of the complex refractive index is related to the absorption coefficient $\alpha(\omega)$.

This is a standard procedure in linear response theory. Section 2.6 of this course presumably covered this derivation. The key result is that the absorption coefficient $\alpha(\omega)$ derived from this model will also show a resonant behavior near $\omega = \omega_0$, and it will have a Lorentzian lineshape.

Page 43:

This page presents the result for the absorption coefficient $\alpha(\omega)$ derived from the driven damped oscillator model discussed on the previous page.

The equation for alpha of omega is:

$$\alpha(\omega) = \frac{Ne^2}{4\epsilon_0 mc} \times \frac{\gamma}{(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}.$$

Let's analyze the terms in this formula:

* N: This is the number density of oscillators (atoms) per unit volume. * e: The elementary charge (magnitude of the electron charge). So e^2 is charge squared. * ϵ_0: The permittivity of free space, a fundamental constant. * m: The mass of the oscillating charge (electron mass). * c: The speed of light in vacuum. * γ: Our familiar damping constant, equal to A_{i} (the spontaneous emission rate or inverse lifetime of the upper state if we connect to a two-level quantum system for absorption from ground to excited state i). * $(\omega - \omega_0)$: The detuning of the driving light frequency ω

from the natural resonance frequency ω_0 of the oscillator. * $\left(\frac{\gamma}{2}\right)^2$: Half the damping rate squared.

The term $\frac{\gamma}{(\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2}$ is a Lorentzian function of ω, peaked at $\omega = \omega_0$, and its FWHM in angular frequency is γ.

The prefactor $\frac{Ne^2}{4\epsilon_0 mc}$ determines the strength of the absorption at resonance.

This prefactor can also be related to the oscillator strength f of the transition, often appearing as $\frac{\pi Ne^2 f}{2\epsilon_0 mc}$ *(normalized lineshape function whose peak is* $2/\gamma$) if using a slightly different lineshape normalization, or $\frac{Ne^2 f}{2\epsilon_0 mc\gamma}$ (Lorentzian normalized to peak value 1 and width γ using a different form). The expression on the slide is correct for the Lorentzian written as $\frac{\gamma}{(\text{detuning})^2+\left(\frac{\gamma}{2}\right)^2}$.

The peak value of this Lorentzian part (when $\omega = \omega_0$) is $\frac{\gamma}{(\gamma/2)^2} = \frac{\gamma}{\gamma^2/4} = \frac{4}{\gamma}$.

So at resonance, $\alpha(\omega_0) = \frac{Ne^2}{4\epsilon_0 mc} \times \frac{4}{\gamma} = \frac{Ne^2}{\epsilon_0 mc\gamma}$.

The second bullet point makes a crucial observation: "Identical Lorentzian form and identical FWHM γ as emission case, confirming reciprocity."

This is extremely important. The absorption lineshape $\alpha(\omega)$ derived from the driven classical oscillator model has exactly the same Lorentzian functional form, and, critically, the same Full Width at Half Maximum γ, as the emission lineshape $I(\omega)$ we derived earlier from the Fourier transform of the freely decaying (undriven) oscillator.

This is a manifestation of a deep principle of reciprocity in physics, related to time-reversal symmetry and detailed balance. The probability of an atom

absorbing a photon of a certain frequency and lineshape is directly related to its probability of emitting a photon of the same frequency and lineshape.

The fact that our simple classical model reproduces this is another testament to its power. So, the natural linewidth γ is the same for both absorption and spontaneous emission for an isolated, stationary atom.

Page 44:

This page provides a graph illustrating the "Shared Lorentzian Profile: Emission & Absorption."

The vertical axis is "Relative Signal Strength," and the horizontal axis is "Angular Frequency (ω)," centered around ω_0.

Two curves are plotted: 1. A blue curve labeled "Emission Profile." 2. A red curve labeled "Absorption Profile," with a note "(Same ω_0 and γ)."

Both curves are Lorentzian in shape.

* They both peak at the same central angular frequency, ω_0.

* They both have the same Full Width at Half Maximum, indicated by a horizontal double-headed arrow labeled γ at the half-maximum height of the (presumably normalized) curves.

The graph visually emphasizes the point made on the previous slide: the lineshape profile (Lorentzian) and the fundamental width (γ, the natural linewidth) are identical for both emission from an excited state and absorption by a ground state (or lower state) atom, assuming we are dealing with isolated, stationary atoms where only natural broadening is present.

The relative heights of the emission and absorption profiles in a real experiment would depend on various factors (like populations N_{i} and N_{k}, oscillator strengths, detection efficiencies, etc.), so the "Relative Signal Strength" axis is general. Here, the absorption profile is drawn slightly lower

in peak height than the emission profile, but this is just an illustrative choice for the plot; the key is that their shapes and widths (characterized by ω_0 and γ) are the same.

This shared profile is a fundamental aspect of light-matter interaction.

Page 45:

Slide 18: Dispersion — Variation of Refractive Index.

We now explore another consequence of the interaction of light with our (driven) classical oscillator:

The first bullet point states: "Real part of the complex response gives refractive index increment."

When we solved for the driven oscillator, we found a complex polarizability, which leads to a complex susceptibility $\chi(\omega)$, and then a complex refractive index $n_{\text{complex}}(\omega) = n'(\omega) + i\,n''(\omega)$.

The imaginary part, $n''(\omega)$, is related to the absorption coefficient $\alpha(\omega)$ (specifically, $\alpha = \frac{2\,\omega}{c}\,n''(\omega)$).

The real part, $n'(\omega)$, is the ordinary refractive index that determines the phase velocity of light in the medium ($v_{\text{phase}} = \frac{c}{n'}$).

The equation given is for the refractive index $n'(\omega)$ itself, or rather, its deviation from unity (since $n = 1$ in vacuum):

$$n'(\omega) = 1 + \frac{Ne^2}{4\varepsilon_0 m\omega_0}\frac{\omega_0 - \omega}{(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}$$

Let's analyze this:

The "1 +" term indicates we are talking about the full refractive index, starting from the vacuum value of 1.

The prefactor $\frac{Ne^2}{4\varepsilon_0 m\omega_0}$ is slightly different from the one for $\alpha(\omega)$. Note the ω_0 in the denominator here instead of c and the overall factor of 2 difference if we compare forms. This is for the increment to n from 1. (A more common form for $n-1$ involves $2\varepsilon_0 m$ in denominator).

The crucial frequency-dependent part is $\frac{\omega_0-\omega}{(\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2}$.

This is not a simple Lorentzian shape like for absorption. It's an "S-shaped" or dispersive curve.

- When $\omega \ll \omega_0$, then $\omega_0-\omega$ is positive and large, while the denominator is large. $n'(\omega)$ will be slightly greater than 1 (normal dispersion).

- When $\omega \gg \omega_0$, then $\omega_0-\omega$ is negative and large, denominator is large. $n'(\omega)$ will be slightly less than 1 (though still positive).

- Near resonance ($\omega \approx \omega_0$), the behavior is dramatic. If ω is slightly less than ω_0, $\omega_0-\omega$ is small positive. $(\omega-\omega_0)^2$ is small. n' can increase significantly. If ω is slightly more than ω_0, $\omega_0-\omega$ is small negative. $(\omega-\omega_0)^2$ is small. n' can decrease significantly. At $\omega=\omega_0$, the numerator $\omega_0-\omega$ is zero, so $n'(\omega_0)=1$ (assuming the prefactor is for $n'-1$, the slide implies this is the *increment*, so at resonance the increment is zero).

The second bullet highlights: " $n'(\omega)$ exhibits steep slope near resonance (anomalous dispersion)."

Indeed, in the region around $\omega=\omega_0$, the term $\omega_0-\omega$ changes sign rapidly, and the denominator is small, leading to a very steep, negative slope for $n'(\omega)$ versus ω. This rapid change of refractive index with frequency near an absorption line is called "anomalous dispersion." (Normal dispersion is when n' increases with ω).

The third bullet points to a fundamental connection: "Relationship between absorption peak and steep dispersion mandated by Kramers-Kronig relations."

The real part $n'(\omega)$ (dispersion) and the imaginary part $n''(\omega)$ (related to absorption $\alpha(\omega)$) of the complex refractive index are not independent. They are related to each other through integral transforms known as the Kramers-Kronig relations. These relations are a mathematical consequence of causality (the response of the medium cannot precede the stimulus). If you know the absorption spectrum $\alpha(\omega)$ over all frequencies, you can, in principle, calculate the dispersion spectrum $n'(\omega)$ at any frequency, and vice-versa. The Lorentzian absorption peak is thus mathematically linked to the S-shaped anomalous dispersion curve.

Page 46:

This page continues the discussion on dispersion with a brief but significant implication.

The single bullet point states: "Such dispersion underlies slow-light and optical switching phenomena." The rapid and strong variation of the refractive index $n'(\omega)$ near an atomic resonance, known as anomalous dispersion, has profound consequences that are exploited in modern optics and photonics.

* **Slow Light:** The group velocity of a light pulse in a dispersive medium is given by $v_{\text{group}} = \frac{c}{n\prime(\omega)+\omega\frac{dn\prime}{d\omega}}$. In regions of steep normal dispersion (large positive $\frac{dn\prime}{d\omega}$), or even more dramatically with specially engineered resonances (e.g., using Electromagnetically Induced Transparency, EIT), the $\frac{dn\prime}{d\omega}$ term can become very large and positive. This can lead to extremely small group velocities, effectively "slowing down" light pulses, sometimes to mere meters per second or even stopping them.

* **Optical Switching:** The refractive index near resonance can be very sensitive to external parameters that might shift the resonance frequency or change the population of atoms (e.g., another light beam, an electric field). If $n'(\omega)$ changes significantly, the phase of the light passing through the medium changes. This can be used to build all-optical switches, where one light beam controls the path or properties of another, by modulating the refractive index of a material.

So, the same underlying physics of the classical (and quantum) oscillator that gives rise to absorption and natural linewidth also leads to dispersion, which is not just a curiosity but an enabling mechanism for advanced optical technologies. The natural lineshape is therefore intrinsically linked to these phenomena as well.

Page 47:

This page shows a diagram illustrating "Slide 18: Dispersion — Variation of Refractive Index," specifically the "Lorentzian Absorption and Corresponding Dispersion Curve."

The vertical axis is labeled "$n'(\omega) - 1$ / Absorption (a.u.)". This means it's plotting both the absorption and the dispersive part of the refractive index (minus 1, representing the contribution from the resonance) on the same arbitrary units scale.

The horizontal axis is "Frequency (ω)," centered at ω_0. Points like $\omega_0 - 2\gamma$, $\omega_0 - \gamma$, $\omega_0 + \gamma$, $\omega_0 + 2\gamma$ are marked, indicating the scale in terms of the linewidth parameter γ.

Two curves are shown:

1. The **Absorption** curve is in red. It's the familiar Lorentzian shape, peaking at $\omega = \omega_0$ and having a width related to γ.

2. The **Dispersion (**$n'(\omega) - 1$) curve is in blue. This is the S-shaped curve characteristic of anomalous dispersion near a resonance. * For $\omega < \omega_0$,

$n'(\omega) - 1$ is positive, meaning $n'(\omega) > 1$. It rises as ω approaches ω_0 from below. * At $\omega = \omega_0$, $n'(\omega) - 1$ is zero, so $n'(\omega) = 1$. * For $\omega > \omega_0$, $n'(\omega) - 1$ is negative, meaning $n'(\omega) < 1$. It dips sharply and then gradually returns towards zero from below as ω increases further.

A gray arrow highlights the region around ω_0 on the blue dispersion curve where the slope $\frac{dn\prime}{d\omega}$ is steeply negative. This region is labeled "Anomalous Dispersion."

This plot beautifully visualizes the Kramers-Kronig relationship in action: where you have an absorption peak (the red Lorentzian), you will inevitably have a corresponding dispersive feature (the blue S-curve) in the refractive index. The two are inextricably linked. The width of the absorption feature (γ) also dictates the frequency range over which the anomalous dispersion is most pronounced.

Page 48:

Slide 19: Practical Limitation — Role of Doppler Broadening

We now encounter "Slide 19: Practical Limitation — Role of Doppler Broadening." So far, our discussion of natural linewidth has implicitly assumed an isolated, *stationary* atom. In real-world scenarios, especially with gases, this is often not the case.

The first bullet point states: "Real gases at temperature T have atoms moving with thermal velocities v approximately $\sqrt{\frac{k_B T}{m}}$."

Atoms in a gas at a finite temperature T are in constant random motion, described by the Maxwell-Boltzmann velocity distribution. A typical measure of their speed v is related to the square root of $\sqrt{\frac{k_B T}{m}}$, where k_B is the Boltzmann constant, T is the absolute temperature, and m is the mass

of the atom. For example, sodium atoms at room temperature have RMS speeds of several hundred meters per second.

The second bullet point explains the consequence of this motion: "Motion shifts resonance by Doppler frequency $\Delta\omega_D = \omega_0 \frac{v}{c}$."

If an atom is moving with a velocity component v along the line of sight of an incoming (or emitted) light wave, it will experience (or produce) a Doppler shift in the frequency of the light. If the atom is moving towards the light source (or detector), the frequency it absorbs (or emits) in its own rest frame, ω_0, will appear higher (blueshifted) in the lab frame. If it's moving away, the frequency will appear lower (redshifted). To first order, the Doppler shift in angular frequency, $\Delta\omega_D$, is given by $\Delta\omega_D = \omega_0 \frac{v}{c}$, where v is the velocity component along the light propagation direction (positive if moving away, negative if moving towards, for absorption of a fixed lab frequency laser, or affects emitted frequency if v is atom's velocity relative to lab) and c is the speed of light. Since there's a distribution of velocities v in the gas, there will be a distribution of Doppler shifts.

The third bullet point gives the "Result: convolution of natural Lorentzian with Gaussian velocity distribution leads to Voigt profile."

Each atom, in its own rest frame, has a natural linewidth described by a Lorentzian profile centered at ω_0. However, due to the thermal motion, an ensemble of atoms will have their resonance frequencies ω_0 (as seen in the lab frame) shifted by the Doppler effect. The distribution of these shifts, arising from the Maxwell-Boltzmann distribution of velocities, is Gaussian. The observed spectral line from the ensemble is therefore a superposition of many Lorentzian profiles, each shifted by a different Doppler amount. The overall lineshape that results from convolving the Lorentzian natural lineshape with the Gaussian Doppler broadening distribution is known as a **Voigt profile**.

In many practical situations, especially for light atoms at moderate to high temperatures, the Doppler broadening (width of the Gaussian) is significantly larger than the natural linewidth (width of the Lorentzian). In such cases, the observed lineshape looks mostly Gaussian, and the underlying natural linewidth can be completely masked. This is a major practical limitation to observing the true natural linewidth in conventional gas-phase spectroscopy.

Page 49:

This page continues the discussion on Doppler broadening and its implications.

The single bullet point states: "Natural profile derived here is observable only with Doppler-free spectroscopy (e.g., saturated absorption, two-photon spectroscopy)." This is a critical point for experimental laser spectroscopy. Because Doppler broadening often dominates in gases, the narrow natural Lorentzian profile we've painstakingly derived is usually hidden within a much broader Doppler profile. To observe and measure the natural linewidth, or to perform spectroscopy with a resolution limited by it, one needs to employ special techniques known as "Doppler-free spectroscopy."

Examples given are:

* **Saturated Absorption Spectroscopy:** This technique uses two counter-propagating laser beams (a strong pump and a weak probe) interacting with the gas. Only atoms with zero velocity component along the beams' axis can interact resonantly with both beams simultaneously. This creates a narrow "Lamb dip" (a dip in the absorption of the probe beam) at the true resonance frequency ω_0, with a width approaching the natural linewidth.

* **Two-Photon Spectroscopy:** If an atom absorbs two photons simultaneously from counter-propagating beams, the Doppler shift from one beam cancels the Doppler shift from the other (for an atom moving along the beam axis, one photon is blueshifted by $\omega_0 \frac{v}{c}$, the other is

redshifted by $\omega_0 \frac{v}{c}$, so the sum of their energies in the atom's frame has the first-order Doppler shift cancel out). This allows for the observation of a Doppler-free signal whose width can be limited by the natural linewidth.

There are other Doppler-free techniques as well (e.g., spectroscopy on collimated atomic beams observed perpendicularly). The development of these techniques was a major advance in laser spectroscopy, allowing physicists to overcome the Doppler limit and probe the intrinsic structure of atomic and molecular lines with unprecedented resolution. Without them, much of what we know about fine details of atomic spectra would be inaccessible.

Page 50:

This page presents a graph titled "Spectral Line Broadening: Natural, Doppler, and Voigt Profiles," visually summarizing the concepts from the last few slides.

The vertical axis is "Normalized Intensity," and the horizontal axis represents frequency, centered at ω_0. The x-axis ranges from -5.0 to 5.0 in some arbitrary units related to the widths.

Three profiles are plotted:

1. **Natural (Lorentzian, Γ_L)**: This is the blue curve. It is the narrowest of the three, sharply peaked at ω_0. This represents the intrinsic lineshape due to the finite lifetime of the excited state, which we've identified as a Lorentzian. Γ_L here would correspond to our 'γ' (or related to it).

2. **Doppler (Gaussian, Γ_G)**: This is the red curve. It is significantly broader than the natural Lorentzian profile in this illustration. It has the characteristic bell shape of a Gaussian distribution, arising from the Maxwell-Boltzmann distribution of atomic velocities. Γ_G represents the width of this Gaussian Doppler profile.

3. **Observed (Voigt, Γ_V)**: This is the purple curve. It is the convolution of the blue Lorentzian and the red Gaussian. In this particular plot, where the Doppler width is shown as larger than the natural width, the Voigt profile looks somewhat like a Gaussian in the core but has broader "wings" (tails) than a pure Gaussian, inherited from the Lorentzian component. The Voigt profile is generally wider than either the pure Lorentzian or pure Gaussian from which it's formed. Γ_V would be its width.

Dashed lines indicate the half-maximum level, and the labels on the graph (e.g., $\Gamma_V = 2.96$, $\Gamma_G = 2.50$, $\Gamma_L = 0.80$) seem to be parameters used for this specific plot, indicating relative widths, perhaps FWHM values. These numbers clearly show $\Gamma_L < \Gamma_G < \Gamma_V$ in this example.

This graph powerfully illustrates why Doppler broadening can be a major challenge. If the red Gaussian is much wider than the blue Lorentzian, the observed purple Voigt profile will largely mimic the Gaussian, and the subtle features of the natural lineshape will be obscured. Doppler-free techniques are designed to effectively "suppress" the red Gaussian contribution, allowing the blue Lorentzian to be revealed.

Page 51:

Slide 20: Example — Sodium D_1 Transition at 589 nm.

Now we move to "Slide 20: Example — Sodium D_1 Transition at 589 nm." This provides concrete numbers to make the concepts more tangible. The Sodium D lines are classic examples in atomic spectroscopy.

- The first bullet, "Data," indicates we're starting with known properties.

- Second bullet: "Wavelength $\lambda = 589.1\,\text{nm}$ (nanometers) implies $\nu_0 = \frac{c}{\lambda} = 5.09 \times 10^{14}\,\text{Hz}$."

The D1 line of sodium has a wavelength of approximately $589.1\,\text{nm}$ (this is actually closer to the average of D1 and D2, D1 is $589.6\,\text{nm}$, D2 is $589.0\,\text{nm}$; let's use the value given).

Using the speed of light $c \approx 3 \times 10^8$ m/s and the wavelength $\lambda = 589.1 \times 10^{-9}$ m, the transition frequency ν_0 is calculated as $\frac{c}{\lambda}$.

$\nu_0 = \frac{3 \times 10^8 \text{ m/s}}{589.1 \times 10^{-9} \text{ m}} \approx 5.092 \times 10^{14}$ Hz. The slide gives 5.09×10^{14} Hz, which is correct. This is a very high frequency, typical for visible light (orange-yellow for sodium D-lines).

- Third bullet: "Lifetime $\tau = 16$ ns $= 16 \times 10^{-9}$ s."

The lifetime of the upper state ($3p^2P_{1/2}$ for D1 line) of the sodium D1 transition is approximately 16 nanoseconds. This is a relatively short lifetime, typical for allowed optical transitions.

- Fourth bullet: "Natural linewidth."

Now we apply our formula $\Delta\nu_{\text{nat}} = \frac{1}{2\pi\tau}$.

$\Delta\nu_{\text{nat}} = \frac{1}{2\pi\tau} = \frac{1}{2\pi \times 16 \times 10^{-9} \text{ s}}$.

Plugging in the values:

$$\frac{1}{2\pi \times 16 \times 10^{-9}} \approx \frac{1}{100.53 \times 10^{-9}} \approx \frac{1}{1.0053 \times 10^{-7}} \approx 0.9947 \times 10^7 \text{ Hz}$$

This is 9.947×10^6 Hz, or 9.947 MHz (Megahertz). The slide gives 9.95×10^6 Hz, which is approximately 10 MHz.

This is correct. So, the natural linewidth of the sodium D1 line is about 10 MHz. This is the minimum possible width this spectral line can have, even if we could perfectly isolate a sodium atom and keep it stationary.

This 10 MHz might sound small, but Doppler broadening for sodium at room temperature is on the order of $1 - 2$ GHz (Gigahertz), which is 100 to 200 times larger! This again emphasizes why Doppler-free techniques are needed to see this natural width.

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This page continues the example of the Sodium D1 transition.

The first bullet point calculates the "Number of optical cycles before amplitude decays by $\frac{1}{e}$":

$N_{\text{cycles}} = \nu_0 \tau'$. Here τ' should be the amplitude decay time constant. The amplitude $x(t)$ decays as $e^{-\gamma t/2}$. The time constant for amplitude decay is $\frac{2}{\gamma}$. We know $\gamma = A_{\text{i}} = \frac{1}{\tau_{\text{lifetime}}}$ (where τ_{lifetime} is the 16 ns energy/population lifetime). So, amplitude decay time constant = $2 \times \tau_{\text{lifetime}} = 2 \times 16\,\text{ns} = 32\,\text{ns}$. $N_{\text{cycles}} = \nu_0 \times (2 \times \tau_{\text{lifetime}}) = (5.09 \times 10^{14}\,\text{Hz}) \times (32 \times 10^{-9}\,\text{s}) = 5.09 \times 32 \times 10^5 = 162.88 \times 10^5 = 1.6288 \times 10^7$ cycles.

However, the slide uses $N_{\text{cycles}} = \nu_0 \tau = (5.09 \times 10^{14}) \times (16 \times 10^{-9})$ approximately 8.1×10^6. Here, $\tau = 16\,\text{ns}$ is the lifetime of the state (power decay time constant, or $1/\gamma$). If the question is "number of optical cycles during one lifetime $\tau = 1/\gamma$", then this calculation is correct: $(5.09 \times 10^{14}\,\text{cycles/sec}) \times (16 \times 10^{-9}\,\text{sec}) = 5.09 \times 16 \times 10^5 = 81.44 \times 10^5 = 8.144 \times 10^6$ cycles. So, during one lifetime τ (where power drops by $\frac{1}{e}$, or population drops by $\frac{1}{e}$), the classical oscillator completes about 8.1 million cycles. This is a very large number.

The second bullet point draws a conclusion: "Confirms γ much, much less than ω_0 by over seven orders of magnitude." Let's check this. $\gamma = \frac{1}{\tau} = \frac{1}{16 \times 10^{-9}\,\text{s}} = 0.0625 \times 10^9\,\text{s}^{-1} = 6.25 \times 10^7\,\text{rad/s}$ (if thinking of γ as angular frequency width). $\omega_0 = 2\pi\nu_0 = 2\pi \times (5.09 \times 10^{14}\,\text{Hz}) \approx 3.198 \times 10^{15}\,\text{rad/s}$. Now, the ratio ω_0/γ:

$$\frac{\omega_0}{\gamma} = \frac{3.198 \times 10^{15}}{6.25 \times 10^7} = \left(\frac{3.198}{6.25}\right) \times 10^8 \approx 0.511 \times 10^8 \approx 5.1 \times 10^7$$

This is a factor of about 51 million. Indeed, ω_0 is vastly larger than γ. This fully justifies our "small damping approximation" ($\gamma \ll \omega_0$) that we used

earlier. The oscillator performs many, many oscillations before its energy (or amplitude) significantly decays. The quality factor Q of such an oscillator, often defined as ω_0/γ, would be very high, around 5×10^7. This means the emitted wavetrain is very long and nearly monochromatic, which corresponds to a narrow line in the frequency domain, as we found (10 MHz width for a 509 THz frequency).

Page 53:

Slide 21: Example — Ultra-Narrow Molecular and Forbidden Lines

Now for "Slide 21: Example — Ultra-Narrow Molecular and Forbidden Lines." This contrasts with the sodium D-line example by looking at systems with much longer lifetimes.

First case: "Vibrational states typical lifetime $\tau = 1$ ms (millisecond)."

Molecular vibrational transitions (e.g., in the infrared) can have lifetimes on the order of milliseconds (10^{-3} s), which is much longer than the nanosecond lifetimes of typical allowed electronic transitions in atoms.

$$\tau = 1\,\text{ms} = 1 \times 10^{-3}\,\text{s}$$

"NATURAL WIDTH":

Using $\Delta\nu_{\text{nat}} = \frac{1}{2\pi\tau}$:

$$\Delta\nu_{\text{nat}} = \frac{1}{2\pi \cdot 1 \times 10^{-3}\,\text{s}} = \frac{1}{0.006283\,\text{s}} \approx 159.15\,\text{Hz}$$

The slide gives "equals 159 Hz, approximately 160 Hz."

This is a remarkably narrow natural linewidth! Compare this to the ~ 10 MHz for the sodium D-line. The 160 Hz width is about 60,000 times narrower, directly reflecting the much longer lifetime (1 ms vs 16 ns, a factor of 62,500 longer).

Such narrow lines are indeed achievable in vibrational spectroscopy if Doppler and collisional broadening can be sufficiently suppressed.

Second case: "Hydrogen $2\,s \rightarrow 1\,s$ two-photon transition."

The $2\,s$ state of hydrogen is metastable.

It cannot decay to the $1\,s$ ground state by emitting a single photon because this would violate electric dipole selection rules (ΔL must be ± 1, but $2\,s$ to $1\,s$ is $\Delta L = 0$).

It can decay by emitting two photons, but this is a much slower process.

"Lifetime $\tau = 0.12\,\text{s}$."

This is an extremely long lifetime for an excited atomic state – $0.12\,\text{s}$! (The actual lifetime of $H(2\,s)$ is about $\frac{1}{8}$ of a second, so $0.12\,\text{s}$ is in the right ballpark).

We'll calculate the width on the next page.

The key takeaway here is that much longer lifetimes lead to much narrower natural linewidths.

Page 54:

Continuing with the Hydrogen $2\,s \rightarrow 1\,s$ two-photon transition example: The lifetime τ was given as 0.12 seconds.

The first bullet asks for the "Width": Using $\Delta\nu_{\text{nat}} = \frac{1}{2\pi\tau}$:

$$\Delta\nu_{\text{nat}} = \frac{1}{2\pi \cdot 0.12\,\text{s}} = \frac{1}{0.75398\,\text{s}} \approx 1.326\,\text{Hz}$$

The slide gives "equals 1.3 Hz." This is an incredibly narrow natural linewidth – just over one Hertz! This is nearly 10 million times narrower than the sodium D-line's natural width. This extremely narrow linewidth is a direct consequence of the very long lifetime of the $2\,s$ state. The two-

photon spectroscopy technique used to observe this transition is also inherently Doppler-free (to first order when using counter-propagating beams), allowing such narrow features to be resolved.

The second bullet point emphasizes the significance: "Such narrow features are pivotal in testing quantum electrodynamics and probing fundamental constants."

Transitions with extremely narrow natural linewidths, like the Hydrogen $1\,s$-$2\,s$ transition, are of immense importance in fundamental physics:

- **Testing QED:** Quantum Electrodynamics is our most precise theory of light-matter interactions. Measuring the frequencies of such transitions with extremely high accuracy provides stringent tests of QED calculations, which include tiny effects like the Lamb shift.

- **Probing Fundamental Constants:** The frequencies of atomic transitions depend on fundamental physical constants (like the Rydberg constant, fine structure constant, proton-to-electron mass ratio). Highly precise measurements of these frequencies can lead to improved values for these constants, or can search for possible slow variations of these constants over cosmological time scales.

- **Atomic Clocks:** Transitions with very narrow linewidths are ideal candidates for optical atomic clocks, which are now the most precise timekeeping devices ever built, far surpassing microwave atomic clocks.

The ability to resolve and accurately measure such ultra-narrow lines, often limited by their natural linewidth, is a triumph of modern laser spectroscopy and atomic physics.

Page 55

This page presents a "Comparison of Natural Linewidths (Logarithmic Scale)" using a bar chart. This is an excellent way to visualize the vast differences in natural linewidths for the examples we've discussed.

The vertical axis is "Natural Linewidth (Hz)" and is on a logarithmic scale, ranging from 0.1 Hz at the bottom to 100 MHz at the top. Each major tick mark represents a factor of 10 increase in linewidth (1 Hz, 10 Hz, 100 Hz, 1 kHz, 10 kHz, 100 kHz, 1 MHz, 10 MHz, 100 MHz). The horizontal axis is "Transition," showing the three cases.

Let's look at the bars:

1. **Sodium D1 Line (**589 nm): The bar for this transition reaches up to a value labeled 9.95 MHz. This corresponds to the ~ 10 MHz natural linewidth we calculated, stemming from its ~ 16 ns lifetime. On the log scale, this is near the top.

2. **Molecular Vibrational States**: The bar for this case (referring to the 1 ms lifetime example) reaches up to a value labeled 159 Hz. This is significantly lower on the logarithmic scale than the Sodium D1 line, reflecting its much narrower linewidth.

3. **Hydrogen** $2s \rightarrow 1s$ (two-photon): The bar for this transition is the shortest, reaching down to a value labeled 1.3 Hz. This is by far the narrowest linewidth of the three, consistent with the very long 0.12 s lifetime of the $2s$ state. It sits very low on the logarithmic scale.

This chart dramatically illustrates the huge range of natural linewidths encountered in atomic and molecular spectroscopy. The linewidth can span many orders of magnitude, directly reflecting the equally vast range of lifetimes of excited states. Understanding this connection allows us to select appropriate transitions for different applications, whether it's for rapid optical pumping (requiring short lifetimes and broad lines) or for ultra-high precision metrology (requiring long lifetimes and extremely narrow lines).

Page 56:

Slide 22: Key Takeaways and Forward Look

We're nearing the end of this section with "Slide 22: Key Takeaways and Forward Look." This slide summarizes the main points we've covered regarding natural linewidth.

First bullet: "Natural linewidth originates from spontaneous emission and is fully characterised by lifetime τ."

This is the fundamental quantum origin. An excited state is not truly stable; it interacts with the vacuum electromagnetic field and spontaneously emits a photon, returning to a lower energy state. This process gives the excited state a finite lifetime, τ. This finite lifetime, via the energy-time uncertainty principle (or through Fourier analysis of a decaying emission), leads directly to a spread in the emitted/absorbed frequency, which is the natural linewidth. The width is inversely proportional to the lifetime $\left(\Delta\nu_{\text{nat}} = \frac{1}{2\pi\tau}\right)$.

Second bullet: "Classical damped oscillator faithfully mirrors quantum behaviour for line-shape and width."

Remarkably, we found that a simple classical model of an electron on a spring, losing energy via radiation (damping), reproduces key features of the quantum system. The solution to the classical equation of motion, when Fourier transformed, yielded a Lorentzian lineshape, and the width of this Lorentzian (γ) could be directly identified with the inverse lifetime $\left(\frac{1}{\tau}\right.$ or $\left.A_{\text{i}}\right)$ from the quantum picture. This analogy is powerful for building intuition.

Third bullet: "Lorentzian profile is universal for isolated, stationary emitters and absorbers."

When natural broadening is the only broadening mechanism present (i.e., for an atom that is isolated from collisions or other external perturbations, and is stationary so there's no Doppler effect), the spectral lineshape is a Lorentzian. This characteristic shape, with its relatively broad wings, is a fingerprint of lifetime broadening.

These three points encapsulate the core physics of natural linewidth.

Page 57:

This final page provides a "Forward Look," indicating what's next in our study of lineshapes.

The bullet point states: "Upcoming sections will incorporate motion (Doppler) and collisional effects to build the complete Voigt profile used in real-world spectroscopy."

We've focused on the natural linewidth, which is the intrinsic, minimum width. However, in many real-world spectroscopic measurements, especially in gases and liquids, other broadening mechanisms are also present and can often dominate:

- **Motion (Doppler Broadening):** As we briefly discussed, the thermal motion of atoms in a gas leads to Doppler shifts, resulting in a Gaussian broadening of the spectral line.

- **Collisional Effects (Pressure Broadening):** Atoms in a gas are not truly isolated; they collide with each other. These collisions can interrupt the process of emission or absorption, effectively shortening the coherent interaction time with the radiation field. This leads to a broadening of the spectral line, which is also typically Lorentzian in shape. The width of this collisional broadening depends on the pressure and temperature of the gas.

When natural broadening (Lorentzian), Doppler broadening (Gaussian), and collisional broadening (Lorentzian) are all significant, the observed lineshape is a convolution of all these effects. Since the convolution of two Lorentzians is another Lorentzian, we effectively have a natural + collisional Lorentzian convolved with a Doppler Gaussian. This combined profile is the Voigt profile we mentioned earlier.

Understanding how to model and analyze Voigt profiles is essential for accurately interpreting spectra from many real-world systems and for

extracting physical parameters like temperature, pressure, and species concentrations.

So, our detailed study of the natural Lorentzian lineshape here serves as a crucial foundation. We'll build upon it by adding these other important broadening mechanisms to develop a more complete picture relevant for practical laser spectroscopy.

And that concludes our lecture on natural linewidth. Thank you.