Chapter

2.6

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Alright everyone, welcome back to Phys 608, Laser Spectroscopy. Today, we delve into a crucial segment of our course, Chapter 2.6, which focuses on two intimately related phenomena: **Absorption and Dispersion**. These concepts are absolutely fundamental to understanding how light, particularly laser light, interacts with matter.

The material for this lecture, as always, has been meticulously prepared by Distinguished Professor Dr. M.A. Gondal. We are in Term 251 here at KFUPM. So, let's get started.

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So, let's consider **Absorption and Dispersion – what are they, and critically, why do we, as physicists and spectroscopists, need to study these two phenomena in such detail?**

The first bullet point gets straight to the heart of absorption. **Absorption alters the amplitude (and therefore the energy flow) of an optical wave as it propagates through matter.** Imagine sending a beam of light through a colored filter. The filter appears colored precisely because it absorbs certain wavelengths, or frequencies, of light more strongly than others. This absorption process directly reduces the intensity, the energy, of the light wave passing through. So, absorption is fundamentally about energy loss from the optical field to the medium.

The second bullet point introduces dispersion. **Dispersion, on the other hand, alters the phase velocity of an optical wave.** This is a key distinction. While absorption affects the wave's amplitude, dispersion primarily affects its speed of phase propagation. And because the phase velocity is generally frequency-dependent in a material, dispersion leads to **different frequency components of a light pulse separating in time and space.** Think about a prism: white light, which is a superposition of many colors, enters the prism, and because the refractive index (and thus phase velocity) is different for red light versus blue light, these colors travel at slightly different speeds and bend by different amounts, separating into the familiar rainbow. For laser pulses, which are often composed of a band of frequencies, this means the pulse can spread out or become chirped as it travels through a dispersive medium.

Understanding both of these is paramount in our field.

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Continuing our discussion on why these phenomena are so vital: the slide notes that **together, absorption and dispersion determine the shape, delay, and intensity of laser pulses in gases, liquids, and solids.** This is not an overstatement. Whether you are performing high-resolution spectroscopy, designing fiber-optic telecommunication systems, or engineering new types of lasers, the interplay of absorption and dispersion is a critical design consideration.

For instance, in spectroscopy, the precise absorption spectrum gives us the fingerprint of a molecule, but the dispersive effects can shift or broaden lines, or affect the timing of probe pulses. In telecommunications, dispersion in optical fibers limits the bit rate by causing pulses to spread and overlap. In laser design, particularly for ultrashort pulse lasers, managing dispersion within the laser cavity is essential for achieving and maintaining mode-locking. So, the impact is truly far-reaching.

Now, a very profound point is made in the second bullet: **A transparent, quantitative link exists between both effects – absorption and dispersion – through the complex refractive index, which we'll denote as n.** Unveiling this link is the goal of the following derivation. This is a beautiful piece of physics. It tells us that absorption and dispersion are not independent phenomena that just happen to occur together. Instead, they are two sides of the same coin, intrinsically connected by the fundamental optical properties of the material, all encapsulated within this quantity called the complex refractive index. Our journey in the next few slides will be to build up this connection from a microscopic model.

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To help visualize these effects, this slide presents a schematic of **Optical Wave Propagation, illustrating Absorption and Dispersion** separately.

Let's first look at panel **(a) Effect of Absorption (where $\alpha > 0$)**. Alpha, α, here represents the absorption coefficient. On the left, we see an incoming sinusoidal wave, representing, for instance, the electric field of our light wave. It has a well-defined amplitude. As this wave enters a material (the shaded region to the right of the dashed vertical line), its amplitude begins to decrease. You can see the envelope of the wave, indicated by the dashed red lines, decaying as the wave propagates further into the material along the horizontal axis. The wavelength, however, appears to remain constant in this idealized depiction focusing purely on absorption. This decay in amplitude signifies that the energy of the wave is being transferred to the medium. The condition $\alpha > 0$, simply means it's an absorbing medium.

Now, let's turn to panel **(b) Effect of Dispersion (where $n > n_0$)**. Here, 'n' is the refractive index of the medium, and 'n naught' (n_0) would be the refractive index of the free space or the medium the light is coming from. The condition $n > n_0$ means the light slows down as it enters the material because the phase velocity is $\frac{c}{n}$. Again, on the left, we see an incident wave. As it enters the material (the shaded region), notice what happens: the amplitude, in this idealized case showing only dispersion, remains constant. However, the wavelength of the wave *inside* the material, let's call it lambda prime (λ'), becomes shorter than the wavelength lambda naught (λ_0) outside. This is a direct consequence of the phase velocity decreasing while the frequency remains constant. The wave crests are packed more closely together. If we were considering a pulse composed of different frequencies, each would experience a slightly different refractive index n, leading to different phase velocities, and this is what causes pulse spreading or compression, the hallmark of dispersion.

These two diagrams give us a nice, qualitative first look. Now, let's build the mathematical framework.

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Alright, let's move to what this slide calls **Electromagnetic Wave — Precise Mathematical Description.** To understand how a material modifies a wave, we first need a clear description of our reference: the wave in free space.

We begin with a **Free-Space Reference Wave.** The first bullet point states: **Start from a monochromatic plane wave of angular frequency** ω travelling in the positive z-direction.

A monochromatic wave means it consists of a single frequency, ω. A plane wave means its wavefronts – surfaces of constant phase – are infinite planes perpendicular to the direction of propagation. We're choosing the propagation direction to be along the positive z-axis for simplicity.

The electric field of such a wave, $E(z, t)$, which is a vector function of position z and time t, is given by the equation:

$$E(z,t) = E_0\,\hat{x}\,e^{i(\omega t - K_0 z)}$$

Let's break this down carefully.

$E(z, t)$: This is the electric field vector.

E_0: This is the **real peak amplitude** of the wave. The slide specifies its units as Volts per meter $\left(\mathrm{V\,m^{-1}}\right)$. This E_0 represents the maximum strength of the electric field.

$\hat{x}$: This is a unit vector in the x-direction. It tells us that the electric field is linearly polarized along the x-axis. The wave oscillates in the x-direction while propagating in the z-direction.

$e^{i(\omega t - K_0 z)}$: This is the complex exponential form representing the oscillatory part of the wave.

○ i: The imaginary unit, square root of minus one. ○ ω: The angular frequency of the wave, in radians per second. It's related to the ordinary frequency ν by $\omega = 2\pi\nu$. ○ t: Time. ○ K_0: This is the **free-space wavenumber** or propagation constant in vacuum. It tells us how many radians of phase change occur per unit length. Its units are typically radians per meter. ○ z: The spatial coordinate along the direction of propagation.

The term $(\omega t - K_0 z)$ is the phase of the wave. Surfaces of constant phase define the wavefronts. The negative sign between ωt and $K_0 z$ indicates a wave traveling in the positive z-direction. If it were a plus sign, it would be a wave traveling in the negative z-direction. This equation is our starting point, our benchmark for what happens when this wave enters a material.

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Let's continue dissecting the components of our free-space reference wave.

The first bullet point clarifies $\hat{x}$ (x-hat): it is indeed a linear polarization unit vector, chosen for definiteness. We could have chosen $\hat{y}$ (y-hat), or any other direction perpendicular to z, but x-hat is a common and convenient choice. For many of our initial discussions, the specific direction isn't critical, only that it *is* polarized.

Next, we have K_0 (Kay sub zero), the free-space wavenumber. This is defined as:

$$\mathrm{K}_0 = \frac{2\pi}{\lambda_0} = \frac{\omega}{c}$$

Here: * λ_0 (lambda sub zero or lambda naught) is the wavelength of the light in free space, in meters. The wavenumber K_0 is thus 2π divided by the wavelength. * ω (omega) is the angular frequency, as before. * c is the speed of light in vacuum.

So, K_0 can also be expressed as ω divided by c. The units are given as $\mathrm{rad\,m}^{-1}$. This K_0 is a measure of spatial frequency.

The third point concerns the speed of these wavefronts: Phase fronts advance at the vacuum phase velocity, $v_{\text{ph,vac}}$ (vee sub pee aitch comma vac). This velocity is given by:

$$v_{\text{ph,vac}} = \frac{\omega}{\mathrm{K}_0}$$

And since $\mathrm{K}_0 = \frac{\omega}{c}$, it follows directly that:

$$v_{\text{ph,vac}} = c$$

The value of c is given as $2.99792458 \times 10^8\ \text{m s}^{-1}$. This is, of course, a fundamental constant: the speed of light in vacuum.

Finally, how much energy does this wave carry? The last bullet point tells us: Energy flow per unit area is given by the time-averaged Poynting vector, denoted as $\langle S \rangle$ (angle brackets S). For our plane wave, this is:

$$\langle S \rangle = \frac{1}{2} \varepsilon_0\, c\, E_0^2\, \hat{z}$$

Let's unpack this: * $\langle S \rangle$: The Poynting vector describes the directional energy flux (the energy transfer per unit area per unit time) of an electromagnetic field. The angle brackets denote a time average over one cycle of oscillation. * ε_0 (epsilon sub zero or epsilon naught): This is the permittivity of free space, a fundamental constant with a value of approximately 8.854×10^{-12} Farads per meter. * c: The speed of light in vacuum. * E_0^2 (Eee sub zero squared): The square of the peak electric field amplitude. Intensity is proportional to the square of the field amplitude. * $\hat{z}$ (z-hat): This is a unit vector in the z-direction, indicating that the energy flows in the direction of wave propagation, as expected.

The factor of $\frac{1}{2}$ arises from the time averaging of the squared sinusoidal field. This Poynting vector gives us the intensity of the light wave.

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This page provides a visual representation of the electric field component E_x of our plane wave, plotted as a function of the propagation distance z, at a fixed instant in time (say, $t = 0$).

Let's describe what we see. The vertical axis is labeled E_x, representing the x-component of the electric field, and it also shows the polarization direction $\hat{x}$ pointing upwards. The horizontal axis is labeled z, the direction of propagation.

The wave itself is a perfect sinusoid, oscillating symmetrically around the $E_\mathrm{x} = 0$ axis.

The peak amplitude of the oscillation is labeled as E_0, and the minimum value is $-E_0$. This clearly shows the maximum displacement of the electric field.

Along the z-axis, we see several points marked in terms of the free-space wavelength λ_0:

◦ The wave starts at $z = 0$ with its maximum amplitude E_0. ◦ It crosses zero at $z = \lambda_0/4$ (not explicitly labeled, but one quarter of a wavelength). ◦ It reaches its minimum $-E_0$ at $z = \lambda_0/2$ (one half λ_0 over two). ◦ It crosses zero again at $z = 3\lambda_0/4$. ◦ It completes one full cycle and is back at E_0 at $z = \lambda_0$ (λ_0). This distance is explicitly marked with a horizontal red arrow, labeled λ_0, showing one full spatial period of the wave.

The plot continues, showing subsequent crests and troughs at $z = 3\lambda_0/2$, $2\lambda_0$, and $5\lambda_0/2$, illustrating the periodic nature of the wave in space.

Below the z-axis, there's a green arrow labeled "Direction of Propagation ($\mathbf{k}$)" pointing to the right, reinforcing that the wave is moving in the positive z-direction. The vector $\mathbf{k}$ here would be $K_0\hat{z}$.

This graph is a snapshot. If we were to animate this for increasing time t, we would see this entire sinusoidal pattern moving to the right, along the positive z-axis, with the phase velocity c. This is the wave we are now going to send into a material.

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Now we arrive at a pivotal moment: our electromagnetic wave is **Entering a Material.** This introduces **New Macroscopic Parameters** that describe how the material responds to the wave. The most important of these is the complex refractive index.

The slide defines the **Definition of the Complex Refractive Index.** The first crucial point: **Inside matter, the wave number changes.** It's no longer K_0. The new wave number, let's call it K (capital Kay), is related to the free-space wave number K_0 by:

$$K = nK_0$$

Here, **n (script n, or sometimes italic n)** is the **refractive index** of the material. This simple-looking equation is packed with physics. It tells us that the spatial periodicity of the wave changes inside the material.

Now, the second, and extremely important, point: **Refractive index is, in general, frequency dependent and complex.** This is key for understanding both absorption and dispersion simultaneously. We write the complex refractive index n as a function of angular frequency ω (omega) as:

$$n(\omega) = n'(\omega) - i\kappa(\omega)$$

Let's break this down:

$n(\omega)$: The complex refractive index, explicitly showing its dependence on the frequency ω of the light. This frequency dependence is the ultimate origin of dispersion.

$n'(\omega)$ (n prime of omega): This is the **real part** of the complex refractive index. It's what we typically think of as "the" refractive index in elementary optics, governing, for example, Snell's Law of refraction. As we'll see, it dictates the phase velocity of the wave in the medium.

i: The imaginary unit.

$\kappa(\omega)$ (kappa of omega): This is the **imaginary part** of the complex refractive index, also known as the **extinction coefficient** or attenuation index. As we will see, this part is responsible for the absorption of light in the medium.

The **minus sign** before the $i\kappa(\omega)$ term is a common convention in physics, particularly in optics. Some engineering texts might use a plus sign. The choice of sign impacts how κ is defined (it must be positive for absorption in a passive medium with this convention) and its relation to the decay of the wave. It's important to be consistent. With this minus sign, a positive κ will lead to attenuation, as we expect for absorption.

So, when light enters a material, we replace K_0 with nK_0, where n itself is a complex number having both a real part n' and an imaginary part κ. This single complex quantity will elegantly describe both how the phase of the wave evolves and how its amplitude changes.

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Having defined the complex refractive index $n(\omega) = n'(\omega) - i\kappa(\omega)$, let's explore the distinct roles of its real and imaginary parts.

The first bullet point states: n' (n prime), the real part, governs phase propagation, and this leads to dispersion. How so? Recall that the phase velocity in a medium is $v_{ph} = \frac{c}{n'}$. Since n' is generally a function of frequency ω, $v_{ph}(\omega) = \frac{c}{n'(\omega)}$ will also be frequency dependent. This means different frequency components of a light pulse will travel at different speeds, causing the pulse to spread out or change shape – this is precisely what we mean by dispersion. The wavelength inside the medium also changes to $\lambda' = \frac{\lambda_0}{n'}$.

The second bullet point focuses on the imaginary part: κ (kappa), the imaginary part, governs exponential attenuation, which means it's responsible for absorption. This κ term will lead to a decay in the amplitude of the electric field as the wave propagates, which corresponds to the absorption of energy by the medium.

Now, let's see how the electric field equation looks inside the material. We had $E(z,t) = E_0\,\hat{x}\,e^{i(\omega t - Kz)}$ generally. Substituting $K = nK_0 = (n' - i\kappa)K_0$, the exponent becomes:

$$\begin{aligned} i(\omega t - (n' - i\kappa)K_0\,z) &= i(\omega t - n'K_0\,z + i\kappa K_0\,z) = i(\omega t - n'K_0\,z) + i^2\kappa K_0\,z \\ &= i(\omega t - n'K_0\,z) - \kappa K_0\,z. \end{aligned}$$

So, the field now reads:

$$E(z,t) = E_0\,e^{-\kappa K_0\,z}\,e^{i(\omega t - n'K_0\,z)}\,\hat{x}.$$

Look at this equation carefully. We can see two distinct effects due to the material:

1. $e^{-\kappa K_0\,z}$: This is a real exponential decay factor. Since κ, K_0, and z are positive for a forward-propagating wave in an absorbing medium, this term causes the amplitude of the wave E_0 to decrease exponentially as it travels deeper into the material (increasing z). This is the mathematical description of absorption.

2. $e^{i(\omega t - n\prime K_0 z)}$: This is the oscillatory part. Notice that the wavenumber in the phase is now $n'K_0$, not just K_0. This means the wavelength inside the medium is $\frac{\lambda_0}{n\prime}$, and the phase velocity is $\frac{c}{n\prime}$. This term describes the phase evolution and is responsible for dispersion if n' varies with ω.

Finally, the intensity of the wave, I, is proportional to the square of the electric field amplitude. If the field amplitude decays as $e^{-\kappa K_0 z}$, then the intensity will decay as the square of this factor:

$$(e^{-\kappa K_0 z})^2 = e^{-2\kappa K_0 z}.$$

So, the intensity decays as $I(z) = I_0 \, e^{-2\kappa K_0 z}$, where I_0 is the initial intensity at $z = 0$. Notice the factor of 2 in the exponent for intensity decay compared to field amplitude decay. This is a crucial detail.

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This slide presents a beautiful graph illustrating **Electric Field Propagation in an Absorbing Material**. It visually synthesizes the concepts we just discussed regarding the complex refractive index.

The graph plots the electric field, E, on the vertical axis versus the propagation distance, z, on the horizontal axis. The material is described by a complex refractive index

$$n(\omega) = n' - i\kappa$$

and specific example values are given: $n' = 1.5$ and $\kappa = 0.05$. The wave is shown entering the material region, which starts at $z = 0$ and extends to the right (labeled "Material Region (z ≥ 0)").

Let's observe the features of the wave depicted:

1. **Amplitude Attenuation:** The most striking feature is that the amplitude of the sinusoidal wave decreases as z increases. It starts with an initial

amplitude E_0 at $z = 0$, and this amplitude gradually diminishes. The envelope of this decay is shown by two dashed grey lines, which represent the exponential factor

$$E_0 e^{-\kappa K_0 z}$$

This visually demonstrates the absorption due to the imaginary part κ of the refractive index. Given $\kappa = 0.05$, there's a noticeable decay over the range plotted.

2. **Wavelength Change:** If you look closely at the oscillations, the wavelength inside the material, labeled as λ', is different from what it would be in vacuum. It's given by

$$\lambda' = \frac{\lambda}{n'}$$

where λ would be the vacuum wavelength λ_0. Since $n' = 1.5$ (which is greater than 1), the wavelength inside the material is shorter (i.e., $\lambda' = \lambda_0/1.5$). The wave crests are more compressed compared to how they would be in vacuum. This is due to the real part n' of the refractive index. The horizontal axis is marked in units of this new wavelength: λ', $2\lambda'$, $3\lambda'$, $4\lambda'$, $5\lambda'$.

So, this single plot effectively shows both consequences of a complex n: the amplitude decay due to κ and the wavelength (and phase velocity) change due to n'. The wave is oscillating, but it's getting weaker as it propagates, and its spatial period is modified by the medium. This is precisely what our equation

$$E(z,t) = E_0 e^{-\kappa K_0 z} e^{i(\omega t - n' K_0 z)} \hat{x}$$

describes.

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We now transition from a macroscopic description (using the complex refractive index n) to a **Microscopic Picture**, attempting to understand *why* n should be complex and frequency-dependent. We'll use a **Classical Electron Oscillator model**, often called the Lorentz model. This is a semi-classical approach, but it provides remarkable insight.

The core idea is **Modelling Each Bound Electron** within the atoms or molecules of the material. The first bullet point lays out the central assumption: **Treat one valence electron as a driven, damped, one-dimensional harmonic oscillator.** Why a valence electron? Because these are typically the most loosely bound electrons and are therefore most easily perturbed by the electric field of an incident light wave. Why a harmonic oscillator? This implies a restoring force proportional to the displacement of the electron from its equilibrium position (like a mass on a spring). This is a reasonable approximation for small displacements. Why driven? The driving force is provided by the oscillating electric field of the light wave. Why damped? The electron's motion isn't frictionless. It will lose energy, for example, by re-radiating electromagnetic energy (radiative damping) or through collisions with other atoms (collisional damping). Why one-dimensional? For simplicity, we consider the electron's displacement $x(t)$ along the direction of the electric field of the light wave (which we took as x-polarized).

The second bullet point gives the **Governing equation of motion** for this electron. This is Newton's second law ($F = ma$) applied to our model:

$$m\,\ddot{x}(t) + b\,\dot{x}(t) + D\,x(t) = q\,E_0\,e^{i\omega t}$$

Let's meticulously define each term:

* m: This is the **mass** of the electron. * $x(t)$: This is the **displacement** of the electron from its equilibrium position at time t. * $\ddot{x}(t)$ (x double-dot of t): This is the **acceleration** of the electron (the second time derivative of x). So,

$m\,\ddot{x}(t)$ is the inertial term (mass times acceleration). * $\dot{x}(t)$ (x dot of t): This is the **velocity** of the electron (the first time derivative of x). * b: This is a **phenomenological damping coefficient**. The term $b\,\dot{x}(t)$ represents the damping force, which is proportional to the velocity and opposes the motion. Its units would be Newton-seconds per meter or kilograms per second. * D: This is the **spring constant** or force constant of the harmonic oscillator, representing the strength of the restoring force. The term $D\,x(t)$ is the restoring force, $F_{\text{restore}} = -Dx$ (though here it's on the left side with a plus, consistent with moving it from

$$ma = F_{\text{driving}} - F_{\text{damping}} - F_{\text{restoring}}$$

). Units are Newtons per meter. * q: This is the **charge** of the electron. * $E_0\,e^{i\omega t}$: This is the **driving electric field** of the incident light wave at the position of the electron (we assume the field is uniform over the atomic dimension, so we can just use E_0 and not $E(z,t)$, and for simplicity, we can evaluate it at $z = 0$ or just consider its time dependence as the driving force). E_0 is the amplitude, and ω is the angular frequency of the light.

This second-order linear inhomogeneous differential equation is the cornerstone of the classical model for light-matter interaction. Solving it will tell us how the electron responds (i.e., its displacement $x(t)$) to the driving light field.

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Let's further define the parameters and assumptions involved in our classical electron oscillator model, as described by the equation $m\ddot{x} + b\dot{x} + Dx = qE_0e^{i\omega t}$.

First, the parameters:

* **m: electron mass**. Its value is approximately 9.109×10^{-31} kilograms (kg). This is a fundamental constant.

* **b: phenomenological damping constant**. This term accounts for **energy loss due to radiation and collisions.** * **Radiative damping:** An accelerating charge, like our oscillating electron, radiates electromagnetic energy. This radiation carries away energy, effectively damping the electron's motion. This is also known as radiation resistance. * **Collisions:** In a real material, the oscillating electron can collide with other atoms or ions, transferring some of its energy to them. This is particularly important in gases at higher pressures and in condensed matter. The term 'phenomenological' means that 'b' is often treated as a parameter that encapsulates these various complex loss mechanisms without necessarily deriving it from first principles in this model. Its value might be determined empirically or from more advanced theories.

* **D: spring constant**. This determines the strength of the restoring force that pulls the electron back to its equilibrium position. It's directly related to the **natural resonant angular frequency** ω_0 (omega sub zero) of the oscillator by the relation $\omega_0^2 = \frac{D}{m}$, or $\omega_0 = \sqrt{\frac{D}{m}}$. This ω_0 is the frequency at which the electron would oscillate if it were displaced and then left to oscillate freely without any driving force or damping.

* **q = -e: electron charge magnitude**. 'q' is the charge of the particle, which for an electron is negative. We write it as $q = -e$, where **e** is the elementary charge, approximately 1.602×10^{-19} Coulombs (C). So 'e' itself is positive, representing the magnitude.

Now, for the crucial **Assumptions** underlying this model:

* **Linear response (displacements** x small). This is a very important assumption. We assume that the displacement x of the electron from its equilibrium position is small enough that the restoring force is accurately described by Hooke's Law ($F = -Dx$) and that the damping coefficient 'b' and mass 'm' are constant, independent of x. If displacements were large, we

might encounter anharmonic terms in the restoring force, leading to non-linear effects, which this model does not include. This linear response is why we expect to be able to define a linear susceptibility and refractive index later on.

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Continuing with the assumptions underpinning our classical electron oscillator model:

The first point here is: **External field uniform over atomic dimension.** This is often referred to as the **dipole approximation**. Atoms have typical dimensions on the order of angstroms (10^{-10} meters). Optical wavelengths are typically hundreds to thousands of nanometers (10^{-7} to 10^{-6} meters). Since the wavelength of light is much larger than the size of the atom, we can assume that the electric field of the light wave is essentially constant across the spatial extent of the electron's orbit or motion. This allows us to describe the interaction using just the electric field $E(t)$ at the "center" of the atom, rather than having to worry about its spatial variation $E(r,t)$ within the atom. This simplifies the driving term in our equation of motion significantly.

The second point is: **Damping** b independent of x. We assume that the damping coefficient 'b' does not depend on the displacement 'x' of the electron. This, like the linear restoring force, is crucial for keeping the differential equation linear. If 'b' depended on 'x' (e.g., $b(x)$), the equation would become non-linear and much harder to solve, and the principle of superposition for responses would break down.

These assumptions, while simplifying, allow the model to capture the essential physics of absorption and dispersion in a remarkably effective way, especially for understanding phenomena near an isolated resonance. We should always keep these assumptions in mind when applying the results of this model.

The three hyphens at the bottom suggest the end of this list of assumptions or perhaps a transition.

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Now we move on to **Solving for the Steady-State Oscillation** of our driven, damped harmonic electron oscillator. We are looking for the long-term behavior of the electron once any initial transient effects (from when the field was first turned on) have died down. This steady-state solution will oscillate at the same frequency ω as the driving electric field.

We use the **Complex-Amplitude Ansatz**. An 'ansatz' is essentially an educated guess for the form of the solution.

* The first bullet states: **Seek a particular solution of the form** $x(t) = x_0 e^{i\omega t}$. Here: * $x(t)$ is the time-dependent displacement of the electron. * x_0 (x sub zero or x naught) is a **complex amplitude**. It's complex because the electron's oscillation might be out of phase with the driving field, and this phase difference, as well as the amplitude of oscillation, will be encoded in x_0. * $e^{i\omega t}$ shows that the electron oscillates at the same angular frequency ω as the driving field.

* The second bullet explains the procedure: **Substitute into the motion equation. Each time derivative multiplies** x_0 by $i\omega$.

Let's do this. Our equation of motion was:

$$m\ddot{x}(t) + b\dot{x}(t) + Dx(t) = qE_0 e^{i\omega t}$$

If $x(t) = x_0 e^{i\omega t}$, then: * The first time derivative is $\dot{x}(t) = \frac{d}{dt}\left(x_0 e^{i\omega t}\right) = i\omega\, x_0 e^{i\omega t}$.

* The second time derivative is $\ddot{x}(t) = \frac{d}{dt}\left(i\omega\, x_0 e^{i\omega t}\right) = (i\omega)^2\, x_0 e^{i\omega t} = -\omega^2\, x_0 e^{i\omega t}$.

Substituting these into the equation of motion:

$$m\left(-\omega^2 x_0 e^{i\omega t}\right) + b\left(i\omega x_0 e^{i\omega t}\right) + D\left(x_0 e^{i\omega t}\right) = qE_0 e^{i\omega t}$$

Now, we can see that the term $e^{i\omega t}$ is common to every term in the equation. Since $e^{i\omega t}$ is never zero, we can divide it out. This leaves us with an algebraic equation for the complex amplitude x_0:

$$m(-\omega^2)x_0 + b(i\omega)x_0 + Dx_0 = qE_0$$

This is a significant simplification! We've transformed a differential equation in time into an algebraic equation for the complex amplitude x_0. Now we just need to solve for x_0.

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Now that we have the algebraic equation for the complex amplitude x_0, which is

$$m(-\omega^2)x_0 + b(i\omega)x_0 + Dx_0 = qE_0,$$

we can solve for x_0.

* The first bullet says: **Rearrange and isolate** x_0. Introduce convenient parameters. We can factor out x_0 from the left-hand side:

$$x_0[-m\omega^2 + ib\omega + D] = qE_0$$

Or, rearranging the terms in the bracket:

$$x_0[D - m\omega^2 + ib\omega] = qE_0$$

Now, we introduce two very important parameters: 1. $\gamma = \frac{b}{m}$ (gamma equals b divided by m). This γ is the **damping rate**, or damping constant per unit mass. It has units of frequency (radians per second or simply per second). It characterizes how quickly oscillations would die out in the absence of a driving force. 2. $\omega_0^2 = \frac{D}{m}$ (omega sub zero squared equals D divided by m). As we discussed before, ω_0 is the **natural undamped resonant angular frequency** of the oscillator.

Using these, we can rewrite the term D by itself as $m\omega_0^2$. And b can be written as $m\gamma$. So the bracket becomes:

$$[m\omega_0^2 - m\omega^2 + im\gamma\omega] = m[\omega_0^2 - \omega^2 + i\gamma\omega].$$

* The second bullet says: **Obtain** x_0. Dividing by $m(\omega_0^2 - \omega^2 + i\gamma\omega)$, we get the expression for x_0:

$$x_0 = \frac{qE_0}{m(\omega_0^2 - \omega^2 + i\gamma\omega)}.$$

This equation is central. It tells us the complex amplitude x_0 of the electron's oscillation in terms of the driving field amplitude E_0, the electron's charge q and mass m, the driving frequency ω, the natural resonant frequency ω_0, and the damping rate γ. Since the denominator is complex, x_0 itself will be complex. The magnitude of x_0 will give the amplitude of the electron's oscillation, and the argument (or phase angle) of x_0 will give the phase of the electron's motion relative to the driving field.

* Now for the **Physical interpretation**: The third bullet point highlights a key aspect: **Denominator magnitude minimum when** $\omega \approx \omega_0 \Rightarrow$ resonance. Let's look at the denominator: $m(\omega_0^2 - \omega^2 + i\gamma\omega)$. Its magnitude is

$$m \cdot \sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}.$$

This magnitude will be smallest when the term $(\omega_0^2 - \omega^2)^2$ is smallest, which occurs when ω^2 is close to ω_0^2, or when the driving frequency ω is close to the natural resonant frequency ω_0. When the denominator's magnitude is at a minimum, the magnitude of x_0 (the amplitude of oscillation) will be at a maximum. This phenomenon is called **resonance**. The electron responds most strongly to the driving field when the field's frequency matches its own natural oscillation frequency. This is a familiar concept from many areas of physics.

Page 16:

Let's continue our physical interpretation of the solution for x_0, particularly focusing on the role of the complex terms. Our expression for the complex amplitude of the electron's oscillation is

$$x_0 = \frac{qE_0}{m(\omega_0^2 - \omega^2 + i\gamma\omega)}.$$

The slide highlights: **Imaginary term** $i\gamma\omega$ introduces phase lag and energy dissipation.

Let's explore this. The term $i\gamma\omega$ in the denominator is purely imaginary.

1. **Phase Lag:** Because of this imaginary term, the complex number $(\omega_0^2 - \omega^2 + i\gamma\omega)$ in the denominator will have a non-zero phase angle (unless $\gamma = 0$ or $\omega = 0$). Consequently, x_0, which is qE_0 divided by this complex number, will also have a phase angle. This means that the electron's oscillation $x(t) = x_0\, e^{i\omega t}$ will generally not be perfectly in phase with the driving field $E(t) = E_0\, e^{i\omega t}$ (assuming E_0 is real for reference). There will be a phase difference, or phase lag (or lead, depending on the sign of $\omega_0^2 - \omega^2$), between the driving force and the electron's response. This phase lag is frequency-dependent and is largest near resonance. If you plot the phase of x_0 versus ω, you'll see it change significantly around ω_0.

2. **Energy Dissipation:** The damping term 'b', and hence $\gamma = \frac{b}{m}$, is fundamentally related to mechanisms that dissipate energy from the oscillating electron system. The power absorbed by the oscillator from the driving field is related to the component of the electron's velocity that is in phase with the driving force. The presence of γ ensures that there is, on average over a cycle, a net transfer of energy from the electric field to the

electron oscillator, which is then dissipated through the damping mechanisms (radiation, collisions). If γ were zero (no damping), then at steady state for $\omega \neq \omega_0$, there would be no continuous energy absorption; the electron would oscillate without loss. On resonance ($\omega = \omega_0$) with $\gamma = 0$, the amplitude x_0 would theoretically go to infinity, which is unphysical and highlights the necessity of damping in any real system.

So, this $i\gamma\omega$ term is not just a mathematical artifact; it's crucial for capturing two vital physical effects: the phase relationship between the response and the drive, and the mechanism by which energy is transferred from the field to the material (which is the heart of absorption). The three hyphens suggest this line of thought concludes for now.

Page 17:

We've successfully modeled the behavior of a single bound electron. Now, the crucial step is to connect this microscopic picture to a macroscopic quantity that describes the material's overall response. This is **From Single Electron to Macroscopic Polarization.** We are **Building the Polarization Vector P.**

- First, let's consider the **Instantaneous dipole moment for one oscillator.**

An electric dipole moment arises when there's a separation of positive and negative charge. Our electron, with charge q, is displaced by $x(t)$ from its equilibrium position (which we can think of as the location of the positive nucleus). So, the induced dipole moment for this single electron, let's call it $p_{el}(t)$ (p sub e l of t), is given by:

$$p_{el}(t) = q\,x(t)$$

Since we found $x(t) = x_0\,e^{i\omega t}$, we can write:

$$p_{el}(t) = q\,x_0\,e^{i\omega t}$$

This $p_{el}(t)$ is a vector, and if $x(t)$ is along the x-direction, then $p_{el}(t)$ is also along the x-direction. Its units would be Coulomb-meters.

- Next, we need to consider that a material contains many such oscillators. The slide introduces the **Number density of identical oscillators:** N (capital N). This N represents the number of these active, oscillating electrons per unit volume. Its units are m^{-3} (inverse cubic meters). We are assuming for now that all these oscillators are identical (same q, m, ω_0, γ) and that they respond independently to the field.

- With these, we can define the **Macroscopic polarization** $P(t)$ (capital P of t). This is defined as the **dipole moment per unit volume.** So, if we have N identical oscillators per unit volume, each with dipole moment $p_{el}(t)$, then the macroscopic polarization is simply N times $p_{el}(t)$:

$$P(t) = N\,p_{el}(t) = N\,q\,x_0\,e^{i\omega t}$$

If the electric field E was polarized along $\hat{x}$ (x-hat), causing a displacement x_0 along $\hat{x}$, then P will also be a vector along $\hat{x}$. The slide writes this as:

$$P(t) = N\,q\,x_0\,e^{i\omega t}\,\hat{x}$$

This implies that x_0 here is the scalar complex amplitude of the x-component of displacement. $P(t)$ is a vector quantity, and its units are Coulomb per square meter (C/m^2), which is dipole moment (C·m) per volume (m^3).

This $P(t)$ is a crucial macroscopic field. It represents the collective response of all the microscopic electron oscillators in the material to the incident electric field. Our next step will be to link this $P(t)$ back to the electric field $E(t)$ that created it, and then to the refractive index.

Page 18:

Now that we have an expression for the macroscopic polarization $P(t)$ arising from our microscopic model, we need to connect it to standard electromagnetic theory.

The first bullet point introduces the **Linear-response definition links** P to the field E:

$$P = \varepsilon_0 \chi(\omega) E$$

Here:

P is the macroscopic polarization vector (we're using complex amplitudes now, so P and E are complex vector amplitudes, $P(\omega)$ and $E(\omega)$).

ε_0 (epsilon naught) is the permittivity of free space.

$\chi(\omega)$ (chi of omega) is the **electric susceptibility** of the material. It's a dimensionless complex quantity that measures how easily the material becomes polarized in response to an applied electric field. Crucially, it's frequency-dependent. This equation assumes a linear, isotropic medium.

The second bullet defines $\chi(\omega)$ further: $\chi(\omega) = \varepsilon_\mathrm{r} - 1: electric susceptibility.$

The third bullet defines $\varepsilon_\mathrm{r} = \frac{\varepsilon}{\varepsilon_0}$: relative dielectric constant.

Finally, the fourth bullet point reminds us of a fundamental relationship from electromagnetism: **Recall relation between refractive index and dielectric constant (in non-magnetic media, where** $\mu_\mathrm{r} \approx 1$ (mu sub r is approximately 1, meaning the relative permeability is close to that of vacuum)):

$$n^2(\omega) = \varepsilon_\mathrm{r}(\omega) = 1 + \chi(\omega)$$

This equation states that the square of the complex refractive index $n(\omega)$ is equal to the complex relative permittivity $\varepsilon_\mathrm{r}(\omega)$, which in turn is equal to 1 plus the complex electric susceptibility $\chi(\omega)$.

Our goal is now clear:

1. We have P from our microscopic model: $P = Nq\, x_0\, e^{i\omega t}\hat{x}$.

2. We have E, the driving field: $E = E_0\, e^{i\omega t}\hat{x}$ (using E_0 as the complex amplitude of E).

3. We can use $P = \varepsilon_0 \chi E$ to find χ in terms of our microscopic parameters (N, q, $\frac{x_0}{E_0}$).

4. Then, we can use $n^2 = 1 + \chi$ to find the complex refractive index n in terms of these microscopic parameters. This will be the grand link!

The three hyphens at the bottom of the slide suggest we are about to embark on this derivation.

Page 19:

This slide is titled **Deriving the Complex Refractive Index**, and it describes the **Algebraic Connection to Oscillator Parameters.** We're about to connect all the pieces.

* The first bullet point instructs us to: **Combine previous expressions, substitute** P and x_0.

Let's recall:

1. From the microscopic model, the polarization is

$$P(t) = Nqx_0\, e^{i\omega t}\hat{x}.$$

If

$$E(t) = E_{0\_\text{field}} e^{i\omega t}\hat{x}$$

(where E_{0_field} is the amplitude of the E-field driving the oscillator), then the complex amplitude of polarization is

$$P_{\text{amp}} = Nqx_0.$$

2. The relation between polarization and field is

$$P_{\text{amp}} = \varepsilon_0 \chi(\omega) E_{0\_\text{field}}.$$

3. So,

$$\chi(\omega) = \frac{P_{\text{amp}}}{\varepsilon_0\, E_{0\_\text{field}}} = \frac{Nqx_0}{\varepsilon_0\, E_{0\_\text{field}}}.$$

4. And we found

$$x_0 = \frac{qE_{0\_\text{field}}}{m(\omega_0^2 - \omega^2 + i\gamma\omega)}.$$

(Here, E_{0_field} is the amplitude of the E field in the expression for x_0).

Substituting x_0 into the expression for $\chi(\omega)$:

$$\chi(\omega) = \frac{Nq}{\varepsilon_0\, E_{0\_\text{field}}} \cdot \frac{qE_{0\_\text{field}}}{m(\omega_0^2 - \omega^2 + i\gamma\omega)}.$$

The E_{0_field} cancels out, as it should, because χ is a material property.

$$\chi(\omega) = \frac{Nq^2}{\varepsilon_0\, m(\omega_0^2 - \omega^2 + i\gamma\omega)}.$$

Now, we use the relation

$$n^2(\omega) = 1 + \chi(\omega).$$

Substituting our $\chi(\omega)$:

$$n^2(\omega) = 1 + \frac{Nq^2}{\varepsilon_0\, m(\omega_0^2 - \omega^2 + i\gamma\omega)}.$$

This is the equation presented on the slide. It's a monumental result! It gives us the square of the complex refractive index, n^2, directly in terms of: * N: number density of oscillators * q: charge of the electron * ε_0: permittivity of free space * m: mass of the electron * ω_0: natural resonant frequency of the oscillators * ω: frequency of the incident light * γ: damping rate of the oscillators

* The second bullet point draws a crucial conclusion: **Note the complex denominator** → n is complex.

Indeed, the term

$$(\omega_0^2 - \omega^2 + i\gamma\omega)$$

in the denominator is complex due to the presence of $i\gamma\omega$. Therefore, the entire second term on the right-hand side is complex. Adding 1 to it means that $n^2(\omega)$ is complex. If n^2 is complex, then n itself must be complex.

And we re-state its form: **Explicitly write** $n(\omega) = n'(\omega) - i\kappa(\omega)$.

This equation for $n^2(\omega)$ is the heart of the classical Lorentz model for optical properties. From it, we can extract the real part $n'(\omega)$ (related to dispersion) and the imaginary part $\kappa(\omega)$ (related to absorption).

Page 20

Now that we have the expression for $n^2(\omega) = 1 + \frac{Nq^2}{\varepsilon_0 m(\omega_0^2 - \omega^2 + i\gamma\omega)}$, let's make some important observations about the behavior of its real and imaginary parts, $n'(\omega)$ and $\kappa(\omega)$, particularly concerning their symmetry with respect to the resonant frequency ω_0.

The slide points out:

* **Important observation:**

* **Real part** $n'(\omega)$ ↔ even function of $\omega - \omega_0$. This statement describes the symmetry of the dispersive part of the refractive index *around* the resonance. If we plot $n'(\omega)$ versus ω, the characteristic "anomalous dispersion" shape it takes around ω_0 is such that if you consider the deviation from some background value, it looks somewhat like an odd function of $(\omega - \omega_0)$.

However, the actual mathematical function for $n'(\omega)$ when extracted from n^2 has a term $(\omega_0^2 - \omega^2)$ in its numerator (as we'll see soon). This term $(\omega_0^2 - \omega^2)$ is an even function of ω if centered at $\omega = 0$, but it's not immediately obvious it's an even function of $(\omega - \omega_0)$.

Let's think about the shape of anomalous dispersion. It typically goes up, then sharply down through resonance, then back up. This shape itself is odd-symmetric around ω_0 if we subtract the value at ω_0 (or far from resonance). The slide says "even function of $\omega - \omega_0$". This might be referring to specific approximations or a different way of looking at it. For instance, the denominator magnitude is

$$(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2,$$

where the $(\omega_0^2 - \omega^2)$ part is symmetric in its deviation from resonance in a particular way.

Often, the absorption profile $\kappa(\omega)$ is an even function (Lorentzian peak) centered at ω_0. The real part $n'(\omega)$ is related to $\kappa(\omega)$ by the Kramers–Kronig relations, which connect an even function to an odd function (and vice versa through Hilbert transforms). So, if $\kappa(\omega)$ is an even function of $(\omega - \omega_0)$, then $n'(\omega)$ minus its value at infinity will be an odd function of $(\omega - \omega_0)$.

The slide states n' is an *even* function of $\omega - \omega_0$. This is a bit unusual for the standard dispersive lineshape. A typical dispersive lineshape is odd around ω_0. Perhaps "even" refers to the fact that the effect is similar for $\omega_0 + \Delta\omega$ and $\omega_0 - \Delta\omega$ in terms of magnitude of change but opposite in sign. Let me proceed with what the slide says and we might clarify this with the explicit forms.

* **Imaginary part** $\kappa(\omega)$ ↔ odd around ω_0 and controls absorption strength. Similarly, $\kappa(\omega)$ is typically a Lorentzian function, which is an *even* function of $(\omega - \omega_0)$, peaking at $\omega = \omega_0$. The slide says κ is *odd* around ω_0. This is also

contrary to the typical Lorentzian absorption profile. Let me reconsider. Perhaps the slide is referring to the numerators before full separation: The term added to 1 to get n^2 is proportional to

$$\frac{1}{(\omega_0^2 - \omega^2) + i\gamma\omega}.$$

To get $n = \sqrt{1+\chi}$, if χ is small, then $n \approx 1 + \frac{\chi}{2}$. So $n'(\omega) - 1 \approx \mathrm{Re}\left(\frac{\chi}{2}\right)$ and $-\kappa(\omega) \approx \mathrm{Im}\left(\frac{\chi}{2}\right)$. χ is proportional to

$$\frac{(\omega_0^2 - \omega^2) - i\gamma\omega}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}.$$

So, $\mathrm{Re}(\chi)$ has $(\omega_0^2 - \omega^2)$ in the numerator. This IS an even function of ω if $\omega_0 = 0$, but an odd function of $(\omega - \omega_0)$ under the near-resonance approximation $\left(\omega_0^2 - \omega^2 \approx 2\omega_0(\omega_0 - \omega)\right)$. And $\mathrm{Im}(\chi)$ has $-\gamma\omega$ in the numerator. This is an odd function of ω. But the absorption coefficient α (and κ) is positive. This slide's statement about even/odd symmetry seems to be non-standard or refers to a specific context not immediately apparent from the general formula of n^2. I will proceed by describing the *typical* shapes and will highlight that these are obtained especially in the near-resonance approximation. Standardly, $\kappa(\omega)$ (absorption) is symmetric (even-like) around ω_0, forming a peak. And $n'(\omega)$ (dispersion) has an antisymmetric (odd-like) shape around ω_0.

This point about symmetry is critical for lineshape analysis, and we'll see it more clearly when we derive the explicit forms for $n'(\omega)$ and $\kappa(\omega)$, especially under the near-resonance approximation. It's possible the slide intends to point towards a symmetry in the mathematical *terms* rather than the final plotted shape directly, or it is using $(\omega - \omega_0)$ in a specific way for the expansion of the denominator terms. We will revisit this when we have the explicit functions.

Let's assume for a moment the slide is correct and there's a nuance I'm missing in this general form. The statement "controls absorption strength" for κ is absolutely correct – a larger κ means stronger absorption.

Page 21:

This slide provides a simple yet effective visual: the **Complex Refractive Index:** $n(\omega) = n'(\omega) - i\kappa(\omega)$ plotted in the complex plane.

Let's break down the diagram:

* It's an Argand diagram. The horizontal axis is the real axis, labeled Re$[n(\omega)]$ or $n'(\omega)$ (Real part of $n(\omega)$ or $n'(\omega)$). * The vertical axis is the imaginary axis, labeled Im$[n(\omega)]$ (Imaginary part of $n(\omega)$). The origin (0,0) is marked. * The complex refractive index $n(\omega)$ is represented as a blue vector starting from the origin and ending at a point in the fourth quadrant. * The real part of this vector, its projection onto the horizontal axis, is $n'(\omega)$. This is shown as a green horizontal vector from the origin to the point $(n', 0)$. * The imaginary part of $n(\omega)$ is $-\kappa(\omega)$ (minus kappa of ω), consistent with our definition $n = n' - i\kappa$. This is shown as a red vertical vector pointing downwards from the point $(n', 0)$ to the tip of the $n(\omega)$ vector. Its length is $\kappa(\omega)$, and it's directed along the negative imaginary axis. The label $-\kappa(\omega)$ is placed on the vertical axis, indicating a point below the origin if n' were zero.

This diagram neatly visualizes how the complex refractive index $n(\omega)$ is composed of its real part $n'(\omega)$ and its imaginary part $-\kappa(\omega)$. The length of the blue vector would be the magnitude $|n(\omega)| = \sqrt{n'^2 + \kappa^2}$, and its angle with the positive real axis would be the argument or phase of $n(\omega)$. As the frequency ω changes, both n' and κ change, so the tip of this blue vector $n(\omega)$ would trace out a path in the complex plane, which is characteristic of the material's resonant behavior.

Page 22:

This slide is titled **Wave Propagation with a Complex** n, and it focuses on **Separating Phase and Amplitude Evolution.** This is essentially a recap and emphasis of what we derived on page 9.

* The first bullet point says: **Write the field again emphasizing individual factors:**

The electric field $E(z,t)$ inside the medium is given as:

$$E(z,t) = E_0 e^{-\kappa K_0 z} \exp[i(\omega t - n' K_0 z)]\hat{x}$$

Let's re-emphasize the parts: * $E_0\hat{x}$: This represents the initial vector amplitude and polarization of the wave at $z = 0$. * $e^{-\kappa K_0 z}$: This is the **amplitude attenuation factor**. It's a real, decaying exponential. κ (kappa) is the extinction coefficient (imaginary part of n), K_0 is the free-space wavenumber. This term shows how the wave's amplitude diminishes as it propagates through the absorbing medium. * $\exp[i(\omega t - n' K_0 z)]$: This is the **phase evolution factor**. It's a complex exponential representing the oscillation. * $n' K_0$: This is the effective wavenumber inside the medium for the phase. n' (n prime) is the real part of the refractive index. This term dictates the wavelength in the medium $\lambda_{\text{med}} = \frac{2\pi}{n' K_0} = \frac{\lambda_0}{n'}$ and the phase velocity.

* The second bullet point gives the **Phase velocity inside the medium,** v_{ph} (vee sub pee aitch):

$v_{ph} = \frac{\omega}{n' K_0}$ Since $K_0 = \frac{\omega}{c}$ (omega over c, the speed of light in vacuum), we can substitute this:

$$v_{ph} = \frac{\omega}{n' \left(\frac{\omega}{c}\right)} = \frac{c}{n'}$$

So, $v_{ph} = \frac{c}{n'}$. This is a fundamental result: the phase velocity of light inside a medium is the speed of light in vacuum divided by the real part of the refractive index of the medium. If $n' > 1$ (as is common for most materials at optical frequencies), then $v_{ph} < c$, meaning light (phase) travels slower in the medium. Since n' can be frequency dependent ($n'(\omega)$), the phase velocity is also frequency dependent, which is the root of dispersion.

* The third bullet point simply says: **Always** 1. This appears to be an incomplete or mistyped fragment, possibly intended to state a condition like $n' \geq 1$ (n prime is greater than or equal to 1) for typical passive media, or $\kappa \geq 0$ (kappa is greater than or equal to 0) for absorption (not gain). Given the context of phase velocity $\frac{c}{n'}$, if it meant n' is always 1, then there would be no change in phase velocity, which is generally not true for materials. If it refers to κ, κ is usually positive for absorption. For our purposes, we'll assume n' is generally different from 1, and κ is positive for absorbing media. The backslash before (1) suggests it might be a remnant of LaTeX formatting.

This slide neatly summarizes how the complex n elegantly separates the description of amplitude decay (absorption via κ) and phase propagation (dispersion via n').

Page 23:

Continuing our discussion of wave propagation with a complex refractive index, this page introduces the concept of the **Intensity attenuation length** $L_{1/e}$ (L sub one over e).

*The first bullet point defines the **Intensity attenuation length*** $L_{1/e}$ (L sub one over e). It is defined by the condition where the electric field amplitude* E has decayed to $1/e$ of its initial value E_0.

So, at $z = L_{1/e}$, we have

$$E(z = L_{1/e}) = \frac{E_0}{e}.$$

We know the field amplitude decays as

$$E_0 e^{-\kappa K_0 z}.$$

Therefore,

$$E_0 e^{-\kappa K_0 L_{1/e}} = \frac{E_0}{e}.$$

This implies

$$e^{-\kappa K_0 L_{1/e}} = e^{-1}.$$

Taking the natural logarithm of both sides (or just comparing exponents):

$$-\kappa K_0 L_{1/e} = -1.$$

Solving for $L_{1/e}$, we get:

$$L_{1/e} = \frac{1}{\kappa K_0}.$$

This is the distance over which the *electric field amplitude* drops to $1/e$ (about 37%) of its initial value.

The slide further expresses this in terms of the free-space wavelength λ_0. Since

$$K_0 = \frac{2\pi}{\lambda_0},$$

we can substitute this:

$$L_{1/e} = \frac{1}{\kappa\left(\frac{2\pi}{\lambda_0}\right)} = \frac{\lambda_0}{2\pi\kappa}.$$

It's very important to note that this $L_{1/e}$ is for the *field amplitude*. Often in spectroscopy, one refers to the absorption length for *intensity*. Since intensity I is proportional to E^2, the intensity decays as

$$e^{-2\kappa K_0 z}.$$

So, the intensity attenuation length, where intensity drops to I_0/e, would be

$$\frac{1}{2\kappa K_0},$$

which is half the length defined here for the field. The definition on the slide is explicitly for the field amplitude E.

* The second bullet point makes a profound statement: **Power loss is intrinsically tied to the same microscopic parameters that set dispersion; this reciprocity becomes clearer on next slides.**

This is a reiteration of the deep connection between absorption κ (related to power loss) and dispersion n' (related to phase velocity). Both n' and κ arise from the same denominator

$$(\omega_0^2 - \omega^2 + i\gamma\omega)$$

in our expression for $n^2(\omega)$, which contains the microscopic parameters ω_0 and γ. This common origin means that if you know the absorption spectrum $\kappa(\omega)$ over a wide range of frequencies, you can, in principle, calculate the dispersion spectrum $n'(\omega)$, and vice-versa. This is formalized by the Kramers-Kronig relations, which we'll likely encounter later.

The three hyphens suggest the end of this section before moving to a new idea.

Page 24:

We now consider a useful simplification. This slide is titled: **Simplification for Dilute Gases, where** $n - 1$ is much, much less than 1. This condition means the refractive index 'n' is very close to unity.

The subtitle is **Linearizing** n^2 around Unity.

* The first bullet point gives a practical context: **For most gases at low pressure,** n' (the real part of n) is approximately 1.000 – 1.005. This means n' is indeed very close to 1. For example, air at STP has $n' \approx 1.00029$. In such cases, $(n' - 1)$ is a very small number. We also assume κ is small.

* The second bullet point shows how to **Expand** $n^2 - 1$ when n is close to 1:

We know that

$$n^2 - 1 = (n + 1)(n - 1)$$

Since $n \approx 1$, the term

$$(n + 1) \approx (1 + 1) = 2$$

So, we can approximate:

$$n^2 - 1 \approx 2(n - 1)$$

This is a very handy linearization.

* The third bullet point says: **Drop terms of order** $(n - 1)^2$. The refractive index becomes... This implies that if we were to do a Taylor expansion, we are keeping only the first-order term in $(n - 1)$. From

$$n^2 - 1 \approx 2(n - 1)$$

we can solve for n:

$$n - 1 \approx \frac{n^2 - 1}{2}$$

So,

$$n \approx 1 + \frac{n^2 - 1}{2}$$

Recall our full expression for n^2 from page 19:

$$n^2(\omega) = 1 + \frac{Nq^2}{\epsilon_0 m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

Therefore,

$$n^2(\omega) - 1 = \frac{Nq^2}{\epsilon_0 m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

Using our approximation $n \approx 1 + \frac{n^2-1}{2}$, we can substitute the expression for $n^2(\omega) - 1$:

$$n(\omega) \approx 1 + \frac{1}{2} \cdot \frac{Nq^2}{\epsilon_0 m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

$$n(\omega) \approx 1 + \frac{Nq^2}{2\epsilon_0 m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

This simplified expression for $n(\omega)$ itself (not $n^2(\omega)$) is extremely useful for dilute gases where n is close to 1. It makes extracting the real part $n'(\omega)$ and imaginary part $\kappa(\omega)$ much more straightforward because $n(\omega) - 1$ is directly given by a single complex fraction.

Page 25:

This page presents the result of the simplification for dilute gases that we just derived.

The equation for the complex refractive index $n(\omega)$ under this approximation ($n - 1 \ll 1$) is:

$$n(\omega) = 1 + \frac{1}{2\epsilon_0 m} \cdot \frac{Nq^2}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

This can be written more compactly as:

$$n(\omega) = 1 + \frac{Nq^2}{2\epsilon_0 m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

This equation directly gives $n(\omega)$. Since $n(\omega) = n'(\omega) - i\kappa(\omega)$, we can also write this as:

$$n'(\omega) - i\kappa(\omega) - 1 = \frac{Nq^2}{2\epsilon_0 m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

So, $n'(\omega) - 1$ is the real part of the fraction on the right, and $-\kappa(\omega)$ is the imaginary part of the fraction on the right.

* The bullet point highlights the **Advantage** of this form: **real and imaginary parts now follow directly by multiplying numerator and denominator with the complex conjugate of the denominator.**

The complex denominator is $\omega_0^2 - \omega^2 + i\gamma\omega$.

Its complex conjugate is $\omega_0^2 - \omega^2 - i\gamma\omega$.

To rationalize the fraction (i.e., make the denominator real), we multiply both the numerator and the denominator by this complex conjugate.

The numerator will become $Nq^2(\omega_0^2 - \omega^2 - i\gamma\omega)$.

The denominator will become

$$2\epsilon_0 m[(\omega_0^2 - \omega^2)^2 - (i\gamma\omega)^2] = 2\epsilon_0 m[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2].$$

This denominator is now purely real.

So, the fraction becomes:

$$\frac{Nq^2(\omega_0^2 - \omega^2 - i\gamma\omega)}{2\epsilon_0 m[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]}$$

From this, we can easily pick out the real and imaginary parts:

$$n'(\omega) - 1 = \text{Real part} = \frac{Nq^2(\omega_0^2 - \omega^2)}{2\epsilon_0 m[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]}$$

$$-\kappa(\omega) = \text{Imaginary part} = \frac{Nq^2(-\gamma\omega)}{2\epsilon_0 m[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]}$$

So, $\kappa(\omega) = \frac{Nq^2\gamma\omega}{2\epsilon_0 m\left[\left(\omega_0^2-\omega^2\right)^2+(\gamma\omega)^2\right]}$.

These explicit expressions are what we will see on the next slide. This approximation makes the algebra much cleaner than trying to take the square root of the full complex $n^2(\omega)$ expression. The three hyphens signal the end of this thought.

Page 26:

This slide, titled **Explicit Expressions — Dispersion Relations**, gives us the fruits of our labor from the previous page, specifically the results of **Isolating** $\kappa(\omega)$ (kappa of omega) and $n'(\omega)$ (n prime of omega) for the case of dilute gases (where $n \approx 1$).

* The first bullet point reminds us of the procedure: **Multiply by the complex conjugate** $\omega_0^2 - \omega^2 - i\gamma\omega$ in numerator & denominator. We conceptually did this on the last slide.

Now, for the explicit expressions:

First, for $\kappa(\omega)$, the imaginary part of n (remember $n = n' - i\kappa$, so κ is defined positively for absorption here):

$$\kappa(\omega) = \frac{Nq^2}{2\varepsilon_0\, m} \cdot \frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

Let's verify the terms:

* $\frac{Nq^2}{2\varepsilon_0\, m}$: This is a prefactor containing the number density N, charge q squared, vacuum permittivity ε_0, and electron mass m. * The second fraction has: * $\gamma\omega$ (gamma omega) in the numerator. γ is the damping rate, ω is the driving

frequency. * $(\omega_0^2-\omega^2)^2+\gamma^2\omega^2$ in the denominator. This is the square of the magnitude of $\omega_0^2-\omega^2+i\gamma\omega$. This term is always positive.

This $\kappa(\omega)$ will be positive if γ and ω are positive, which corresponds to absorption. The shape of $\kappa(\omega)$ as a function of ω will be a resonant peak, a Lorentzian lineshape, centered near ω_0.

Next, for $n'(\omega)$, the real part of n:

$$n'(\omega)=1+\frac{Nq^2}{2\varepsilon_0\, m}\cdot\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

* It starts with **1 + ...**, because we derived $n(\omega)-1$. * The prefactor $\frac{Nq^2}{2\varepsilon_0\, m}$ is the same as for $\kappa(\omega)$. * The second fraction has: * $\omega_0^2-\omega^2$ (omega sub zero squared minus omega squared) in the numerator. This term changes sign as ω passes through ω_0. It's positive if $\omega<\omega_0$, zero if $\omega=\omega_0$, and negative if $\omega>\omega_0$. * The denominator $(\omega_0^2-\omega^2)^2+\gamma^2\omega^2$ is the same as for $\kappa(\omega)$.

The shape of $(n'(\omega)-1)$ as a function of ω will be the characteristic "anomalous dispersion" curve, passing through zero at $\omega=\omega_0$ (since the numerator $\omega_0^2-\omega^2$ becomes zero there, assuming $\omega_0\neq 0$).

These two equations for $\kappa(\omega)$ and $n'(\omega)$ are fundamental results from the classical Lorentz model in the $n\approx 1$ approximation. They explicitly show how absorption (via κ) and dispersion (via n') depend on frequency and the microscopic parameters of the material.

Page 27:

Having derived the explicit expressions for $\kappa(\omega)$ and $n'(\omega)$, let's reflect on their significance.

- The first bullet point states: **These two coupled formulas are called the (classical) dispersion relations.** Indeed, $\kappa(\omega)$ and $n'(\omega)$ are not independent. They arise from the real and imaginary parts of the same underlying complex susceptibility $\chi(\omega)$ (or complex refractive index $n(\omega)$). The term "dispersion relations" more generally refers to the Kramers-Kronig relations, which are integral relationships connecting the real and imaginary parts of any causal linear response function. What we have here are the specific algebraic forms derived from our particular classical oscillator model. They embody that interconnectedness.

- The second bullet point highlights a **Key insight: same denominator ⇒ absorption peak and dispersion inflection share identical spectral width** γ. Let's look back at the expressions for $\kappa(\omega)$ and $(n'(\omega) - 1)$ from the previous page. Both have the exact same denominator: $(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2$.
    - For $\kappa(\omega)$, this denominator, along with the $\gamma\omega$ term in the numerator, gives rise to a Lorentzian absorption peak. The width of this peak (e.g., Full Width at Half Maximum, FWHM) is determined by the damping constant γ.
    - For $(n'(\omega) - 1)$, this same denominator governs how rapidly n' changes with frequency around the resonance ω_0. The region where n' changes most dramatically (the "inflection" region of the S-shaped curve) also has a spectral width characterized by γ.

This is a profound consequence: the physical process that causes damping and gives a certain width to the absorption line (γ) is the *exact same* process that determines the spectral range over which the refractive index changes rapidly (dispersion). They are intrinsically linked. You cannot have sharp absorption without associated rapid dispersion over a similar frequency range, and vice-versa. This is a manifestation of causality in the system's response.

This shared denominator and the role of γ in dictating the spectral features of both absorption and dispersion is a central takeaway from this model.

Page 28:

This slide presents a graph visualizing the **Classical Dispersion Relations** – that is, plots of $\kappa(\omega)$ and $n'(\omega) - 1$ versus frequency, based on the formulas we derived.

Let's describe the graph:

* The horizontal axis is normalized frequency, ω/ω_0 (omega over omega sub zero). The resonance occurs at $\omega/\omega_0 = 1$. The axis ranges roughly from 0.5 to 1.5.

* The vertical axis serves for both $\kappa(\omega)$ and $n'(\omega) - 1$. The scale ranges from approximately -1.20 to +1.20. A horizontal line at zero is clearly visible.

* The center of the graph, where $\omega/\omega_0 = 1$, is marked with a vertical dashed line, indicating the resonant frequency.

There are two curves plotted:

1. The **red curve** is labeled $\kappa(\omega)$ (kappa of omega). * This curve represents the absorption. It has a characteristic **Lorentzian shape**: a symmetric bell-shaped peak. * It is centered precisely at $\omega/\omega_0 = 1$, meaning absorption is strongest when the driving frequency ω matches the natural resonant frequency ω_0 of the electron oscillators. * The peak value is positive, as expected for absorption. * The width of this peak (the Full Width at Half Maximum, or FWHM) is related to the damping parameter γ. A larger γ would result in a broader, flatter peak, while a smaller γ would give a taller, sharper peak.

2. The **blue curve** is labeled $n'(\omega) - 1$ (n prime of omega minus one). * This curve represents the dispersive part of the refractive index (relative to vacuum). It has the characteristic S-shape of **anomalous dispersion** around the resonance. * Far below resonance ($\omega/\omega_0 \ll 1$), $n'(\omega) - 1$ is small and positive, meaning $n' > 1$ (normal dispersion region, though this plot doesn't go far enough to fully show that asymptotic behavior). * As ω approaches ω_0

from below, $n'(\omega) - 1$ increases. * Precisely at resonance ($\omega/\omega_0 = 1$), $n'(\omega) - 1$ passes through zero, meaning $n'(\omega_0) = 1$ (in this simplified model for $n \approx 1$, and assuming ω_0 itself isn't zero). This occurs because the numerator term ($\omega_0^2 - \omega^2$) in the expression for $n'(\omega) - 1$ becomes zero at $\omega = \omega_0$. * As ω increases beyond ω_0, $n'(\omega) - 1$ becomes negative, meaning $n' < 1$ in this region. It reaches a minimum value. * Far above resonance ($\omega/\omega_0 \gg 1$), $n'(\omega) - 1$ approaches zero again from the negative side. *The region where n'* decreases* with increasing frequency (around $\omega/\omega_0 = 1$) is termed "anomalous dispersion." The slope $\frac{dn\prime}{d\omega}$ is negative here. * The points of maximum positive and negative slope of $n'(\omega)$ occur roughly at the half-maximum points of the $\kappa(\omega)$ curve.

This graph beautifully illustrates the coupled nature of absorption ($\kappa(\omega)$) and dispersion ($n'(\omega)$). Where absorption is strong and changing rapidly, dispersion is also strong and changing rapidly. The characteristic shapes are a direct consequence of the underlying physics of a damped, driven oscillator.

The number "251)" at the bottom right seems to be a stray part of a reference or term indicator.

Page 29:

Now we consider another important approximation: the **Near-Resonance Approximation**. This applies when the driving frequency ω is very close to the resonant frequency ω_0, specifically when the magnitude of their difference, $|\omega - \omega_0|$, is much, much less than ω_0 itself. This allows for further simplification of our expressions for $\kappa(\omega)$ and $n'(\omega)$, leading to what are often called **Lorentzian Absorption and Derivative-Like Dispersion** forms.

- The first step is to **Expand** $\omega_0^2 - \omega^2$ under this near-resonance condition:

$$\omega_0^2 - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega)$$

Since $\omega \approx \omega_0$, we can approximate $\omega_0 + \omega \approx \omega_0 + \omega_0 = 2\omega_0$.

So, $\omega_0^2 - \omega^2 \approx (\omega_0 - \omega)(2\omega_0) = 2\omega_0(\omega_0 - \omega)$.

The slide has written this as approximately $2\omega(\omega_0 - \omega)$, which is also valid since $\omega \approx \omega_0$. This form is convenient.

Also, in the term $\gamma\omega$ that appears, we can approximate $\omega \approx \omega_0$, so $\gamma\omega \approx \gamma\omega_0$.

- The second bullet instructs us to: **Insert into previous relations; obtain compact Lorentzian forms.**

Let's use the expressions for $\kappa(\omega)$ and $n'(\omega)$ from page 26:

$$\kappa(\omega) = \frac{Nq^2}{2\epsilon_0 m} \cdot \frac{\gamma\omega}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}$$

$$n'(\omega) = 1 + \frac{Nq^2}{2\epsilon_0 m} \cdot \frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}$$

Let's work on the denominator first:

$$(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2$$

Using $\omega_0^2 - \omega^2 \approx 2\omega_0(\omega_0 - \omega)$ and $\gamma\omega \approx \gamma\omega_0$:

$$\begin{aligned} \text{Denom} &\approx (2\omega_0(\omega_0 - \omega))^2 + (\gamma\omega_0)^2 = 4\omega_0^2(\omega_0 - \omega)^2 + \gamma^2\omega_0^2 \\ &= \omega_0^2[4(\omega_0 - \omega)^2 + \gamma^2] \end{aligned}$$

This can be rewritten as:

$\omega_0^2 \cdot 4 \cdot \left[(\omega_0 - \omega)^2 + \left(\frac{\gamma}{2}\right)^2\right]$ or $\omega_0^2 \cdot 4 \cdot \left[(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2\right]$ since $(\omega_0 - \omega)^2 = (\omega - \omega_0)^2$.

Now for $\kappa(\omega)$:

Numerator term: $\gamma\omega \approx \gamma\omega_0$.

So,

$$\kappa(\omega) \approx \frac{Nq^2}{2\epsilon_0\, m} \cdot \frac{\gamma\omega_0}{\omega_0^2 \cdot 4 \cdot \left[(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2\right]}$$

$$\kappa(\omega) \approx \frac{Nq^2\gamma\omega_0}{8\epsilon_0\, m\omega_0^2} \cdot \frac{1}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}$$

$$\kappa(\omega) \approx \frac{Nq^2\gamma}{8\epsilon_0\, m\omega_0} \cdot \frac{1}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}$$

To match the standard Lorentzian form which often has $\Gamma/2$ or $\gamma/2$ in the numerator (where Γ or γ is FWHM), the slide presents:

$$\kappa(\omega) = \frac{Nq^2}{8\epsilon_0\, m\omega_0} \cdot \frac{\gamma}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}$$

This form implies that γ here is indeed the FWHM of the Lorentzian peak for $\kappa(\omega)$ when plotted against ω. The $\left(\frac{\gamma}{2}\right)$ is the Half Width at Half Maximum (HWHM).

Now for $n'(\omega)$:

Numerator term: $\omega_0^2 - \omega^2 \approx 2\omega_0(\omega_0 - \omega)$.

$$n'(\omega) - 1 \approx \frac{Nq^2}{2\epsilon_0\, m} \cdot \frac{2\omega_0(\omega_0-\omega)}{\omega_0^2 \cdot 4 \cdot \left[(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2\right]}$$

$$n'(\omega) - 1 \approx \frac{Nq^2 \cdot 2\omega_0(\omega_0-\omega)}{8\epsilon_0\, m\omega_0^2} \cdot \frac{1}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}$$

$$n'(\omega) - 1 \approx \frac{Nq^2(\omega_0-\omega)}{4\epsilon_0\, m\omega_0} \cdot \frac{1}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}$$

So, the slide gives:

$$n'(\omega) = 1 + \frac{Nq^2}{4\epsilon_0\, m\omega_0} \cdot \frac{(\omega_0 - \omega)}{(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}$$

This can also be written as:

$$n'(\omega) = 1 - \frac{Nq^2}{4\epsilon_0\, m\omega_0} \cdot \frac{(\omega - \omega_0)}{(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}$$

This form for $n'(\omega) - 1$ is proportional to

$$\frac{(\omega - \omega_0)}{(\omega - \omega_0)^2 + \left(\frac{\gamma}{2}\right)^2},$$

which has the shape of the derivative of a Lorentzian function, hence "derivative-like dispersion."

These near-resonance forms are extremely common in spectroscopy because many experiments probe frequencies very close to an atomic or molecular resonance. The parameter γ is the linewidth (FWHM) of the absorption.

Page 30:

Let's make some **Observations** based on these near-resonance expressions for $\kappa(\omega)$ and $n'(\omega)$ that we just derived:

$$\kappa(\omega) = [\text{Constant}_1] \cdot \frac{\gamma}{(\omega - \omega_0)^2 + (\gamma/2)^2}$$

$$n'(\omega) = 1 + [\text{Constant}_2] \cdot \frac{\omega_0 - \omega}{(\omega - \omega_0)^2 + (\gamma/2)^2}$$

* First observation: $\kappa(\omega)$ is symmetric, peaks at ω_0. Looking at the expression for $\kappa(\omega)$, the frequency dependence is in the term $(\omega - \omega_0)^2$ in the denominator. Since it's squared, $\kappa(\omega)$ will have the same value for a detuning $\omega - \omega_0 = +\Delta\omega$ as for $\omega - \omega_0 = -\Delta\omega$. Thus, the lineshape is symmetric around ω_0.

The denominator is minimized when $(\omega - \omega_0)^2 = 0$, i.e., when $\omega = \omega_0$. At this point, $\kappa(\omega)$ reaches its maximum value. So, the absorption peak is centered at the resonant frequency ω_0. This is the standard Lorentzian profile.

* Second observation: $n'(\omega)$ is antisymmetric, crosses $n' = 1$ at ω_0. Let's look at

$$n'(\omega) - 1 = [\text{Constant}_2] \cdot \frac{\omega_0 - \omega}{(\omega - \omega_0)^2 + (\gamma/2)^2}.$$

The denominator

$$(\omega - \omega_0)^2 + (\gamma/2)^2$$

is symmetric around ω_0.

The numerator term $\omega_0 - \omega$ changes sign as ω passes through ω_0.

If $\omega < \omega_0$, then $\omega_0 - \omega$ is positive.

If $\omega > \omega_0$, then $\omega_0 - \omega$ is negative.

If $\omega = \omega_0$, then $\omega_0 - \omega$ is zero.

Therefore, $n'(\omega) - 1$ is an antisymmetric function of $\omega - \omega_0$.

When $\omega = \omega_0$, the numerator $\omega_0 - \omega$ becomes zero, so $n'(\omega_0) - 1 = 0$. This means $n'(\omega_0) = 1$.

This behavior corresponds to the anomalous dispersion shape we saw in the graph on page 28.

* Third observation: **Extremal dispersion slopes at** $\omega_\text{m} = \omega_0 \pm \gamma/2$. The "dispersion slope" refers to $\frac{dn\prime}{d\omega}$. If you take the derivative of $n'(\omega)$ with respect to ω and set it to zero to find the extrema of the slope (i.e., points of inflection of $n'(\omega)$ itself, or where $\frac{dn\prime}{d\omega}$ is max/min), you'll find these points occur at $\omega = \omega_0 + \gamma/2$ and $\omega = \omega_0 - \gamma/2$.

These frequencies, $\omega_0 \pm \gamma/2$, are precisely where the absorption profile κ(ω)$\kappa(\omega)$ drops to half of its maximum value (the Half Width at Half Maximum points). This again underscores the intimate connection between the features of the absorption and dispersion curves. The region of steepest change in refractive index coincides with the shoulders of the absorption peak.

These observations are fundamental characteristics of the interaction of light with a resonant medium under this classical model. The three hyphens signal the end of these observations.

Page 31:

We now shift our focus to practically relating what we've learned about κ (kappa), the imaginary part of the refractive index, to the commonly measured **absorption coefficient** α (alpha). This slide is titled **Connecting Absorption Coefficient** α to κ.

The subtitle is **Intensity Versus Field Amplitude.**

- The first bullet point reminds us of the definition of **Time-averaged intensity for a plane wave:**

$$I = \frac{1}{2}\epsilon_0 c|E|^2$$

Where I is the intensity (Power/Area), ϵ_0 is the permittivity of free space, c is the speed of light in vacuum, and $|E|$ is the magnitude of the electric field amplitude. We saw this on page 6. For a wave propagating in a medium where the real part of n is n', the speed of light is $\frac{c}{n'}$, and the permittivity can be effectively $n'^2\epsilon_0$ if we are careful with magnetic properties (or use Poynting vector $E \times H$). More simply, I is proportional to $|E|^2$.

- The second bullet point states: **Propagating field contains factor** $e^{-2\kappa K_0 z}$ for intensity (square of amplitude).

We found that the electric field amplitude $E(z)$ decays as

$E_{\text{amp}}(z) = E_{\text{amp}}(0)\, e^{-\kappa K_0 z}$ (from page 9).

Since intensity I is proportional to the square of the field amplitude (i.e., $I \propto |E_{\text{amp}}|^2$), the intensity will decay as:

$$I(z) \propto \left[E_{\text{amp}}(0)\, e^{-\kappa K_0 z}\right]^2 = \left[E_{\text{amp}}(0)\right]^2 e^{-2\kappa K_0 z}$$

So,

$$I(z) = I(0)\, e^{-2\kappa K_0 z}$$

The factor in the exponent for intensity decay is indeed $-2\kappa K_0 z$. This factor of 2 is critical.

- The third bullet point then says: **Define absorption coefficient** α [m^{-1}]:

The absorption coefficient α is typically defined via the differential equation for intensity decay:

$$\frac{dI}{dz} = -\alpha I$$

This equation states that the rate of decrease of intensity with propagation distance z is proportional to the intensity I itself, with α being the proportionality constant. The units of α are inverse length, commonly m^{-1} (inverse meters). Integrating this definition gives the Beer-Lambert law:

$$I(z) = I(0)\, e^{-\alpha z}$$

Our task now is to compare this standard definition

$$I(z) = I(0)\, e^{-\alpha z}$$

with the form we derived from the complex refractive index,

$$I(z) = I(0)\, e^{-2\kappa K_0 z}$$

to find the direct relationship between α and κ.

Page 32:

Continuing from the definition of the absorption coefficient α, we have the Beer-Lambert law: If $\frac{dI}{dz} = -\alpha I$, then integrating this gives $I(z) = I_0 e^{-\alpha z}$, where I_0 is the intensity at $z = 0$.

Now, the slide says: **Direct comparison yields...** We have two expressions for $I(z)$:

1. From the definition of α: $I(z) = I_0 e^{-\alpha z}$ 2. From the complex refractive index $n = n' - i\kappa$: $I(z) = I_0 e^{-2\kappa K_0 z}$ (as shown on page 31)

Comparing the exponents of these two expressions, we must have: $-\alpha z = -2\kappa K_0 z$

Dividing by $-z$ (assuming $z \neq 0$), we get: $\alpha(\omega) = 2\, K_0 \kappa(\omega)$

This is the fundamental relationship between the absorption coefficient $\alpha(\omega)$ and the extinction coefficient $\kappa(\omega)$ (the imaginary part of n). Remember K_0 is the free-space wavenumber.

Since $K_0 = \frac{2\pi}{\lambda_0}$ (where λ_0 is the free-space wavelength), we can also write this as: $\alpha(\omega) = 2 \times \left(\frac{2\pi}{\lambda_0}\right) \times \kappa(\omega)$ So, $\alpha(\omega) = \frac{4\pi}{\lambda_0}\kappa(\omega)$.

This is a very important result. It provides a direct way to calculate the measurable absorption coefficient α if we know κ from our model, or vice-versa.

* The final bullet point states: **Therefore** $\alpha(\omega)$ inherits the Lorentzian profile of $\kappa(\omega)$. Since $\alpha(\omega)$ is just $\kappa(\omega)$ multiplied by a factor ($2\,K_0$ or $\frac{4\pi}{\lambda_0}$) which is positive and, for a given λ_0, constant (or slowly varying if λ_0 in K_0 is taken as wavelength in medium, but here K_0 is free space), the spectral shape of $\alpha(\omega)$ will be identical to the spectral shape of $\kappa(\omega)$. If $\kappa(\omega)$ has a Lorentzian lineshape (as predicted by our near-resonance model), then $\alpha(\omega)$ will also have a Lorentzian lineshape, with the same resonant frequency ω_0 and the same linewidth γ.

The three hyphens indicate the end of this derivation. This connection is essential for experimental spectroscopy, as α is often what is directly measured.

Page 33:

Now we arrive at the **Beer-Lambert Law**, which is a cornerstone for practical measurement of absorption in spectroscopy. This slide discusses **Implementing** α in Experiments.

The first bullet point sets up the scenario: **Given a sample length** L:

Imagine we have a sample of material (e.g., a cuvette filled with a solution, or a gas cell) of a known thickness or path length L. We shine light through it. Let $I_{\text{inc}}(\omega)$ be the incident intensity of light at a particular angular frequency ω, just before it enters the sample. Let $I_{\text{trans}}(\omega)$ be the transmitted intensity of light at that same frequency ω, after it has passed through the length L of the sample. According to the Beer-Lambert law ($I(z) = I_0 e^{-\alpha z}$), the relationship between these is:

$$I_{\text{trans}}(\omega) = I_{\text{inc}}(\omega) e^{-\alpha(\omega) L}$$

Here, $\alpha(\omega)$ is the absorption coefficient of the sample material at frequency ω.

The second bullet point describes a common method for data analysis: **Taking natural logarithm gives linear relation for data fitting:**

Let's rearrange the Beer-Lambert equation:

$$\frac{I_{\text{trans}}(\omega)}{I_{\text{inc}}(\omega)} = e^{-\alpha(\omega)L}$$

Or,

$$\frac{I_{\text{inc}}(\omega)}{I_{\text{trans}}(\omega)} = e^{+\alpha(\omega)L}$$

Now, take the natural logarithm (ln) of both sides:

$$\ln\left[\frac{I_{\text{inc}}(\omega)}{I_{\text{trans}}(\omega)}\right] = \alpha(\omega)L$$

This quantity $\ln\left[\frac{I_{\text{inc}}}{I_{\text{trans}}}\right]$ is often called the **absorbance** (or optical density, though absorbance is more common in this form). This equation is extremely useful. If we measure $I_{\text{inc}}(\omega)$ and $I_{\text{trans}}(\omega)$ at a given frequency ω, and we know the sample length L, we can directly calculate $\alpha(\omega)$. Alternatively, if we vary L and measure the corresponding I_{trans} for a fixed ω, a plot of $\ln\left[\frac{I_{\text{inc}}}{I_{\text{trans}}}\right]$ versus L should yield a straight line passing through the origin, with a slope equal to $\alpha(\omega)$.

The third bullet point mentions **Advantages:** (which will likely be detailed on the next page).

This logarithmic form is highly advantageous for experimental determination of $\alpha(\omega)$.

Page 34:

Continuing with the advantages and limitations of using the Beer-Lambert law for measuring the absorption coefficient $\alpha(\omega)$.

- The first advantage: **Directly extract** $\alpha(\omega)$ from slope. As mentioned, if you plot the absorbance, $\ln\left[\frac{I_{inc}(\omega)}{I_{trans}(\omega)}\right]$, versus the path length L (by using samples of different thicknesses, for instance), the slope of the resulting straight line is $\alpha(\omega)$. More commonly, if L is fixed and known, and you measure I_{inc} and I_{trans}, you can calculate $\alpha(\omega) = \frac{1}{L}\ln\left[\frac{I_{inc}}{I_{trans}}\right]$. If you do this as a function of frequency ω (e.g., by tuning your laser or using a spectrometer), you can obtain the entire absorption spectrum $\alpha(\omega)$.

- The second advantage: **Works for broadband or tunable-laser measurements.** This technique is versatile.

    - With a **tunable laser**, you can measure $I_{inc}(\omega)$ and $I_{trans}(\omega)$ point-by-point at different frequencies ω to map out $\alpha(\omega)$.

    - With a **broadband source** (like a lamp) and a spectrometer, you can measure the entire $I_{inc}(\omega)$ and $I_{trans}(\omega)$ spectra simultaneously and then calculate $\alpha(\omega)$ across that spectral range.

- Now, a crucial **Limitation: assumes linear absorption** ⇒ valid only at low intensities (no saturation). The Beer-Lambert law, and our derivation of $\alpha(\omega)$ from the classical oscillator model, fundamentally assumes that the absorption coefficient $\alpha(\omega)$ is a property of the material and the frequency of light, but *independent of the intensity of the light*. This is the regime of **linear absorption**. However, if the light intensity becomes very high (e.g., with powerful lasers), this assumption can break down.

    - **Saturation:** At high intensities, a significant fraction of the absorbing atoms or molecules can be excited to their upper energy state. If the upper state becomes heavily populated, there are fewer ground-state absorbers available, and the material becomes more transparent. The absorption coefficient α effectively decreases with increasing intensity. This is called **saturation** of the absorption.

The Beer-Lambert law in its simple form (α independent of I) is therefore only valid at low light intensities where saturation effects are negligible. We will explore non-linear optics and saturation in later parts of the course.

The three hyphens signal the end of this discussion. It's vital to remember this limitation when applying Beer-Lambert law in experiments.

Page 35:

This slide, titled **Power Absorbed in a Finite Volume Element**, moves from the coefficient α to considering the actual **Energy Dissipation** within the material.

* The first bullet point asks us to: **Consider small slab Δz** of cross-section area A. The volume of this slab is then $\Delta V = A\Delta z$.

Imagine a beam of light propagating through this small elementary volume of material.

* The second bullet point quantifies the **Power removed from the beam within ΔV**:

Let $I(\omega)$ be the intensity of the light at frequency ω incident on this slab. The change in intensity dI as light passes through a thickness dz is given by

$$dI = -\alpha(\omega)I(\omega)dz.$$

The power incident on the face of the slab is

$$P_{in} = I(\omega)A.$$

The power exiting the slab is

$$P_{out} = (I(\omega) + dI)A = (I(\omega) - \alpha(\omega)I(\omega)\Delta z)A.$$

The power absorbed within the slab ΔV, let's call it $\Delta P_{\text{abs}}(\omega)$, is the difference $P_{in} - P_{out}$:

$$\Delta P_{\text{abs}}(\omega) = I(\omega)A - [I(\omega)A - \alpha(\omega)I(\omega)A\Delta z].$$

$$\Delta P_{\text{abs}}(\omega) = \alpha(\omega)I(\omega)A\Delta z.$$

Since $\Delta V = A\Delta z$, we can write this as:

$$\Delta P(\omega) = \alpha(\omega)I(\omega)\Delta V$$

This equation tells us that the power absorbed per unit frequency interval within a small volume element ΔV is equal to the absorption coefficient $\alpha(\omega)$ times the incident intensity $I(\omega)$ times the volume ΔV. This makes intuitive sense: more absorption (larger α), higher intensity (larger I), or larger volume (larger ΔV) all lead to more power being absorbed.

* The third bullet point considers a common scenario: **For a broadband continuum** $I(\omega)$ and a narrow atomic line $\alpha(\omega)$:

Imagine our incident light is not monochromatic but has a spectral intensity $I(\omega)$ that is spread over a range of frequencies (a continuum). And suppose the material has an absorption feature $\alpha(\omega)$ that is very narrow in frequency (like an atomic absorption line). To find the total power absorbed by this narrow line from the broadband light, we would need to integrate over frequency, which is what the next page will likely address.

Page 36:

Continuing from the previous slide, where we considered a broadband continuum $I(\omega)$ incident on a material with a narrow absorption line $\alpha(\omega)$. To find the total power ΔP (delta P) absorbed by this line within the volume element ΔV, we need to integrate the per-frequency power absorption over all frequencies.

The equation for ΔP is given as:

$$\Delta P = \left(\int_{-\infty}^{+\infty} \alpha\,(\omega) I(\omega)\, d\omega \right) \Delta V$$

This expression means we multiply the spectral absorption coefficient $\alpha(\omega)$ by the spectral intensity $I(\omega)$ at each frequency ω, integrate this product over all frequencies (from minus infinity to plus infinity, though practically it's over the range where $\alpha(\omega)I(\omega)$ is non-zero), and then multiply by the volume element ΔV. This gives the total power absorbed in ΔV from all spectral components of the light that interact with the absorption line.

* An important simplification is noted in the first bullet point: **If** $I(\omega)$ varies slowly across the line, treat it as constant and pull outside integral.

Often, an atomic or molecular absorption line $\alpha(\omega)$ is very narrow compared to the spectral width over which the incident intensity $I(\omega)$ varies. For example, if you're using a lamp source, its spectrum $I(\omega)$ might be quite flat over the tiny frequency range where $\alpha(\omega)$ is significant.

In such cases, we can approximate $I(\omega)$ by its value at the center of the absorption line, say $I(\omega_0)$, and take it outside the integral:

$$\Delta P \approx I(\omega_0) \left(\int_{-\infty}^{+\infty} \alpha\,(\omega)\, d\omega \right) \Delta V$$

The integral $\int \alpha(\omega)\, d\omega$ is then the **integrated absorption coefficient** for the line. This quantity is often related to the oscillator strength of the transition and the number density of absorbers.

* The second bullet point mentions **Applications:**

Understanding and calculating absorbed power is crucial for several reasons:

1. **Quantify heat load in gas cells:** When a gas in a cell absorbs laser power, that energy is typically converted into heat. This can raise the temperature and pressure of the gas, which might be undesirable (e.g., causing line

broadening or cell damage) or desirable (e.g., in photoacoustic spectroscopy where the heating causes a measurable sound wave). Calculating ΔP helps predict and manage this heat load.

2. **Determine oscillator strengths:** As mentioned, the integrated absorption coefficient is fundamentally related to the transition probability or oscillator strength (often denoted 'f') of the atomic/molecular transition. By measuring the integrated absorption, one can work backwards to determine these fundamental microscopic parameters.

The three hyphens signal the end of this discussion.

Page 37:

This slide introduces a very illustrative concept in spectroscopy: **Spectral "Hole" Burning Illustration.** It asks the question: **What Happens to the Transmitted Spectrum?**

Imagine we have an incident light beam that has a relatively flat and broad spectrum, like white light or light from certain types of lamps or broadband lasers. Now, suppose this light passes through a material that has a very narrow absorption line at a specific frequency.

* The first bullet point describes this: **Narrow absorption line (with a width** $\delta\omega$, delta omega) imprints a dip in an otherwise flat input continuum. The material will absorb light very strongly only within that narrow frequency range $\delta\omega$ corresponding to its absorption line. Outside this range, it's transparent. So, if you look at the spectrum of the light *after* it has passed through the material, you'll see the original flat continuum but with a "hole" or a "dip" at the frequency of the absorption line. This is because the light at those specific frequencies has been selectively removed by absorption.

* The second bullet states: **Depth and width of the "hole" directly reflect** $\alpha(\omega)$. The shape of this hole in the transmitted spectrum is not arbitrary. * The

depth of the hole at any frequency ω within the absorption line is determined by the value of $\alpha(\omega)$ at that frequency (and the path length L, via $e^{-\alpha(\omega)L}$). Where $\alpha(\omega)$ is largest (at the peak of the absorption line), the hole will be deepest. * The **width** of the hole will correspond to the width $\delta\omega$ of the absorption line $\alpha(\omega)$. So, by carefully measuring the shape of this spectral hole, we can directly deduce the absorption spectrum $\alpha(\omega)$ of the material.

* The third bullet considers a specific case: **For extremely wide continuum (where the continuum bandwidth** $\Delta\omega$ (capital Delta omega) is much, much greater than the absorption linewidth $\delta\omega$ (delta omega)), total transmitted power hardly changes, but spectral shape is modified. If the input light spectrum is very broad, and the absorption line removes only a tiny sliver of frequencies from it, the total integrated power of the transmitted light might be almost the same as the total incident power. However, the crucial point is that the *spectral distribution* of that power has been altered – specific frequencies have been attenuated, creating the hole. This is the essence of absorption spectroscopy: looking for these characteristic missing pieces in the spectrum to identify and quantify the absorber.

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Continuing our discussion on spectral hole burning:

* The first bullet point considers the converse scenario: **Conversely, if** $\delta\omega \gg \Delta\omega$ ($\delta\omega$ is much, much greater than capital $\Delta\omega$), the continuum is uniformly attenuated—no visible hole.

Here, $\delta\omega$ is the width of the absorption line, and $\Delta\omega$ is the width of the incident light's continuum. If the absorption line is very broad ($\delta\omega$ is large) compared to the spectral width of the incoming light ($\Delta\omega$ is small – e.g., if we're using a relatively narrow laser line as our "continuum"), then the absorption coefficient $\alpha(\omega)$ will be essentially constant across the entire narrow bandwidth $\Delta\omega$ of the incident light.

In this case, all frequency components within the incident $\Delta\omega$ experience roughly the same amount of absorption. The result is that the entire input spectrum is attenuated by a similar factor, without a distinct "hole" appearing within its profile. The shape of the narrow $\Delta\omega$ spectrum would be preserved, just its overall intensity would be reduced.

Now, the slide previews what's coming next:

Next 3 Slides – 3 plots:

* **a) input flat spectrum,** (This will show the incident light before it interacts with the absorber.) * **b) output with narrow hole,** (This will show the transmitted spectrum when a narrow absorption line interacts with a broad input spectrum – the classic hole burning case.) * **c) output when continuum narrower than absorption line showing uniform attenuation.** (This will illustrate the case where the input light is spectrally narrow compared to a broad absorption feature.)

Let's look at these plots.

Page 39:

Here is the first of the three illustrative plots, labeled **(a) Input Flat Continuum & Narrow Transmission Profile.**

The graph plots **Intensity / Transmission** on the vertical axis, ranging from 0.0 to 1.0, versus **Angular Frequency (**ω) on the horizontal axis, which ranges from 0 to 1000 in arbitrary units.

There are two curves shown:

1. **Input Continuum (**$I_{\text{inc}}(\omega)$): This is represented by a **solid blue horizontal line** near the top of the plot (around a value of 0.8, let's say, though the vertical axis is normalized for transmission later). The legend indicates this is

"Input Continuum ($I_{\text{inc}}(\omega)$)". Being flat means the incident light has equal intensity at all frequencies within this range.

2. **Transmission ($e^{-\alpha(\omega)L_{\text{narrow}}}$)**: This is represented by a **dashed red curve**. The legend indicates this is "Transmission ($e^{-\alpha(\omega)L_{\text{narrow}}}$)". This curve represents the transmission factor $T(\omega) = e^{-\alpha(\omega)L}$ of a material that has a narrow absorption line. * Far away from the absorption resonance, $\alpha(\omega)$ is very small, so $T(\omega) \approx e^0 = 1$. The red dashed curve is at a value of 1.0 on the vertical axis in these regions. * Around a central frequency ω (labeled on the axis, let's call it ω_0), $\alpha(\omega)$ becomes large due to the narrow absorption line. In this region, $T(\omega) = e^{-\alpha(\omega)L}$ drops significantly, forming a downward "dip" or "valley." The shape of this dip mirrors the shape of the narrow absorption line $\alpha(\omega)$ (specifically, it's $e^{-\alpha}$). * The title above these curves is **"**$I(\omega)$ and Narrow $T(\omega)$".

This slide effectively sets up the components: a broadband input $I_{\text{inc}}(\omega)$ and a transmission function $T(\omega)$ that has a narrow feature. The next slide will show what happens when this $I_{\text{inc}}(\omega)$ passes through a material with this $T(\omega)$.

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This is the second plot, labeled **(b) Transmitted Spectrum: Narrow Hole Burning.**

Again, the graph plots **Intensity / Transmission** on the vertical axis (0.0 to 1.0) versus **Angular Frequency (**ω) on the horizontal axis (0 to 1000 arbitrary units).

There is one curve shown:

- A **solid green line**, which is labeled in the legend as **"Transmitted (**$I_{\text{trans}}(\omega)$) with Hole". This curve represents the spectrum of the light *after* it has passed through the absorbing material. It is the product of the input flat continuum (the blue line from slide 39) and the narrow transmission profile (the red dashed line from slide 39).

$$I_{\text{trans}}(\omega) = I_{\text{inc}}(\omega) \times T(\omega) = I_{\text{inc}}(\omega) \times e^{-\alpha(\omega)L}$$

- Where the material was transparent ($T(\omega) \approx 1$, far from resonance), the transmitted spectrum $I_{\text{trans}}(\omega)$ is simply equal to the input spectrum $I_{\text{inc}}(\omega)$ (the flat part). - Where the material had its narrow absorption ($T(\omega) < 1$, around the resonance ω_0), the transmitted intensity is reduced, forming a distinct **"hole"** or **"dip"** in the otherwise flat spectrum. - The shape of this hole (its depth and width) directly reflects the characteristics of the narrow absorption line $\alpha(\omega)$ and the path length L.

The title above the plot is **"I(**ω) = I(ω) · e". This seems to be a shorthand or a slight typo. It should likely represent $I_{\text{trans}}(\omega) = I_{\text{inc}}(\omega) \times e^{-\alpha(\omega)L}$, or qualitatively $I_{\text{output}} = I_{\text{input}} \times$ Transmission. The key is that the green curve is the *result* of the input passing through the absorber.

This plot perfectly illustrates the phenomenon of spectral hole burning: a narrow absorption feature "burns a hole" into a broader incident spectrum. This is the basis of absorption spectroscopy.

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This is the third illustrative plot, labeled **(c) Transmitted Spectrum: Narrow Input, Broad Absorption (Uniform Attenuation).** The axes are the same: **Intensity / Transmission** (vertical, 0.0 to 1.0) versus **Angular Frequency (**ω**)** (horizontal, 0 to 1000 arbitrary units).

Two curves are shown here, overlaid:

1. **Narrow Input Continuum (**$I_{\text{inc}}(\omega)$**)**: This is represented by a **solid blue line**. Unlike the previous plots, this input spectrum is not flat and broad. Instead, it's a **narrow peak** centered at some frequency ω (labeled on the axis). This

could represent, for example, the spectrum of a single-mode laser. The legend confirms: "Narrow Input Continuum ($I_{\text{inc}}(\omega)$)".

2. **Transmitted ($I_{\text{trans}}(\omega)$)** - Uniform Attenuation: This is represented by a **solid green line**. It has the *same shape* as the blue input peak but is **uniformly reduced in amplitude**. The legend confirms: "Transmitted ($I_{\text{trans}}(\omega)$) - Uniform Attenuation".

The title above these curves is **"Narrow** $I(\omega)$ & Uniform Attenuation".

This plot illustrates the scenario where the absorption feature of the material, $\alpha(\omega)$, is very broad compared to the spectral width of the incident light, $\Delta\omega_{\text{input}}$. Or, equivalently, $\alpha(\omega)$ is nearly constant across the narrow range of frequencies contained in $I_{\text{inc}}(\omega)$.

In this case, all frequency components within the narrow input peak experience essentially the same absorption coefficient α. Therefore, they are all attenuated by the same factor $e^{-\alpha L}$.

The result is that the transmitted spectrum $I_{\text{trans}}(\omega)$ retains the same spectral shape as the input $I_{\text{inc}}(\omega)$, but its overall intensity is reduced. There is no "hole" burned *within* the input spectral profile because the absorption is not selective enough across that narrow profile. It's just uniform attenuation.

These three plots (slides 39, 40, 41) provide a very clear visual summary of how the relative spectral widths of the incident light and the absorption feature determine the appearance of the transmitted spectrum.

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We now return to a more fundamental question concerning our classical oscillator model. This slide is titled: **Physical Origin of Damping Constant** γ (gamma). It asks: **Where Does the Energy Go?** Recall that our equation of motion $m\ddot{x} + b\dot{x} + Dx = qE$ included a damping term $b\dot{x}$, and we defined $\gamma =$

$\frac{b}{m}$ as the damping rate. This γ appeared in our expressions for κ and n', determining the width of the absorption line and the dispersive features. So, what are the physical processes that contribute to this 'b' or 'γ'?

* The first bullet point discusses **Radiative damping: "Oscillating charge re-emits electromagnetic radiation; classical Larmor formula predicts..."** An electron, when it oscillates, is an accelerating charge. According to classical electrodynamics, any accelerating charge radiates electromagnetic waves, thereby losing energy. This energy loss acts as a damping force on the electron's motion. This is an intrinsic damping mechanism, present even for an isolated atom in vacuum. The classical Larmor formula can be used to calculate the power radiated by an accelerating charge, and from this, one can derive an effective damping constant due to radiation. The slide gives the prediction for this radiative damping rate, γ_{rad} (gamma sub rad):

$$\gamma_{\text{rad}} = \frac{e^2 \omega_0^2}{6\pi\epsilon_0 m c^3}$$

Let's break down the terms: * **e**: elementary charge (magnitude). * ω_0: the resonant angular frequency of the oscillator. The radiated power depends on the frequency of oscillation. * ϵ_0 (epsilon naught): permittivity of free space. * **m**: mass of the electron. * **c**: speed of light in vacuum. * π (pi): the mathematical constant pi. This γ_{rad} provides a fundamental lower limit to the linewidth of an atomic transition, often called the natural linewidth.

* The second bullet point discusses **Collisional damping: "In gases, electron momentum transferred to neighboring atoms/molecules during collisions."** In a real material, especially in gases at finite pressures or in condensed matter, the oscillating electron (or the atom/molecule it belongs to) is not isolated. It will frequently collide with other atoms or molecules. During these collisions, the phase of the electron's oscillation can be randomly interrupted, or energy can be transferred from the oscillating

electron to the collision partner (e.g., increasing its kinetic energy). These processes effectively remove energy from the coherently driven oscillation and contribute to the damping term b. Collisional damping is often strongly dependent on pressure (in gases) and temperature. At high pressures, it can become the dominant broadening mechanism for spectral lines.

So, the total damping constant γ that we use in our model is generally a sum of contributions from various such processes: $\gamma = \gamma_{\text{rad}} + \gamma_{\text{coll}} + \cdots$ (other mechanisms).

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Continuing our discussion on the origins and interpretations of the damping constant γ, or more broadly, the linewidth of spectral features.

The first bullet point connects our classical γ to the quantum picture: **Intrinsic (natural) linewidth in quantum picture relates to excited-state lifetime** τ (tau): $\gamma = \frac{1}{\tau}$.

In a quantum mechanical description of an atom, if an electron is excited to an upper energy state, that state will have a finite lifetime, τ, before the electron spontaneously decays back to a lower state, emitting a photon. This finite lifetime τ is related to the transition probability.

The Heisenberg uncertainty principle, in the form $\mathit{\Delta} E \mathit{\Delta} t \geq \frac{\hbar}{2}$, implies that a state with a finite lifetime τ (so $\mathit{\Delta} t \approx \tau$) will have an uncertainty in its energy $\mathit{\Delta} E$. This energy uncertainty translates into a spread of frequencies for photons emitted or absorbed involving this state, leading to a **natural linewidth**.

This natural linewidth, often denoted $\mathit{\Gamma}_{\text{nat}}$ or γ_{nat}, is given by $\gamma_{\text{nat}} = \frac{1}{\tau}$. (Sometimes, depending on definition, it might be A_{ul}, the Einstein A coefficient for spontaneous emission from upper to lower level).

The radiative damping γ_{rad} we saw from the classical Larmor formula is the classical counterpart to this quantum mechanical natural linewidth. For many transitions, the classical and quantum results are in reasonable agreement or can be formally related. So, the lifetime of the excited state is a fundamental contributor to γ.

The second bullet point considers damping in more complex environments: **In condensed matter, additional damping from phonon coupling, inhomogeneous broadening, impurity scattering.**

In solids or liquids (condensed matter), the environment around an absorbing atom or molecule is much more complex than in a dilute gas.

* **Phonon coupling:** The electronic states of an atom/ion in a solid can couple to the vibrational modes of the crystal lattice (phonons). This interaction can lead to energy exchange and dephasing of the electronic oscillation, contributing significantly to the damping and broadening of spectral lines, often with a strong temperature dependence.

Inhomogeneous broadening: *In many real materials, especially solids, not all absorbing centers (atoms, molecules, ions) are in identical local environments*. *There might be slight variations in local crystal fields, strains, or proximity to defects*. *These variations can cause the resonant frequency* ω_0 to be slightly different for different absorbers in the ensemble. Even if each individual absorber has a narrow homogeneous linewidth (due to natural lifetime, phonon coupling, etc.), the superposition of all these slightly shifted lines results in a much broader observed *or* inhomogeneous* linewidth for the bulk material. Our simple model assumes all oscillators have the same ω_0; inhomogeneous broadening violates this.

* **Impurity scattering:** If there are impurities or defects in the material, the oscillating electrons (or excitons, etc.) can scatter off these impurities, losing energy or phase coherence, which also contributes to damping.

The three hyphens signal the end of this list.

Understanding these various contributions to γ is crucial because the linewidth and lineshape are key observables in spectroscopy, providing rich information about the absorber and its environment.

Page 44

While the classical electron oscillator model has provided us with tremendous insight into absorption and dispersion, it is, after all, a classical model. It's important to understand its **Limitations**. This slide asks: **When and Why Does It Fail?**

* First limitation: **Cannot predict discrete allowed transition frequencies;** ω_0 must be inserted from experiment or quantum theory.

In our classical model, the natural resonant frequency ω_0 (omega sub zero) is an input parameter. We essentially say, "Let's assume the electron is bound with a restoring force that gives it this resonant frequency." The model itself doesn't tell us *why* ω_0 has a particular value, or why only certain ω_0 values might be allowed.

In contrast, **quantum theory** (e.g., solving the Schrödinger equation for an atom) predicts that atoms and molecules have discrete, quantized energy levels. Transitions between these levels (say, E_1 and E_2) occur at specific frequencies given by the Bohr condition:

$$\omega = \frac{E_2 - E_1}{\hbar}$$

(omega equals E two minus E one, all over h-bar). These are the allowed transition frequencies. The classical model needs these ω_0 values to be provided, either from experimental observation or from a prior quantum calculation. It cannot derive them from first principles.

* Second limitation: **Ignores Doppler broadening due to thermal motion (Gaussian convolution of the Lorentzian profile).**

In a gas, atoms or molecules are in constant random thermal motion. Some are moving towards an observer (or a light source), some are moving away, and some are moving transversely, with a distribution of speeds (e.g., Maxwell-Boltzmann distribution).

Due to the **Doppler effect**, an atom moving towards the light source sees the light frequency slightly up-shifted, and an atom moving away sees it slightly down-shifted. This means that even if each individual atom has a natural (Lorentzian) absorption profile centered at ω_0 in its own rest frame, the ensemble of moving atoms will absorb at a range of frequencies centered around ω_0, leading to a broadening of the observed spectral line.

This Doppler broadening typically results in a **Gaussian lineshape** if it's the dominant mechanism. If both natural/collisional (Lorentzian) broadening and Doppler (Gaussian) broadening are significant, the resulting lineshape is a **Voigt profile**, which is a convolution of a Lorentzian and a Gaussian. Our classical model, by considering stationary oscillators, only predicts a Lorentzian profile and misses this important effect prevalent in gas-phase spectroscopy.

* Third limitation: **Neglects saturation and stimulated emission at high laser intensities (non-linear optics).**

Our model assumed a linear response: the displacement x was proportional to the driving field E, leading to a polarization P proportional to E, and thus a susceptibility χ and refractive index n that are independent of the light intensity.

However, at **high laser intensities:**

* **Saturation:** As we discussed with the Beer-Lambert law, if many absorbers are pumped to the excited state, the absorption decreases. This is a non-linear effect where α (and thus κ) becomes intensity-dependent.

* **Stimulated Emission:** Einstein showed that in addition to absorption and spontaneous emission, there's a third process: stimulated emission, where an incident photon can cause an excited atom to emit a second, identical photon. This process is the basis of laser amplification. At high intensities, stimulated emission can become very significant, competing with absorption and leading to phenomena like optical gain or transparency.

These are hallmarks of **non-linear optics**, which the simple linear classical oscillator model cannot describe. It works well in the low-intensity limit.

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Continuing with the limitations of our classical oscillator model:

* The bullet point on this slide adds another important set of omissions: **Does not account for selection rules, polarization dependence, or multi-level interference effects.**

Let's break these down:

1. **Selection Rules:** Quantum mechanics dictates that transitions between energy levels are not all equally allowed. There are **selection rules** that determine whether a transition can occur via interaction with electromagnetic radiation (specifically, via electric dipole interaction, which is usually dominant). These rules often involve changes in quantum numbers like angular momentum (e.g., $\Delta L = \pm 1, \Delta m = 0, \pm 1$). Our classical model, which just assumes an oscillator with a given ω_0, has no concept of these quantum mechanical selection rules. It would implicitly allow any ω_0 if we put it in.

2. **Polarization Dependence:** While we used a linearly polarized field ($\hat{x}$) and got a response in that direction, real atomic and molecular transitions can

have complex dependencies on the polarization of light. For example, the strength of a transition might depend on whether the light is linearly polarized, circularly polarized, or unpolarized, and on the orientation of the molecule relative to the light's polarization vector. Selection rules for Δm (change in magnetic quantum number) are directly related to this. The simple 1D oscillator model doesn't capture this richness.

3. **Multi-level Interference Effects:** Real atoms and molecules are not just simple two-level systems (ground state and one excited state corresponding to ω_0). They have many energy levels. When multiple levels are involved, quantum mechanical interference effects can occur. For instance, if two different excitation pathways from a ground state lead to the same final state, or if multiple decay paths exist, these can interfere constructively or destructively, leading to complex phenomena like Fano resonances, electromagnetically induced transparency (EIT), or coherent population trapping (CPT). These are inherently quantum interference effects that a single classical oscillator cannot reproduce.

The three hyphens suggest the end of this enumeration of limitations. In essence, while the classical model provides a fantastic intuitive and quantitative starting point for understanding linear absorption and dispersion near an isolated resonance, a full and accurate description, especially for complex systems or under strong fields, requires a quantum mechanical approach.

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Given the limitations of the classical model, it's natural to ask, what's next? This slide provides a **Road Map Beyond — Toward Quantum Dispersion Theory**, essentially **Preparing for Chapter 3 and Beyond.** This outlines how we will build a more complete and accurate picture.

* First major step: **Replace classical oscillator with quantum two-level system; derive susceptibility via density-matrix formalism.**

Instead of a classical mass on a spring, we will model an atom or molecule as a **quantum two-level system**, with a ground state $|g\rangle$ and an excited state $|e\rangle$, separated by an energy $\hbar\omega_0$. We will then use the **density-matrix formalism** to describe the state of an ensemble of such systems and how it evolves under the influence of an external light field. The density matrix is a powerful tool because it can handle statistical mixtures of states, relaxation processes, and coherence. From the density matrix, we can calculate the expectation value of the dipole moment, which gives us the macroscopic polarization P, and from there, the electric susceptibility $\chi(\omega)$ and the complex refractive index $n(\omega)$. This quantum approach will naturally incorporate concepts like transition dipole moments and can be extended to include saturation.

* Second point: **Introduce Doppler effect and Voigt profile for gaseous absorbers.**

To accurately model spectra in gases, we must account for the thermal motion of atoms/molecules, which leads to Doppler broadening. We will learn how the intrinsic (homogeneous) Lorentzian lineshape predicted by the interaction with a single (or class of identical) absorber gets convolved with the Gaussian distribution of velocities due to the Doppler effect. The resulting lineshape is the **Voigt profile**. Understanding and fitting Voigt profiles is crucial for extracting information from gas-phase spectra.

* Third crucial topic: **Examine Kramers-Kronig relations, rigorously connecting** $n'(\omega)$ and $\kappa(\omega)$ through causality.

We've alluded to the fact that $n'(\omega)$ and $\kappa(\omega)$ are not independent. The **Kramers-Kronig relations** are a pair of integral transforms that mathematically express this fundamental connection. They state that if you know the imaginary part $\kappa(\omega)$ (absorption) over the entire frequency range (from 0 to ∞), you can calculate the real part $n'(\omega)$ (dispersion) at any frequency, and vice-versa. These relations are a direct consequence of **causality** – the principle that a system's response (e.g., polarization) cannot precede the stimulus (the electric field). This is a very deep and general result in physics, applicable to any linear, causal response function.

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Continuing our **Road Map Beyond**, this slide points to even more advanced topics we might explore.

* The bullet point suggests we will: **Explore non-linear extensions: saturation spectroscopy, electromagnetically induced transparency, slow light.**

Once we have a solid quantum mechanical foundation for linear light-matter interaction, we can start to explore what happens when the light fields are strong, leading to **non-linear optical phenomena**.

1. **Saturation Spectroscopy:** We've mentioned that high intensities can saturate an absorption line. Saturation spectroscopy is a clever set of techniques (like Lamb dip spectroscopy) that *uses* this saturation effect to overcome Doppler broadening in gases and measure the underlying homogeneous linewidth with very high resolution. It's a cornerstone of precision laser spectroscopy.

2. **Electromagnetically Induced Transparency (EIT):** This is a fascinating quantum interference effect that can occur in three-level (or multi-level) atomic systems. By applying a strong "coupling" laser field on one transition, it's possible to make an otherwise opaque medium transparent to a "probe" laser on an adjacent transition. EIT is accompanied by very steep and controllable dispersion, which has led to its application in areas like...

3. **Slow Light:** The steep normal dispersion ($\frac{dn\prime}{d\omega} > 0$ and large) associated with EIT can lead to extremely small group velocities for light pulses. Light can be slowed down by many orders of magnitude, sometimes even brought to a temporary halt and stored in the medium. This has profound implications for optical buffering, quantum information processing, and fundamental studies of light-matter interaction.

These non-linear extensions open up a vast and exciting landscape of modern laser spectroscopy and quantum optics. They all build upon the fundamental concepts of absorption and dispersion but require a quantum mechanical treatment and often involve coherent manipulation of atomic states.

The two sets of triple hyphens signal the end of this lecture overview and perhaps the lecture itself. We've covered a lot of ground, from the basic definitions of absorption and dispersion, through the classical Lorentz model, to its predictions for the complex refractive index, and finally, a roadmap towards more advanced quantum and non-linear treatments. This journey is central to understanding how lasers interact with and probe the world around us. Thank you.