Second solution (Arfken 9.6)

Linear Independence of Solutions

The Wronskian

$$W[\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)] = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}$$

1. If the Wronskian is not equal to zero, then the set of functions φ_n is therefore linearly independent.
2. If the Wronskian vanishes at isolated values of the argument, this does not necessarily prove linear dependence (unless the set of functions has only two functions). However, if the Wronskian is zero over the entire range of the variable, the functions φ_n are linearly dependent over this range.

Example 9.6.1 Linear Independence

The solutions of the linear oscillator equation (9.84) are $\varphi_1 = \sin\omega x$, $\varphi_2 = \cos\omega x$. The Wronskian becomes

$$\begin{vmatrix} \sin\omega x & \cos\omega x \\ \omega\cos\omega x & -\omega\sin\omega x \end{vmatrix} = -\omega \neq 0.$$

These two solutions, φ_1 and φ_2, are therefore linearly independent. For just two functions this means that one is not a multiple of the other, which is obviously true in this case.

You know that

$$\sin\omega x = \pm\left(1 - \cos^2\omega x\right)^{1/2},$$

but this is **not** a **linear** relation, of the form of Eq. (9.111). ■

```
In[11]:= Wronskian[y1_, y2_][t_] := y1[ω t] ∂t y2[ω t] - y2[ω t] ∂t y1[ω t]

In[12]:= Wronskian[Sin, Cos][t] // Simplify

Out[12]= -ω
```

Examples 9.6.2 Linear Dependence

For an illustration of linear dependence, consider the solutions of the one-dimensional diffusion equation. We have $\varphi_1 = e^x$ and $\varphi_2 = e^{-x}$, and we add $\varphi_3 = \cosh x$, also a solution. The Wronskian is

$$\begin{vmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{vmatrix} = 0.$$

The determinant vanishes for all x because the first and third rows are identical. Hence e^x, e^{-x}, and $\cosh x$ are linearly dependent, and, indeed, we have a relation of the form of Eq. (9.111):

$$e^x + e^{-x} - 2\cosh x = 0 \qquad \text{with } k_\lambda \neq 0.$$ ■

Theorem: If $y_1(x)$ and $y_2(x)$ are two solutions of the differential equation:

$$y''+P(x)y'+Q(x)y=0 \qquad (1)$$

In the interval $[a,b]$ and if inside the interval $P(x)$ and $Q(x)$ are continuous functions, then either the Wronskian is identically zero inside the interval or not zero at any point in the interval.

Prove: Start with the Wronskian

$$W\left[y_1(x),y_2(x)\right]=y_1y_2'-y_2y_1'$$

$$\Rightarrow \frac{dW}{dx}=y_1y_2''-y_2y_1'' \qquad (2)$$

Since $y_1(x)$ and $y_2(x)$ are two solutions of (1), then

$$y_1''=-P(x)y_1'-Q(x)y_1, \qquad (3)$$

$$y_2''=-P(x)y_2'-Q(x)y_2 \qquad (4)$$

Substituting from (3) and (4) in (2), we get:

$$\frac{dW}{dx}=-P(x)W \qquad (5)$$

Integrating in the interval $[a,b]$, $a<x<b$

$$W(x)=Ae^{-\int_a^x P(x)dx} \qquad (6)$$

Where $A=W(a)$ is a constant.

The function $\int_a^x P(x)dx$ can not be zero within the interval $[a,b]$. If $W(a)\neq 0$, then $W(x)$ is not zero at any point of the interval. If $W(a)=0$ then $W(x)$ is zero everywhere inside the interval.

With the help of the equation $W\left[y_1(x),y_2(x)\right]=y_1y_2'-y_2y_1'=y_1^2\frac{d}{dx}\left(\frac{y_2}{y_1}\right)$, we can calculate the second solution from equation (6) as:

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right)=W(a)\frac{e^{-\int_a^x P(x)dx}}{y_1^2} \qquad (7)$$

Integrating in the interval $[a',x]$, $a<a'<x$, we get

$$\left(\frac{y_2}{y_1}\right)\Bigg|_{a'}^{x}=W(a)\int_{a'}^{x}\frac{e^{-\int_a^x P(u)du}}{y_1^2(v)}dv \qquad (8)$$

$$\Rightarrow \boxed{y_2(x)=W(a)y_1(x)\int_{a'}^{x}\frac{e^{-\int_a^v P(u)du}}{y_1^2(v)}dv} \qquad (9)$$

Example: For the equation $y''(x)+y(x)=0$, we have $p(x)=0$. Let one of the solution be $y_1(x)=\sin(x)$, then the second one will be:

$$y_2(x) = \sin(x)\int^x \frac{e^{-\int_a^x 0 dt}}{\sin^2(t)}dt = \sin(x)\int^x \frac{1}{\sin^2(t)}dt = \sin(x)\left[-\cot(x)\right] = -\cos(x)$$

The second solution is linearly independent (not a linear multiple) of $\sin(x)$

Example: For second order differential equation in the form:

$$\frac{d^2y}{dx^2} + P(x)\frac{d^2y}{dx^2} + Q(x)y = 0,$$

It was fund that $P(x) = \frac{2}{1+2x}$ and the first solution is

$$y_1 = x(1 - 2x + 4x^2 - 8x^3 + 16x^4 + \cdots).$$

a- Find the second solution y_2.

[Hint: use $y_2 = y_1 \int \frac{e^{-\int P(x')dx'}}{y_1^2}$,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots]$$

```
y = x/(1 + 2 x)
x/(1 + 2 x)
pI = e^(-∫ 2/(1+2 x) dx)
1/(1 + 2 x)
yI = ∫ pI/y^2 dx // Simplify
-1/x + 2 Log[x]
y2=y yI  // ExpandAll
-1/(1 + 2 x) + (2 x Log[x])/(1 + 2 x)
```

H.W. For the diff. equation $x(x-1)y''(x) + 3xy'(x) + y(x) = 0$,

1- find the roots of the indicial equation

2- prove that the first solution is: $y_1(x) = \frac{x}{(1-x)^2}$ and the second one is: $y_1(x) + y_1(x)\ln(x)$.

H.W. solve the differential equation: $(1-x^2)y'' - 2xy' + 2y = 0$.

Answer:

$$y(x) = C_1 x + C_o(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \cdots - \frac{x^{2i}}{(2i-1)} - \cdots).$$

Frobenius method for the solution of ordinary differential equations

The Frobenius method is used to solve the following differential equation.

$$\frac{d^2y(x)}{dx^2}+\frac{b(x)}{x}\frac{dy(x)}{dx}+\frac{c(x)}{x^2}y=0 \quad \Rightarrow \quad y''+P(x)y'+Q(x)y=0 \qquad [23]$$

In this equation $b(x)$ and $c(x)$ are analytic at $x=0$. Note that the conventional power series method cannot be used for this equation because the coefficients of $\frac{dy(x)}{dx}$ and *y* are not analytic at $x=0$.

In the Frobenius method the solution is written as follows.

$$y(x)=x^r\sum_{n=0}^{\infty}a_n x^n=\sum_{n=0}^{\infty}a_n x^{n+r} \qquad [24]$$

Here the a_n are unknown coefficients and the value of r is also unknown. The value of r is determined during the solution procedure so that $a_0\neq 0$. We can differentiate the series for $y(x)$ in equation [24] two times to obtain.

$$\frac{dy}{dx}=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \qquad and \qquad \frac{d^2y}{dx^2}=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} \quad [25]$$

We assume that $b(x)$ and are function of *x*, simple polynomials or series expressions that have the following general forms.

$$b(x)=\sum_{n=0}^{\infty}b_n x^n \qquad c(x)=\sum_{n=0}^{\infty}c_n x^n \qquad [26]$$

We can substitute equations [24], [25], and [26], into equation [23] and multiply the result by x^2 to obtain the following equation.

$$\begin{gathered}x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\\ x\left(\sum_{n=0}^{\infty}b_n x^n\right)\left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}c_n x^n\right)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0\end{gathered} \qquad [27]$$

We can combine the x^2 and x terms outside the sums with the x terms inside the sums as follows.

$$\begin{gathered}\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}+\\ \left(\sum_{n=0}^{\infty}b_n x^n\right)\left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}\right)+\left(\sum_{n=0}^{\infty}c_n x^n\right)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0\end{gathered} \qquad [28]$$

We can then write out the first few terms in each series or product of series to give

$$\begin{aligned}&r(r-1)a_0x^r+(1+r)ra_1x^{r+1}+(2+r)(1+r)a_2x^{r+2}+\cdots\\ &+b_0ra_0x^r+[b_1ra_0+b_0(1+r)a_1]x^{r+1}+[b_2ra_0+b_1(1+r)a_1+b_0(2+r)a_2]x^{r+2}+\cdots\\ &+c_0a_0x^r+[c_1a_0+c_0a_1]x^{r+1}+[c_2a_0+c_1a_1+c_0a_2]x^{r+2}+\cdots=0\end{aligned} \qquad [29]$$

As in the conventional power series solution, we require the coefficient of each term in the power series to vanish to satisfy equation [28] or [29]. Starting with the lowest power of x, x^r, we require that the following coefficient be zero.

$$[r(r-1)+b_0 r+c_0]a_0=0 \qquad [30]$$

In the Frobenius method, we want to keep $a_0 \neq 0$, thus we require that

$$r(r-1)+b_0 r+c_0=0 \qquad [31]$$

This yields a quadratic equation in r which is called the **indicial equation**. We find two possible values of r from the conventional solution of the quadratic equation.

$$r=\frac{1-b_0 \pm \sqrt{(1-b_0)^2-4c_0}}{2} \qquad [32]$$

In the Frobenius method, the original solution in equation [24], with two different values of *r*, can provide two different solutions

a. **If the two values** of r found from equation [32] **are different and their difference is not an integer**. These two solutions form a basis for all solutions to equation [23]. These are

$$y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_n x^n \qquad and \qquad y_2(x)=x^{r_2}\sum_{n=0}^{\infty}A_n x^n \qquad [33]$$

The coefficients for the two solutions, a_n for $y_1(x)$ and A_n for $y_2(x)$, are different. These coefficients are found in the same way that the coefficients were found in the usual power series equation, once the values of r are determined.

b. **If there is a double r**oot for r **or if the two values of** r **differ by an integer**, it is necessary to have a second solution that has a different form. For a double root, $r=r_1=r_2$, we have the two following solutions.

$$y_1(x)=x^{r}\sum_{n=0}^{\infty}a_n x^n \qquad and \qquad y_2(x)=y_1(x)\ln(x)+\sum_{n=1}^{\infty}A_n x^n \qquad [34]$$

c- **If the two roots**, r_1 and r_2 differ by an integer, the two possible solutions are written as follows.

$$y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_n x^n \qquad and \qquad y_2(x)=ky_1(x)\ln(x)+x^{r_2}\sum_{n=0}^{\infty}A_n x^n \qquad [35]$$

In the last expression the roots are defined such that $r_1>r_2$; the value of *k* may be zero in this case. Note that the a_n coefficients and the A_n coefficients are different; also for the double root, the summation for $y_2(x)$ starts at $n=1$ instead of $n=0$.

Summary

Can the power series method solve any differential equation? The answer is the Fuchs' theorem.
"If we are expanding about an ordinary point or at worst about a regular singularity, the series substitution approach will yield at least one solution. Whether we get one or two distinct solutions depends on the roots of the indicial equation.

1. If the two roots of the indicial equation are equal, $r_1=r_2$, we can obtain only one solution by this series substitution method.
2. If the two roots differ by a non-integral number, $r_1-r_2 \neq \text{Integer}$, two independent solutions may be obtained.
3. If the two roots differ by an integer, $r_1-r_2=\text{Integer}$, the larger of the two will yield a solution.

The smaller may or may not give a solution, depending on the behavior of the coefficients.
In the linear oscillator equation we obtain two solutions; for Bessel's equation, we get only one solution.

In this section we learn how to extend series solutions to a class of differential equations that appear at first glance to diverge in our region of interest.
Let's consider the equation:

$$2x^2 y'' + 7x(x+1)y' - 3y = 0 \qquad (1)$$

and we are interested in finding the series solution to this equation in the vicinity of $x = 0$.
Rearrange equation (1) in the form:

$$y'' + \frac{7x(x+1)}{2x^2} y' - \frac{3}{2x^2} y = 0 \qquad (2)$$

In the vicinity of $x = 0.$, it appears that this equation is undefined and will not yield meaningful solutions to the equation (1) near 0.

The necessary conditions for solving equations of the form of (2)

However, the ***method of Frobenius*** provides us with a method of adapting our series solutions techniques to solve equations like this if certain conditions hold. Let's consider now those conditions.
First, we will write our second order differential equation as:

$$y'' + P(x)y' + Q(x)y = 0 \qquad (3)$$

When transformed into this form, our original equation in (1) has

$$P(x) = \frac{7x(x+1)}{2x^2}, \qquad Q(x) = -\frac{3}{2x^2} y$$

The theorem of Frobenius shows that if **both** $(x-x_0)P(x)$ **and** $(x-x_0)^2 Q(x)$ have meaningful series solutions around x_0 then a series solution to the differential equation can be found.
Let's apply this theorem to eq. (2) to see if the conditions of this theorem hold:
We want to find a series solution in the neighborhood of $x_0 = 0$, so $(x-x_0) = x$. Then, we construct the terms *xP(x) and* $x^2 Q(x)$ and see if they have are well behaved at $x = 0$. It is easy to show that **both** *xP(x)* **and** $x^2 Q(x)$ are well behaved at $x_0=0$ and will have power series that converge near 0. Thus, the conditions of the Theorem of Frobenius are met, and we can find a power series solution for (1).

The Method of Frobenius

If the conditions described in the previous section are met, then we can find at least one solution to a second order differential equation by assuming a solution of the form:

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad (4)$$

where *r* and a_n are constants to be determined, and $n = 0, 1, 2, 3,\ldots$ While *n* is always an integer, there is no such constraint on *r*. We will find that *r* may have negative or fractional values.
Eq. (4) looks very familiar to our earlier work with series solutions, and we should expect that we will have to derive recursion relations to find the exact values of a_n. The obvious difference is that we also have to determine the values of the exponent *r*; it is these factors of x^r that allow us to find

solutions to our differential equations. (Note that other authors and sources will use a staggeringly diverse array of symbols to represent the exponent that we have called *r*. There seems to be no standard for this; do not be confused, they are all dummy variables referring to the same concept. But we will use *r* throughout.)

The indicial equation and the values of r

The first step in using the method of Frobenius is to determine the values of *r* that allow us to solve the differential equation. We do this by exploiting the fact that this method produces a series where the first non zero term is a_0.

We substitute our assumed form of the series solution (4) into our original equation and determine the proper values of *r* by setting $n=0$; in other words, we find an expression that will ensure the a_0 term is non-zero. Let's apply this technique to our original equation (1). Substituting (4) into (1) gives us:

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+7x^2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+7x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-3\sum_{n=0}^{\infty}a_n x^{n+r}=0 \qquad (5)$$

Let's look carefully at (5). Notice that unlike our earlier work with series solutions, we do not change the lower limit of summation when we differentiate our solution. This is because of the presence of the factor x^r; as noted above, the value of *r* is chosen so that the first non zero term of the expansion is the a_0 term, so we do not change the lower limit of summation upon differentiation as we did in our prior work with series solutions. Now, let's rewrite (5) by consolidating all powers of *x* inside the summation signs, and get:

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}+7\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1}+7\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}-3\sum_{n=0}^{\infty}a_n x^{n+r}=0 \qquad (6)$$

We see immediately that three of the summations are to the power of $n+r$; and that only one summation (the second one in (6)) is to a different power. We know that we want to re-index this sum so that it is also to the power of $n+r$¸ and when we do this, eq (6) becomes:

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}+7\sum_{n=1}^{\infty}(n+r-1)a_{n-1} x^{n+r}+7\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}-3\sum_{n=0}^{\infty}a_n x^{n+r}=0 \qquad (7)$$

Now it is appropriate to change the lower limit of the second summation since we have to re-index each term that involves *n*. We are now in a position to determine the values of *r* for this equation. Remember that *r* must be chosen so that the a_0 term is non zero. We can "strip out" the $n=0$ term in (7) and solve for the values of *r*. Setting $n= 0$ in (7) gives us:

Now it is appropriate to change the lower limit of the second summation since we have to re-index each term that involves *n*. We are now in a position to determine the values of *r* for this equation. Remember that *r* must be chosen so that the a_0 term is non zero. We can "strip out" the $n=0$ term in (7) and solve for the values of *r*. Setting $n= 0$ in (7) gives us:

$$[2r(r-1)+7r-3]a_0=0 \Rightarrow (2r^2+5r-3)a_0=0 \qquad (8)$$

Since we know that a_0 is non-zero, eq. (8) implies:

$$2r^2+5r-3=0 \qquad (9)$$

Equation (9) is called the **indicial equation**, and its solution gives us the values of *r* we use in finding our solution in the form of (4).

Solving (9) is easily done by factoring:

$$(2r-1)(r+3)=0 \quad (10)$$

and our values of r are $r = ½$ and $r = -3$; we will see that our solutions to the differential equation will involve factors of $x^{1/2}$ and x^{-3}. In fact, we can derive the indicial equation even before re-indexing summations. We look for those summations that involve the lowest power of x, and use those terms with n=0 to derive the indicial equation. In equation (6) above, the first, third and fourth sums are the terms with the lowest power of x, and these are the terms that we use to derive the indicial equation.

Finding the recursion relation The next step in this method is a familiar one to us, namely, finding the recursion relation that will enable us to determine the values of a_n. We begin by referring back to eq. (7):

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + 7\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r} + 7\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - 3\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

We know that since we have all our exponents to the same power, we can combine like terms to produce:

$$[2(n+r)(n+r-1)+7(n+r)-3]a_n + 7(n+r-1)a_{n-1} = 0 \qquad (11)$$

or equivalently:

$$a_n = \frac{-7(n+r-1)}{[2(n+r)-1][(n+r)+3]}\; a_{n-1} \qquad (12)$$

And this is the **recursion relation** for this particular differential equation.

We can make a couple of important points about recursion relations in the method of Frobenius. First, you will have to use this recursion relation twice; once to determine values of a_n when $r = ½$, and a second time to determine values of a_n when $r = -3$.
Second, notice the denominator in (12). It is highly recommended that you factor the coefficient of a_n in writing your final recursion relation. Review the steps in deriving the indicial equation (10). You set $n = 0$ in deriving (10). If you replace r with $n+r$, you will have determined the coefficient of a_n for all values of n (including non zero values). Look now at the denominator in (12); the factors in the denominator are those you would derive by replacing r with $(n+r)$ in (10).
Of course, it is critical that you determine the indicial equation correctly; else you will propagate your errors all the way through the problem.

Finding the values of a_n

Now that we have a recursion relation, we can calculate the coefficients a_n. As noted above, we have to do this separately for each value of r.
It is standard to start with the greater value of r.
When $r = ½$:

$$a_n = \frac{-7(n+r-1)a_{n-1}}{[2(n+r)-1][(n+r)+3]} = \frac{-7(n-1/2)a_{n-1}}{[2(n+1/2)-1][n+7/2]} = \frac{-7(2n-1)a_{n-1}}{2n(2n+7)} \quad for\ n \geq 1 \quad (13)$$

This recursion relation gives us the following values:

$$a_1 = \frac{-7a_0}{18}, \quad a_2 = -\frac{21a_1}{44} = \frac{147}{792}a_0 \qquad (14)$$

And we can write the first solution to the differential equation as:

$$y_1 = a_0 x^{1/2}\left(1 - \frac{7}{18}x + \frac{147}{792}x^2 + \ldots\right) \qquad (15)$$

When r = - 3:

We go back to the original recursion relation (12) and begin as:

$$a_n = \frac{-7(n+r-1)a_{n-1}}{[2(n+r)-1][(n+r)+3]} = \frac{-7(n-4)a_{n-1}}{[2(n-3)-1][n]} = \frac{-7(n-4)a_{n-1}}{n(2n-7)} \; \textit{for } n \ge 1 \qquad (16)$$

This recursion relation yields the following coefficients:

$$a_1 = \frac{-21}{5}a_0, \quad a_2 = -\frac{7a_1}{3} = \frac{49}{5}a_0, \quad a_3 = \frac{-7a_0}{3} = \frac{-343}{15}a_0 \qquad (17)$$

And the solution corresponding to this value of r is then given by:

$$y_2 = a_0 x^{-3}\left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3 \ldots\right) \qquad (18)$$

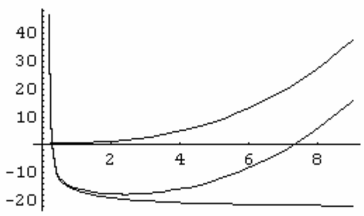
The general solution to the differential equation

We have in essence found the complete solution to our original equation in (1). We then write the general solution as:

$$y = c_1 y_1 + c_2 y_2 = c_1 x^{1/2}\left(1 - \frac{7}{18}x + \frac{147}{792}x^2 + \ldots\right) + c_2 x^{-3}\left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3\right) \; (19)$$

There is not much new in (19); the only point to make is to take explicit note of the coefficients. Above we calculated coefficients in the two cases (i.e., where r = ½ and when r = -3) and called the coefficients in both cases a_n as is our custom. In writing the final solution (19), we just have to remember that y_1 and y_2 are independent solutions to the original differential equation, so we need to keep in mind that the coefficients c_1 and c_2 are likely to be different.

```
In[24]:= y1 = x^(1/2) (1 - 7/18 x + 147/792 x^2);
In[25]:= y2 = x^-3 (1 - 21/5 x + 49/5 x^2 - 343/15 x^3);
In[28]:= Plot[{y1, y2, y1 + y2}, {x, 0.2, 9}]
```



Note: The second solution will usually diverge at the origin because of the logarithmic factor and the negative powers of x in the series. For this reason y2(x) is often referred to as the **irregular solution**. The first series solution, y1(x), which usually converges at the origin, is called the **regular solution**.

Example: For the diff. equation:

$$x(x-1)y''(x)+3xy'(x)+y(x)=0,$$

One finds $P(x)=\dfrac{3}{(x-1)}$, $Q(x)=\dfrac{1}{x(x-1)}$, $\Rightarrow$ $x=0$ is a regular singular point.

With the power series:

$$y(x)=x^r\sum_{n=0}^{\infty}a_n x^n=\sum_{n=0}^{\infty}a_n x^{n+r}\ , \quad \frac{dy}{dx}=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \text{ and}$$

$$\frac{d^2y}{dx^2}=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

The indicial equation will be:

$$\sum_{n=0}(n+r)(n+r-1)a_n x^{n+r}-\sum_{n=0}(n+r)(n+r-1)a_n x^{n+r-1}+3\sum_{n=0}(n+r)a_n x^{n+r}+\sum_{n=0}a_n x^{n+r}=0$$

The indicial equation

We use the above equation to find the indicial equation. As described before, the indicial equation provides us with the values of *r* that will solve the differential equation and will ensure that the first non zero term of the power series solution is the a_0 term. We can find the indicial equation by setting n equal to zero in those terms of the above equation bearing the lowest power of *x*. In this case, there is only one such term: the second one. (The exponent in the second term is $n+r-1$, in all other terms the exponent is $n+r$.)

Setting n = 0 in the second term produces the indicial equation:

$$r(r-1)=0$$

for which the roots are easily seen to be $r=0,1$. This is an example of CASE III. where the difference in the roots of the indicial equation is an integer.

The recursion relation

We know that the method of Frobenius vill produce at least one solution for the original equation. To produce the recursion relation we need to re-index only the second summation. Doing so in the typical way, we can proceed by writing the recursion relation in the form:

$$a_{n+1}=\frac{(n+r)(n+r-1)+3(n+r)+1}{(n+r)(n+r+1)}a_n=\frac{(n+r+1)^2}{(n+r)(n+r+1)}a_n=\frac{(n+r+1)}{(n+r)}a_n$$

Use MATHEMATICA to check the simplification:

```
A=(n+r)(n+r-1)+3(n+r)+1
 1+3 (n+r)+(-1+n+r) (n+r)
 B= (n+r+1)^2
 (1+n+r)^2
 A-B //Simplify
 0
```

The first solution:

By now we are very adept at using recursion relations to find coefficients, and doing so allows us to write: For $r=1$, we have

$$a_{n+1}=\frac{(n+r+1)}{(n+r)}a_n=\frac{(n+2)}{(n+1)}a_n$$

So we have the coefficients:

$$a_1=2a_0\,;\;\; a_2=\frac{3}{2}a_1=3a_0\,;\;\; a_3=\frac{4}{3}a_2=4a_0$$

And the first solution will be:

$$y_1=x^r\sum_{n=0}^{\infty}a_nx^n=a_0x^1(1+2x+3x^2+4x^3+...nx^{n-1})=\frac{x}{(1-x)^2}$$

This is the first solution with $a_0=1$. We have to check for the second solution.

The second solution:

For $r=0$, we have

$$a_{n+1}=\frac{(n+r+1)(n+r+1)}{(n+r)(n+r+1)}a_n=\frac{(n+r+1)}{(n+r)}a_n=\frac{(n+1)}{n}a_n$$

$\therefore\ \lim_{n\to 0}a_{n+1}=\infty$. So, we can not have a second solution through the usual way. To find the second solution, we can use: $y_2(x)=y_1(x)\int^x\frac{e^{-\int^x P(x')dx'}}{y_1^2(x'')}dx''$.

First integral:

$$\int P(x')dx'=\int\frac{3}{(x'-1)}dx'=\frac{1}{(x-1)^3}$$

Second :

$$y_2=y_1\int\frac{\frac{1}{(z-1)^3}}{\left[\frac{z}{(1-z)^2}\right]^2}dz=-y_1(x)\int\frac{1-z}{z^2}dz\Bigg|_x=\frac{x}{(1-x)^2}\left(\frac{1+x\ln x}{x}\right)=\frac{1+x\ln x}{(1-x)^2}=\frac{y_1}{x}+y_1\ln x$$

Another method for the second solution:

Start with: $a_{n+1} = \frac{(n+r+1)}{(n+r)} a_n$, one finds

$$a_1 = \frac{(r+1)}{(r)} a_0, \quad a_2 = \frac{(r+2)}{(r+1)} a_1 = \frac{(r+2)}{(r)} a_0, \quad a_3 = \frac{(r+3)}{(r+2)} a_2 = \frac{(r+3)}{(r)} a_0$$
$$\vdots$$
$$a_n = \frac{(r+n)}{(r+n-1)} a_2 = \frac{(r+n)}{(r)} a_0$$

Then,

$$y(x) = a_0 x^r \left\{ 1 + \frac{(r+1)}{(r)} x + \frac{(r+2)}{(r)} x^2 + \cdots \right\}$$

In the case of $r = 0$ we have singular coefficients. To remove the singularity one has to replace $a_o = b_o r$ in the above equation:

$$y(x) = b_0 x^r \left\{ r + (r+1) x + (r+2) x^2 + \cdots \right\}$$

Differentiate w.r.t. r

$$\frac{dy(x)}{dr} = b_0 \left(x^r \ln x \right) \left\{ r + (r+1) x + (r+2) x^2 + \cdots \right\} + b_0 x^r \left\{ 1 + (1+1) x + (1+2) x^2 + \cdots \right\}$$

Put $r = 0$ in the last equation:

$$y_2 = \left. \frac{dy(x)}{dr} \right|_{r=0} = b_0 \left(x^0 \ln x \right) \underbrace{\left\{ 0 + (0+1) x + (0+2) x^2 + \cdots \right\}}_{x y_1 / x} + b_0 x^0 \underbrace{\left\{ 1 + 2x + 3x^2 + \cdots \right\}}_{y_1 / x}$$

$$y_2 = \left. \frac{dy(x)}{dr} \right|_{r=0} = \left(y_1 \ln x \right) + \frac{y_1}{x}$$

Example: For the differential equation

$$x^2 y'' + x y' - x y = 0$$

The indicial equation becomes:

$$r(r-1) + r = 0 \quad \Rightarrow \quad r = 0, 0$$

Since the two roots of the indicial equation are zero, we know we can find one solution of the form:

$$y_1 = x^0 \sum a_n x^n$$

The exponent of *x* arises from *r* = 0, and we have to find the values of a_n from the recursion relation:

$$a_n = \frac{a_{n-1}}{(n+r)^2} = \frac{a_{n-1}}{n^2} \quad when \ r = 0$$

This recursion relation leads to a first solution of:

$$y_1(x) = a_0 (1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \ldots)$$

To find the second solution, we make use of $y_2(x) = W(a) y_1(x) \int_{a'}^{x} \frac{e^{-\int_a^v P(u)du}}{y_1^2(v)} dv$ with $P(x) = 1/x$, so we have that:

$$\exp[-\int P(x)dx] = \exp[-\int \frac{1}{x} dx] = \exp[-\ln x] = \frac{1}{x}$$

Then

$$y_2(x) = y_1(x)\left[\int \frac{1/x}{a_0^2(1+x+x^2/4+x^3/36+...)^2} dx\right]$$

looks painful!. So, we can use the expansion $1/(1+x+x^2/4+x^3/36+\ldots)^2$ in power series. Recall that:

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3...$$

So that we can expand:

$$\frac{1}{(y_1(x))^2} = \frac{1}{(1+x+x^2/4+x^3/36+..)^2} = 1 - 2x + \frac{5}{2}x^2 - \frac{23}{9}x^3 + ...$$

With this expansion, our second solution becomes:

$$y_2(x) = y_1(x)\left[\int \frac{1-2x+5x^2/2-23x^3/9...}{x} dx\right] = y_1(x)[\ln x - 2x + 5x^2/4 - 23x^3/27...]$$
$$= y_1(x)\ln x + a_0(-2x - 3x^2/4 - 11x^3/108 - ...)$$

Example 1: for the Bessel differential equation

$$x^2 \frac{d^2 y(x)}{dx^2} + x\frac{dy(x)}{dx} + x^2 y = 0$$

Find the second independent solution.

Answer: Here

$$y_1(x) = J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2}\left(\frac{x}{2}\right)^{2n} = 1 - \frac{x^2}{4} + \frac{x^4}{64} + \frac{x^6}{2304} + \cdots, \qquad P(x) = \frac{1}{x},$$

Thus

$$y_2(x) = y_1(x)\int^x \frac{e^{-\int_a^x P(x_1)dx_1}}{y_1^2(x)}dx = J_0(x)\int^x \frac{e^{-\int_a^x \frac{dx_1}{x_1}}}{J_0^2(x)}dx = J_0(x)\int^x \frac{e^{-[\ln(x)-\ln(a)]}}{J_0^2(x)}dx$$

$$= J_0(x)\int^x \frac{\frac{a}{x}}{J_0^2(x)}dx = aJ_0(x)\int^x \frac{1}{xJ_0^2(x)}dx$$

Then

$$y_2(x) = \int^x \frac{1}{xJ_0^2(x)}dx = \int^x \frac{1}{x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + \frac{x^6}{2304} + \cdots\right)^2}dx$$

$$= \int^x \left(\frac{1}{x} + \frac{x}{2} + \frac{5x^3}{32} + \cdots\right)dx = \ln x + \frac{1}{4}x^2 + \frac{5}{128}x^4 + \frac{23}{3456}x^6 + \cdots$$

Therefore

$$y_2(x) = J_0(x)\ln(x) + J_0(x)\left[\frac{1}{4}x^2 + \frac{5}{128}x^4 + \frac{23}{3456}x^6 + \cdots\right]$$

Where, the overall constant "a" has been put equal to unity.