Integral Representation (Arfken chapter 11, P. 679)

Bessel's differential equation:

$$x^2 y''(x) + xy'(x) + \left\{x^2 - n^2\right\} y = 0$$

has the power series solution in the form:

$$y(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!} \left(\frac{x}{2}\right)^{n+2j} = J_n(x)$$

The generating function for $J_n(x)$

The function $g(x,t) = e^{(t-t^{-1})\frac{x}{2}}$ is called the generating function for J_n and the following shows the fact that:

$$g(x,t) = e^{(t-t^{-1})\frac{x}{2}} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

Since

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

then

$$e^{(t-t^{-1})\frac{x}{2}} = e^{t\frac{x}{2} - \frac{x}{2t}} = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{xt}{2}\right)^i \sum_{j=0}^{\infty} \frac{1}{j!}\left(-\frac{x}{2t}\right)^j$$

$$= \sum_{i,j=0}^{\infty} \frac{(-1)^j}{i!\,j!\,2^{i+j}} x^{i+j} t^{i-j}$$

Let $i - j = n$, $\Rightarrow \quad n \equiv \{-\infty, \infty\}$, then:

$$e^{(t-t^{-1})\frac{x}{2}} = \sum_{n=-\infty}^{\infty} \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!2^{2j+n}} (x)^{2j+n} \right) t^n = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

Comment: If now $n < 0$, we still have the coefficient of t^n for a fixed value of *j* given by:

$$\frac{(-1)^j}{j!(j-n)!2^{2j-n}} (x)^{2j-n}$$

But, now the requirement that $j \geq 0$ with $j = i - n$ is satisfied for all values of *i*. Hence the total coefficient of t^n is just:

$$\sum_{j=0}^{\infty} \frac{(-1)^{j-n}}{j!(j-n)!2^{2j-n}} (x)^{2j-n} = (-1)^{-n} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(p+n)!2^{2p+n}} (x)^{2p+n}, \qquad p = j - n$$

$$= (-1)^{-n} J_n(x) = J_n(x)$$

H. W. Prove the following Recurrence relations for $J_n(x)$

1- $\frac{d}{dx}\left\{x^n J_n(x)\right\} = x^n J_{n-1}(x)$

2- $\frac{d}{dx}\left\{x^{-n} J_n(x)\right\} = -x^{-n} J_{n+1}(x)$

Three terms Recurrence relations

3- $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

4- $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$

5- $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

6- $2nJ_n(x) = x\left\{J_{n+1}(x) + J_{n-1}(x)\right\}$

Proof:

1- Start with the definition $J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!}\left(\frac{x}{2}\right)^{n+2j}$, multiply both sides with x^n and rearrange, one gets:

$$x^n J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j\, 2^n}{j!(n+j)!}\left(\frac{x}{2}\right)^{2n+2j}$$

Differentiate the last equation w.r.t. *x*,

$$\frac{d}{dx}\left(x^n J_n(x)\right) = \frac{d}{dx}\left(\sum_{j=0}^{\infty} \frac{(-1)^j\, 2^n}{j!(n+j)!}\left(\frac{x}{2}\right)^{2n+2j}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j\, 2^n}{j!(n+j)!}\left[\left(\not{2}n + \not{2}j\right)\left(\frac{x}{2}\right)^{2n+2j-1} \frac{1}{\not{2}}\right]$$

$$= x^n \sum_{j=0}^{\infty} \frac{(-1)^j\, 2^n\,(n+j)}{j!(n+j)!}\left(\frac{x}{2}\right)^{n+2j-1} = x^n \sum_{j=0}^{\infty} \frac{(-1)^j\, 2^n}{j!(n+j-1)!}\left(\frac{x}{2}\right)^{n+2j-1}$$

$$= x^n J_{n-1}(x)$$

2- Same like 1

3- From (2) $\underbrace{\frac{d}{dx}\left\{x^{-n} J_n(x)\right\}}_{-nx^{-n-1}J_n(x)+x^{-n}J_n'(x)} = -x^{-n} J_{n+1}(x)$, multiply both sides with x^{n+1} and rearrange, we get

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

4- From (1) $\underbrace{\frac{d}{dx}\left\{x^{n} J_n(x)\right\}}_{nx^{n-1}J_n(x)+x^{n}J_n'(x)} = x^{n} J_{n-1}(x)$, multiply both sides with x^{-n+1} and rearrange, we get

$$xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$$

5- Adding 3 and 4, we have the result.

6- Subtracting 3 and 4, we reach the result.

Comments:

$$J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!}\left(\frac{x}{2}\right)^{n+2j}, \qquad J_{-n}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(-n+j)!}\left(\frac{x}{2}\right)^{-n+2j}$$

1- The series converge for all finite x.

2- If n is not an integer, the solutions are linearly independent.

3- If n is an integer, they are linearly dependent, and in particular $J_{-m}(x) = (-1)^m J_m(x)$

4- Because of the linearly dependence of $J_{-m}(x)$ on $J_m(x)$, we introduce a second linearly independent function

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)},$$

known as Neumann Function or the Bessel Function of the second kind. Conventionally, we choose as our linearly independent function $J_\nu(x)$ and $N_\nu(x)$ even if ν is not an integer.

Integral representations of Bessel functions

A particularly useful and powerful way of treating Bessel functions employs integral representations.

If we return to the generating function $g(x,t) = e^{(t-t^{-1})\frac{x}{2}}$, and expand it, we get:

$$e^{(t-t^{-1})\frac{x}{2}} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

$$= J_0(x) + t^1 J_1(x) + t^2 J_2(x) + \cdots + t^{-1} \underbrace{J_{-1}(x)}_{(-)^1 J_1(x)} + t^{-2} \underbrace{J_{-2}(x)}_{(-)^2 J_2(x)} + \cdots$$

$$= J_0(x) + \left(t - \frac{1}{t}\right) J_1(x) + \left(t^2 + \frac{1}{t^2}\right) J_2(x) + \left(t^3 - \frac{1}{t^3}\right) J_3(x) + \cdots$$

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Now, let $t = e^{i\theta}$, we get:

$t^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta), \quad t^{-n} = e^{-in\theta} = \cos(n\theta) - i\sin(n\theta),$

Adding and subtracting the above equations, we have:

$\left(t^n + \frac{1}{t^n}\right) = 2\cos(n\theta), \quad \left(t^n - \frac{1}{t^n}\right) = 2i\sin(n\theta),$

Then

$$e^{(t-t^{-1})\frac{x}{2}} = e^{ix\sin(\theta)} = \cos\left[x\sin(\theta)\right] + i\sin\left[x\sin(\theta)\right] \qquad \text{(I)}$$

and

$$e^{(t-t^{-1})\frac{x}{2}} = J_0(x) + \underbrace{\left(t - \frac{1}{t}\right)}_{2i\sin(\theta)} J_1(x) + \underbrace{\left(t^2 + \frac{1}{t^2}\right)}_{2\cos(2\theta)} J_2(x) + \underbrace{\left(t^3 - \frac{1}{t^3}\right)}_{2i\sin(3\theta)} J_3(x) + \cdots \qquad \text{(II)}$$

Equate the real and imaginary parts in (I) and (II), one finds:

$$\cos\left[x\sin(\theta)\right] = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\theta),$$

$$\sin\left[x\sin(\theta)\right] = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\sin\left[(2n-1)\theta\right]$$

These are Jacobi's series.

H.W. Show that:

$$\cos(x) = J_0(x) - 2J_2(x) + 2J_4(x) - \cdots,$$

$$\sin(x) = 2J_1(x) - 2J_3(x) + 2J_5(x) - \cdots$$

Standard formulae

in which n and m are **positive** integers (zero is excluded),[5] we obtain

$$\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\cos n\theta\,d\theta=\begin{cases}J_n(x), & n \text{ even},\\ 0, & n \text{ odd},\end{cases} \qquad (11.27)$$

$$\frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta)\sin n\theta\,d\theta=\begin{cases}0, & n \text{ even},\\ J_n(x), & n \text{ odd}.\end{cases} \qquad (11.28)$$

If these two equations are added together,

$$\begin{aligned} J_n(x) &= \frac{1}{\pi}\int_0^{\pi}\left[\cos(x\sin\theta)\cos n\theta+\sin(x\sin\theta)\sin n\theta\right]d\theta \\ &= \frac{1}{\pi}\int_0^{\pi}\cos(n\theta-x\sin\theta)\,d\theta, \qquad n=0,1,2,3,\ldots. \end{aligned} \qquad (11.29)$$

As a special case, Equation (11.29) gives:

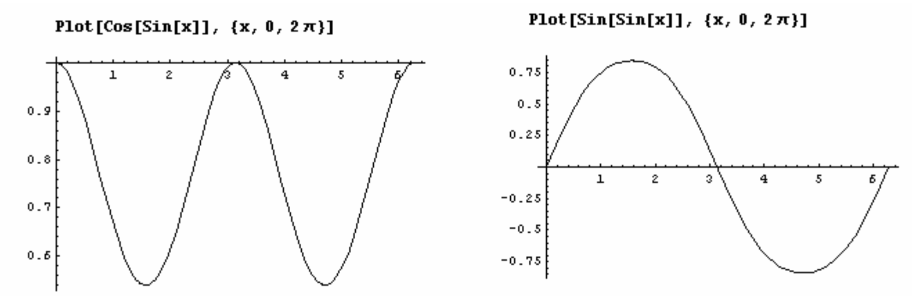
$$J_0(x)=\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\,d\theta. \qquad (11.30)$$

Noting that $\cos(x\sin\theta)$ repeats itself in all four quadrants, we may write Eq. (11.30) as

$$J_0(x)=\frac{1}{2\pi}\int_0^{2\pi}\cos(x\sin\theta)\,d\theta. \qquad (11.30a)$$

On the other hand, $\sin(x\sin\theta)$ reverses its sign in the third and fourth quadrants, so

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(x\sin\theta)\,d\theta=0. \qquad (11.30b)$$



Adding Eq. (11.30a) and i times Eq. (11.30b), we obtain the complex exponential representation

$$J_0(x)=\frac{1}{2\pi}\int_0^{2\pi}e^{ix\sin\theta}\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}e^{ix\cos\theta}\,d\theta. \qquad (11.30c)$$

Example: Evaluate the indefinite integral $I=\int x\,(\ln x)\,J_o(x)\,dx$

$$I=\int x\,(\ln x)\,J_o(x)dx=\int(\ln x)\underbrace{\left[xJ_o(x)\right]}_{\frac{\partial}{\partial x}xJ_1(x)}dx=\int(\ln x)\,d\left[xJ_1(x)\right]$$

$$=x\,(\ln x)\,J_1(x)-\int\frac{x}{x}\underbrace{J_1(x)}_{-\frac{\partial}{\partial x}J_0(x)}dx=x\,(\ln x)\,J_1(x)+J_0(x)$$

Example 1. *Evaluate the integral*

$$\int_0^\infty e^{-ax}J_0(bx)\,dx, \qquad a>0, \quad b>0.$$

Replacing $J_0(bx)$ by its integral representation (5.10.8), we find that

$$\begin{aligned}\int_0^\infty e^{-ax}J_0(bx)\,dx &= \int_0^\infty e^{-ax}\,dx\,\frac{2}{\pi}\int_0^{\pi/2}\cos(bx\sin\varphi)\,d\varphi \\ &= \frac{2}{\pi}\int_0^{\pi/2}d\varphi\int_0^\infty e^{-ax}\cos(bx\sin\varphi)\,dx \\ &= \frac{2}{\pi}\int_0^{\pi/2}\frac{a\,d\varphi}{a^2+b^2\sin^2\varphi},\end{aligned}$$

where the absolute convergence of the double integral justifies reversing the order of integration. Evaluating the last integral, we have

$$\int_0^\infty e^{-ax}J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}, \qquad a>0, \quad b>0. \qquad (5.15.1)$$

Example . *Evaluate Weber's integral*

$$\int_0^\infty e^{-a^2x^2}J_\nu(bx)x^{\nu+1}\,dx, \qquad a>0, \quad b>0, \quad \operatorname{Re}\nu > -1.$$

Replacing $J_\nu(bx)$ by its series expansion and integrating term by term, we find that

$$\begin{aligned}\int_0^\infty e^{-a^2x^2}J_\nu(bx)x^{\nu+1}\,dx &= \int_0^\infty e^{-a^2x^2}x^{\nu+1}\,dx\sum_{k=0}^\infty\frac{(-1)^k(bx/2)^{\nu+2k}}{k!\Gamma(k+\nu+1)} \\ &= \sum_{k=0}^\infty\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{b}{2}\right)^{\nu+2k}\int_0^\infty e^{-a^2x^2}x^{2\nu+2k+1}\,dx \\ &= \sum_{k=0}^\infty\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{b}{2}\right)^{\nu+2k}\frac{1}{2a^{2\nu+2k+2}}\int_0^\infty e^{-t}t^{\nu+k}\,dt \\ &= \frac{b^\nu}{(2a^2)^{\nu+1}}\sum_{k=0}^\infty\frac{(-b^2/4a^2)^k}{k!},\end{aligned}$$

where reversing the order of integration and summation is again justified by an absolute convergence argument. Summing the last series, we have

$$\int_0^\infty e^{-a^2x^2}J_\nu(bx)x^{\nu+1}\,dx = \frac{b^\nu}{(2a^2)^{\nu+1}}e^{-b^2/4a^2}, \qquad a>0,\ b>0, \quad \operatorname{Re}\nu > -1.$$

H.W. Show that $\int_0^y \frac{x\sin(ax)}{\sqrt{y^2-x^2}}dx = \frac{\pi y}{2}J_1(ay)$

Answer: Use: $\sin\left[x\sin(\varphi)\right] = 2J_1(x)\sin(\varphi) + 2J_3(x)\sin(3\varphi) + \cdots$

$$\int_0^\pi \sin\left[x\sin(\varphi)\right]\sin(\varphi)d\varphi = 2J_1(x)\underbrace{\int_0^\pi \sin(\varphi)\sin(\varphi)d\varphi}_{\pi/2} + 2J_3(x)\underbrace{\int_0^\pi \sin(3\varphi)\sin(\varphi)d\varphi}_{0} + \underbrace{\cdots}_{0}$$

$$\pi J_1(x) = \int_0^\pi \underbrace{\sin\left[x\sin(\varphi)\right]\sin(\varphi)}_{F(\varphi)=F(\pi-\varphi)}d\varphi = 2\int_0^{\pi/2}\sin\left[x\sin(\varphi)\right]\sin(\varphi)d\varphi$$

$$\pi J_1(ay) = 2\int_0^{\pi/2} \sin\left[a\, y\, \sin(\varphi)\right]\sin(\varphi)\, d\varphi$$

Changing the variable $y\,\sin(\varphi) = x \quad \Rightarrow \quad dx = y\,\cos(\varphi)\,d\varphi$

$$d\varphi = \frac{dx}{y\,\cos(\varphi)} = \frac{dx}{\sqrt{y^2 - x^2}}$$

y0=BesselJ[0,z];y1=BesselJ[1,z];y2=BesselJ[2,z];y3=BesselJ[3,z];

```
Plot[{y0, y1, y2, y3}, {z, 0.01, 14}, Frame → True,
 PlotStyle → {GrayLevel[0], Dashing[{0.01}], Dashing[{0.03}], Dashing[{0.05}]},
 PlotLegend → {"ν=0", "ν=1", "ν=2", "ν=3"}, LegendPosition → {0.99, -0.2},
 FrameLabel → {r, Jι}]
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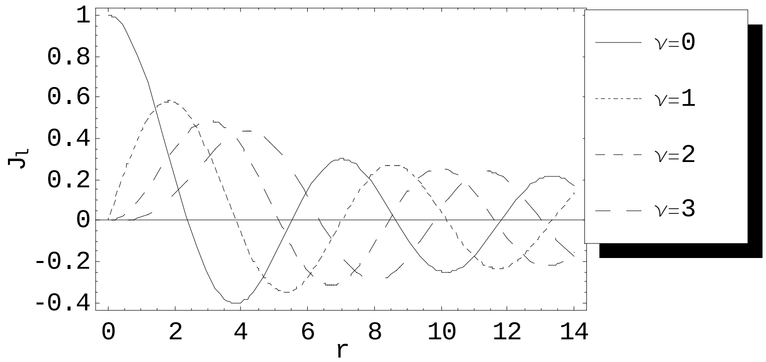

y0=BesselY[0,z];y1=BesselY[1,z];y2=BesselY[2,z];y3=BesselY[3,z];

```
Plot[{y0, y1, y2}, {z, 0.00, 12}, Frame → True,
 PlotStyle → {GrayLevel[0], Dashing[{0.01}], Dashing[{0.03}]},
 PlotLegend → {"ν=0", "ν=1", "ν=2"}, LegendPosition → {0.99, -0.2},
 FrameLabel → {r, ηι}]
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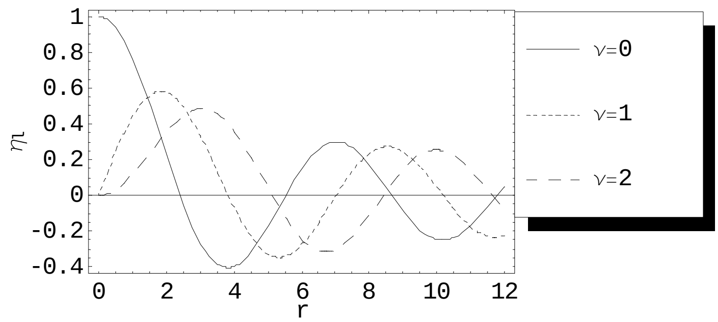


```
(* Find the roots    *)
<< NumericalMath`BesselZeros`
BesselJZeros[0,5]
{2.40483,5.52008,8.65373,11.7915,14.9309}
```