Green's function
Eigenfunctions Expansion
(Exercise 9.7.5, Jackson 3.9)

In this section we will obtain an expression for the Green's function of the inhomogeneous equation:

$$L(x)\,y(x) - \lambda\, y(x) = f(x) \tag{1}$$

in terms of the eigenfunctions $\phi_n(x)$ of the homogeneous equation:

$$L(x)\,\phi_n(x) = \lambda_n \phi_n(x)\,. \tag{2}$$

which have the same boundary conditions as $y(x)$.

To solve (1), we expand $y(x)$ and $f(x)$ in eigen functions of the operator L in (1), and one obtains:

$$y(x) = \sum_n c_n \phi_n(x), \tag{3a}$$

$$f(x) = \sum_n d_n \phi_n(x), \quad \Rightarrow \quad d_n = \langle \phi_n | f \rangle \tag{3b}$$

for some choice of the constants c_n and d_n. Substituting (3a, 3b) into Eq. (1) gives

$$\sum_n c_n \underbrace{L(x)\phi_n(x)}_{\lambda_n \phi_n(x)} - \lambda \sum_n c_n \phi_n(x) = \sum_n d_n \phi_n(x)$$

$$\Rightarrow \quad \sum_n c_n \left(\lambda_n - \lambda\right)\phi_n(x) = \sum_n d_n \phi_n(x) \tag{4}$$

To determine the coefficients c_n we multiply Eq. (4) by one of the eigenfunctions, $y_m(x)$ say, and integrate use the orthogonality of the eigenfunctions, this gives:

$$c_n = \frac{d_n}{\lambda_n - \lambda} = \frac{\langle \phi_n | f \rangle}{\lambda_n - \lambda} \qquad \left(\because d_n = \langle \phi_n | f \rangle\right)$$

Substituting for c_n into Eq. (3a) gives

$$y(x) = \sum_n \frac{\phi_n(x)\langle \phi_n | f \rangle}{\lambda_n - \lambda} = \sum_n \frac{\phi_n(x)}{\lambda_n - \lambda}\int_a^b \phi_n(x')\, f(x')\,dx'$$

$$= \int_a^b G(x, x', \lambda)\, f(x')\,dx'$$

where

$$G(x, x', \lambda) \equiv G(x, x') = \sum_n \frac{\phi_n(x')\phi_n(x)}{\lambda_n - \lambda} \tag{5}$$

Notice the symmetric property of the function: $G(x, x')$ in the argument x and x'. Furthermore, $G(x, x')$ only depends on the eigenfunctions of the corresponding homogeneous equation, i.e. on the boundary conditions and L. It is independent of $f(x)$ and so can be computed once and for all, and then applied to any $f(x)$ just by doing the integral:

$$y(x) = \int_{-\infty}^{\infty} G(x, x') f(x')\,dx'$$

Example: A string of length ℓ is vibrating with frequency ω. The equation and boundary conditions are:

$$\frac{d^2u(x)}{dx^2}+k^2u(x)=0, \qquad u(0)=u(\ell)=0, \quad k=\frac{\omega}{c}$$

Determine the Green's function.

Solution:

Start with the eigenvalue equation: $\frac{d^2\phi_n(x)}{dx^2}=\lambda_n\,\phi_n(x), \quad \lambda_n=-n^2$ and using the boundary conditions $\phi_n(0)=\phi_n(\ell)=0$, we have:

The eigenvalues $\lambda_n=-\left(\frac{n\pi}{\ell}\right), \qquad n=1,2,3,\cdots$

Normalized eigenfunction: $u_n=\sqrt{\frac{2}{\ell}}\sin\left(\frac{n\pi x}{\ell}\right)$

The Green's function

$$G(x,x')=\sum_n\frac{u_n(x')u_n(x)}{\lambda_n-\lambda}=\frac{2}{\ell}\sum_n\frac{\sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{n\pi x'}{\ell}\right)}{k^2-\left(\frac{n\pi}{\ell}\right)^2}$$

Example, Find the solution of the differential equation:

$$\frac{d^2y(x)}{dx^2}=f(x), \qquad f(x)=\sin(2x) \tag{A}$$

in the interval $0\le x\le\pi$, with the boundary conditions $y(0)=y(\pi)=0$.

Answer: Start with the eigenvalue equation: $\frac{d^2\phi_n(x)}{dx^2}=\lambda_n\,\phi_n(x), \quad \lambda_n=-n^2$ and using the boundary conditions $\phi_n(0)=\phi_n(\pi)=0$, we have:

$$\phi_n(x)=\sqrt{\frac{2}{\pi}}\sin\left(nx\right), \qquad n=1,2,3,\cdots \tag{B}$$

and

$$G(x,x')=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx')\sin(nx)}{\lambda_n}$$

Then:

$$y(x)=\int_0^{\pi}G(x,x')f(x')dx'=\frac{2}{\pi}\int_0^{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx')\sin(nx)}{-n^2}\sin(2x')dx'$$

$$=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{-n^2}\underbrace{\int_0^{\pi}\sin(nx')\sin(2x')dx'}_{\frac{\pi}{2}\delta_{n,2}}=\frac{2}{\pi}\frac{\pi}{2}\frac{\sin(2x)}{-2^2}=-\frac{1}{2}\sin(x)\cos(x)=-\frac{1}{4}\sin(2x)$$

This is because of the orthogonality of the $\sin(nx)$ in the interval from 0 to π only the $n=2$ term contributes and the integral for this case is $\pi/2$.

```
solution = DSolve[{y''[x] == Sin[2 x], y[0] == 0, y[π] == 0}, y[x], x]

{{y[x] → -1/2 Cos[x] Sin[x]}}
```

in agreement with Eq. (C).

Example, Find the solution of the differential equation $\dfrac{d^2y(x)}{dx^2}+\dfrac{1}{4}y(x)=\sin(2x)$

Answer:

$$y(x)=\int_0^{\pi} G(x,x')f(x')dx'=\frac{2}{\pi}\int_0^{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx')\sin(nx)}{\lambda_n}\sin(2x')dx', \qquad \lambda_n=\frac{1}{4}-n^2$$

$$=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{\frac{1}{4}-n^2}\underbrace{\int_0^{\pi}\sin(nx')\sin(2x')dx'}_{\frac{\pi}{2}\delta_{n,2}}=\frac{2}{\pi}\frac{\pi}{2}\frac{\sin(2x)}{\frac{1}{4}-2^2}=-\frac{8}{15}\sin(x)\cos(x)=-\frac{4}{15}\sin(2x)$$

Because of the orthogonality of the $\sin(nx)$ in the interval from 0 to π only the n = 2 term contributes, and the integral for this case is $\frac{\pi}{2}$.

```
In[3]:= solution =
         DSolve[{y''[x] + 1/4 y[x] == Sin[2 x], y[0] == 0, y[π] == 0},
           y[x], x]  // FullSimplify

Out[3]= {{y[x] → -8/15 Cos[x] Sin[x]}}
```

H. W. Do example 10.5.1 in Arfken.

H. W. Find the solution of the differential equation:

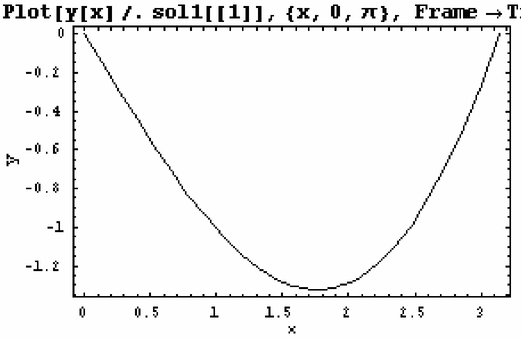
$$\frac{d^2y(x)}{dx^2}+\frac{1}{4}y(x)=f(x), \qquad f(x)=\frac{x}{2}$$

in the interval 0 ≤ x ≤ π, with the boundary conditions $y(0)=y(\pi)=0$.

Answer:

a- The differential equation recipe gives: $y(x)=2x-2\pi\sin(x/2)$

```
DE = y''[x] + 1/4 y[x] - x / 2 == 0
-x/2 + y[x]/4 + y''[x] == 0
sol1 = DSolve[{DE, y[0] == 0, y[π] == 0}, y, x]
{{y → Function[{x}, -2 (-x + π Sin[x/2])]}}
Plot[y[x] /. sol1[[1]], {x, 0, π}, Frame → True, FrameLabel → {x, y}]
```



b- Direct and Green function gives: $y(x) = 2x - 2\pi\sin(x/2)$

c- Eigenfunction gives: $y(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sin(nx)}{n(\frac{1}{4}-n^2)}$.

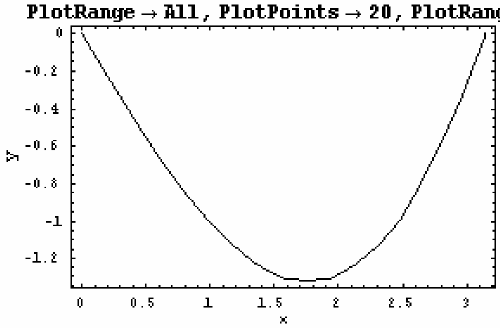
Note:

$$y(x) = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin nx}{\frac{1}{4}-n^2}\int_0^{\pi} x'\sin nx'\,dx'.$$

,

$$\int_0^{\pi} x'\sin nx'\,dx' = \left[-\frac{x'\cos nx'}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx'}{n}\,dx' = -\frac{\pi\cos n\pi}{n} + \left[\frac{\sin nx'}{n^2}\right]_0^{\pi} = -(-1)^n\frac{\pi}{n}.$$

```
f[x_, m_] := Sum[(-1)^(n+1)/(n (1/4 - n^2)) Sin[n x], {n, 1, m, 1}]
Do[Plot[f[x, m], {x, 0, π}, Frame → True, FrameLabel → {x, y},
  PlotRange → All, PlotPoints → 20, PlotRange → All], {m, 30, 30, 5}]
```



Example: For the D.E.

$y''(x) + y(x) = f(x)$ such that $y(0) = y(\pi/2) = 0$,

calculate $G(x, x')$ and solve for $y(x)$. Plot both functions.

Answer:

$$G(x,x')=\begin{cases} A(x')\sin(x) & (0\le x<x') \\ B(x')\cos(x) & (x'<x\le \pi/2) \end{cases}$$

$$\frac{dG(x,x')}{dx}=\begin{cases} A(x')\cos(x) & (0\le x<x') \\ -B(x')\sin(x) & (x'<x\le \pi/2) \end{cases}$$

$$G(x,x')=\begin{cases} -\cos(x')\sin(x) & (0\le x<x') \\ -\sin(x')\cos(x) & (x'<x\le \pi/2) \end{cases}$$

$$y(x)=-\cos(x)\int_0^x \sin(x')f(x')dx'-\sin(x)\int_x^{\pi/2}\cos(x')f(x')dx'$$

For the functions:

1- $\csc(x) \Rightarrow y(x)=-x\cos(x)+\sin(x)\ln[\sin(x)]$ -x Cos[x]+Log[Sin[x]] Sin[x]

2- $\sin(2x) \Rightarrow y(x)=$ $-\frac{2}{3}$ Cos[x]^3 Sin[x] $-\frac{2}{3}$ Cos[x] Sin[x]^3

3- $\sec(x) \Rightarrow y(x)=$ Cos[x] Log[Cos[x]] $-\frac{1}{2}$ (π - 2 x) Sin[x]

4- x $\Rightarrow y(x)=$ -Cos[x] (-x Cos[x] + Sin[x]) $-\frac{1}{2}$ Sin[x] (π - 2 Cos[x] - 2 x Sin[x])

Note that:

1- Using the Green's function may seem to be a complicated way to proceed, especially for certain choices of $f(x)$. However, you should realize that the "elementary" derivation of the solution may not be so simple in other cases, and you should note that the Green's function applies to all possible choices of the function on the RHS, $f(x)$. Furthermore, we will see in the next section that one can often get a closed form expression for G, rather than an infinite series. It is then much easier to find a closed form expression for the solution.

2- The orthogonal function expansion method can easily be extended to two and three dimensions. For example if $\{u_n(x)\};\{v_n(x)\}$ and $\{w_n(x)\}$ denote the complete functions in the x, y, and z directions respectively, then the three dimensional Green's function can be written as:

$$G(\vec{x},\vec{x}')=-4\pi\sum_{n,m,l}\frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{\ell\pi z}{c}\right)\sin\left(\frac{\ell\pi z'}{c}\right)}{\lambda_{n,m,\ell}}$$

Where $\lambda_{n,m,\ell}=-\{\alpha_n^2+\beta_m^2+\gamma_\ell^2\}=-\left\{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2+\left(\frac{\ell\pi}{c}\right)^2\right\}$

$$\frac{d^2}{dx^2}u_l(x)=-\alpha_l u_l(x),\quad \frac{d^2}{dy^2}v_m(x)=-\beta_m v_m(y),\quad \text{and}\quad \frac{d^2}{dz^2}w_n(z)=-\gamma_n w_n(z).$$

See Eq. 3.167 in Jackson for an example. In Jackson, he used

$$\frac{\partial^2}{\partial x^2}G(x,x')=-4\pi\delta(x-x').$$ and $$\frac{d^2}{dx^2}u_n(x)=-\alpha_n u_n(x),$$

Example: A hollow, grounded, conducting cube with sides of length *a* has a charge density

$$\rho = \rho_o \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)\sin\left(\frac{\pi z}{a}\right)$$

placed within its interior ($0 < x < a$, $0 < y < a$, $0 < z < a$).

a. Using the method of eigenfunction expansion, show that the Green's function

$$G(\vec{x},\vec{x}\,') = \left(\frac{32}{\pi a}\right)\sum_{n,m,l}\frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sin\left(\frac{m\pi y'}{a}\right)\sin\left(\frac{\ell\pi z}{a}\right)\sin\left(\frac{\ell\pi z'}{a}\right)}{n^2+m^2+\ell^2}$$

b. Find the electrostatic potential $V(x,y,z)$ inside the cube.

Solution:

a- The eigenfunction is: $\Psi_{n,m,\ell}(\vec{x}) = \sqrt{\frac{8}{abc}}\sin(\alpha_n x)\sin(\beta_m y)\sin(\gamma_\ell z)$

And the eigenvalues are: $\lambda_{n,m,\ell} = -\left\{\alpha_n^2+\beta_m^2+\gamma_\ell^2\right\} = -\left\{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2+\left(\frac{\ell\pi}{c}\right)^2\right\}$

$$G(\vec{x},\vec{x}\,') = -4\pi\sum_{n,m,l}\frac{\Psi_{n,m,\ell}(\vec{x})\Psi_{n,m,\ell}(\vec{x}\,')}{\lambda_{n,m,\ell}}$$

Here, $a=b=c\equiv a$, then $\lambda_{n,m,\ell} = -\left\{\alpha_n^2+\beta_m^2+\gamma_\ell^2\right\} = -\left(\frac{\pi}{a}\right)^2\left\{n^2+m^2+\ell^2\right\}$, and

$$G(\vec{x},\vec{x}\,') = -4\pi\left(\frac{a}{\pi}\right)^2\left(\frac{8}{a^3}\right)\sum_{n,m,l}\frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sin\left(\frac{m\pi y'}{a}\right)\sin\left(\frac{\ell\pi z}{a}\right)\sin\left(\frac{\ell\pi z'}{a}\right)}{-\left(n^2+m^2+\ell^2\right)}$$

b- Inside the cube:

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_o}\int_V \rho(\vec{x}\,')G(\vec{x},\vec{x}\,')d\tau'$$

$$V(x,y,z) = \frac{32\rho_o}{4\pi^2 a\varepsilon_o}\int_0^a\int_0^a\int_0^a dx'dy'dz'\sin\left(\frac{\pi x'}{a}\right)\sin\left(\frac{\pi y'}{a}\right)\sin\left(\frac{\pi z'}{a}\right)$$

$$\times\sum_{n,m,l}\frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sin\left(\frac{m\pi y'}{a}\right)\sin\left(\frac{\ell\pi z}{a}\right)\sin\left(\frac{\ell\pi z'}{a}\right)}{\left(n^2+m^2+\ell^2\right)}$$

From orthogonality, the only non-vanishing integrals are for $n=m=\ell=1$

$$V(x,y,z) = \frac{32\rho_o}{4\pi^2 a\varepsilon_o}\left(\frac{1}{3}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sin\left(\frac{\ell\pi z}{a}\right)\underbrace{\int_0^a\int_0^a\int_0^a dx'dy'dz'\sin^2\left(\frac{\pi x'}{a}\right)\sin^2\left(\frac{\pi y'}{a}\right)\sin^2\left(\frac{\pi z'}{a}\right)}_{(a/2)^3}$$

$$= \frac{1}{3}\frac{a^2}{\pi^2}\frac{\rho_o}{\varepsilon_o}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sin\left(\frac{\ell\pi z}{a}\right)$$