

Chapter 29

Magnetic Fields Due to Currents

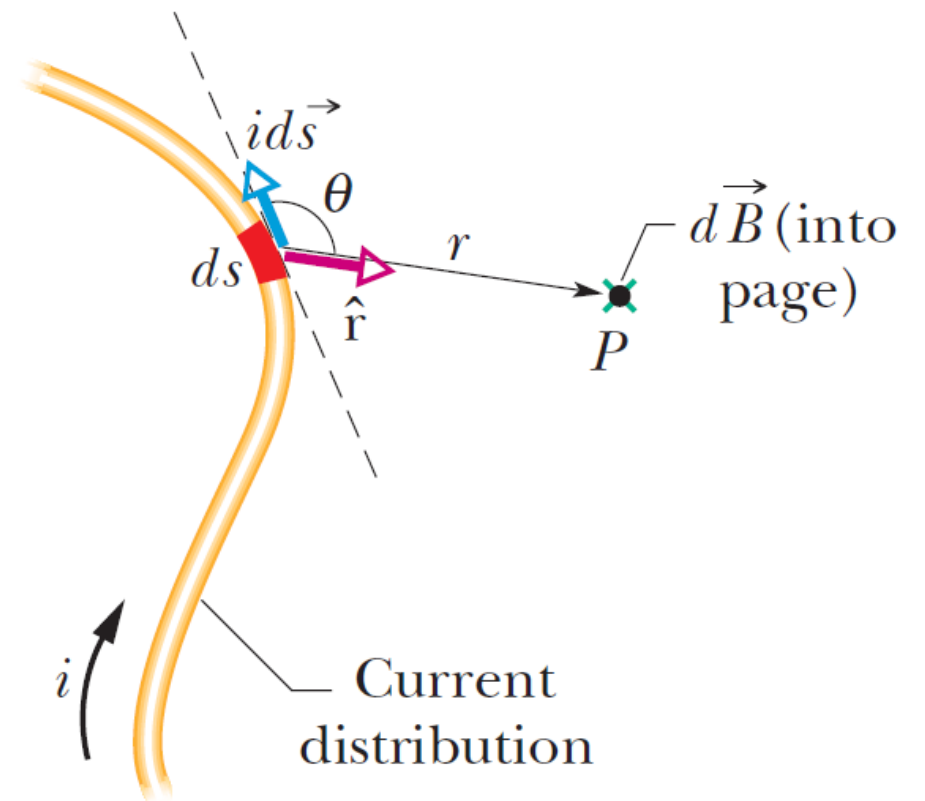
1. Calculating the Mag. Field Due to a Current

A moving charged particle produces a magnetic field around itself. Consequently, a current of moving charged particles produces a magnetic field around the current.

The figure shows a wire of arbitrary shape, carrying a current i . The magnitude of the magnetic field $d\vec{B}$ produced at point P at distance r by the current-length element $i d\vec{s}$ is

$$dB = \frac{\mu_0}{4\pi} \frac{id\vec{s} \sin \theta}{r^2},$$

where θ is the angle between the directions of $d\vec{s}$ and \hat{r} , a unit vector that points from ds toward P .



1. Calculating the Mag. Field Due to a Current

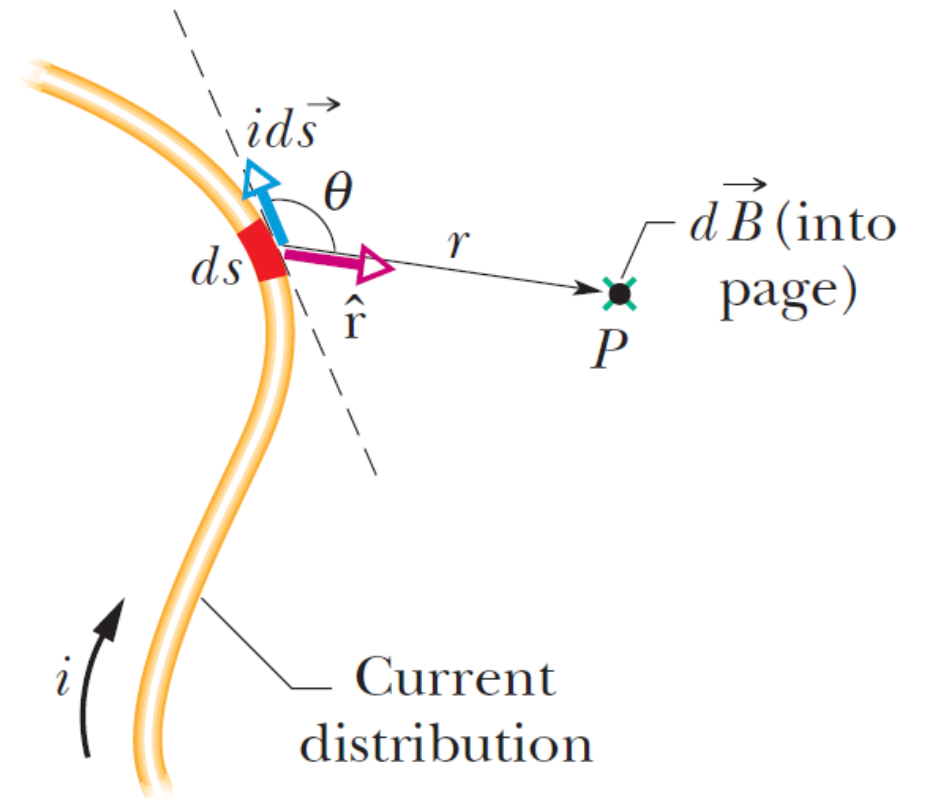
$$dB = \frac{\mu_0}{4\pi} \frac{id\vec{s} \sin \theta}{r^2}.$$

The constant μ_0 is called the **permeability constant**, whose value is defined to be

$$\mu_0 = 4\pi \times 10^{-7} \text{ T} \cdot \text{m/A}.$$

The direction of $d\vec{B}$ is that of the cross product $d\vec{s} \times \hat{r}$. We therefore rewrite the equation above in vector form as

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i d\vec{s} \times \hat{r}}{r^2}.$$

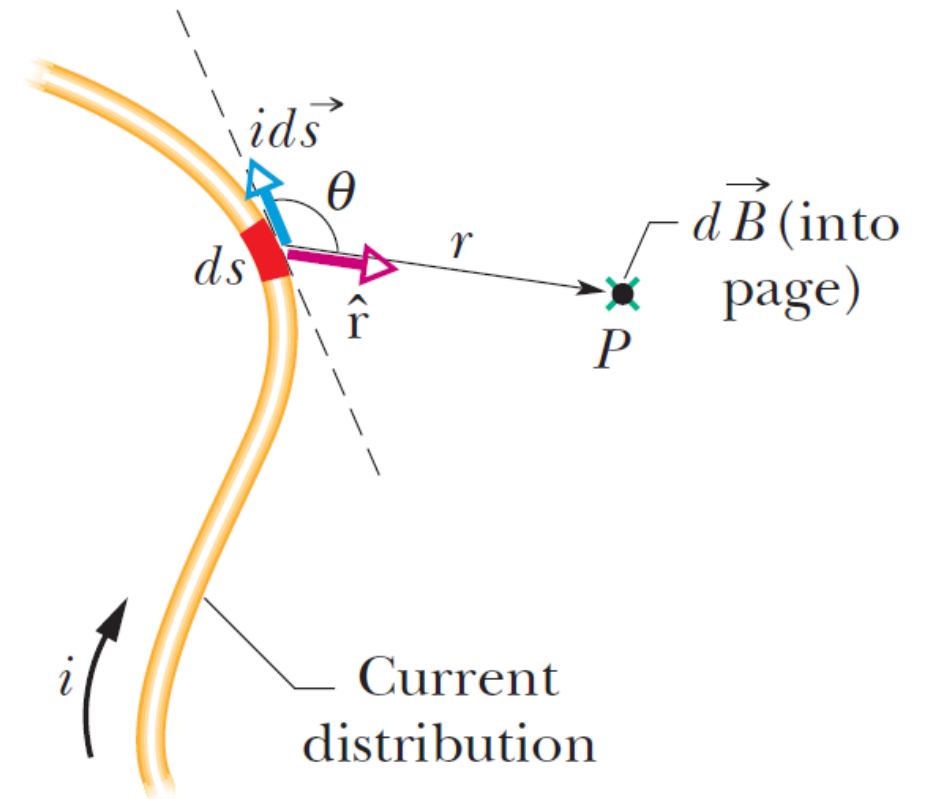


1. Calculating the Mag. Field Due to a Current

$$d\vec{B} = \frac{\mu_0 i}{4\pi} \frac{d\vec{s} \times \hat{r}}{r^2}.$$

This equation is known as **Biot-Savart law**.

We can use this law to calculate the net magnetic field produced at a point by various current distributions.



1. Calculating the Mag. Field Due to a Current

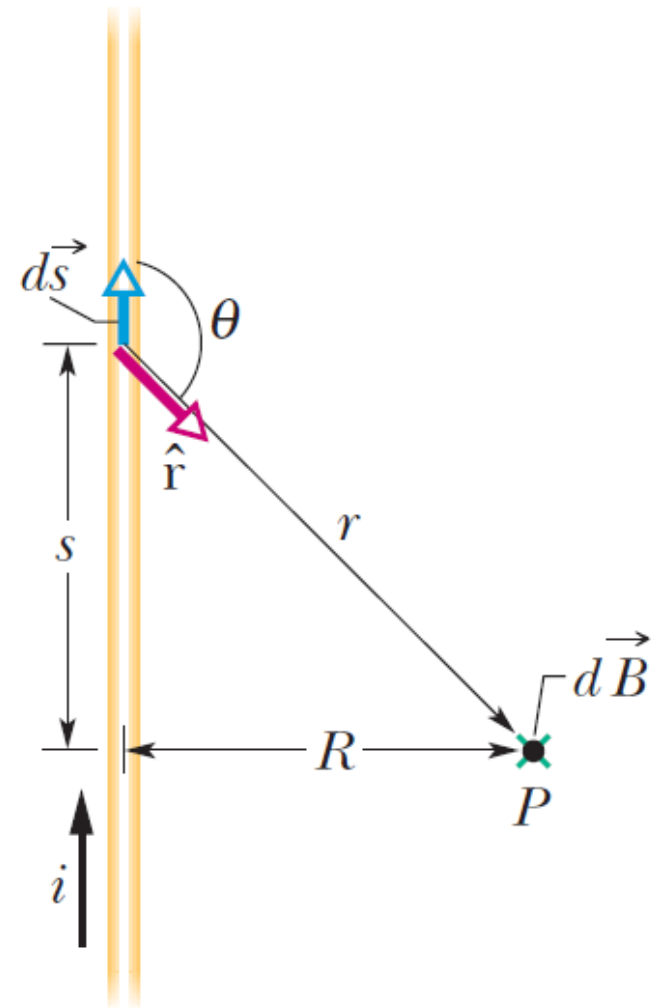
Magnetic Field Due to a Current in a Large Straight Wire

We can use Biot-Savart law to prove that the magnitude of the magnetic field at a perpendicular distance R from the end of a long, **semi-infinite** straight wire carrying a current i is given by

$$B = \frac{\mu_0 i}{4\pi R} \quad (\text{semi-infinite straight wire}).$$

The magnetic field due to an infinite wire is then

$$B = \frac{\mu_0 i}{2\pi R} \quad (\text{infinite straight wire}).$$



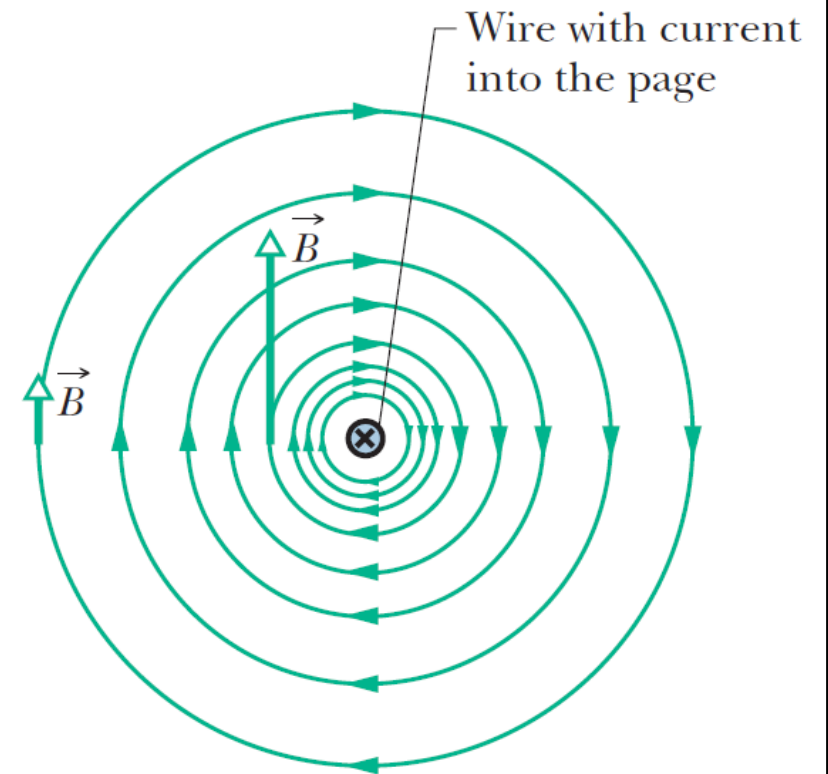
1. Calculating the Mag. Field Due to a Current

Magnetic Field Due to a Current in Large Straight Wire

The field lines of \vec{B} form concentric circles around the wire, as shown in the figure.

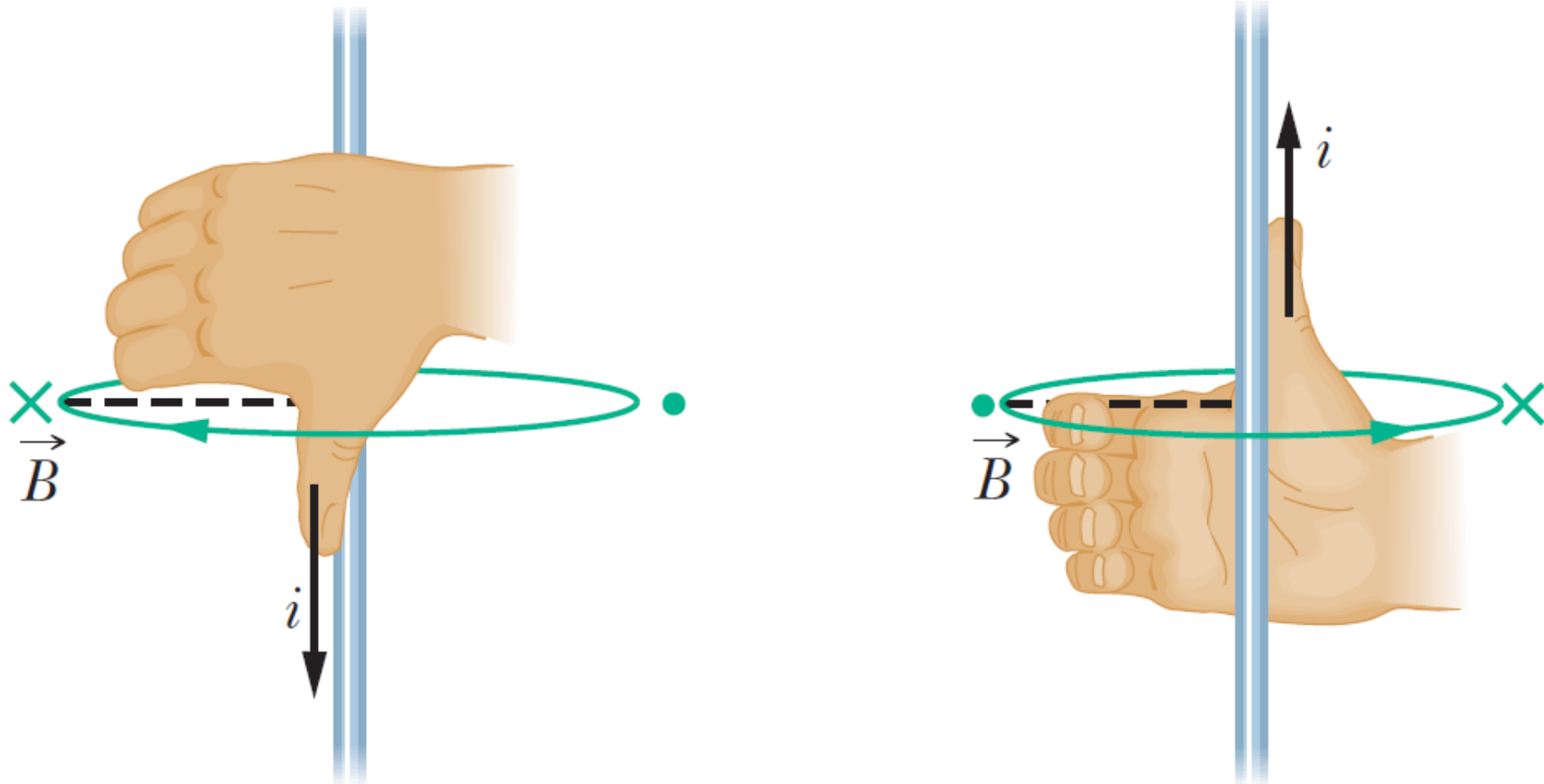
To find the direction of the magnetic field set up by a current-carrying element, we use the following right-hand rule:

Curled–straight right-hand rule: Grasp the element in your right hand with your extended thumb pointing in the direction of the current. Your fingers will then naturally curl around in the direction of the magnetic field lines due to that element.



1. Calculating the Mag. Field Due to a Current

Magnetic Field Due to a Current in Large Straight Wire



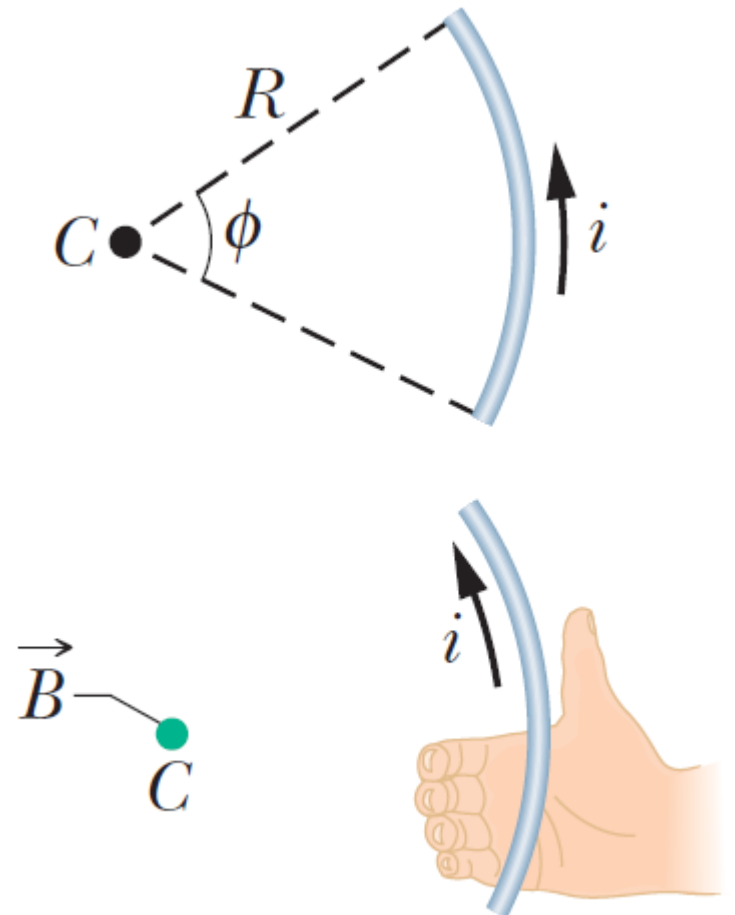
1. Calculating the Mag. Field Due to a Current

Magnetic Field Due to a Current in a Circular Arc of Wire

We can also use Biot-Savart law to show that the magnitude of the magnetic field at the center of a circular arc of radius R and central angle ϕ (in radians), carrying current i , is

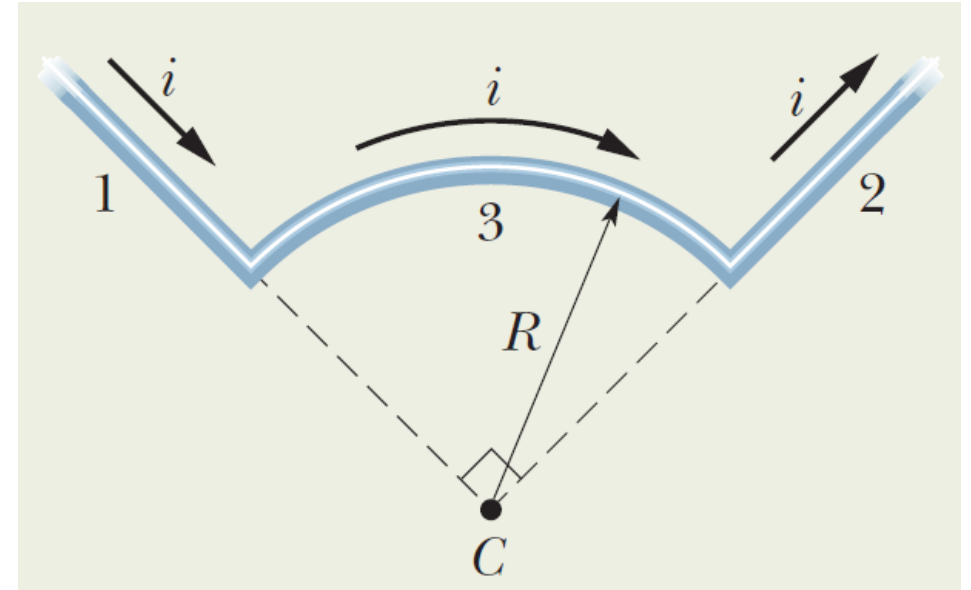
$$B = \frac{\mu_0 i \phi}{4\pi R}.$$

The direction of the field at the center of the circular arc can be found using the right-hand rule.



1. Calculating the Mag. Field Due to a Current

Example 1: The wire in the figure carries a current i and consists of a circular arc of radius R and central angle $\pi/2$ rad, and two straight sections whose extensions intersect the center C of the arc. What magnetic field \vec{B} (magnitude and direction) does the current produce at C ?



1. Calculating the Mag. Field Due to a Current

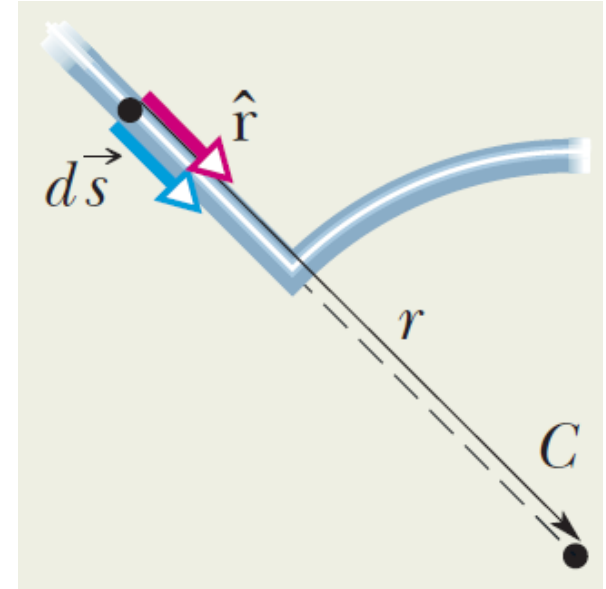
For segment 1, the angle θ between $d\vec{s}$ and \vec{r} is zero. Therefore,

$$dB_1 = \frac{\mu_0 i ds \sin 0}{4\pi r^2} = 0.$$

Thus, $B_1 = 0$. Similarly, $B_2 = 0$, because the angle θ between $d\vec{s}$ and \vec{r} , for segment 2, is 180° .

Segment 3 is a circular arc of angle $\phi = \frac{\pi}{2}$ rad. Thus,

$$B_3 = \frac{\mu_0 i \phi}{4\pi R} = \frac{\mu_0 i (\pi/2)}{4\pi R} = \frac{\mu_0 i}{8R}.$$

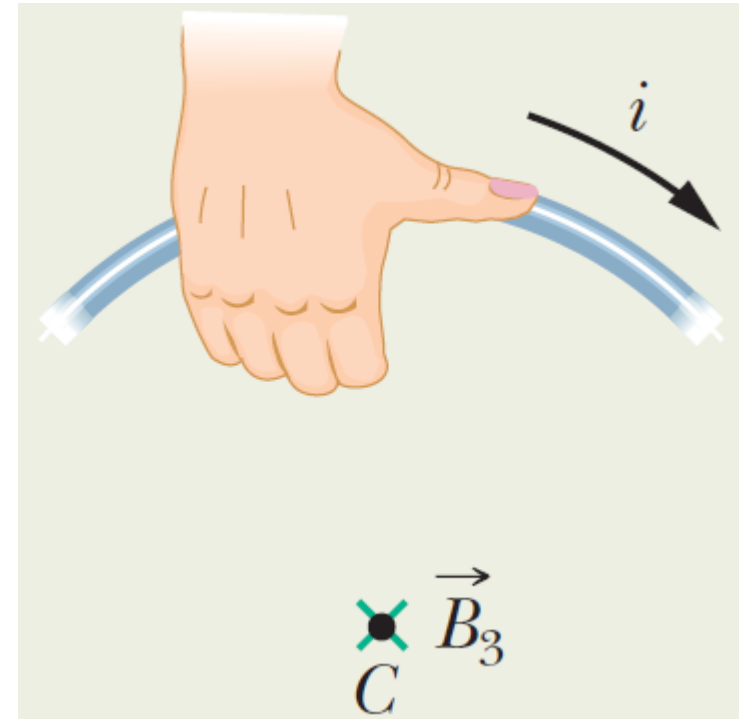


1. Calculating the Mag. Field Due to a Current

The magnitude of the magnetic field at point C is

$$B = B_1 + B_2 + B_3 = \frac{\mu_0 i}{8R}.$$

The direction of \vec{B} is into the plane.



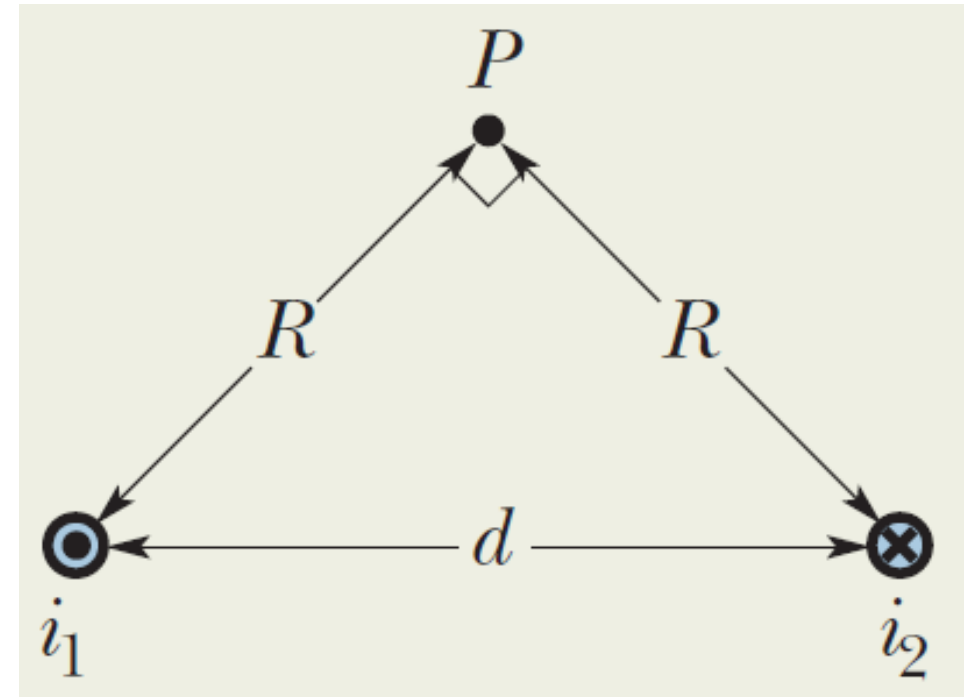
1. Calculating the Mag. Field Due to a Current

Example 2: The figure shows two long parallel wires carrying currents i_1 and i_2 in opposite directions. What are the magnitude and direction of the net magnetic field at point P ? Assume the following values: $i_1 = 15$ A, $i_2 = 32$ A, and $d = 5.3$ cm.

The currents generate magnetic fields \vec{B}_1 and \vec{B}_2 at point P with magnitudes

$$B_1 = \frac{\mu_0 i_1}{2\pi R} \quad \text{and} \quad B_2 = \frac{\mu_0 i_2}{2\pi R}.$$

We can replace R with $\sqrt{2}(d/2) = d/\sqrt{2}$.

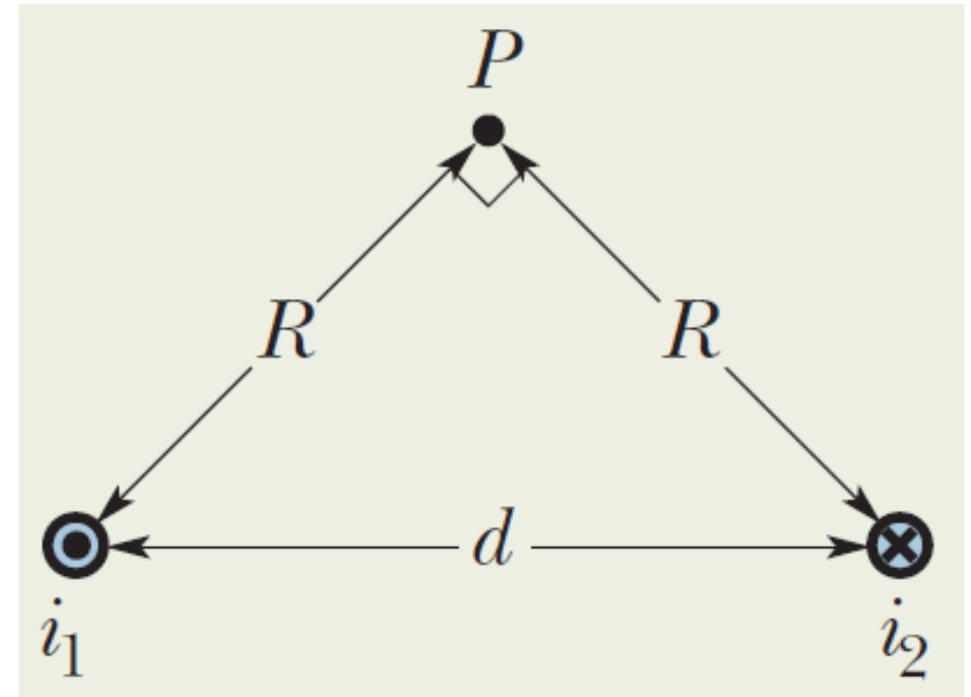


1. Calculating the Mag. Field Due to a Current

With this replacement, B_1 and B_2 become

$$B_1 = \frac{\sqrt{2}\mu_0 i_1}{2\pi d} \quad \text{and} \quad B_2 = \frac{\sqrt{2}\mu_0 i_2}{2\pi d}.$$

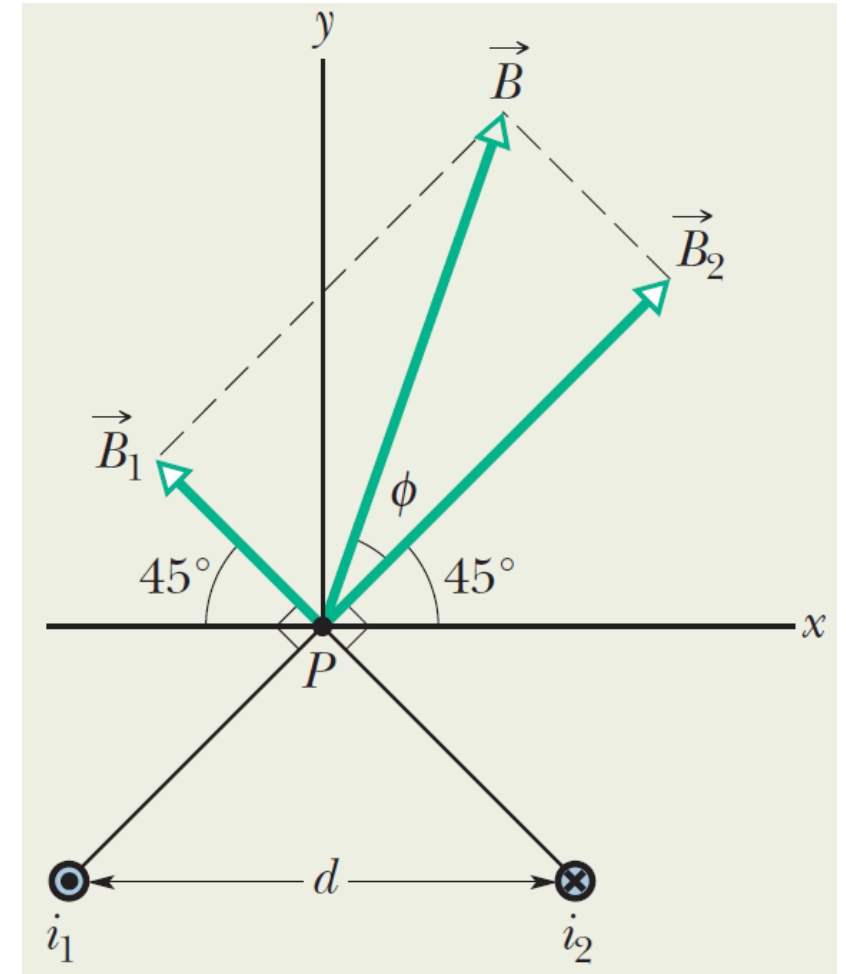
We need to add \vec{B}_1 and \vec{B}_2 vectorially to find the resultant magnetic field at P .



1. Calculating the Mag. Field Due to a Current

The directions of \vec{B}_1 and \vec{B}_2 at point P are shown in the figure. The magnitude of the resultant field is

$$\begin{aligned} B &= \sqrt{B_1^2 + B_2^2} = \sqrt{\left(\frac{\sqrt{2}\mu_0 i_1}{2\pi d}\right)^2 + \left(\frac{\sqrt{2}\mu_0 i_2}{2\pi d}\right)^2} \\ &= \frac{\sqrt{2}\mu_0}{2\pi d} \sqrt{i_1^2 + i_2^2} \\ &= \frac{\sqrt{2}(4\pi \times 10^{-7} \text{T} \cdot \text{m/A})}{2\pi(0.053 \text{ m})} \sqrt{(15\text{A})^2 + (32\text{A})^2} \\ &= 190 \mu\text{T}. \end{aligned}$$



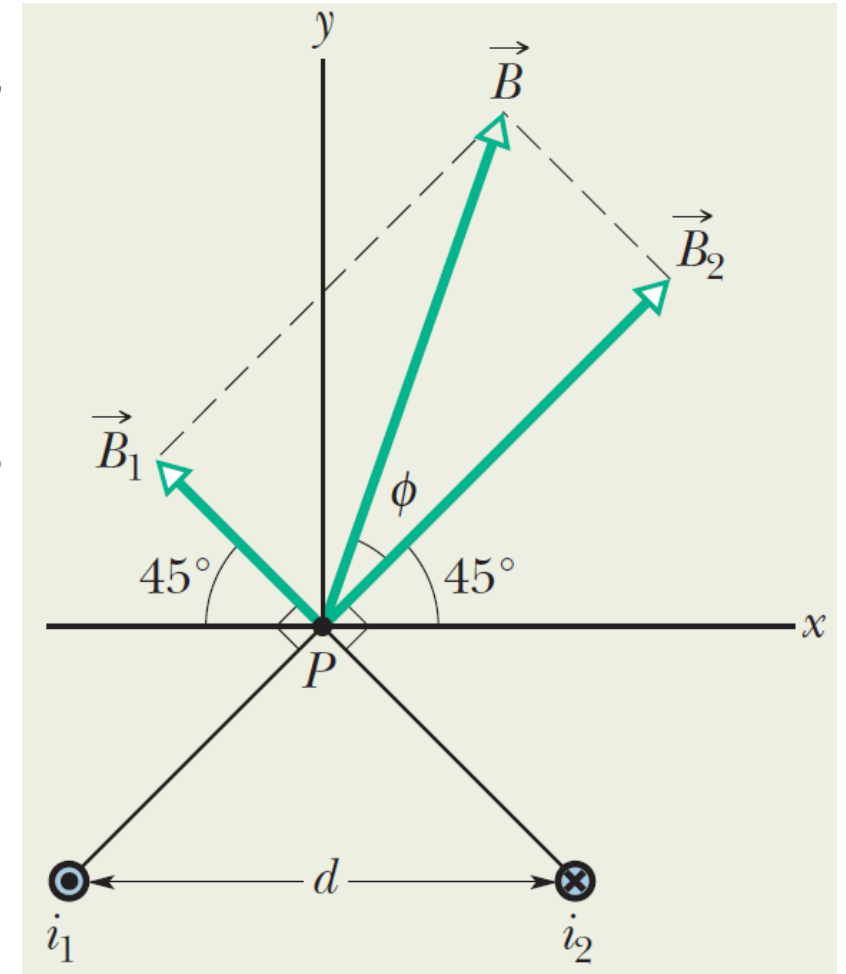
1. Calculating the Mag. Field Due to a Current

The angle ϕ between the directions of \vec{B} and \vec{B}_2 is given by

$$\phi = \tan^{-1} \frac{B_1}{B_2} = \tan^{-1} \frac{i_1}{i_2} = \tan^{-1} \frac{15 \text{ A}}{32 \text{ A}} = 25^\circ.$$

The angle between the directions of \vec{B} and the x-axis is then

$$\phi + 45^\circ = 70^\circ.$$



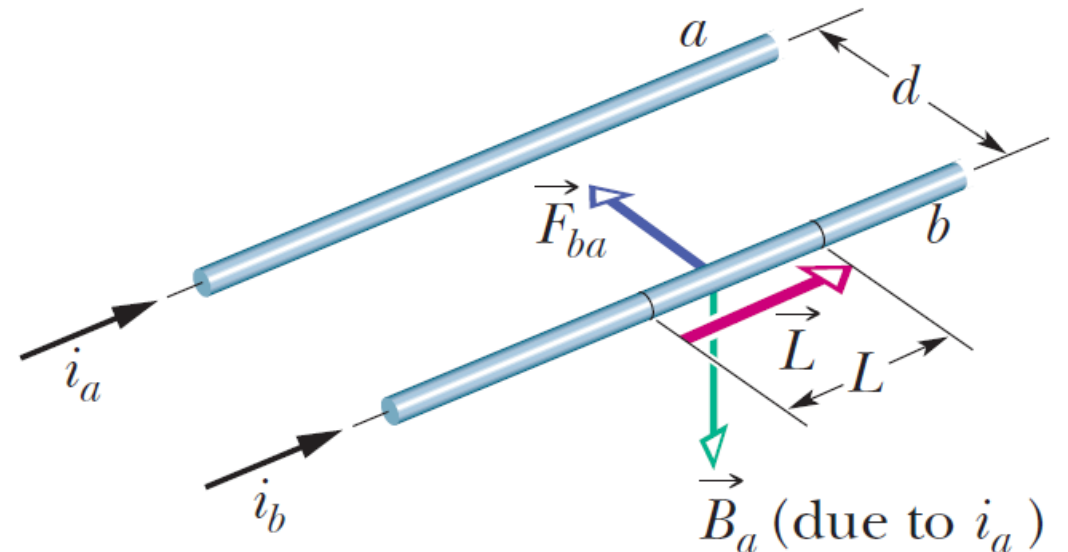
2. Force Between Two Parallel Current

The figure shows two long wires, separated by a distance d and carrying currents i_a and i_b . We want to study the forces on these wires due to each other.

Let us first inspect the force on wire b due to the current in wire a . The magnetic field \vec{B}_a is responsible for the force on wire b . The magnitude of \vec{B}_a at every point on wire b is

$$B_a = \frac{\mu_0 i_a}{2\pi d}.$$

\vec{B}_a is directed downward.



2. Force Between Two Parallel Current

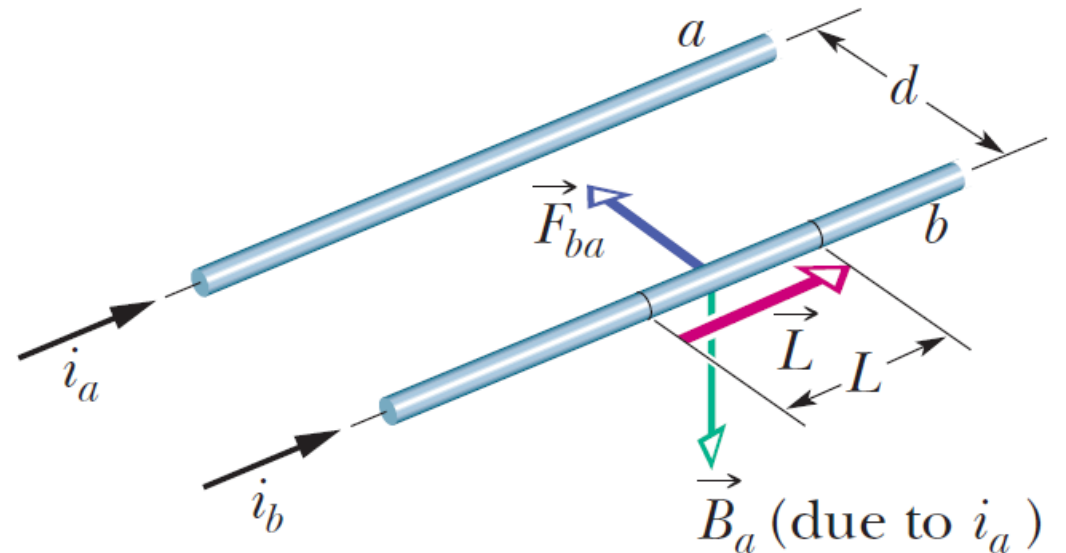
The force \vec{F}_{ba} on a length L of wire b due to the external magnetic field \vec{B}_a is

$$\vec{F}_{ba} = i_b \vec{L} \times \vec{B}_a.$$

where \vec{L} is the length vector of wire b . Because \vec{L} and \vec{B}_a are perpendicular to each other, we can write

$$F_{ba} = i_b L B_a \sin 90^\circ = \frac{\mu_0 L i_a i_b}{2\pi d}.$$

The direction of \vec{F}_{ba} is the direction of the cross product $\vec{L} \times \vec{B}_a$, which is toward wire a .

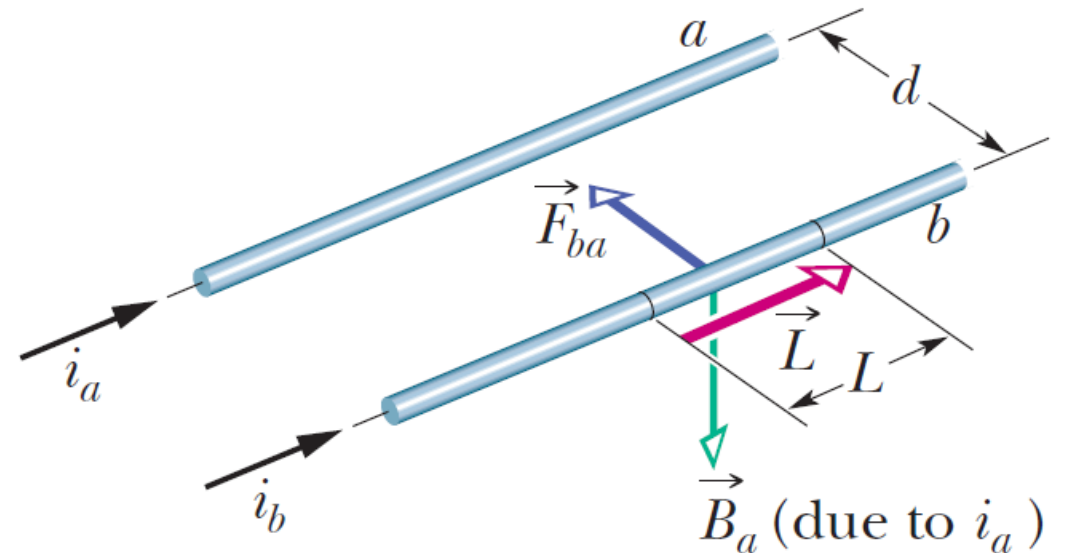


2. Force Between Two Parallel Current

The general procedure for finding the force on a current-carrying wire is this:

To find the force on a current-carrying wire due to a second current-carrying wire, first find the field due to the second wire at the site of the first wire. Then find the force on the first wire due to that field.

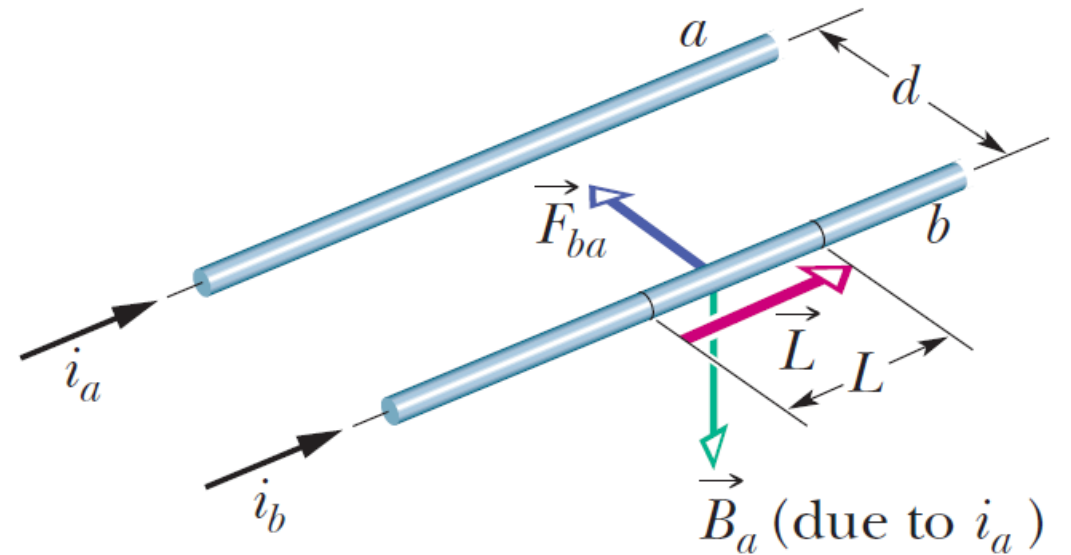
Using similar analysis, we can show that the force on wire a due to the current in wire b is toward wire b . Therefore, the two wires with parallel currents attract each other.



2. Force Between Two Parallel Current

Similarly, if the two currents were antiparallel, we could show that the two wires repel each other.

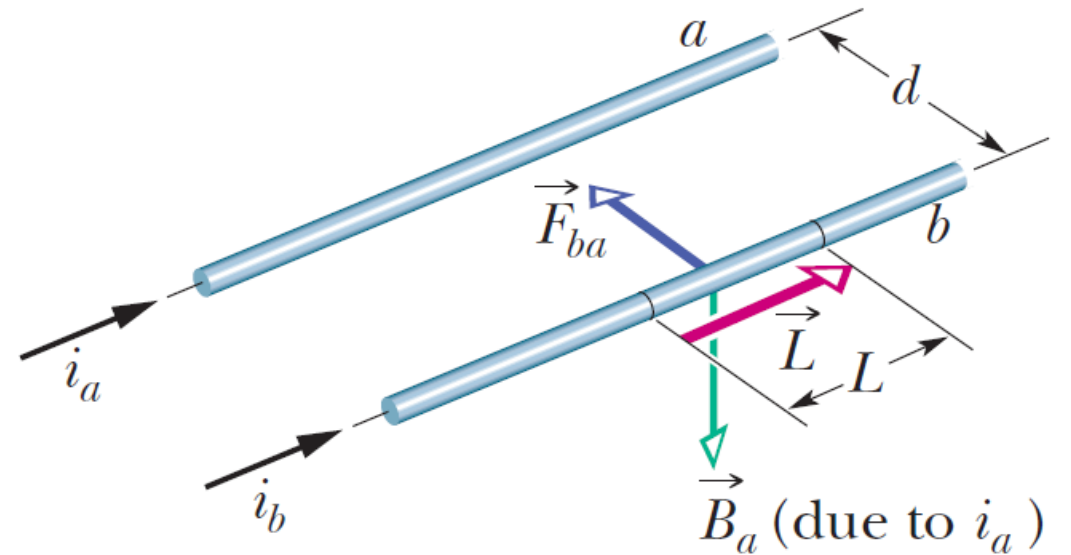
Thus, parallel currents attract each other, and antiparallel currents repel each other.



2. Force Between Two Parallel Current

The force acting between currents in parallel wires is the basis for the definition of the ampere.

The ampere is that constant current which, if maintained in two straight, parallel conductors of infinite length, of negligible circular cross section, and placed 1 m apart in vacuum, would produce on each of these conductors a force of magnitude 2×10^{-7} newton per meter of wire length.

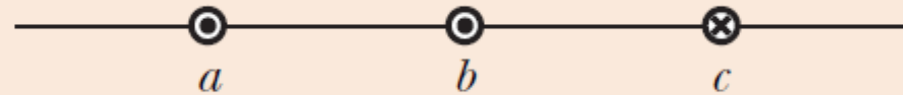


2. Force Between Two Parallel Current



CHECKPOINT 1

The figure here shows three long, straight, parallel, equally spaced wires with identical currents either into or out of the page. Rank the wires according to the magnitude of the force on each due to the currents in the other two wires, greatest first.



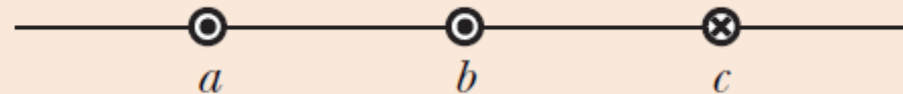
$$F_{12} = \frac{\mu_0 L i_1 i_2}{2\pi d}$$

2. Force Between Two Parallel Current



CHECKPOINT 1

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b, c, a

$$F_{12} = \frac{\mu_0 L i_1 i_2}{2\pi d}$$

3. Ampere's Law

Ampere's law reads

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}.$$

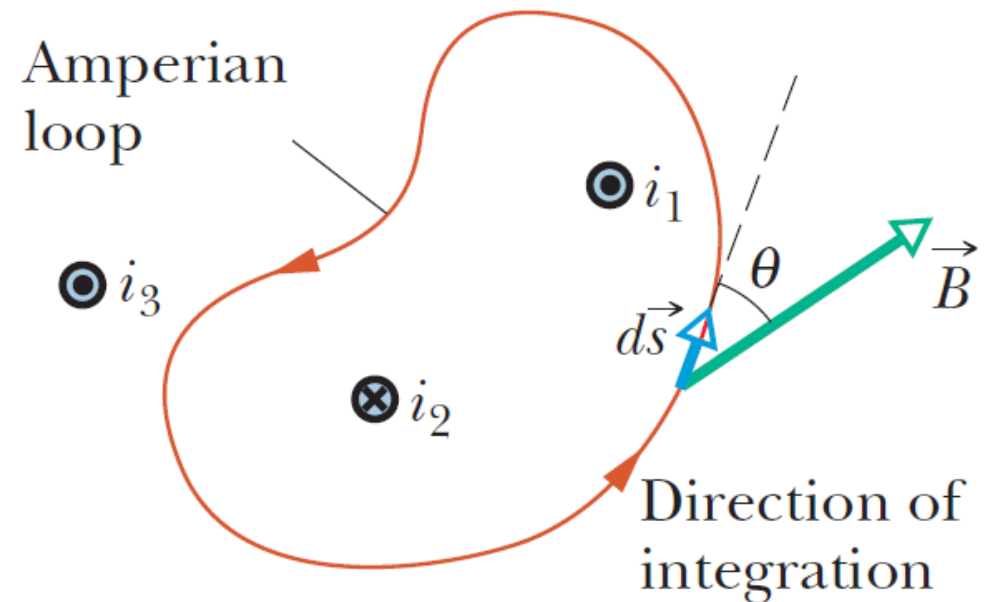
The loop on the integral sign means that the dot product $\vec{B} \cdot d\vec{s}$ is to be integrated around a closed loop, called an Amperian loop.

The current i_{enc} is the *net* current encircled by that closed loop.

3. Ampere's Law

To better understand the meaning of the scalar product $\vec{B} \cdot d\vec{s}$ and its integral, let us apply Ampere's law to the general situation of the figure. The figure shows cross sections of three straight wires and an arbitrary Amperian loop. The counterclockwise direction of integration shown on the loop is chosen arbitrarily.

We start by dividing the loop into differential vector elements $d\vec{s}$ that are everywhere directed along the tangent to the loop in the direction of integration.



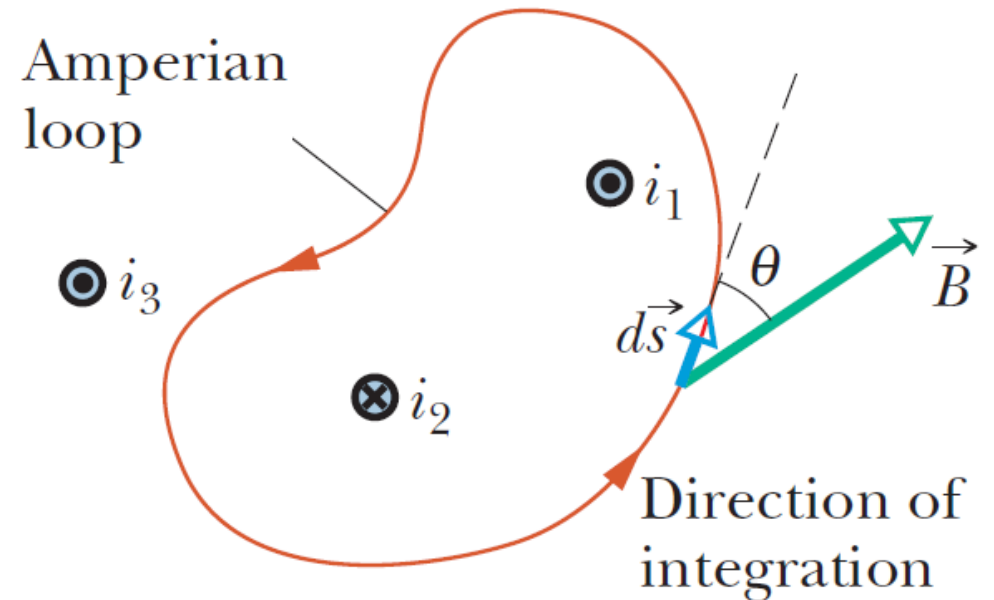
3. Ampere's Law

The magnetic field at $d\vec{s}$ due to each current is in the plane of the figure. Thus, their net magnetic field \vec{B} at $d\vec{s}$ must also be in that plane.

Ampere's law can be written as

$$\oint \vec{B} \cdot d\vec{s} = \oint B \cos \theta ds = \mu_0 i_{\text{enc.}}$$

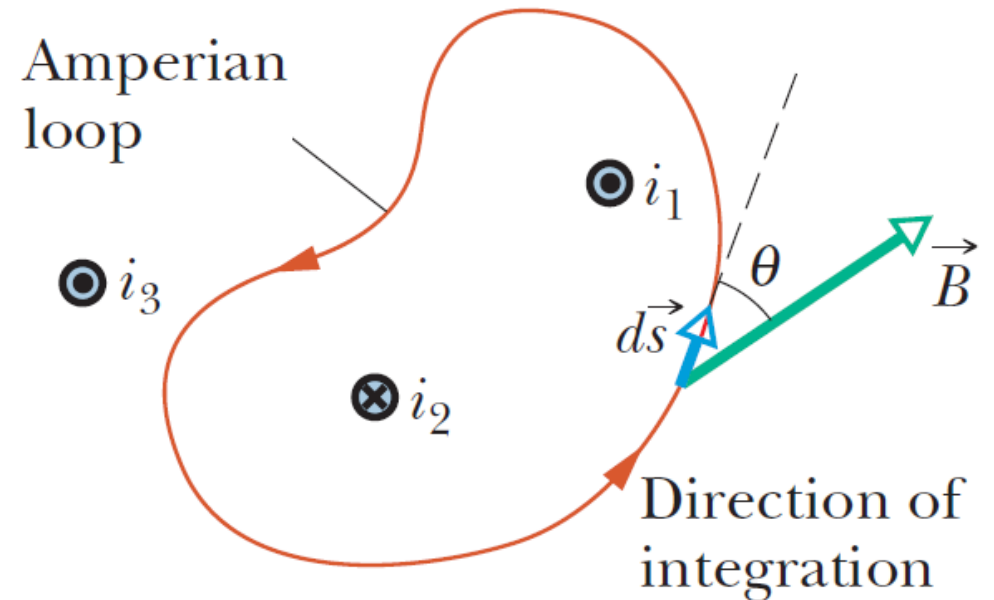
We can interpret the dot product $\vec{B} \cdot d\vec{s}$ as being the product of a length ds of the Amperian loop and the field component $B \cos \theta$ tangent to the loop.



3. Ampere's Law

Then, we can interpret the integral as being the summation of all such products around the entire loop.

We do not need to know the direction of \vec{B} to perform the integration. Instead, we arbitrarily assume \vec{B} to be in the direction of integration. We then use the following right-hand rule to assign a plus sign or a minus sign to each of the currents that make up the net encircled current i_{enc} .

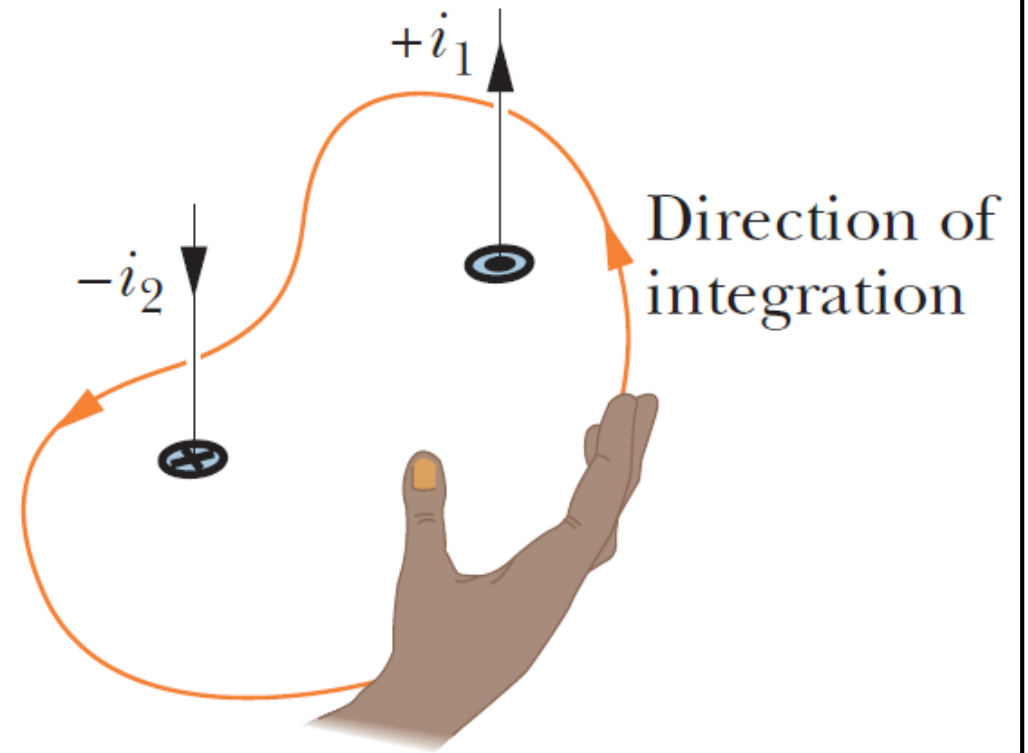


3. Ampere's Law

Curl your right hand around the Amperian loop, with the fingers pointing in the direction of integration. A current through the loop in the general direction of your outstretched thumb is assigned a plus sign, and a current generally in the opposite direction is assigned a minus sign.

With the indicated counterclockwise direction of integration, the net current encircled by the loop is

$$i_{\text{enc}} = i_1 - i_2.$$

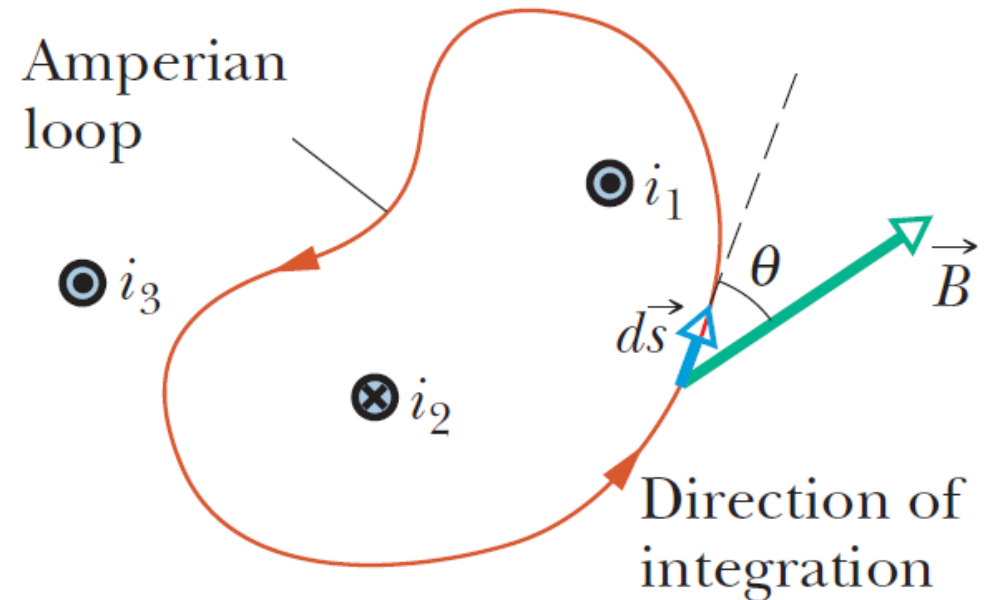


3. Ampere's Law

We then can write

$$\oint B \cos \theta ds = \mu_0(i_1 - i_2).$$

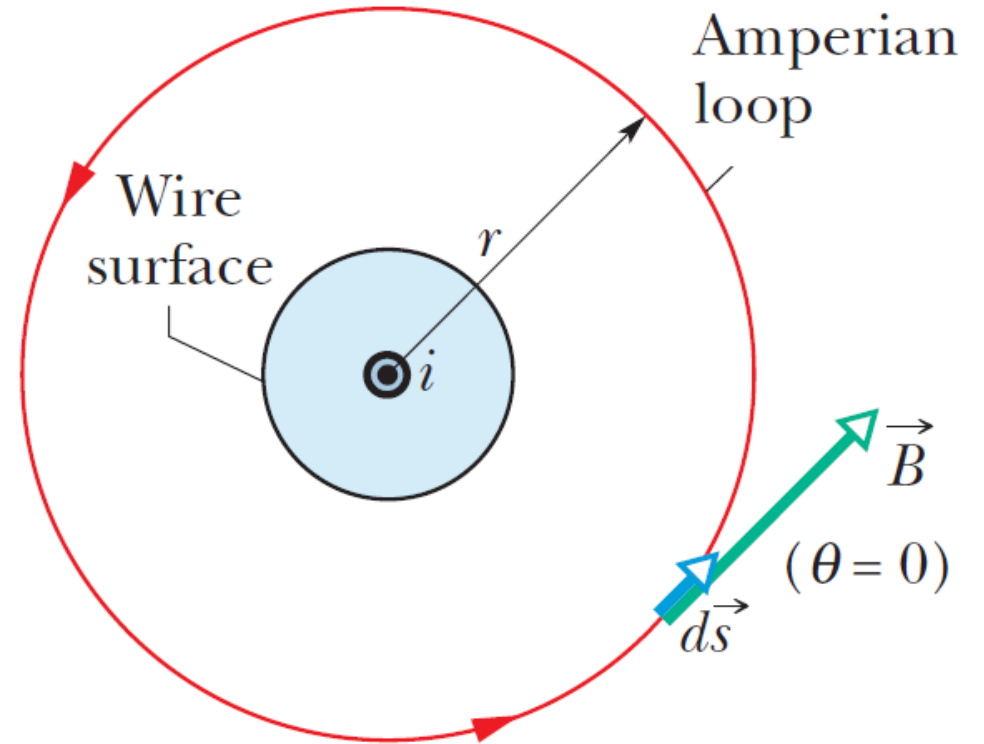
We will next apply Ampere's law to two situations in which symmetry allows us to evaluate the integral and find the magnetic field.



3. Ampere's Law

Magnetic Field Outside a Long Straight Wire with Current

The figure shows a long straight wire that carries current i out of the page. The magnetic field \vec{B} has a cylindrical symmetry about the wire. Thus, we encircle the wire with a concentric circular Amperian loop of radius r . The magnetic field \vec{B} then has the same magnitude B at every point on the loop. We integrate counterclockwise, so that $d\vec{s}$ has the direction shown in the figure. Note that \vec{B} is tangent to the loop at every point along the loop, as is $d\vec{s}$.

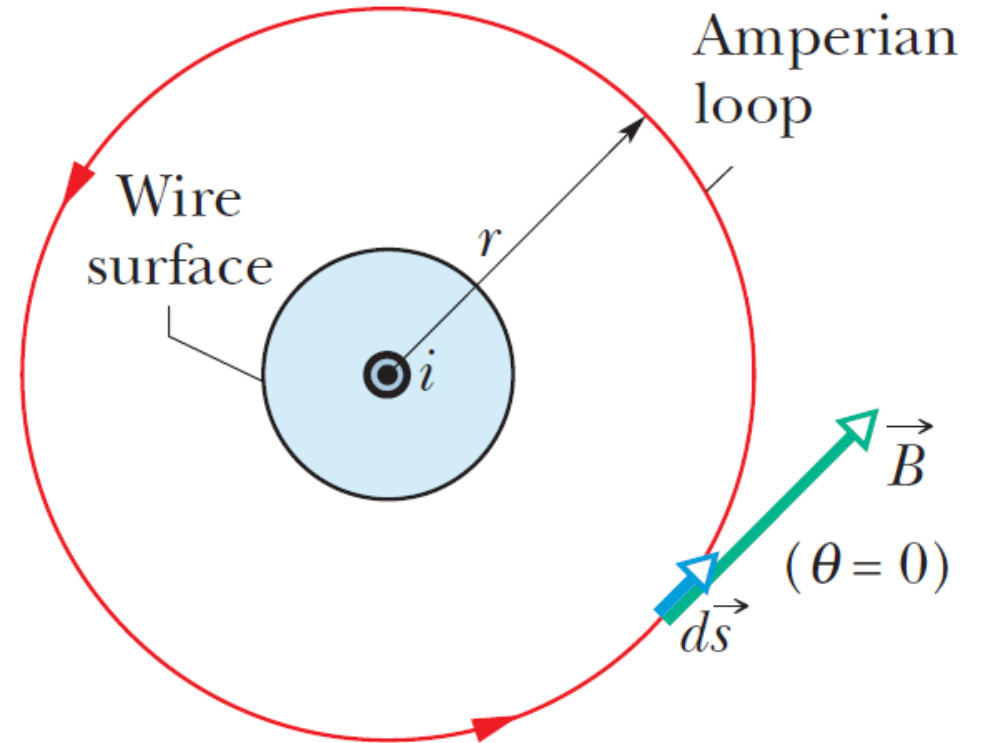


3. Ampere's Law

Magnetic Field Outside a Long Straight Wire with Current

Thus, \vec{B} and $d\vec{s}$ are either parallel or antiparallel along the loop. We will assume that \vec{B} and $d\vec{s}$ are parallel. The angle θ between them is then 0. The integration becomes

$$\oint \vec{B} \cdot d\vec{s} = \oint B \cos \theta ds = B \oint ds = B(2\pi r).$$



3. Ampere's Law

Magnetic Field Outside a Long Straight Wire with Current

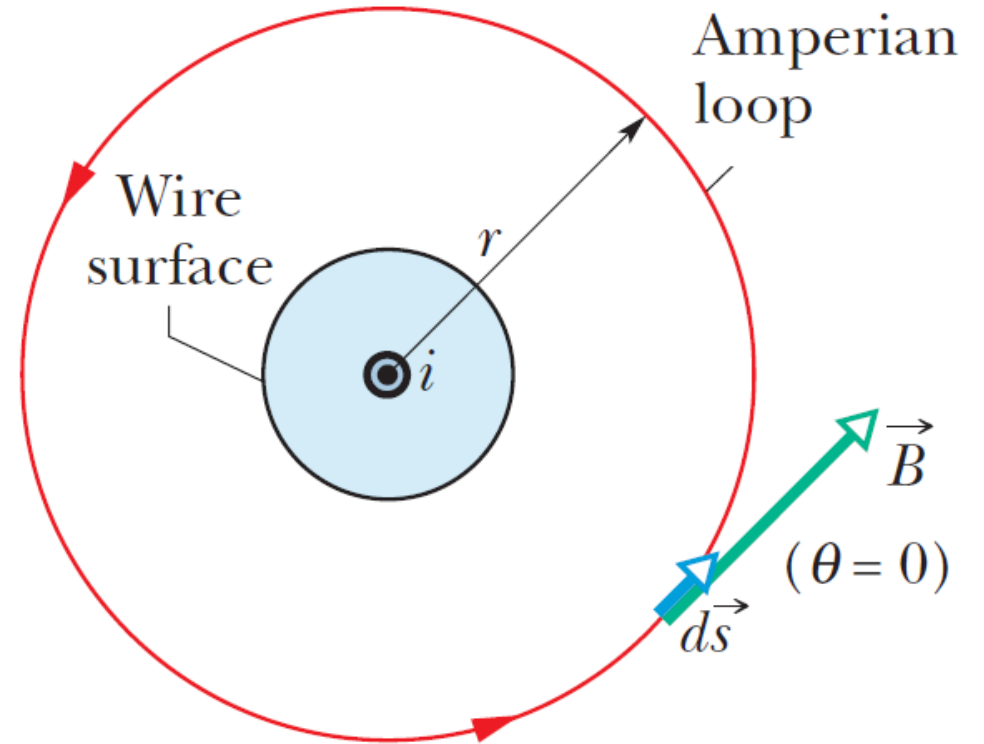
The right-hand rule gives us a plus sign for the current.

Ampere's law becomes

$$B(2\pi r) = \mu_0 i.$$

or

$$B = \frac{\mu_0 i}{2\pi r}.$$



3. Ampere's Law

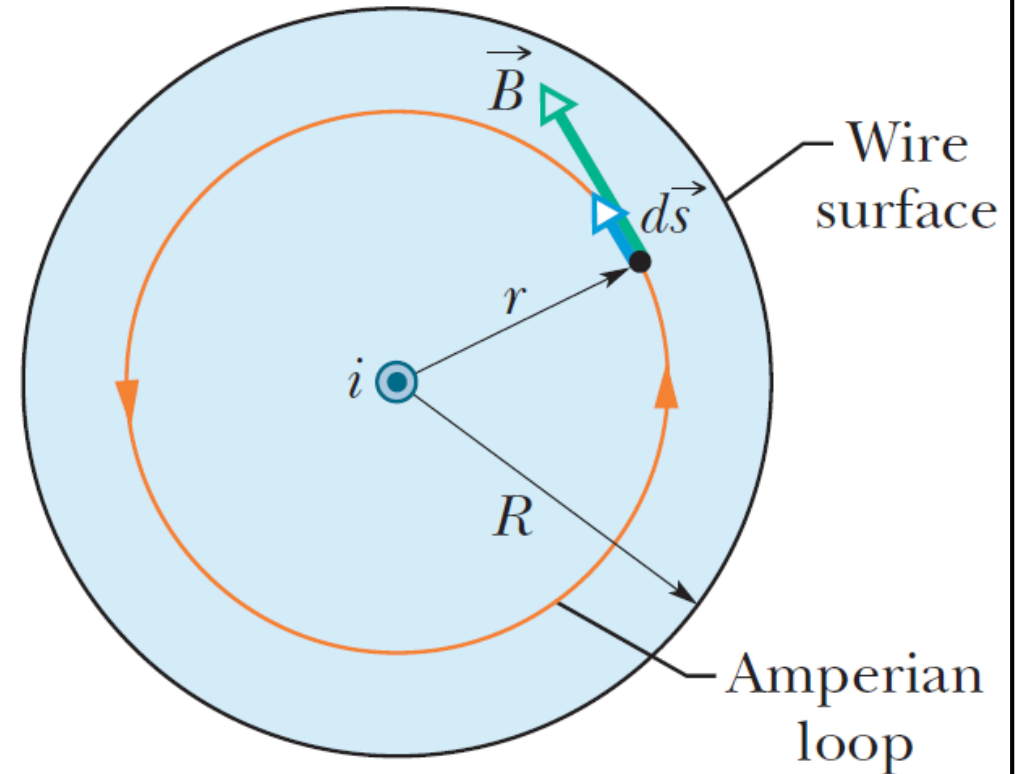
Magnetic Field Inside a Long Straight Wire with Current

To find the magnetic field at points inside the wire shown, we can again use an Amperian loop of radius r , where now $r > R$. As in the previous subsection, the left side of Ampere's law gives

$$\oint \vec{B} \cdot d\vec{s} = B(2\pi r).$$

The current encircled by the loop is

$$i_{\text{enc}} = JA_{\text{loop}} = \frac{i}{A_{\text{wire}}} A_{\text{loop}} = i \frac{\pi r^2}{\pi R^2} = i \frac{r^2}{R^2}.$$



3. Ampere's Law

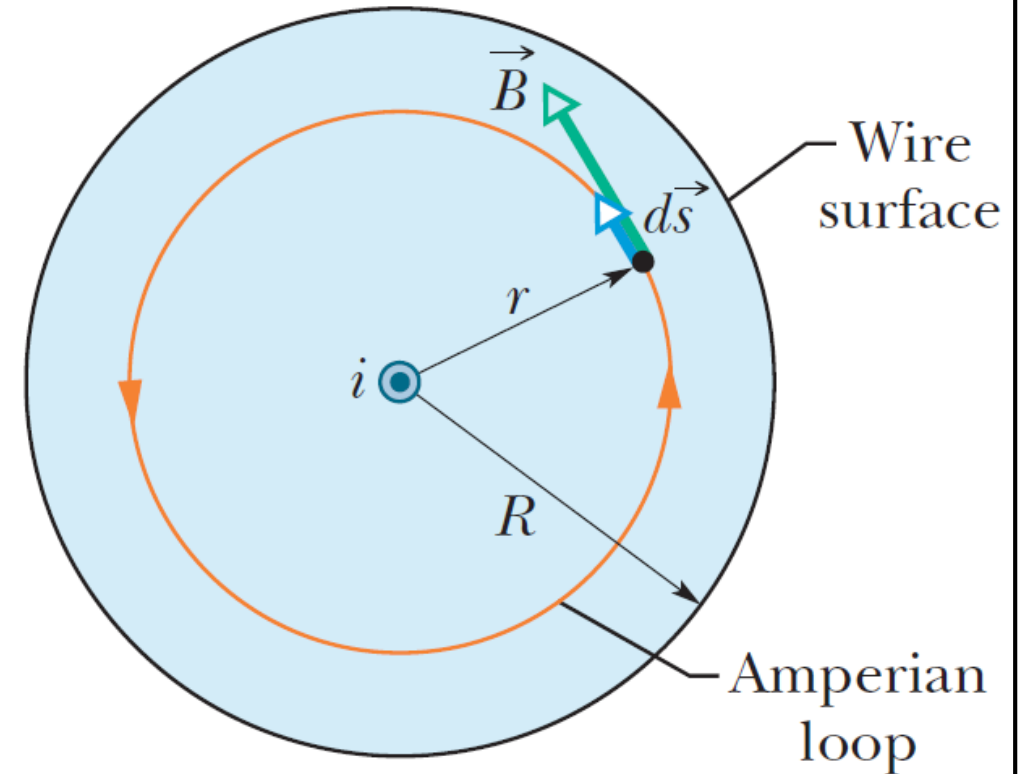
Magnetic Field Inside a Long Straight Wire with Current

Ampere's law becomes

$$B(2\pi r) = \mu_0 i \frac{r^2}{R^2},$$

or

$$B = \frac{\mu_0 i}{2\pi R^2} r.$$

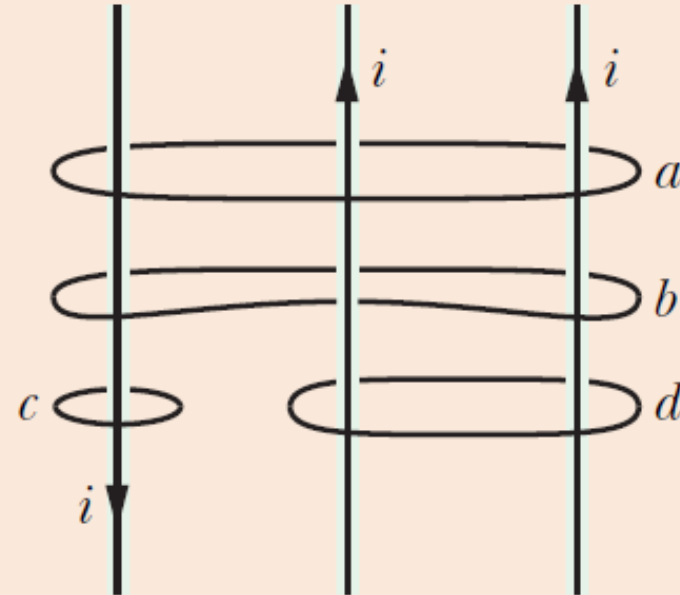


3. Ampere's Law



CHECKPOINT 2

The figure here shows three equal currents i (two parallel and one antiparallel) and four Amperian loops. Rank the loops according to the magnitude of $\oint \vec{B} \cdot d\vec{s}$ along each, greatest first.



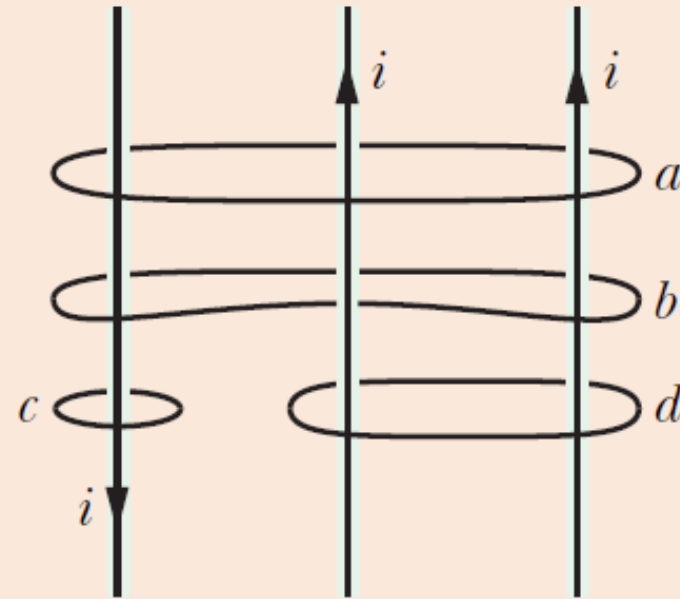
$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}$$

3. Ampere's Law



CHECKPOINT 2

The figure here shows three equal currents i (two parallel and one antiparallel) and four Amperian loops. Rank the loops according to the magnitude of $\oint \vec{B} \cdot d\vec{s}$ along each, greatest first.

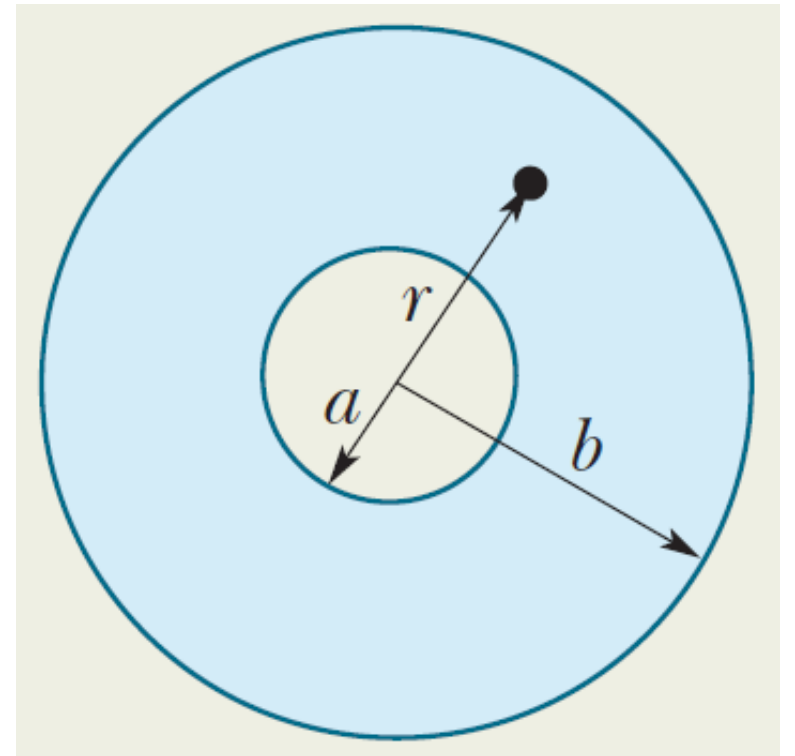


$d, a \ \& \ c$ tie, b

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}$$

3. Ampere's Law

Example 3: The figure shows the cross section of a long conducting cylinder with inner radius $a = 2.0$ cm and outer radius $b = 4.0$ cm. The cylinder carries a current out of the page, and the magnitude of the current density in the cross section is given by $J = cr^2$, with $c = 3.0 \times 10^6$ A/m⁴ and r in meters. What is the magnetic field at the dot in the figure, which is at radius $r = 3.0$ cm from the central axis of the cylinder?



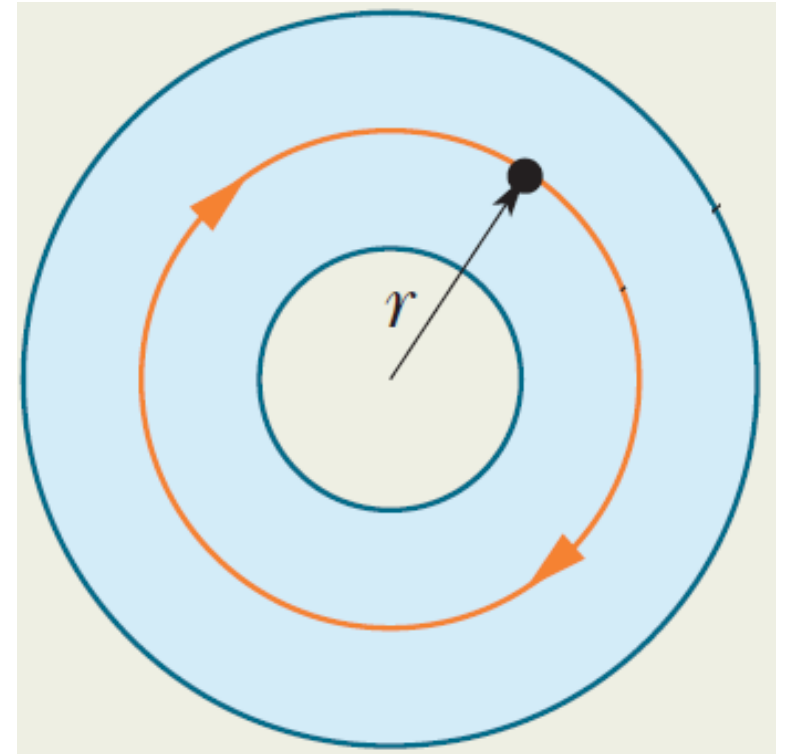
3. Ampere's Law

We first draw the Amperian loop shown in the figure. The left side of Ampere's law is then

$$\oint \vec{B} \cdot d\vec{s} = B(2\pi r).$$

The encircled current is

$$\begin{aligned} i_{\text{enc}} &= \int J da = \int_a^r cr^2(2\pi r dr) = 2\pi c \int_a^r r^3 dr. \\ &= \frac{\pi c(r^4 - a^4)}{2}. \end{aligned}$$



3. Ampere's Law

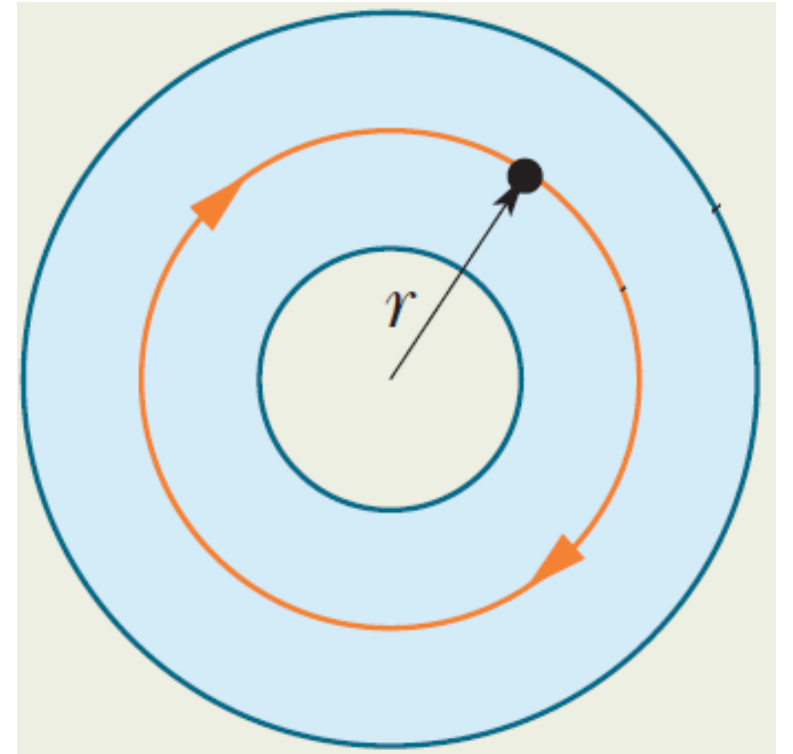
We should take i_{enc} is negative, according to the right-hand rule.

Thus, Ampere's law gives us that

$$B(2\pi r) = -\mu_0 \frac{\pi c(r^4 - a^4)}{2}.$$

Solving for B and substituting give

$$\begin{aligned} B &= -\frac{\mu_0 c}{4r} (r^4 - a^4) \\ &= -\frac{\left(4\pi \times 10^{-7} \text{ T} \cdot \frac{\text{m}}{\text{A}}\right) (3.0 \times 10^6 \text{ A/m}^4)}{4(0.003 \text{ m})} \\ &\quad \times [(0.003 \text{ m})^4 - (0.002 \text{ m})^4] \\ &= -2.0 \times 10^{-5} \text{ T}. \end{aligned}$$

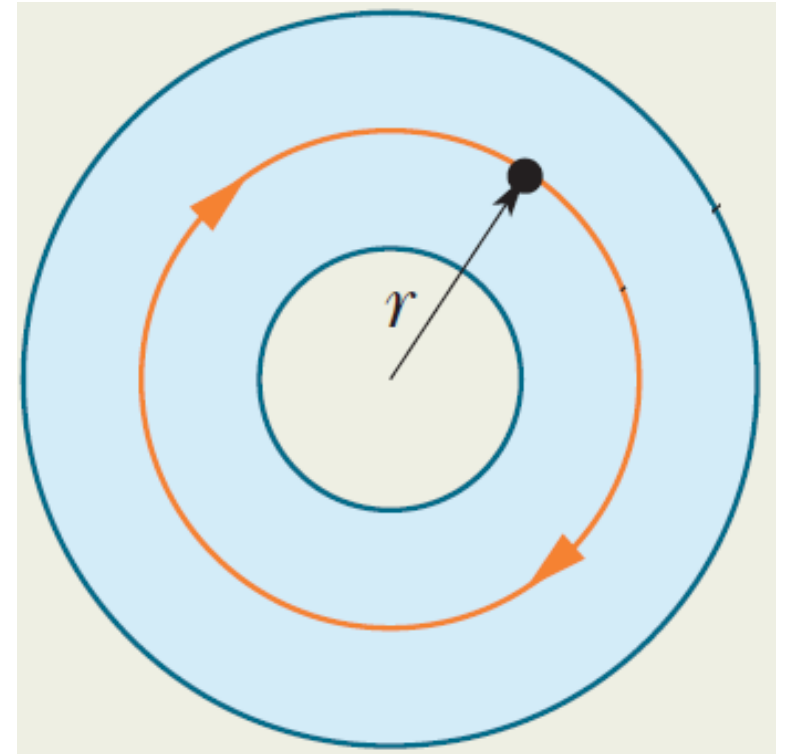


3. Ampere's Law

Thus, the magnetic field at $r = 3.0$ cm is

$$B = 2.0 \times 10^{-5} \text{ T.}$$

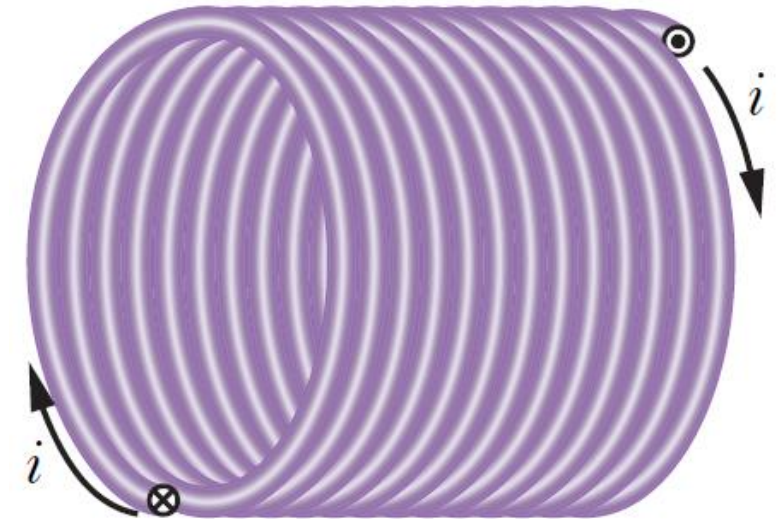
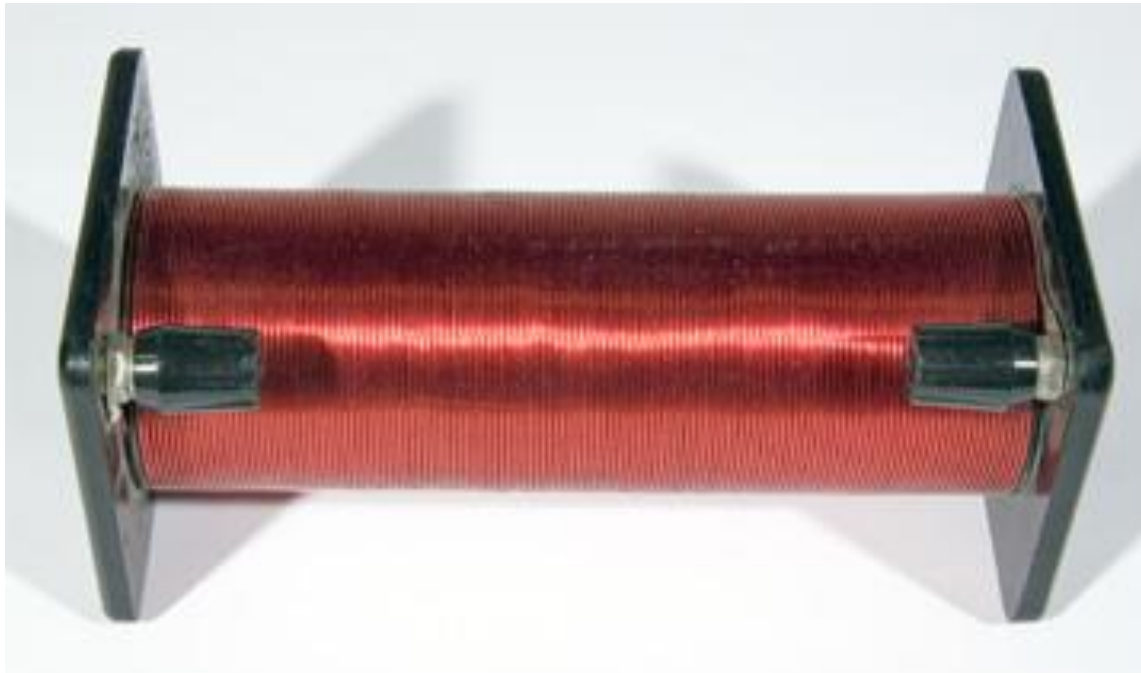
The direction of \vec{B} is opposite to the direction of integration. Thus, \vec{B} is counterclockwise.



4. Solenoids and Toroids

Magnetic Field of a Solenoid

The figure shows a **solenoid**, which is a long, tightly wound helical coil of wire. We assume that the length of the solenoid is much greater than its diameter.



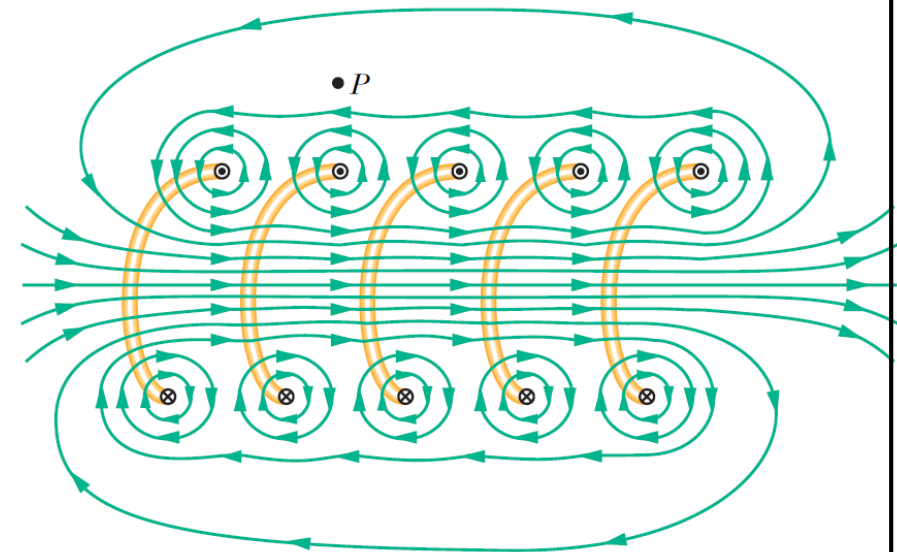
4. Solenoids and Toroids

Magnetic Field of a Solenoid

This figure shows a section through a portion of a stretched-out solenoid. The solenoid's magnetic field is the vector sum of the fields produced by the individual turns.

For points very close to a turn, the wire behaves magnetically like a long straight wire, and the field lines there are concentric circles. The field between adjacent turns tends to cancel.

Additionally, the field inside the solenoid is approximately parallel to the solenoid axis.

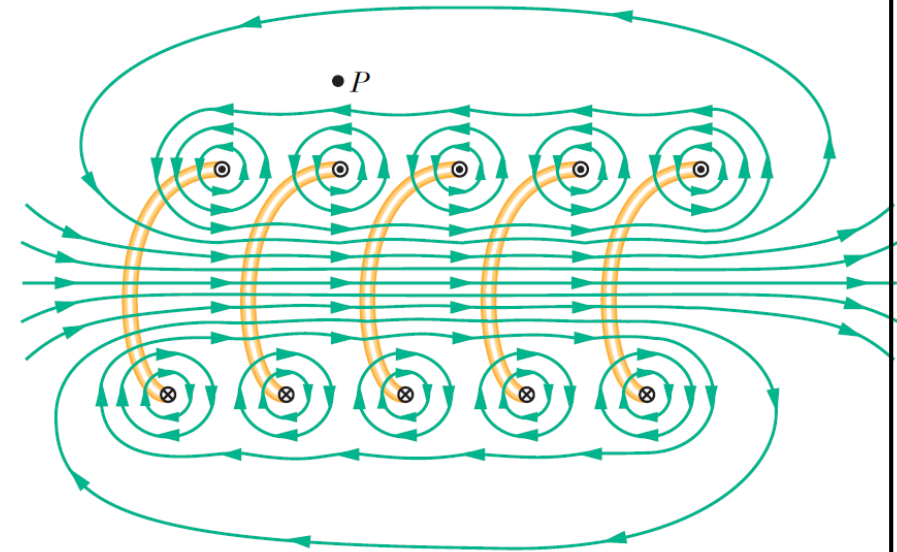


4. Solenoids and Toroids

Magnetic Field of a Solenoid

In the limiting case of **an ideal solenoid**, which is infinitely long and consists of tightly packed turns of square wire, the field inside the coil is uniform and parallel to the solenoid axis.

The magnetic field set up by the upper parts of the turns tends to cancel the field set up there by the lower parts of the turns. For an ideal solenoid, the magnetic field outside the solenoid is zero.

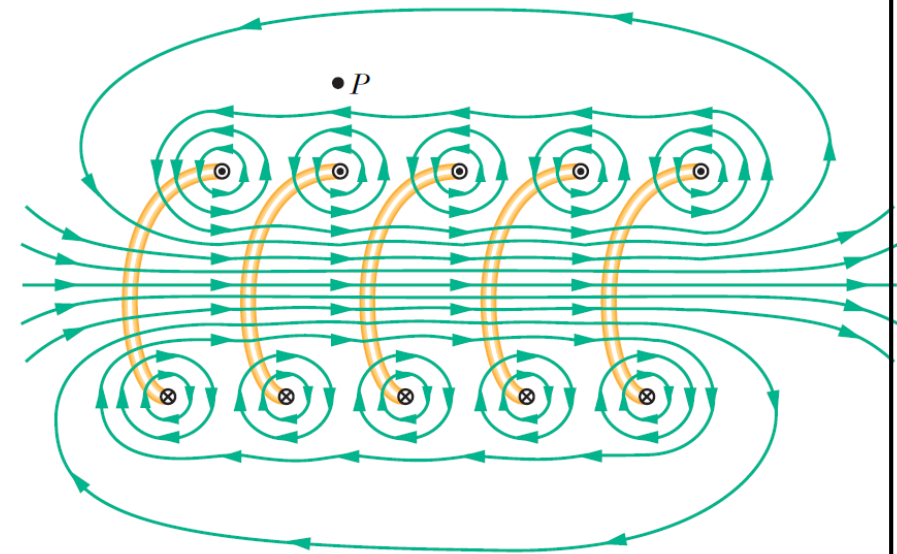


4. Solenoids and Toroids

Magnetic Field of a Solenoid

For a real solenoid, the external magnetic field can be excellently approximated to be zero at external points that are not at either end of the solenoid.

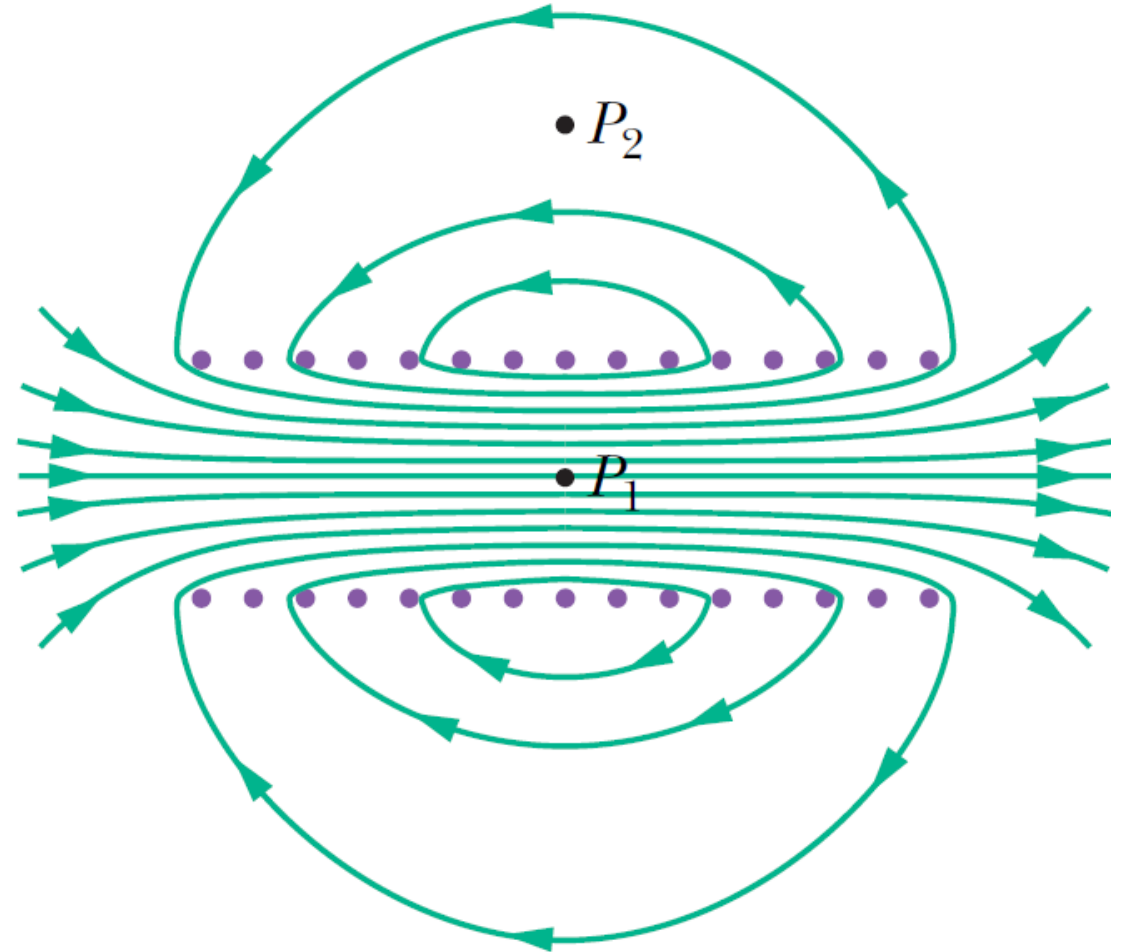
The direction of the magnetic field along the solenoid axis is given by a curled-straight right-hand rule: Grasp the solenoid with your right hand so that your fingers follow the direction of the current in the windings; your extended right thumb then points in the direction of the axial magnetic field.



4. Solenoids and Toroids

Magnetic Field of a Solenoid

The figure shows the lines of the magnetic field for a real solenoid.



4. Solenoids and Toroids

Magnetic Field of a Solenoid

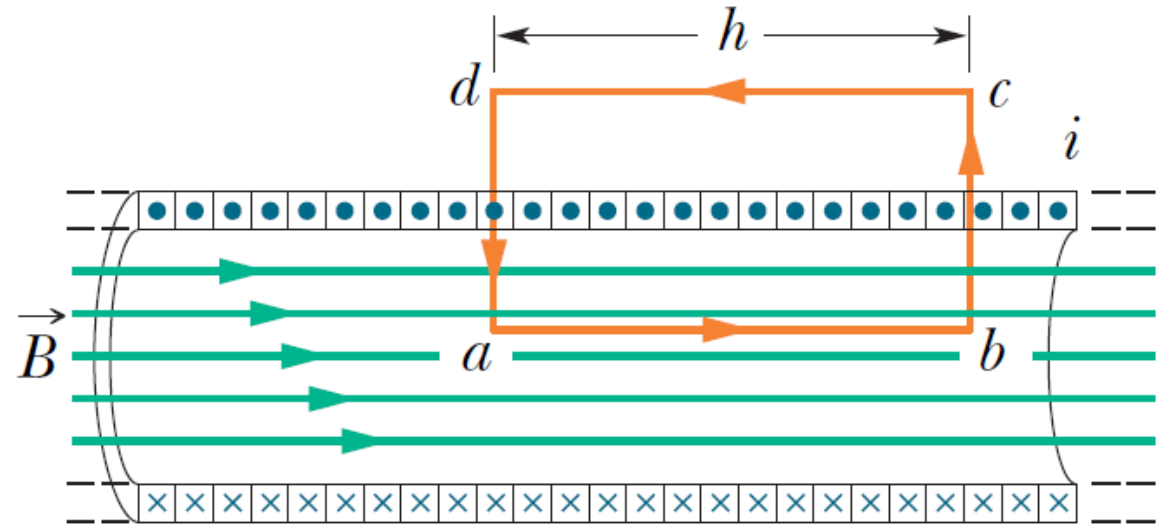
To find the magnetic field \vec{B} for an ideal solenoid, we use Ampere's law:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}$$

For the rectangular Amperian loop shown,

$$\oint \vec{B} \cdot d\vec{s} = Bh.$$

Let n be the number of turns per unit length of the solenoid.



4. Solenoids and Toroids

Magnetic Field of a Solenoid

The encircled current is then

$$i_{\text{enc}} = i(nh).$$

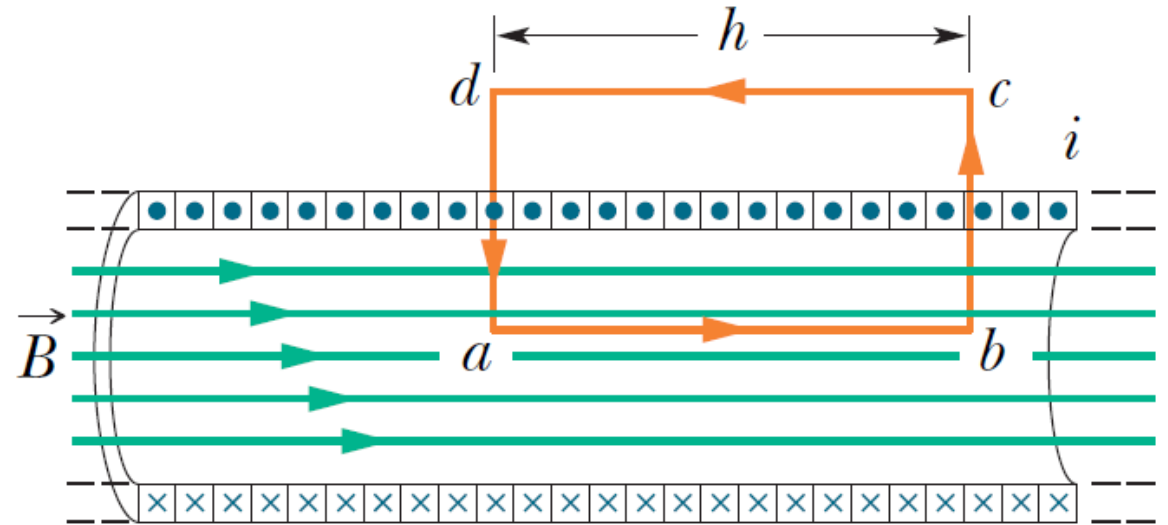
Ampere's law gives

$$Bh = \mu_0 i n h,$$

or

$$B = \mu_0 i n.$$

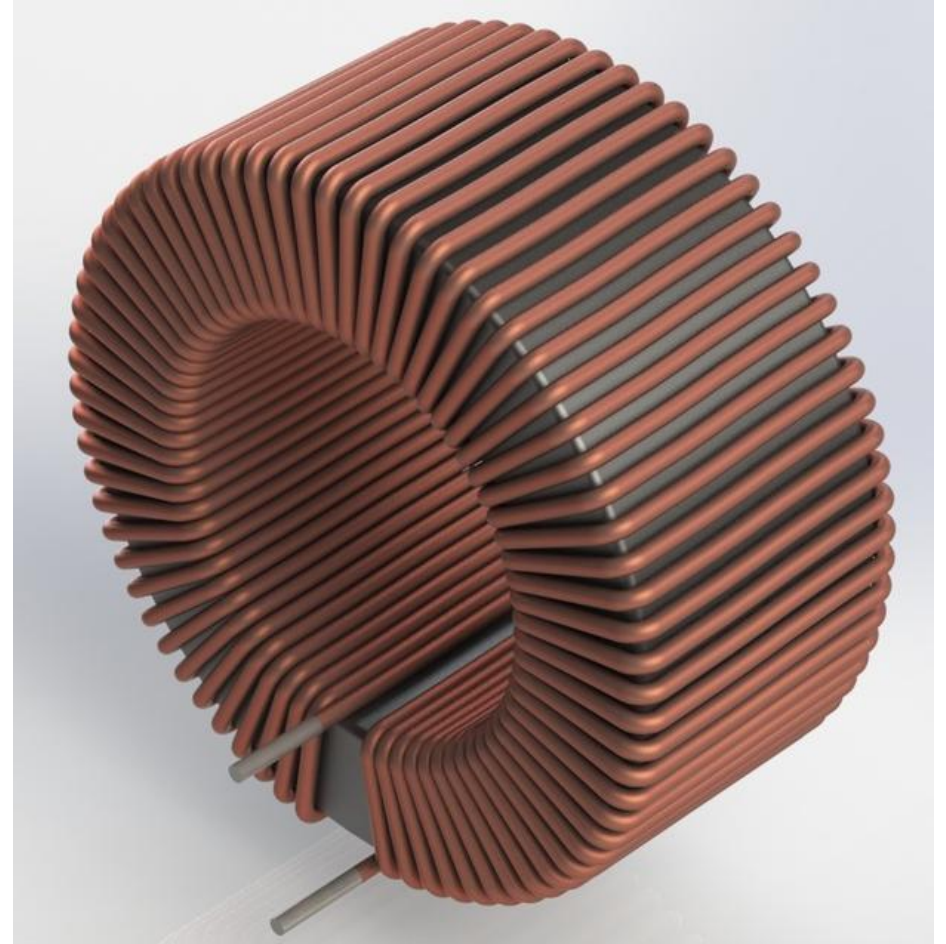
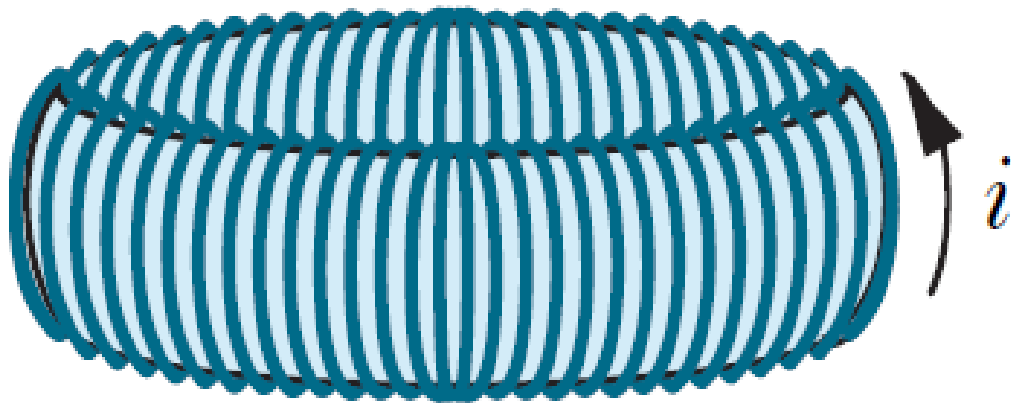
Note that \vec{B} does not depend on the length or diameter of the solenoid.



4. Solenoids and Toroids

Magnetic Field of a Toroid

The figure shows a **toroid**, which is a ring form of a solenoid.

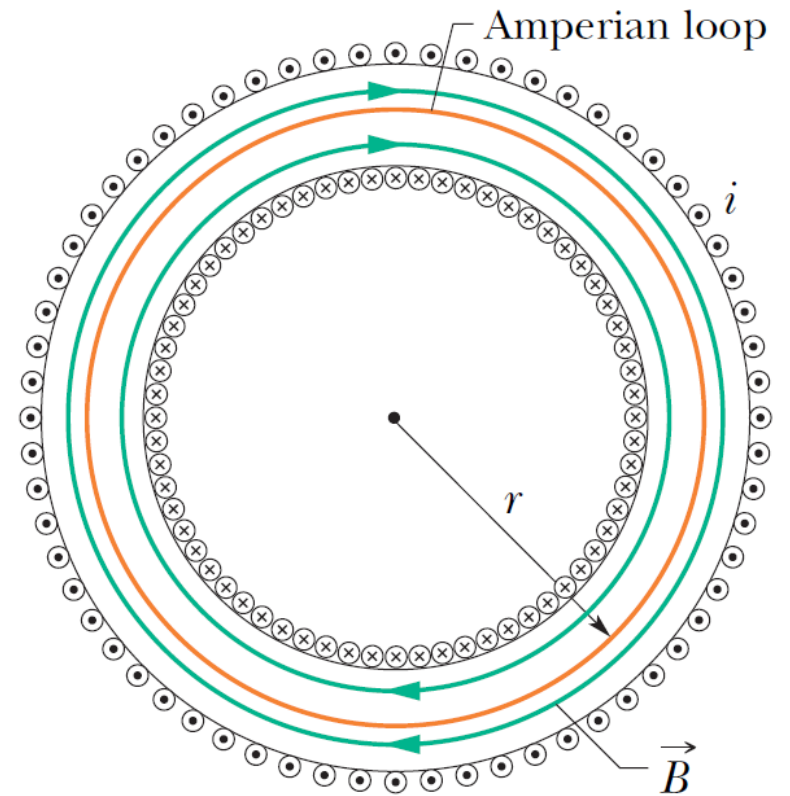


4. Solenoids and Toroids

Magnetic Field of a Toroid

We want to find the magnetic field \vec{B} inside an ideal toroid, using Ampere's law and the symmetry of the toroid.

From symmetry, we conclude that the field lines are concentric circles inside the toroid. We choose a concentric circle of radius r as an Amperian loop and traverse it clockwise.



4. Solenoids and Toroids

Magnetic Field of a Toroid

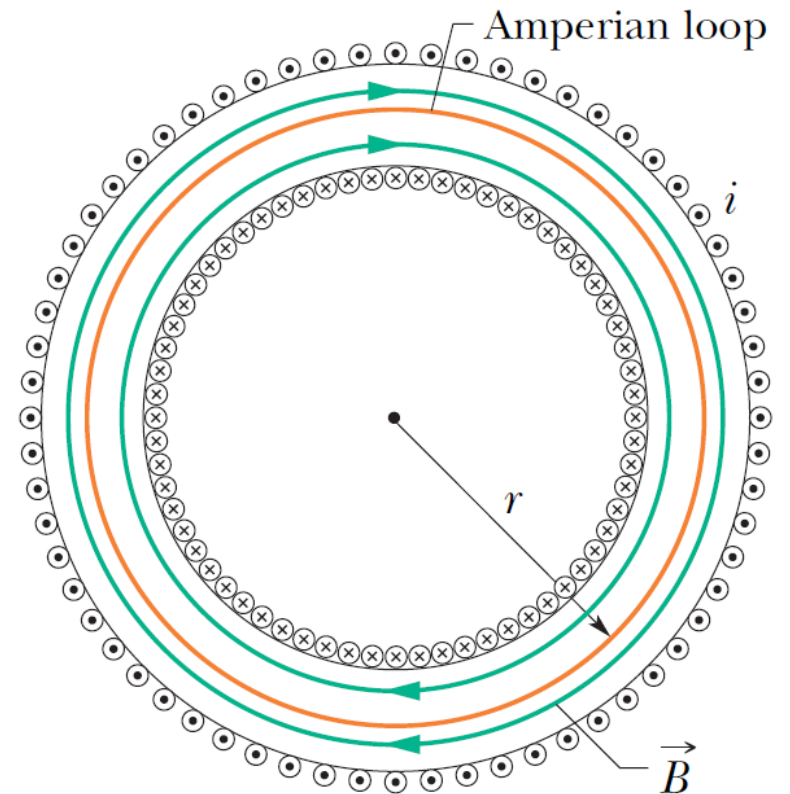
Ampere's law yields

$$B(2\pi r) = \mu_0 i N,$$

where i is the current in the toroid windings and N is the total number of the turns. Thus,

$$B = \frac{\mu_0 i N}{2\pi r}.$$

Unlike in the case of a solenoid, the magnetic field is not constant over a cross section of a toroid.

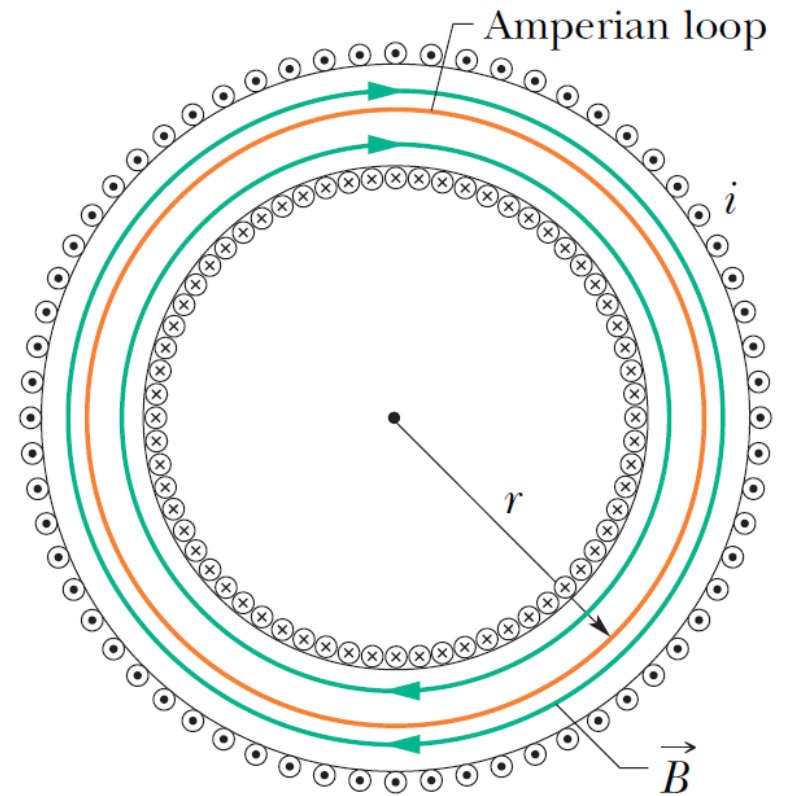


4. Solenoids and Toroids

Magnetic Field of a Toroid

It can be shown by direct application of Ampere's law that the magnetic field is zero outside an ideal toroid.

The direction of the field is given by a curled-straight right-hand rule: Grasp the toroid with the fingers of your right hand curled in the direction of the current in the windings; your extended right thumb points in the direction of the magnetic field.



4. Solenoids and Toroids

Example 4: A solenoid has length $L = 1.23$ m and inner diameter $d = 3.55$ cm and it carries a current $i = 5.57$ A. It consists of five close-packed layers, each with 850 turns along length L . What is B at its center?

$$\begin{aligned} B &= \mu_0 i n = \mu_0 i \left(5 \times \frac{N}{L} \right) = \left(4\pi \times 10^{-7} \text{ T} \cdot \frac{\text{m}}{\text{A}} \right) (5.57 \text{ A}) \left(5 \times \frac{850 \text{ turns}}{1.23 \text{ m}} \right) \\ &= 24.2 \text{ mT}. \end{aligned}$$

5. A Current-Carrying Coil as a Magnetic Dipole

So far we have studied the magnetic fields produced by currents in a long straight wire, solenoid and toroid. We now turn our attention to the magnetic field produced by a coil carrying a current.

We saw that a coil behaves as a magnetic dipole. A torque $\vec{\tau}$ given by

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

acts on a coil if we place it in an external magnetic field.

5. A Current-Carrying Coil as a Magnetic Dipole

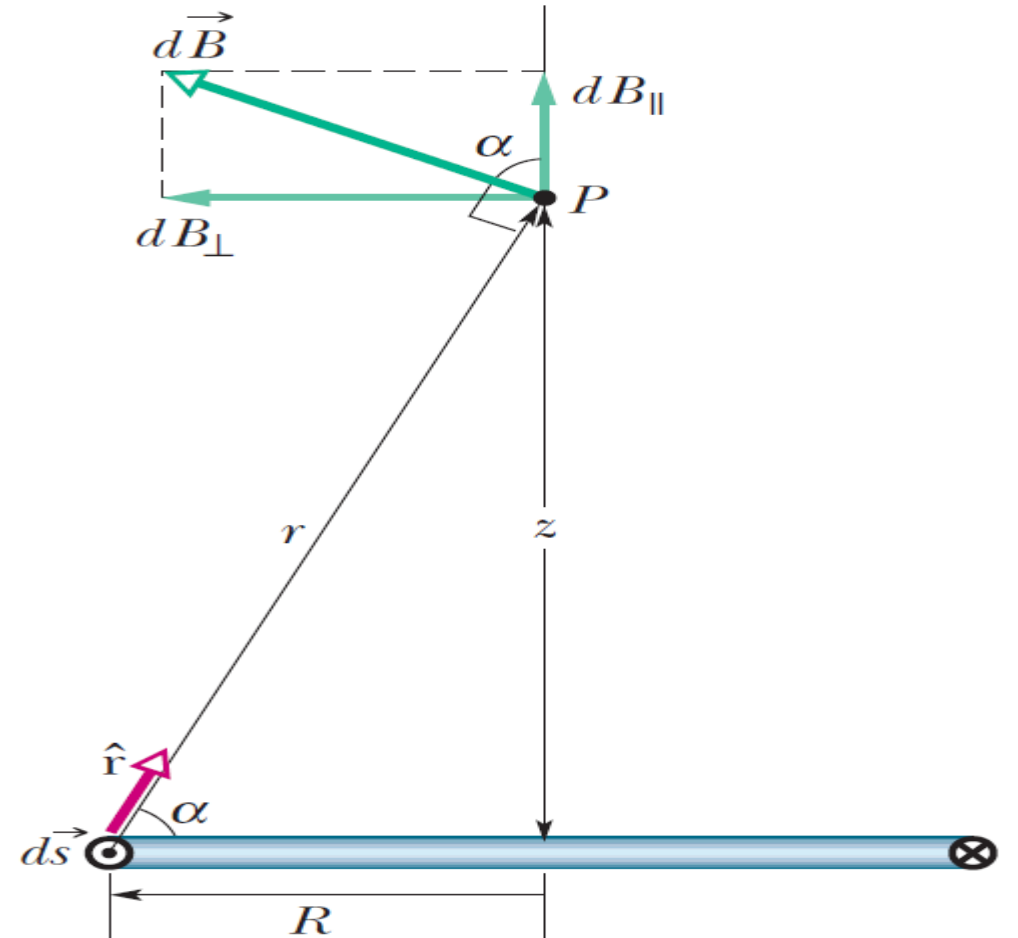
Magnetic Field of a Coil

Unfortunately, the symmetry of the does not allow us to use Ampere's law; we need to use Biot-Savart law.

The magnitude of the magnetic field at any point on the loop's perpendicular central axis (z axis) is given by

$$B(z) = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}},$$

where R is the radius of the loop. The direction of \vec{B} is the same as that of $\vec{\mu}$.



5. A Current-Carrying Coil as a Magnetic Dipole

Magnetic Field of a Coil

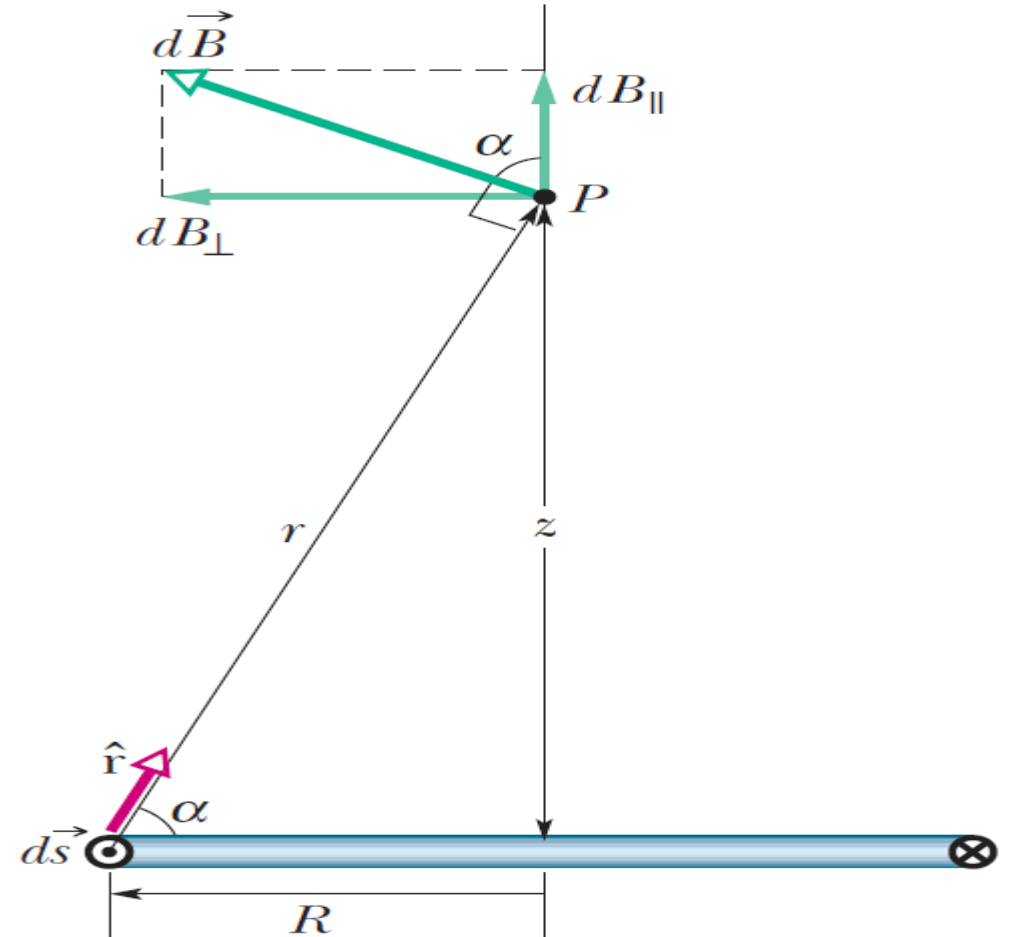
For axial points far from the loop, where $z \gg R$, B can be approximated as

$$B(z) \approx \frac{\mu_0 i R^2}{2z^3}.$$

For a coil of N turns, we can write

$$B(z) = \frac{\mu_0 NiA}{2\pi z^3}.$$

where $A = \pi R^2$, the area of the coil.



5. A Current-Carrying Coil as a Magnetic Dipole

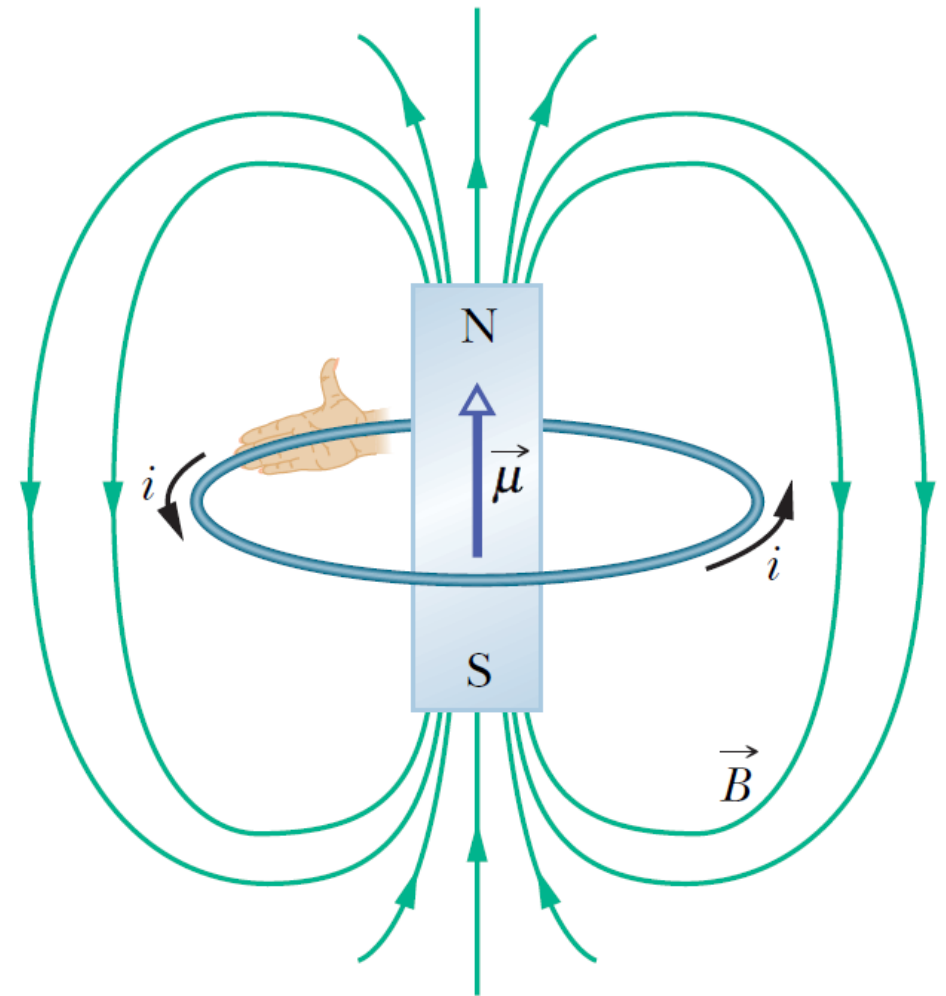
Magnetic Field of a Coil

In terms of $\vec{\mu}$, we can recast $B(z)$ as

$$B(z) = \frac{\mu_0}{2\pi} \frac{\vec{\mu}}{z^3}.$$

We can regard a current-carrying coil as a magnetic dipole in two ways:

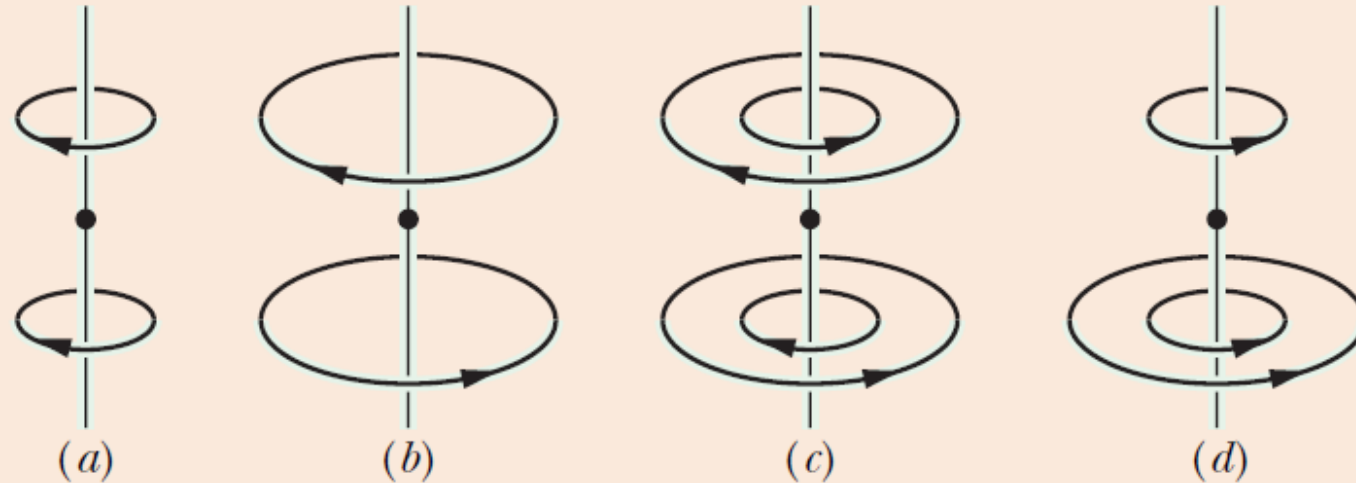
- (1) It experiences a torque when we place it in an external magnetic field.
- (2) Its magnetic field resembles that of a bar magnet.



5. A Current-Carrying Coil as a Magnetic Dipole

✓ CHECKPOINT 3

The figure here shows four arrangements of circular loops of radius r or $2r$, centered on vertical axes (perpendicular to the loops) and carrying identical currents in the directions indicated. Rank the arrangements according to the magnitude of the net magnetic field at the dot, midway between the loops on the central axis, greatest first.

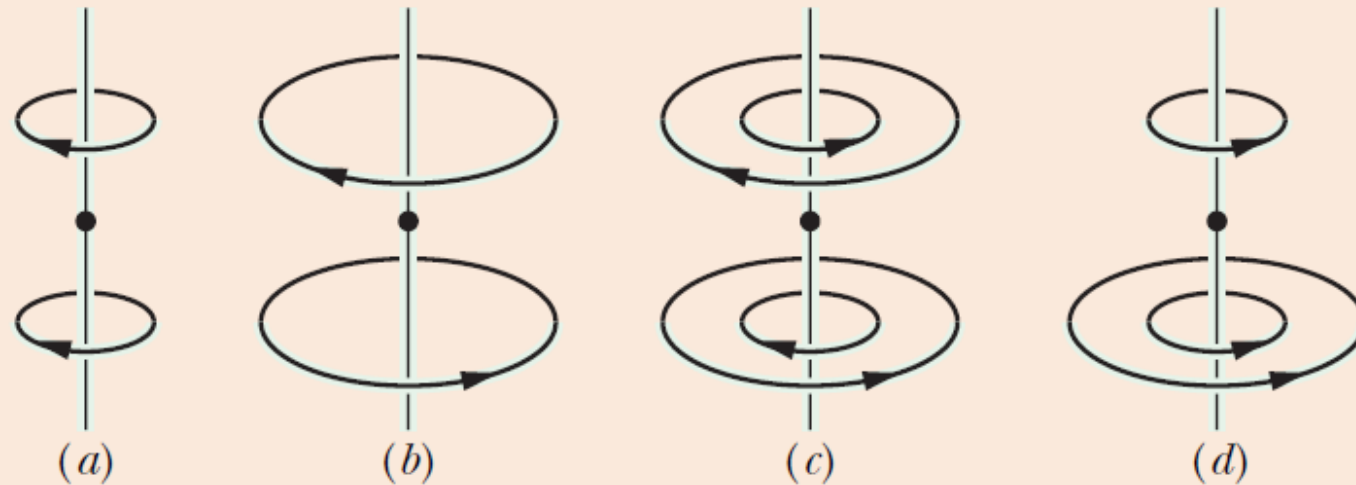


$$B(z) = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}}$$

5. A Current-Carrying Coil as a Magnetic Dipole

✓ CHECKPOINT 3

The figure here shows four arrangements of circular loops of radius r or $2r$, centered on vertical axes (perpendicular to the loops) and carrying identical currents in the directions indicated. Rank the arrangements according to the magnitude of the net magnetic field at the dot, midway between the loops on the central axis, greatest first.



d, a, b & c tie.