Chapter 23

Gauss' Law

1. Introduction

Evaluating the electric field of a charge distribution is usually difficult.

If a certain charge distribution is symmetric enough, we then can easily find the electric field with the help of Gauss' law.

Gauss' law considers an imaginary closed surface enclosing a charge distribution, called the **Gaussian surface**. The Gaussian surface mimics the symmetry of the charge distribution. For example, if the charge distribution is spherical, we enclose the charge with a spherical Gaussian surface.

Gauss' law relates the electric fields at points on a (closed) Gaussian surface to the net charge enclosed by that surface.

1. Introduction

We can also use Gauss' law in reverse. If we know the electric field on a Gaussian surface, we can find the net charge enclosed by the surface.

To know how much charge is enclosed, we first need to know how much electric field is intercepted by the Gaussian surface.

A measure of intercepted field is called the **flux**.

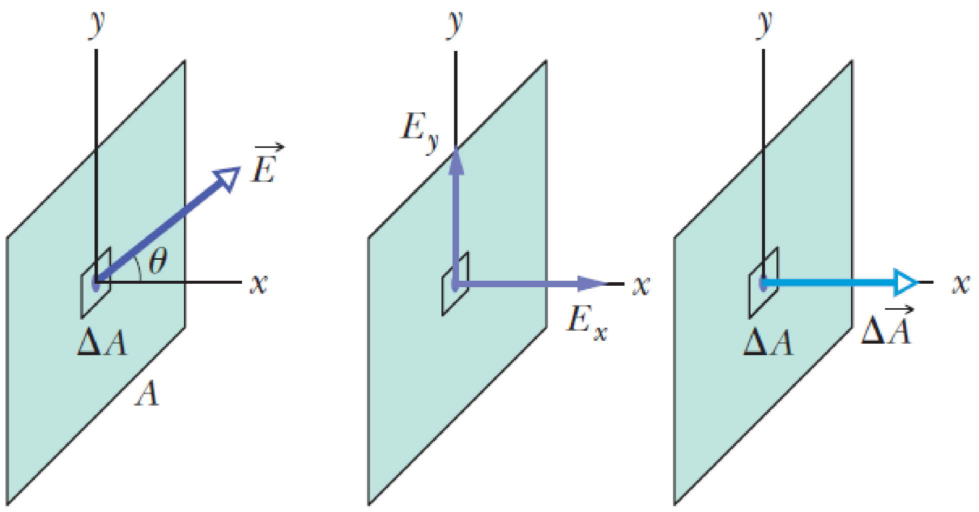
2. Flux of an Electric Field

The electric flux Φ through a surface is the amount of electric field that pierces the surface.

The area vector $d\vec{A}$ for an area element (patch element) on a surface is a vector that is perpendicular to the element and has a magnitude equal to the area dA of the element.

The electric flux $d\Phi$ through a patch element with area vector $d\vec{A}$ is given by

$$d\Phi = (E\cos\theta)dA.$$



2. Flux of an Electric Field

We can rewrite the last expression as a dot product:

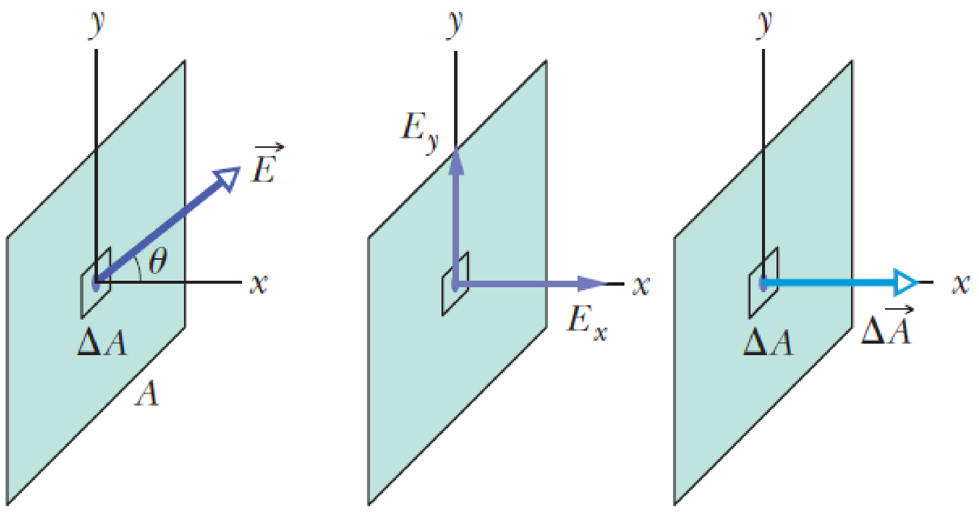
$$d\Phi = \vec{E} \cdot d\vec{A}.$$

The total flux through a surface is given by

$$\Phi = \int \vec{E} \cdot d\vec{A},$$

where the integration is carried over the surface.

The flux of the electric field is a scalar. Its SI unit is $\mathrm{N \cdot m^2/C}$.



2. Flux of an Electric Field

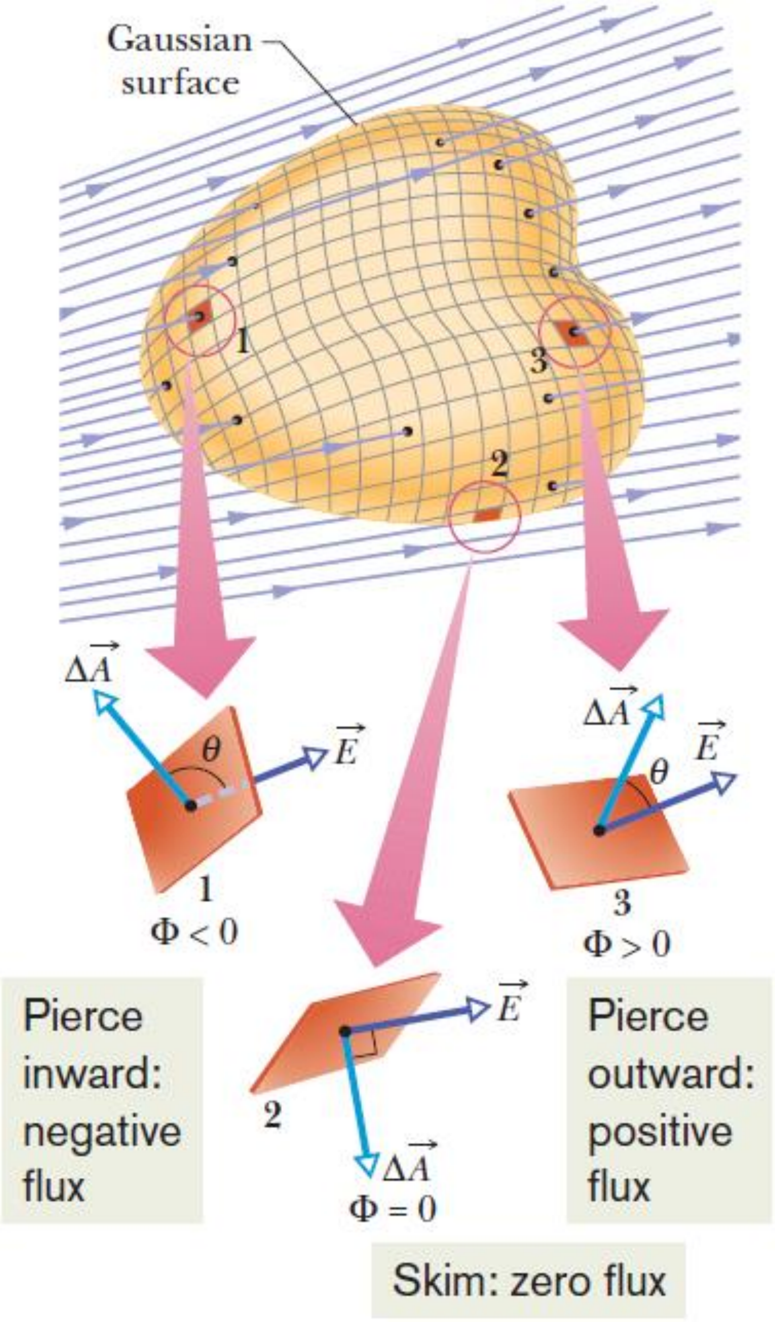
Consider the **closed** Gaussian surface shown in the figure. An inward piercing field is negative flux. An outward piercing field is positive flux. A skimming field is zero flux.

The net flux of the electric field through the surface is

$$\Phi = \sum \vec{E} \cdot \Delta\vec{A}.$$

In the limit that $\Delta\vec{A} \rightarrow d\vec{A}$ we get that

$$\Phi = \oint \vec{E} \cdot d\vec{A}.$$

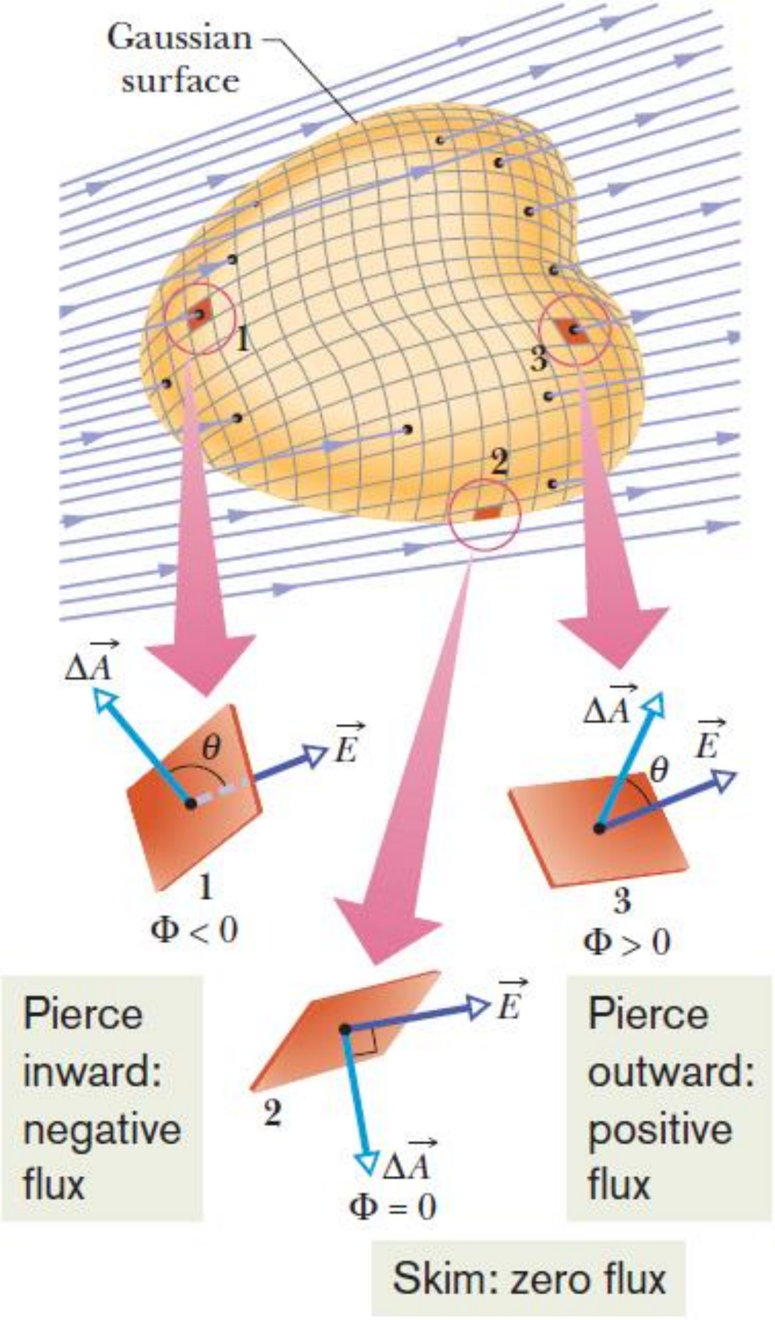


2. Flux of an Electric Field

Recall that the magnitude of the electric field is proportional to the number of electric field lines per unit area. Thus, the flux is proportional to the number of electric field lines passing through area $d\vec{A}$.

We can therefore interpret the electric flux as follows:

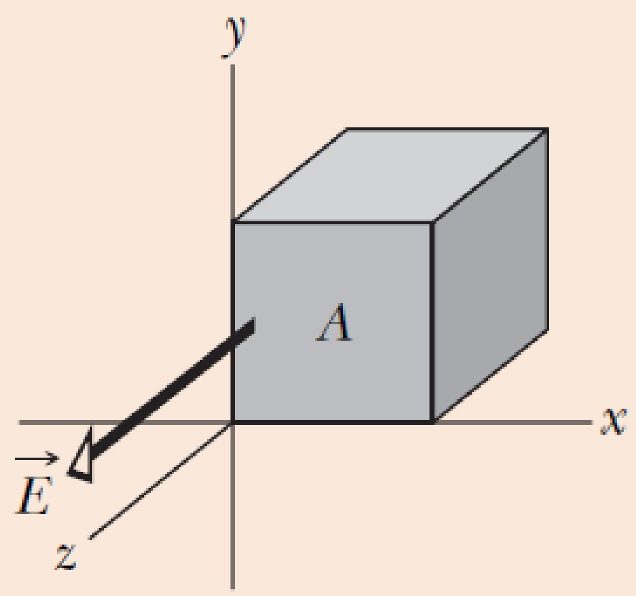
The electric flux Φ through a Gaussian surface is proportional to the net number of electric field lines passing through that surface.



2. Flux of an Electric Field

CHECKPOINT 1

The figure here shows a Gaussian cube of face area A immersed in a uniform electric field $\vec{E}$ that has the positive direction of the z axis. In terms of E and A, what is the flux through (a) the front face (which is in the xy plane), (b) the rear face, (c) the top face, and (d) the whole cube?



(a) EA

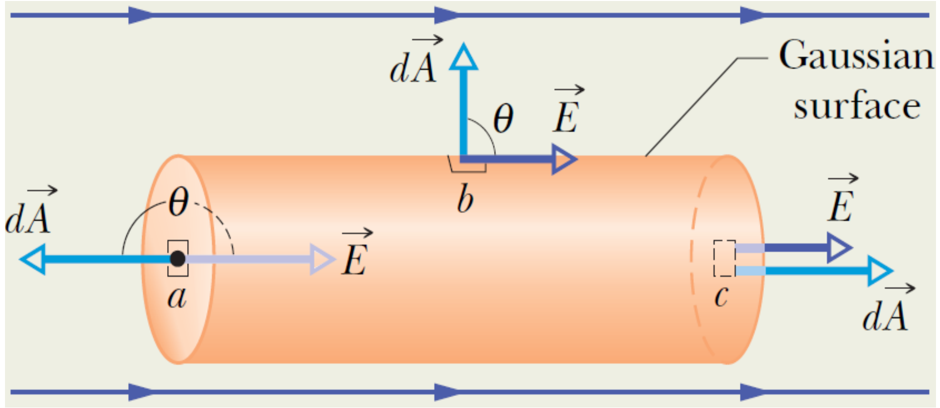
(b) $-EA$

(c) 0

(d) 0

3. Flux of an Electric Field

Example 1: the figure shows a Gaussian surface in the form of a cylinder of radius *R* immersed in a uniform electric field, with the cylinder axis parallel to the field. What is the flux Φ of the electric field through this closed surface?



We can do the integration by writing the flux as the sum of three terms: the integrals over the left cylinder cap a, the cylindrical surface b, and the right cap c.

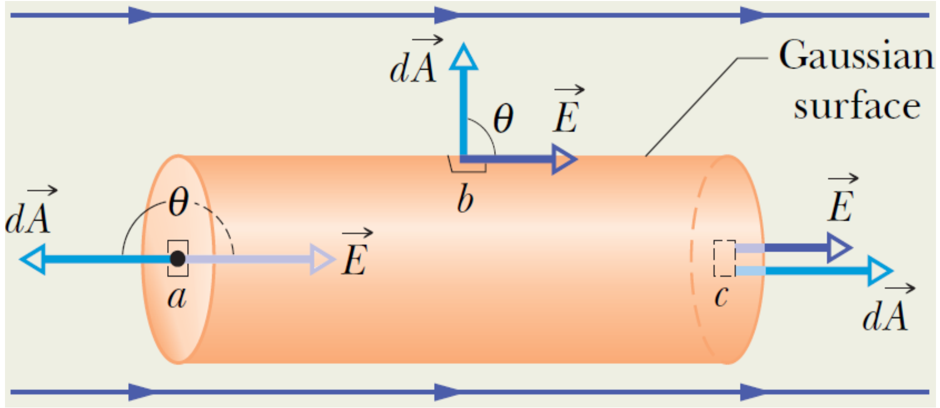
3. Flux of an Electric Field

$$\Phi = \oint \vec{E} \cdot d\vec{A}$$

$$= \int_a \vec{E} \cdot d\vec{A} + \int_b \vec{E} \cdot d\vec{A} + \int_c \vec{E} \cdot d\vec{A}.$$

$$\int_a \vec{E} \cdot d\vec{A} = \int_a E(\cos\pi)dA = -E\int_a dA$$

$$= -EA.$$

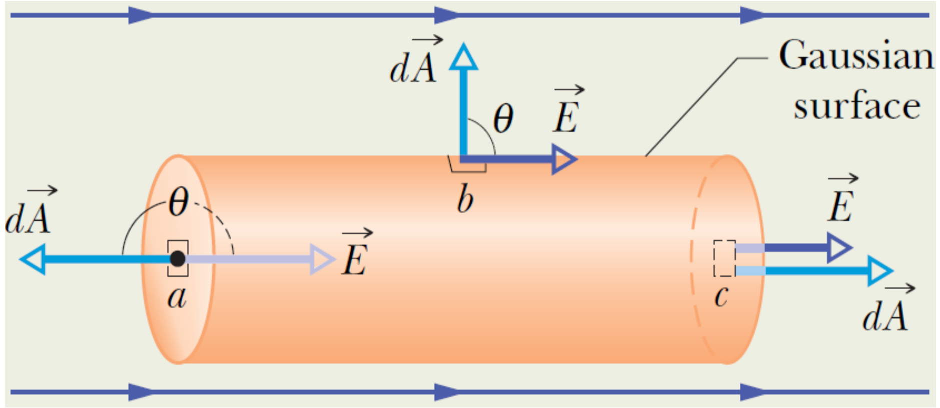


3. Flux of an Electric Field

$$\int_b \vec{E} \cdot d\vec{A} = \int_b E(\cos \pi/2) dA = 0.$$

$$\int_c \vec{E} \cdot d\vec{A} = \int_c E(\cos 0) dA = E \int_c dA$$

$$= EA.$$

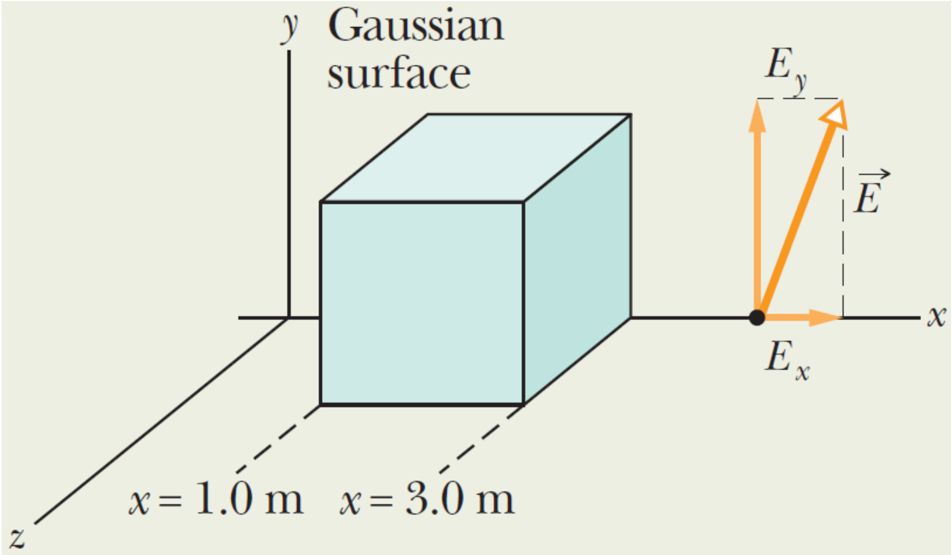


Thus,

$$\Phi = -EA + 0 + EA = 0.$$

3. Flux of an Electric Field

Example 2: A *nonuniform* electric field given by $\vec{E} = 3.0x\,\hat{\imath} + 4.0\,\hat{\jmath}$ pierces the Gaussian cube shown in the figure. (E is in newtons per coulomb and *x* is in meters.)



(a) What is the electric flux through the right face?

$$\Phi_R = \int_R \vec{E}\cdot d\vec{A} = \int_R (3.0x\,\hat{\imath} + 4.0\,\hat{\jmath})\cdot(dA\,\hat{\imath})$$

$$= 3.0\int_R x\,dA = 3.0\int_R (3.0)\,dA = 9.0A$$

$$= (9.0)(4.0) = 36\ \mathrm{N\cdot m^2/C}.$$

3. Flux of an Electric Field

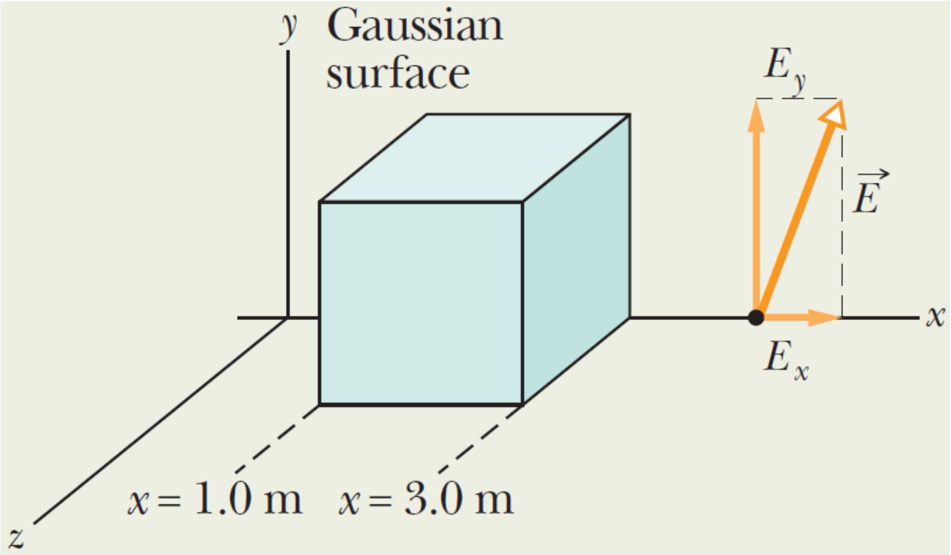
(b) What is the electric flux through the left face?

$$\Phi_L = \int_L \vec{E} \cdot d\vec{A} = \int_L (3.0x\,\hat{\imath} + 4.0\,\hat{\jmath}) \cdot (-dA\,\hat{\imath})$$

$$= -3.0 \int_L x\,dA = -3.0 \int_L (1.0)\,dA$$

$$= -3.0A = -(3.0)(4.0)$$

$$= -12\ \mathrm{N \cdot m^2/C}.$$



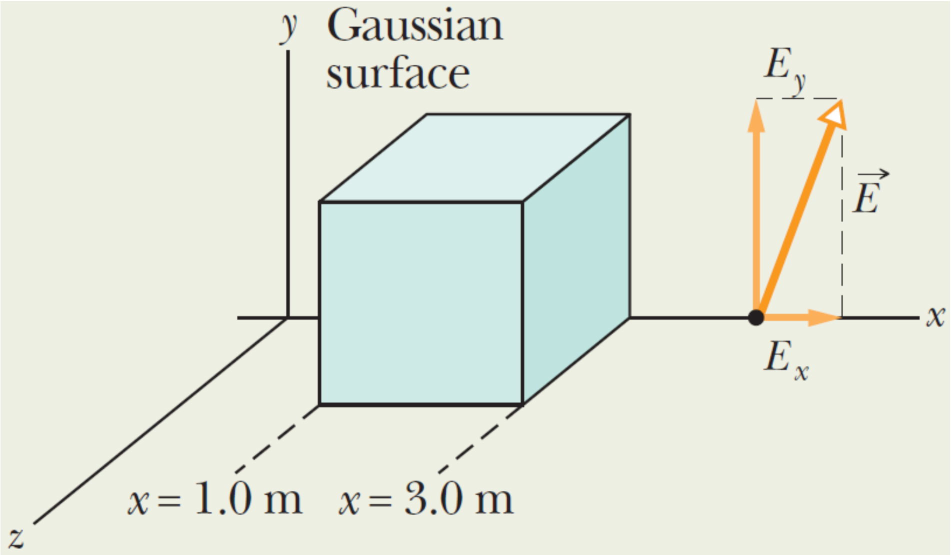
3. Flux of an Electric Field

(c) What is the electric flux through the top face?

$$\Phi_T = \int_T \vec{E} \cdot d\vec{A} = \int_T (3.0x\,\hat{\imath} + 4.0\,\hat{\jmath}) \cdot (dA\,\hat{\jmath})$$

$$= 4.0 \int_T dA = 4.0A = 4.0(4.0)$$

$$= 16\ \mathrm{N \cdot m^2/C}.$$



4. Gauss' Law

Gauss' law relates the net flux Φ of an electric field through a closed surface (a Gaussian surface) to the *net* charge q_{enc} that is *enclosed* by that surface:

$$\Phi = \frac{q_{\text{enc}}}{\varepsilon_0}.$$

Using the expression for Φ we get

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}.$$

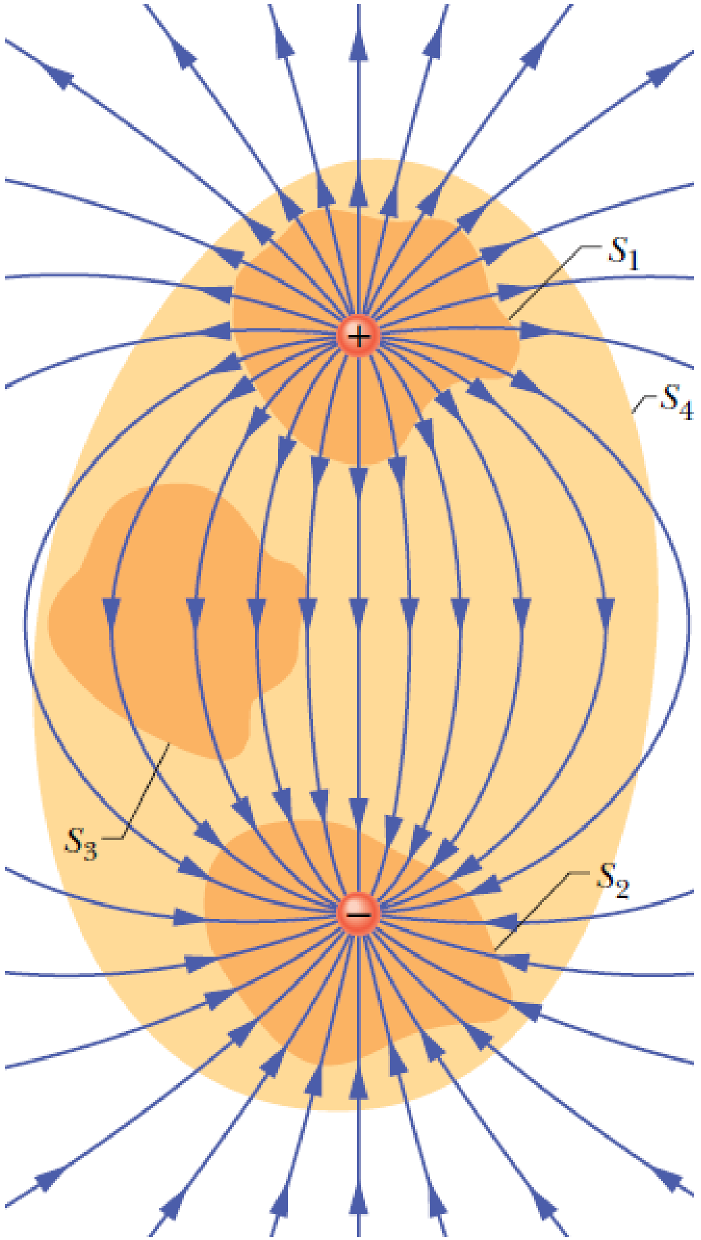
4. Gauss' Law

Let us apply Gauss' law to the charge(s) shown in the figure.

Surface S_1: The electric field lines point outward for all points on the surface. The flux is thus positive and so is the net charge within the surface (If Φ is positive, q_{enc} must be positive too).

Surface S_2: The electric field lines point inward for all points on the surface. The flux is thus negative and so is the net charge within the surface.

Surface S_3: The electric field lines pass entirely through the surface. The flux is thus zero. This surface encloses no charge and thus $q_{\text{enc}} = 0$. Gauss' law too requires that the net flux is zero.

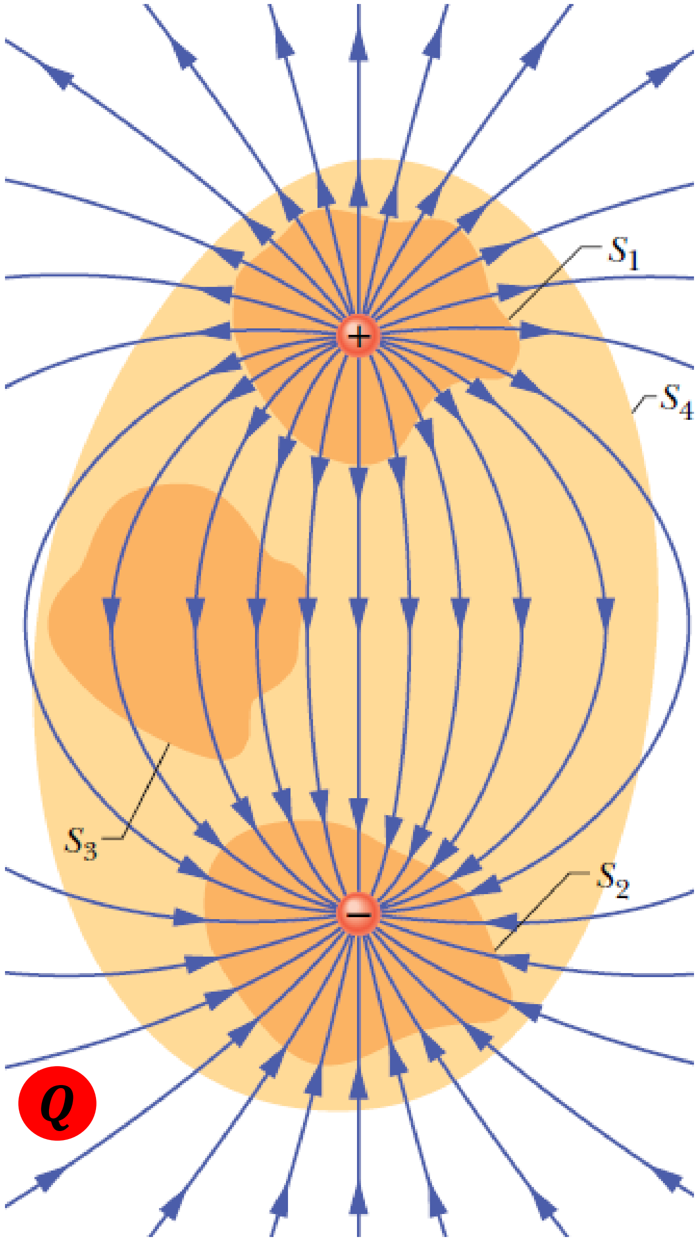


4. Gauss' Law

Surface S_4: This surface encloses zero net charge. Gauss' law requires that the net flux of the electric field through this surface is zero. There are as many field lines leaving the surface as entering it.

Q: What happens if we were to bring charge Q up close to S_4?

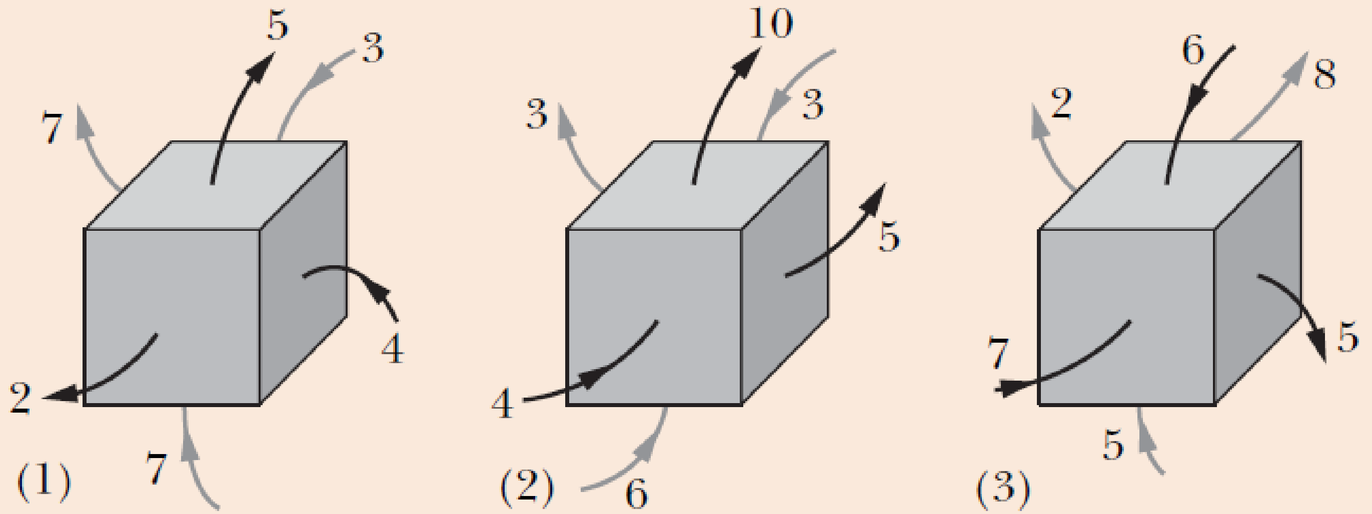
A: The pattern of the electric field lines would change, but the net flux for each of the four Gaussian surface would not change. The field lines associated with the added charge Q would pass entirely through each of the four surfaces. Q would not enter Gauss' law since Q lies outside all four Gaussian surfaces.



4. Gauss' Law

CHECKPOINT 2

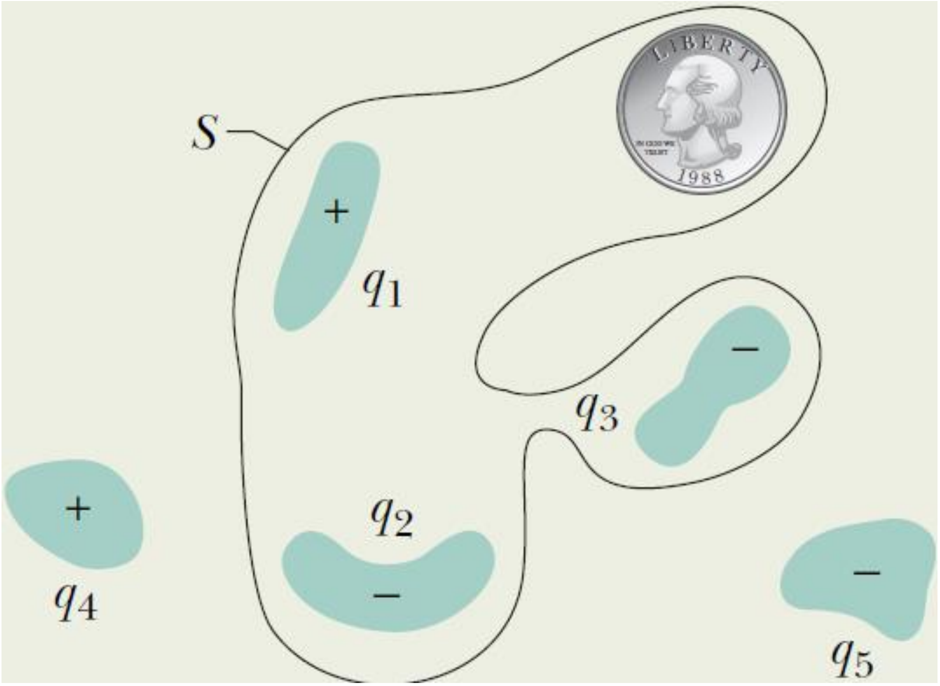
The figure shows three situations in which a Gaussian cube sits in an electric field. The arrows and the values indicate the directions of the field lines and the magnitudes (in $N \cdot m^2/C$) of the flux through the six sides of each cube. (The lighter arrows are for the hidden faces.) In which situation does the cube enclose (a) a positive net charge, (b) a negative net charge, and (c) zero net charge?



(a) 2
(b) 3
(c) 1

4. Gauss' Law

Example 3: The figure shows five charged lumps of plastic and an electrically neutral coin. The cross section of a Gaussian surface S is indicated. What is the net electric flux through the surface if $q_1 = q_4 = +3.1\ \text{nC}$ $q_2 = q_5 = -5.9\ \text{nC}$, and $q_3 = -3.1\ \text{nC}$?



$$\Phi = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{q_1 + q_2 + q_3}{\varepsilon_0}$$

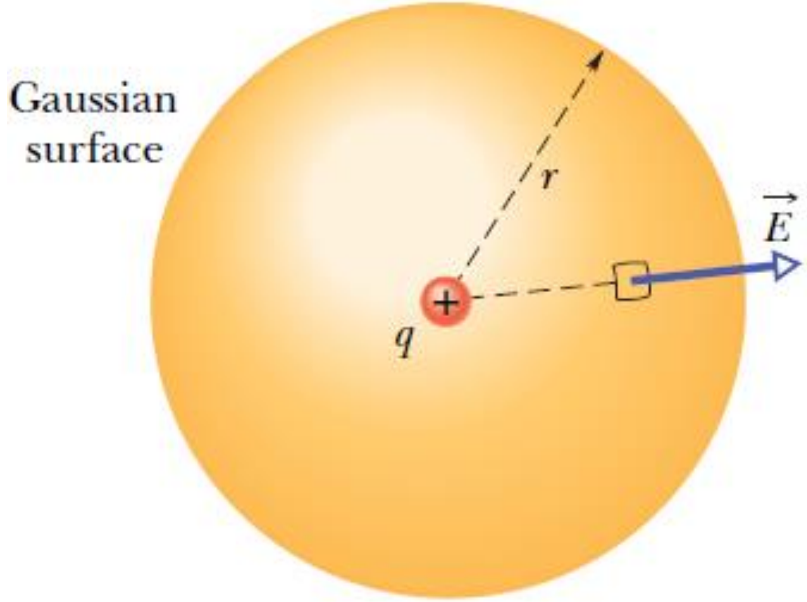
$$= \frac{3.1\ \text{nC} - 5.9\ \text{nC} - 3.1\ \text{nC}}{8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2}$$

$$= -670\ \text{N}\cdot\text{m}^2/\text{C}.$$

5. Gauss' Law and Coulomb's Law

Here we want to derive Coulomb's law from Gauss' law. For the Gaussian surface in the figure

$$\oint \vec{E} \cdot d\vec{A} = \oint E dA \cos 0 = \oint E dA = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{q}{\varepsilon_0}.$$



E is constant over the surface and we can take it out of the integral:

$$E \oint dA = \frac{q}{\varepsilon_0}.$$

5. Gauss' Law and Coulomb's Law

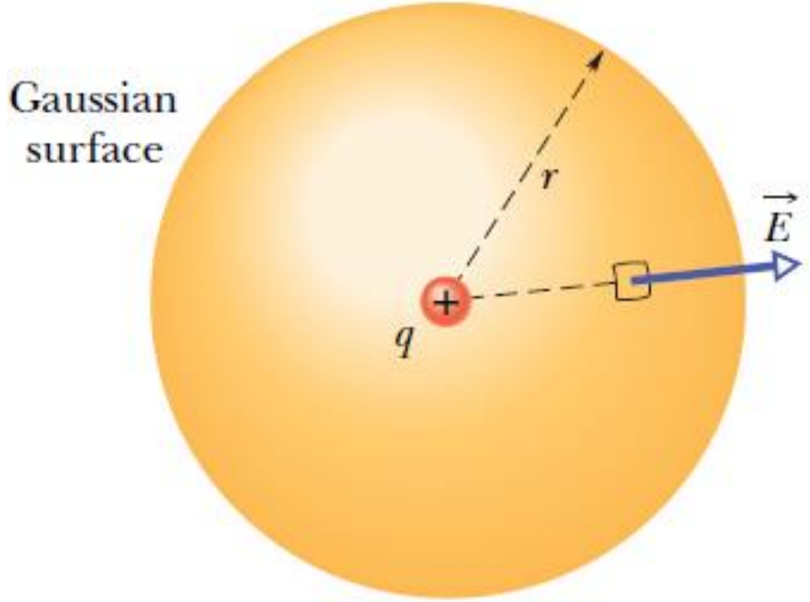
The integral is just the sum of all differential areas dA on the sphere and thus is just the surface area $4\pi r^2$. This gives

$$E(4\pi r^2) = \frac{q}{\varepsilon_0},$$

or

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2},$$

which is Coulomb's law.



5. Gauss' Law and Coulomb's Law

CHECKPOINT 3

There is a certain net flux Φ_i through a Gaussian sphere of radius r enclosing an isolated charged particle. Suppose the enclosing Gaussian surface is changed to (a) a larger Gaussian sphere, (b) a Gaussian cube with edge length equal to r, and (c) a Gaussian cube with edge length equal to $2r$. In each case, is the net flux through the new Gaussian surface greater than, less than, or equal to Φ_i?

(a) Equal to Φ_i.

(b) Equal to Φ_i.

(c) Equal to Φ_i.

$$\Phi = \frac{q}{\varepsilon_0}$$

6. A Charged Isolated Conductor

Gauss' law can be used to prove an important theorem about conductors:

"If an excess charge is placed on an isolated conductor, that amount of charge will move entirely to the surface of the conductor. None of the excess charge will be found within the body of the conductor."

This sounds reasonable because charges with the same sign repel one another and get as far away from one another as they can.

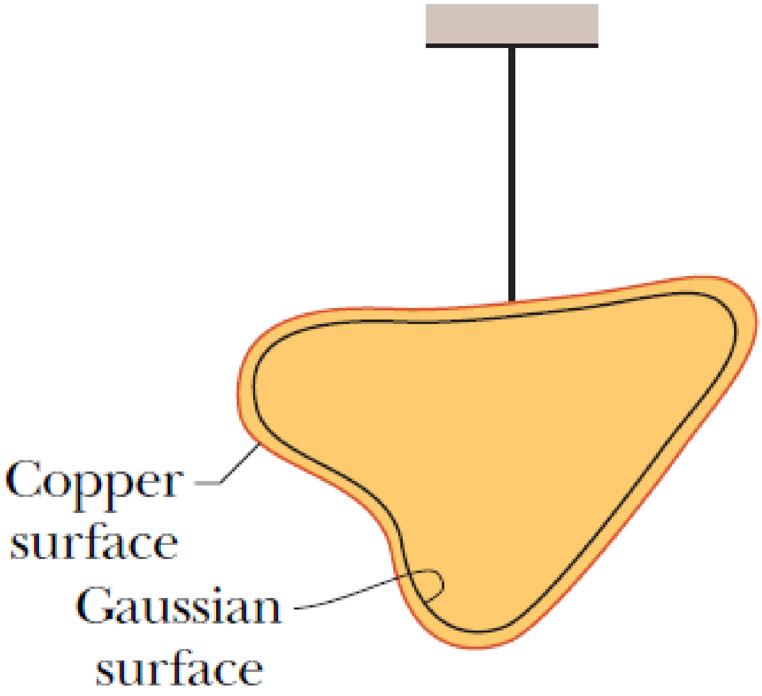
Let us use Gauss' law to prove this speculation.

6. A Charged Isolated Conductor

The figure shows an isolated copper lump having an excess charges q. We place a Gaussian surface just inside the surface of the conductor.

The electric field inside this conductor must be zero. If this were not so, the field would move the conduction electrons, generating a continuous current. Since there is no such current in isolated conductors, the field inside is zero.

The electric field is zero for all points on the Gaussian surface. The flux through the Gaussian surface must be zero too. The excess charge is not inside the Gaussian surface, it must lie on the surface of the conductor.

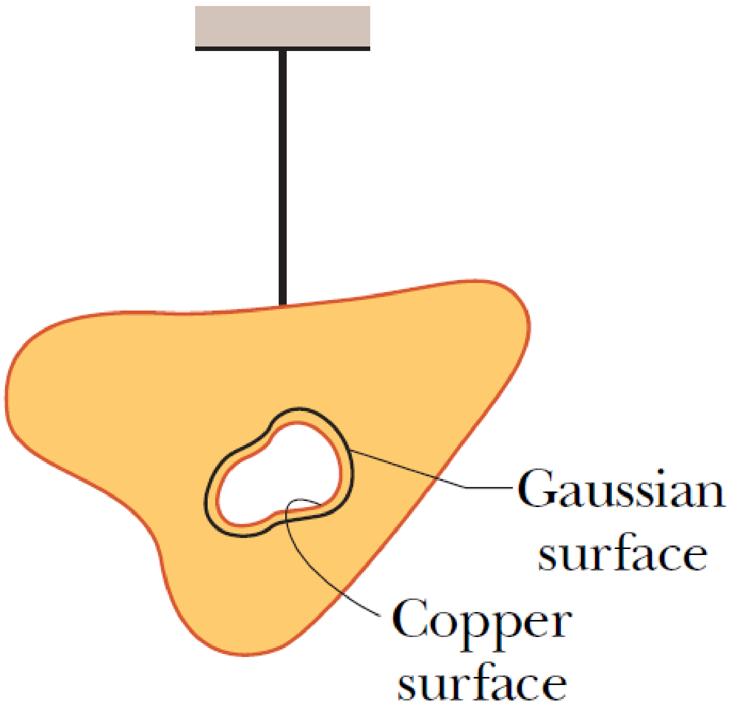


6. A Charged Isolated Conductor

<u>An Isolated Conductor with a Cavity</u>

Consider the previous figure but now with a cavity that is totally within the conductor. It is reasonable to say that when we scoop out the electrically neutral material to form the cavity we do not change the charge distribution. Let us prove this conclusion quantitatively using Gauss' law.

We draw a Gaussian surface surrounding the cavity. Because $\vec{E} = 0$ inside the conductor, the flux through this Gaussian surface is zero. We conclude that the net charge on the cavity walls is zero. All the excess charge remains on the outer surface of the conductor.



6. A Charged Isolated Conductor

The Conductor Removed

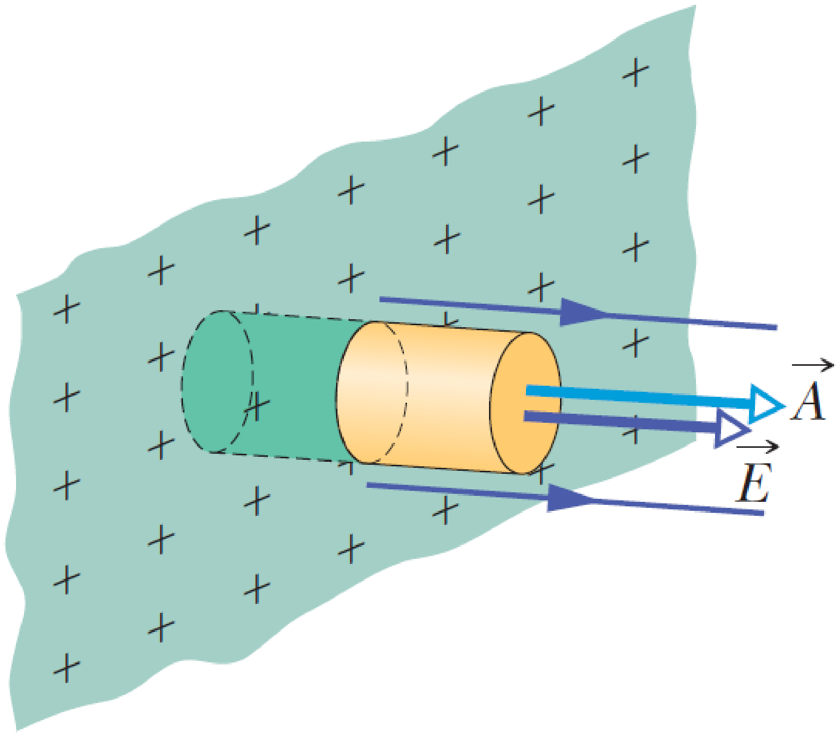
Suppose that we remove the conductor completely and leave only the charges. The electric field would not change and would remain zero inside this thin shell and would remain unchanged for all external points. The electric field is set up completely by the charges and not by the conductors.

6. A Charged Isolated Conductor

The External Electric Field

We have seen that excess charge on an isolated conductor move entirely to the surface. The charge distribution (or the charge density $\sigma = q/A$) is uniform only when the conductor is spherical. Generally, this nonuniformity makes the determination of the electric field set up by the surface charge very difficult.

Luckily, we can determine the electric field just outside the surface easily using Gauss' law. Consider the Gaussian surface shown in the figure.

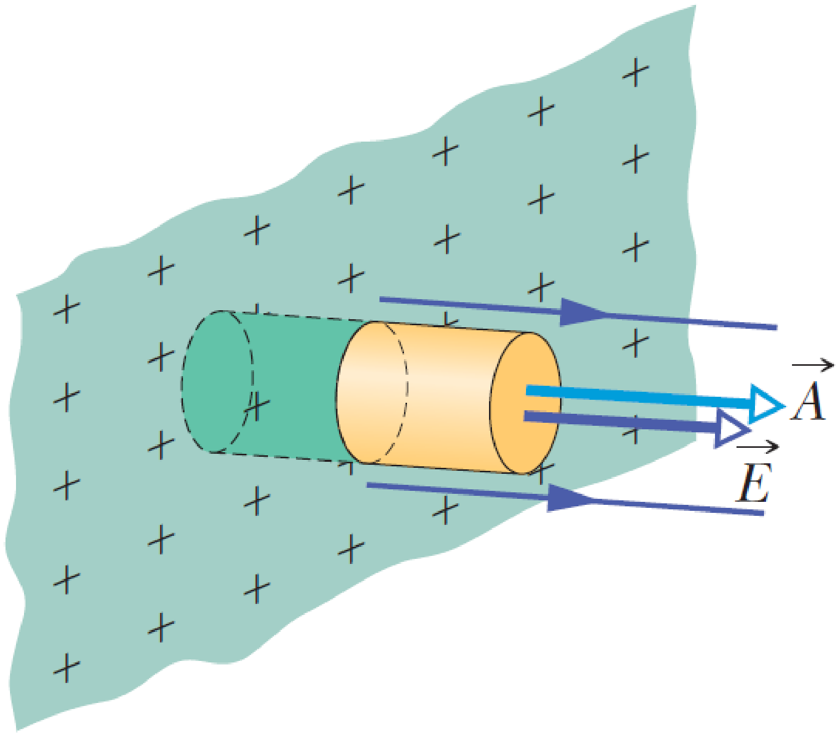


6. A Charged Isolated Conductor

<u>The External Electric Field</u>

The electric field $\vec{E}$ at and just outside the conductor surface must be perpendicular to the conductor surface. Otherwise, the electric field would have a component along the conductor surface, causing the charges to move.

The flux through the internal cap of the Gaussian surface is zero since there is no field inside the conductor. The flux through the curved surface of the cylinder is zero too because the electric field is parallel to the curved surface.



6. A Charged Isolated Conductor

<u>The External Electric Field</u>

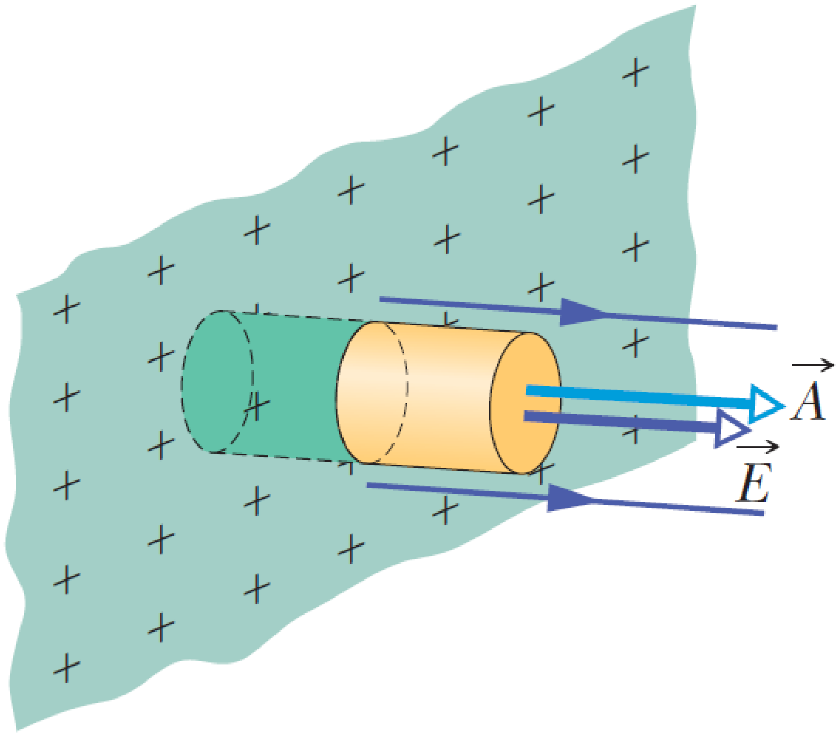
We assume that the cap is small enough that the electric is constant over the outer cap. The flux though the cap is then EA.

The charge q_{enc} enclosed by the Gaussian surface is σA, where σ is the charge per unit area. Gauss' law becomes

$$EA = \frac{\sigma A}{\varepsilon_0},$$

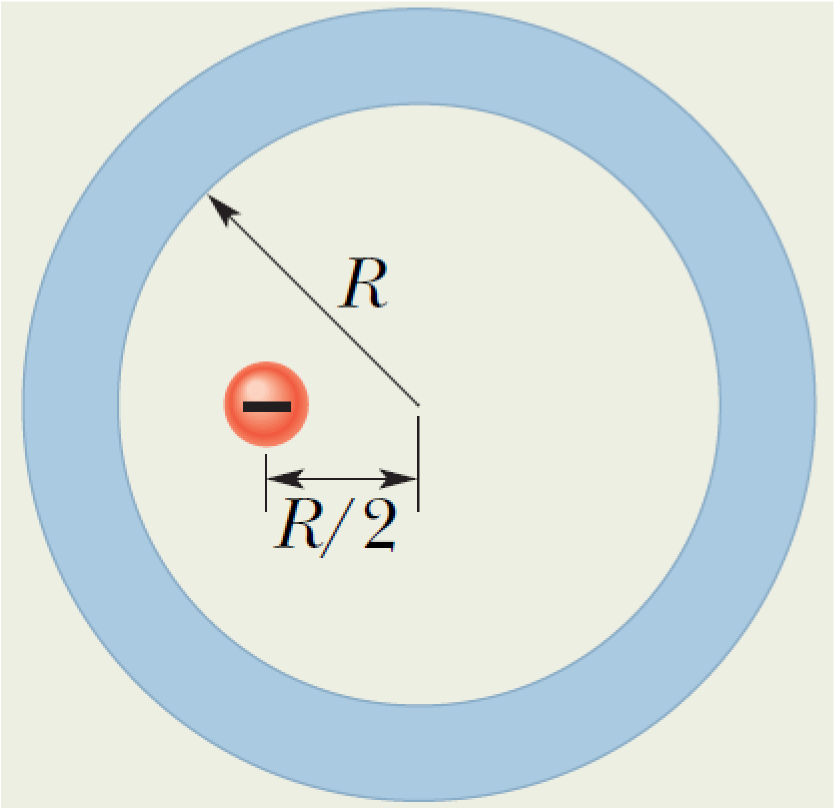
or

$$E = \frac{\sigma}{\varepsilon_0}.$$



6. A Charged Isolated Conductor

Example 4: The figure shows a cross section of a spherical metal shell of inner radius R. A point charge of $-5.0\,\mu\text{C}$ is located at a distance $R/2$ from the center of the shell. If the shell is electrically neutral, what are the (induced) charges on its inner and outer surfaces? Are those charges uniformly distributed?

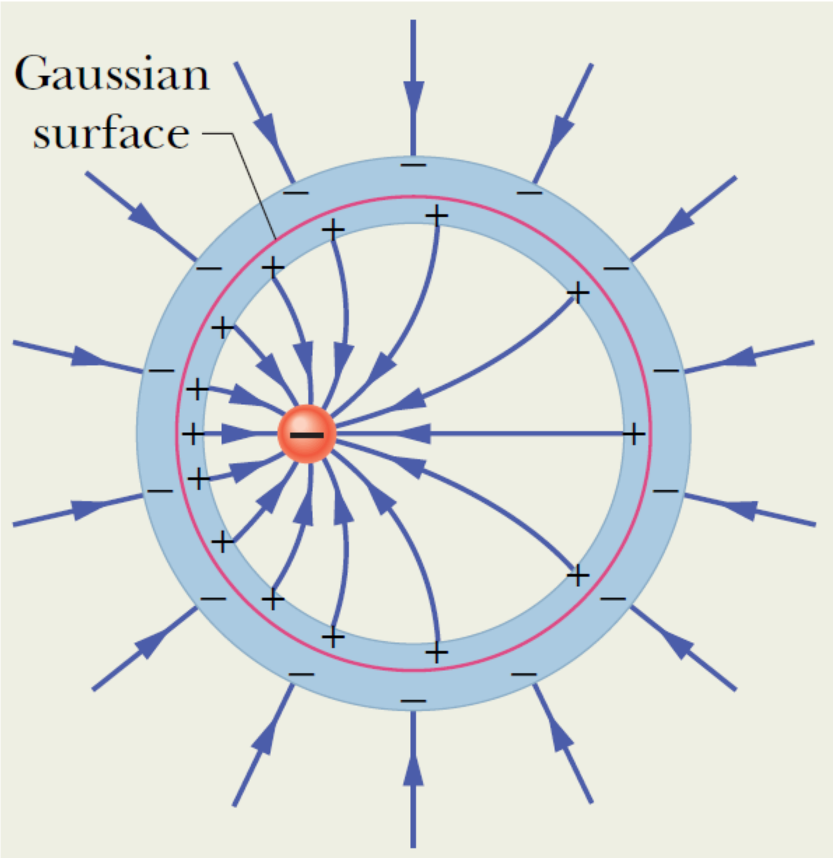


6. A Charged Isolated Conductor

The electric field must be zero inside the metal. This makes the flux through the Gaussian surface shown in the figure zero.

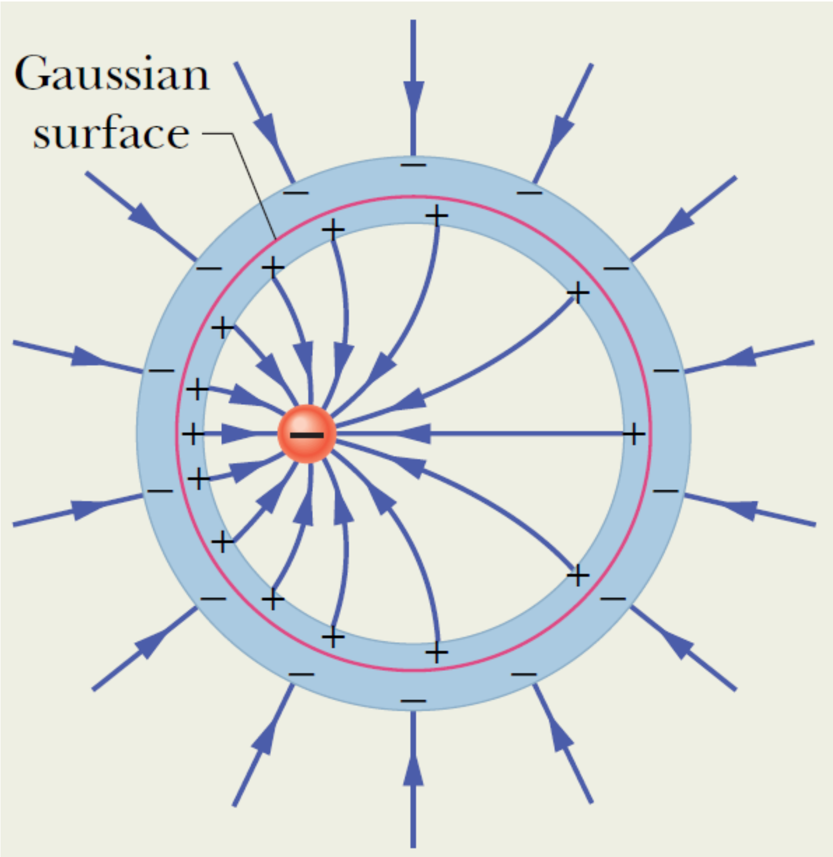
With a point charge of $-5.0\,\mu C$ within the shell, a charge of $+5.0\,\mu C$ must lie on the inner wall of the shell in order that the net enclosed charge be zero. This positive charge is not uniformly distributed.

The induced charge on the outer surface must be $-5.0\,\mu C$. This distribution of negative charge is uniform because the shell is spherical and because the skewed distribution of positive charge on the inner wall cannot produce an electric field in the shell to affect the distribution of charge on the outer wall.



6. A Charged Isolated Conductor

What is the field pattern inside and outside the shell?

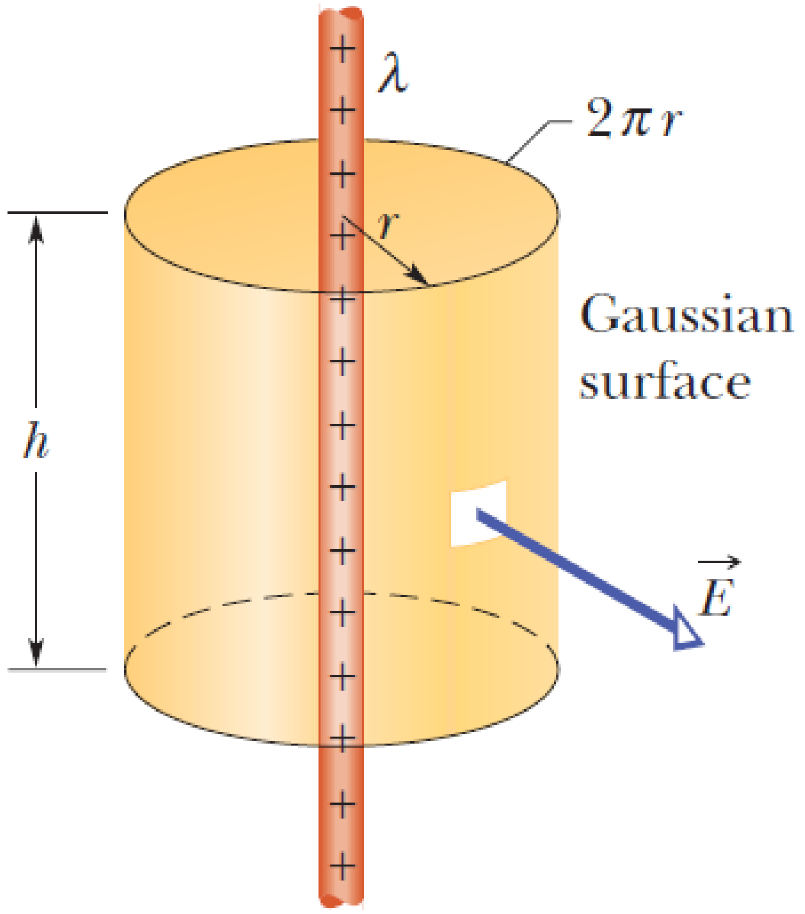


The field lines inside and outside the shell are shown approximately in the figure. All the field lines intersect the shell and the point charge perpendicularly. Inside the shell the pattern of field lines is skewed because of the skew of the positive charge distribution. Outside the shell the pattern is the same as if the point charge were centered and the shell were missing.

7. Applying Gauss' Law: Cylindrical Symmetry

The figure shows a section of an infinitely long cylindrical plastic rod with linear charge density λ. We want to find an expression for the electric field $\vec{E}$ at a distance r from the axis of the rod.

We choose a cylindrical Gaussian surface of radius r and length h, coaxial with the rod. The symmetry of the problem implies that $\vec{E}$ has the same magnitude at every point on the Gaussian surface, and the direction of $\vec{E}$ is radially outward for positive λ.



7. Applying Gauss' Law: Cylindrical Symmetry

The flux of $\vec{E}$ through the Gaussian surface is

$$\Phi = EA\cos\theta = E(2\pi rh)\cos 0 = E(2\pi rh).$$

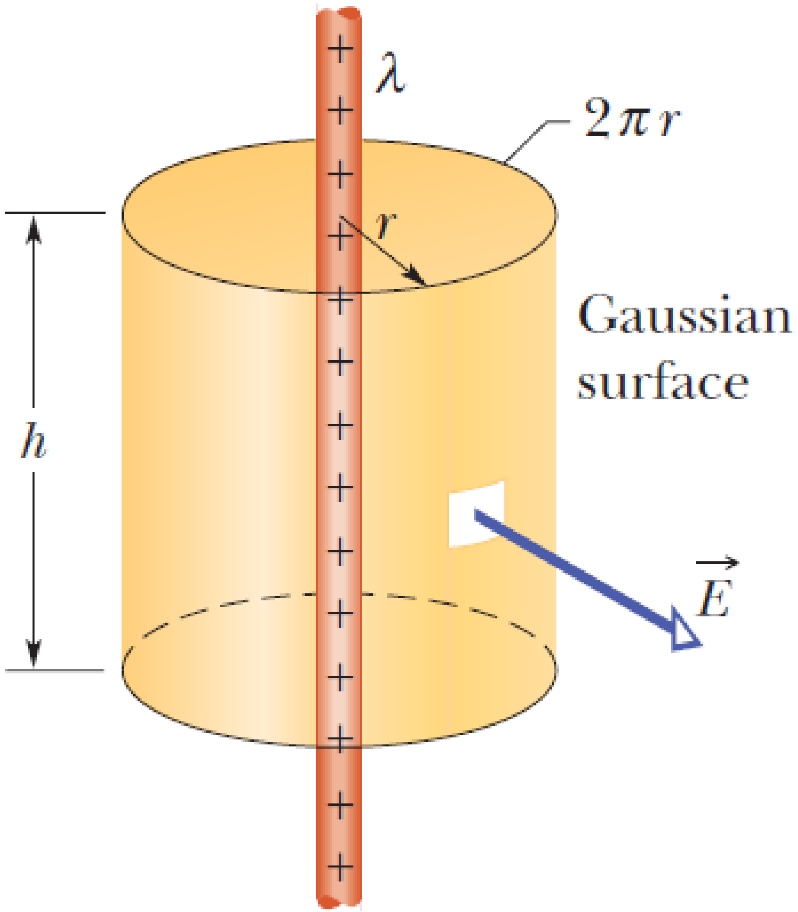
The charge enclosed by the Gaussian surface is λh. Gauss' law becomes

$$E(2\pi rh) = \frac{\lambda h}{\varepsilon_0},$$

yielding

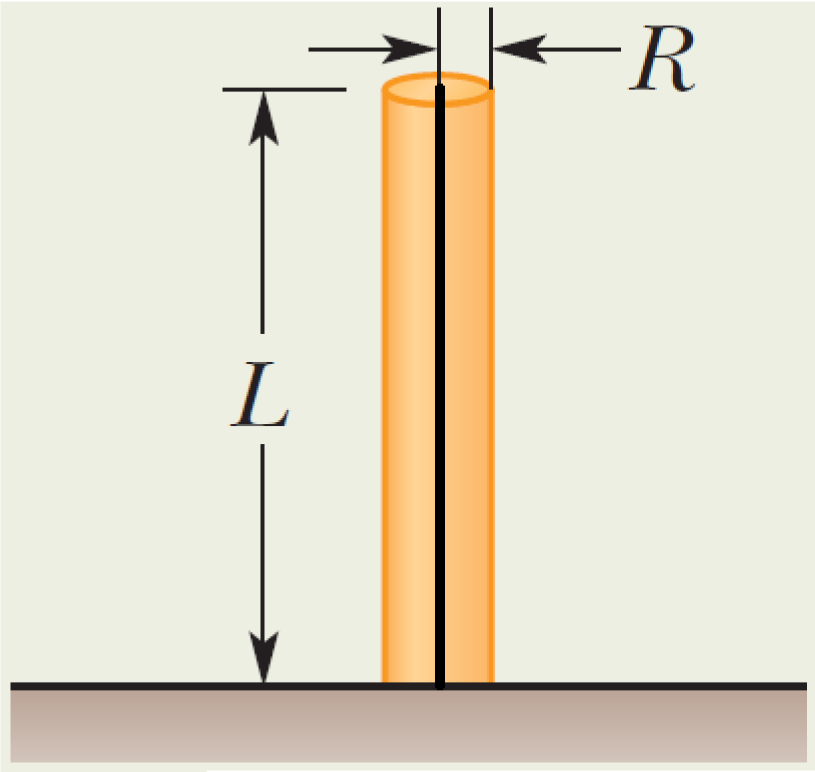
$$E = \frac{\lambda}{2\pi\varepsilon_0 r}.$$

Note that the electric field here is proportional to $1/r$!



7. Applying Gauss' Law: Cylindrical Symmetry

Example 5: The figure shows a narrow vertical cylinder of height $L = 1.8\ \text{m}$ and radius $R = 0.10\ \text{m}$. Assume that charge Q is uniformly distributed along a thin wire along the cylinder's axis. What value of Q would have put the air at the cylinder's surface to have electric field $E_c = 2.4\ \text{MN/C}$?



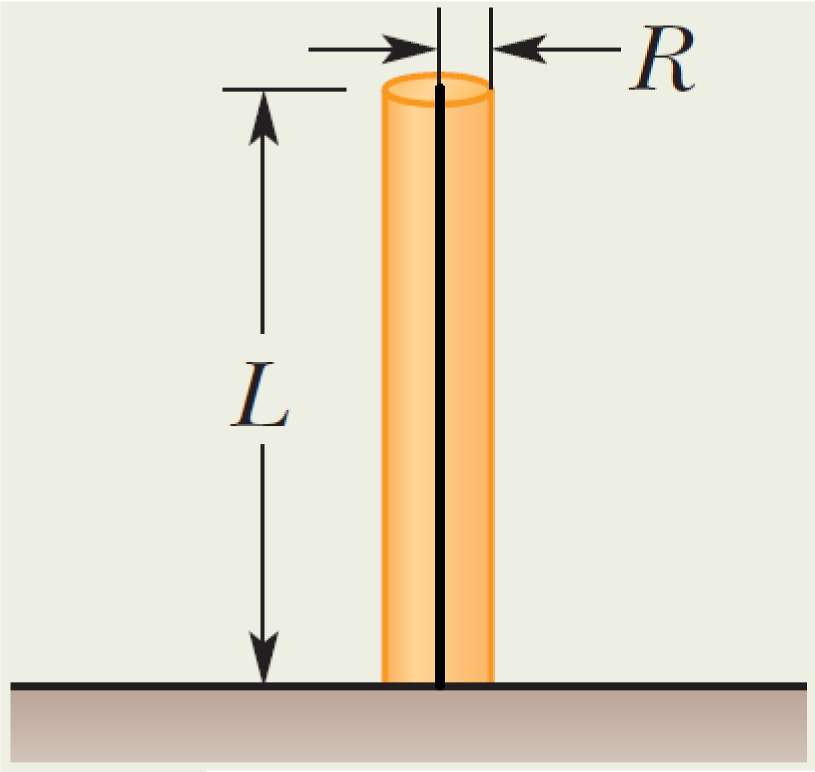
Because $R \ll L$, we can approximate the charge distribution as a long line of charge. Thus, for points that are not too near the ends (compared with R),

$$E = \frac{\lambda}{2\pi\varepsilon_0 R} = \frac{Q/L}{2\pi\varepsilon_0 R}.$$

7. Applying Gauss' Law: Cylindrical Symmetry

Solving for Q and substituting gives

$$Q = 2\pi\varepsilon_0 RLE = 2\pi\varepsilon_0 RLE_c$$
$$= 2\pi(8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2})$$
$$\times (0.10\ \mathrm{m})(1.8\ \mathrm{m})(2.4\ \mathrm{MN/C})$$
$$= 24\ \mu\mathrm{C}.$$

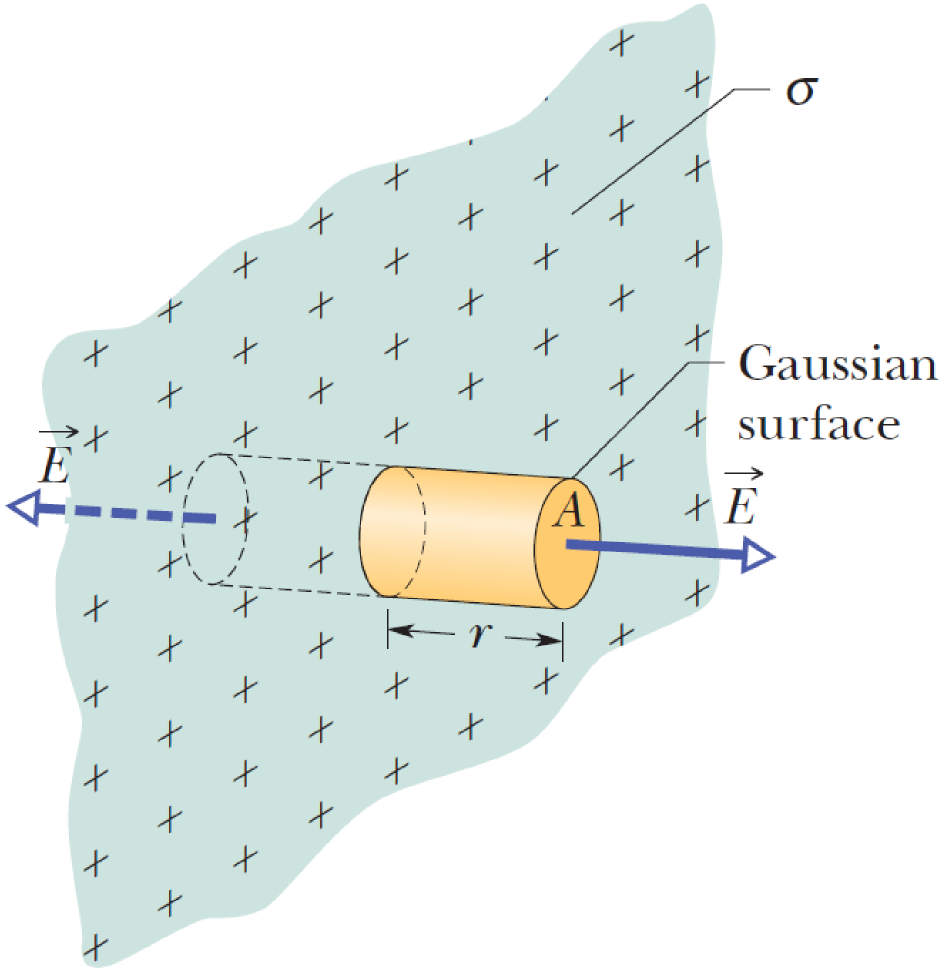


8. Applying Gauss' Law: Planar Symmetry

<u>Nonconducting Sheet</u>:

The figure shows a portion of a thin infinite, nonconducting sheet with a uniform (positive) surface charge density σ. We want to find the electric field $\vec{E}$ a distance r in front of the sheet.

A Gaussian surface is shown. From the symmetry of the problem, $\vec{E}$ must be perpendicular to the sheet, directed away from the surface.



8. Applying Gauss' Law: Planar Symmetry

<u>Nonconducting Sheet</u>:

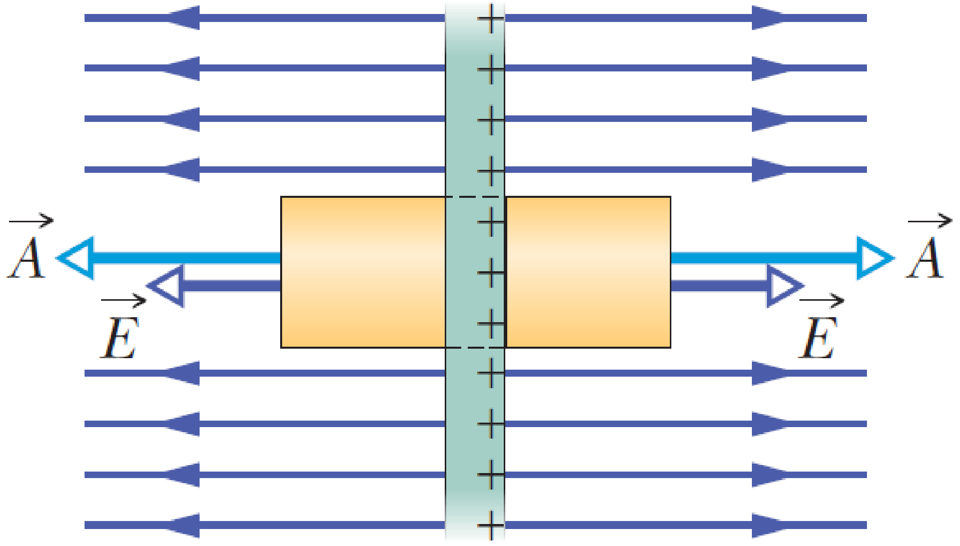
The flux through each of the end caps of the Gaussian surface is EA. Gauss' law reads

$$(EA + EA) = \frac{\sigma A}{\varepsilon_0},$$

or

$$E = \frac{\sigma}{2\varepsilon_0}.$$

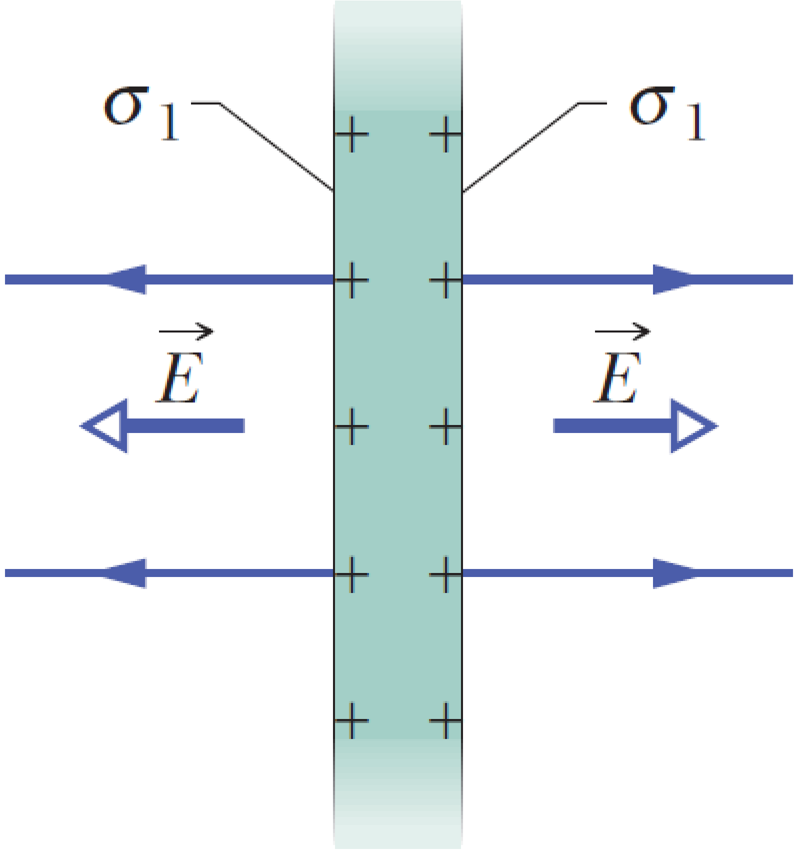
The electric field is independent of the distance to the sheet!



8. Applying Gauss' Law: Planar Symmetry

<u>Two Conducting Plates</u>:

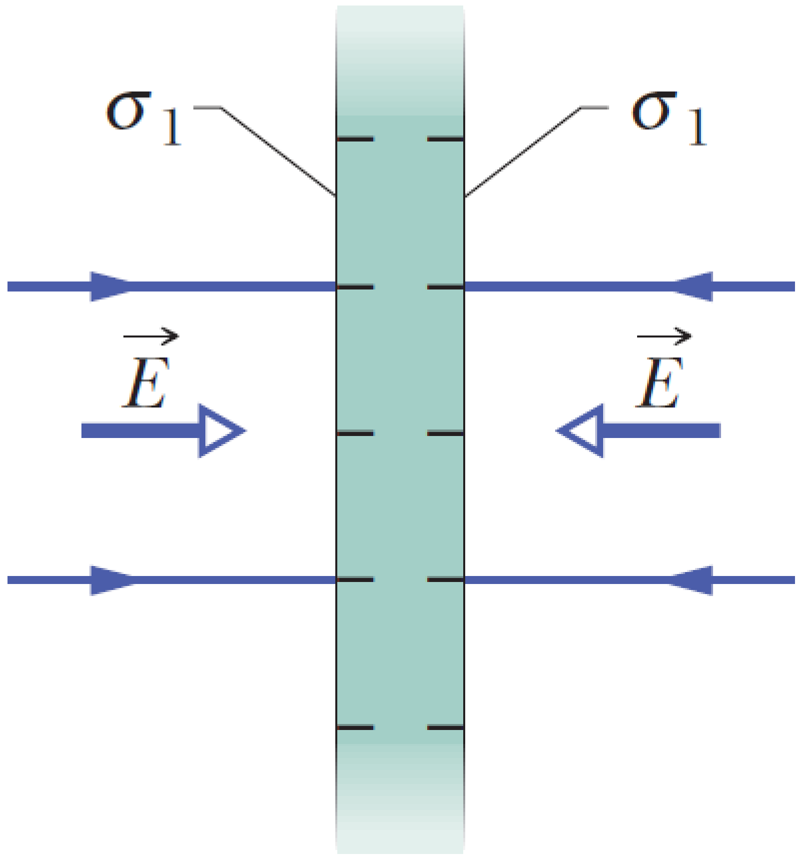
Consider the conducting plate shown in the figure. The magnitude of the electric field outside the plate is $E = \sigma_1/\varepsilon_0$. It points away from the plane.



8. Applying Gauss' Law: Planar Symmetry

Two Conducting Plates:

The electric field has the same magnitude when the charge density is negative. The electric field points toward the plate.



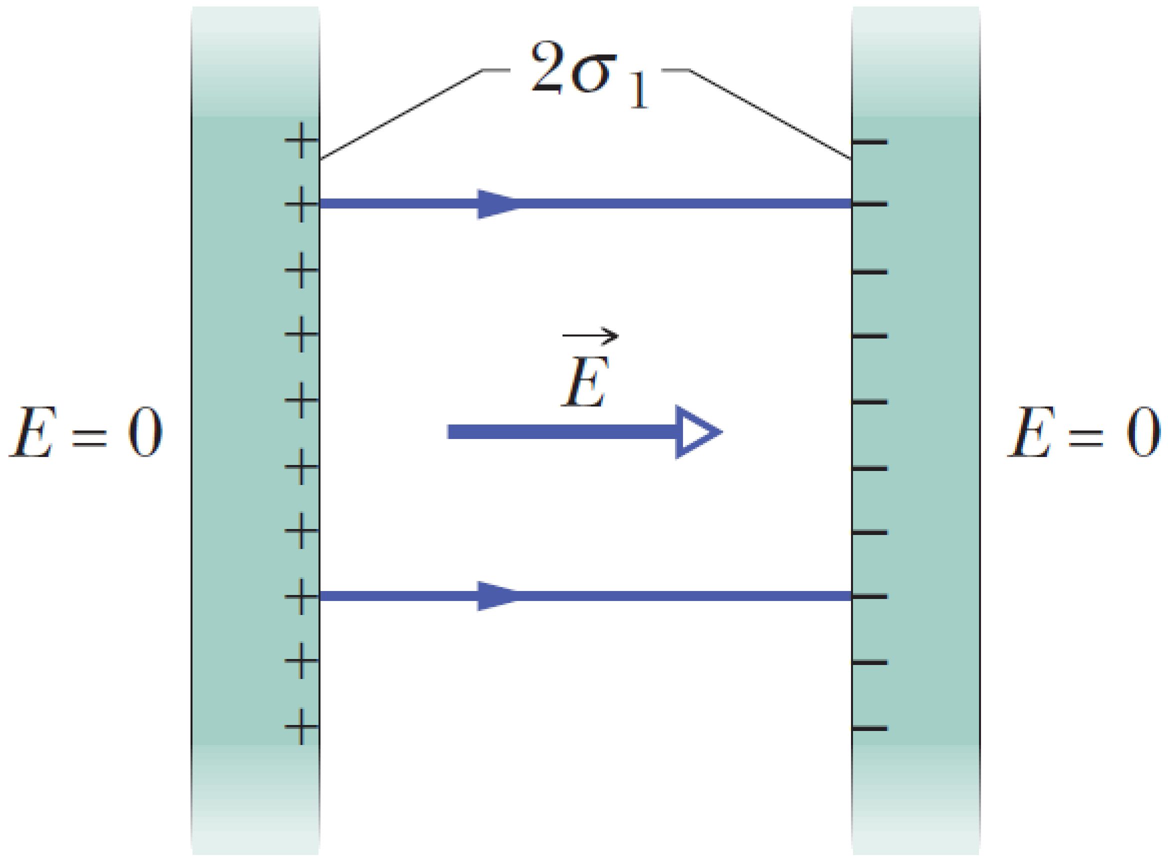
8. Applying Gauss' Law: Planar Symmetry

Two Conducting Plates:

When the two plates are brought near each other, all the excess charge move onto the inner faces of the plates, as in the figure. The electric field at any point between the plates becomes

$$E = \frac{2\sigma_1}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}.$$

The electric field is directed away from the positive plate, toward the negative plate. The electric field is zero to the left and right of the plates.

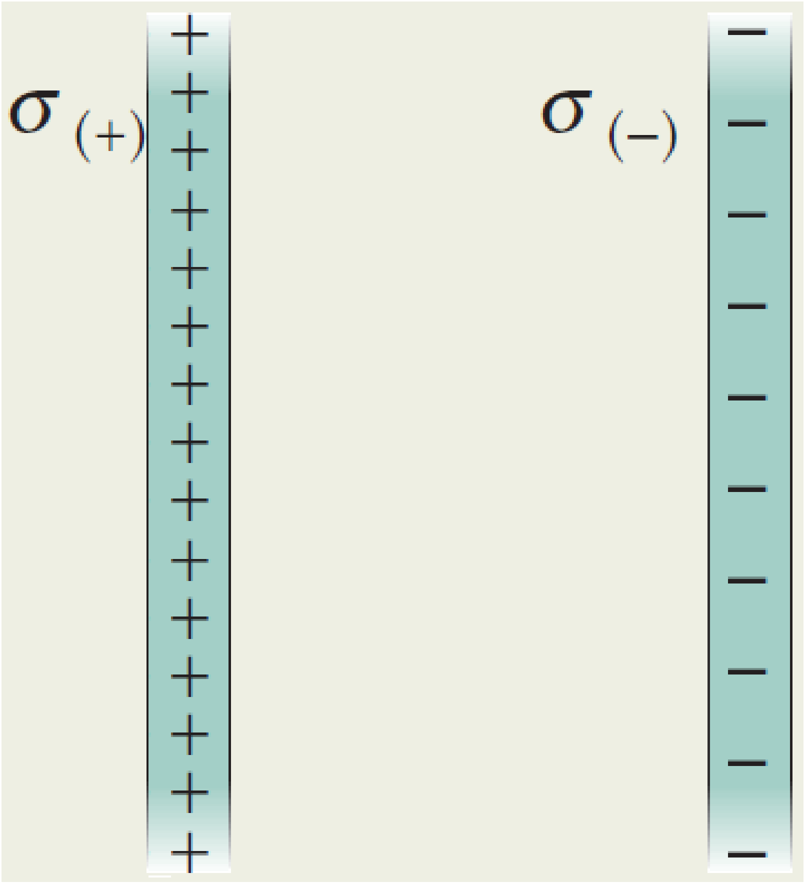


8. Applying Gauss' Law: Planar Symmetry

Example 6: The figure shows portions of two large, parallel, nonconducting sheets, each with a fixed uniform charge. The magnitudes of the surface charge densities are $\sigma_{(+)} = 6.8\,\mu\mathrm{C/m^2}$ for the positively charged sheet and $\sigma_{(-)} = 4.3\,\mu\mathrm{C/m^2}$ for the negatively charged sheet.

Find the electric field $\vec{E}$ (a) to the left of the sheets, (b) between the sheets, and (c) to the right of the sheets.

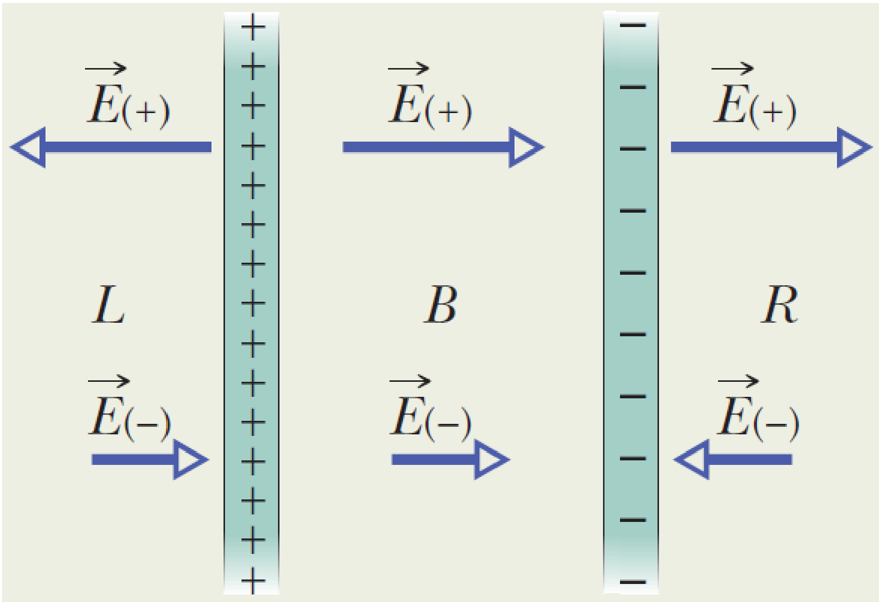
We first calculate the electric field due to each sheet and then use the superposition principle to find the net electric field $\vec{E}$.



8. Applying Gauss' Law: Planar Symmetry

The electric field field $\vec{E}_{(+)}$ due to the positive sheet is directed *away* from the sheet and has the magnitude

$$E_{(+)} = \frac{\sigma_{(+)}}{2\varepsilon_0} = \frac{6.8 \times 10^{-6}\ \mathrm{C/m^2}}{(2)8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}}$$
$$= 3.84 \times 10^5 \mathrm{N/C}.$$



The electric field field $\vec{E}_{(-)}$ due to the negative sheet is directed *toward* the sheet and has the magnitude

$$E_{(-)} = \frac{\sigma_{(-)}}{2\varepsilon_0} = \frac{4.3 \times 10^{-6}\ \mathrm{C/m^2}}{(2)8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}}$$
$$= 2.43 \times 10^5 \mathrm{N/C}.$$

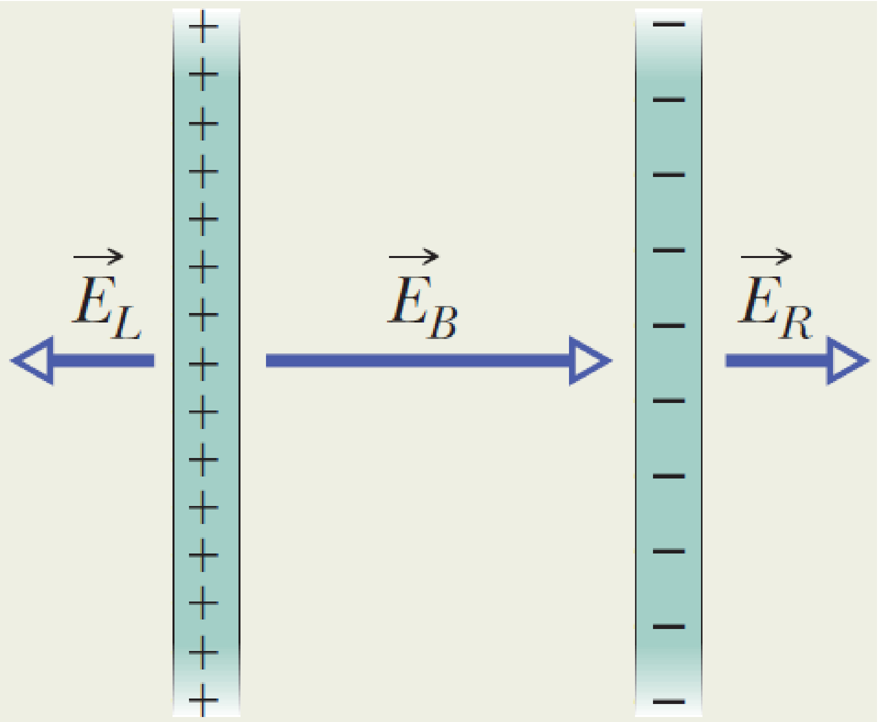
8. Applying Gauss' Law: Planar Symmetry

The magnitude of the net field $\vec{E}_B$ between the sheets is

$$E_B = E_{(+)} + E_{(-)}$$
$$= 3.84 \times 10^5 \text{ N/C} + 2.43 \times 10^5 \text{ N/C}$$
$$= 6.3 \times 10^5 \text{ N/C.}$$

The magnitudes of the left and right net fields $\vec{E}_L$ and $\vec{E}_R$ are

$$E_L = E_R = E_{(+)} - E_{(-)}$$
$$= 3.84 \times 10^5 \text{N/C} - 2.43 \times 10^5 \text{ N/C}$$
$$= 1.4 \times 10^5 \text{ N/C.}$$



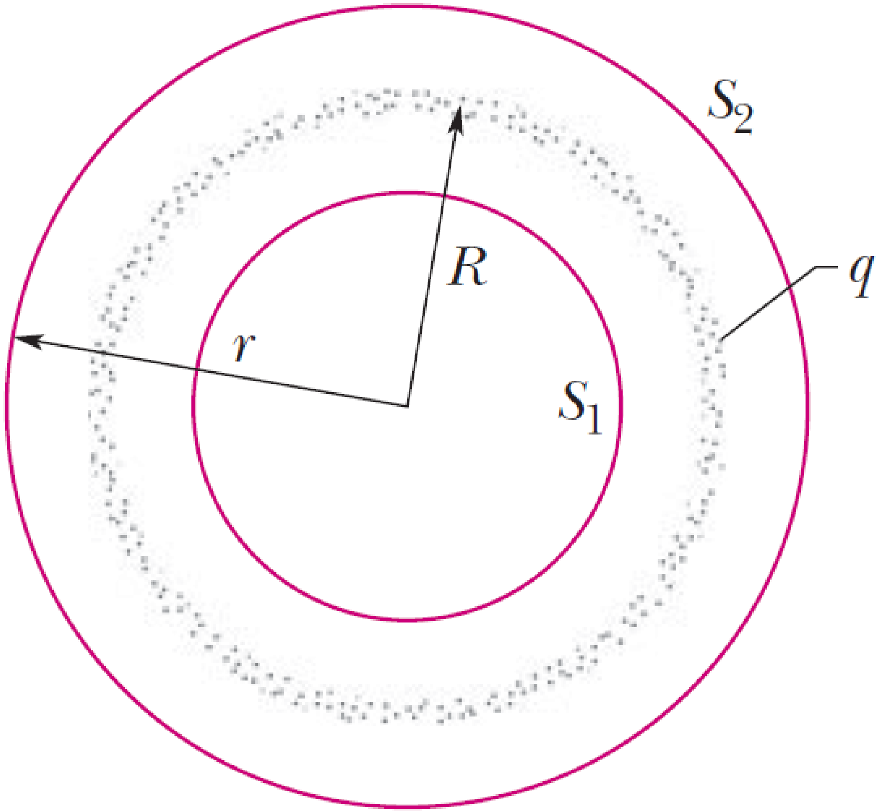
9. Applying Gauss' Law: Spherical Symmetry

We want to prove the shell theorem, presented before, using Gauss' law.

"A shell of uniform charge attracts or repels a charged particle that is outside the shell as if all the shell's charge were concentrated at the center of the shell."

"If a charged particle is located inside a shell of uniform charge, there is no electrostatic force on the particle from the shell."

Consider the charged spherical shell of total charge q and radius R.



9. Applying Gauss' Law: Spherical Symmetry

Applying Gauss' law to surface S_2 we find that

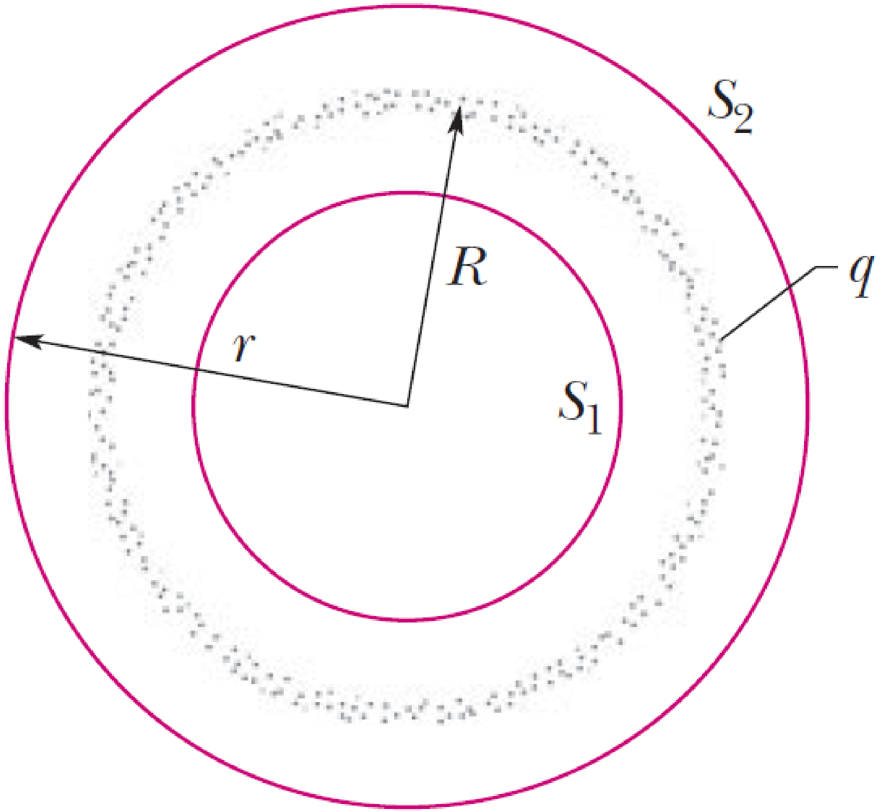
$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}. \qquad (r \geq R)$$

This proves the first shell theorem.

Applying Gauss' law to surface S_1 leads to

$$E = 0, \qquad (r < R)$$

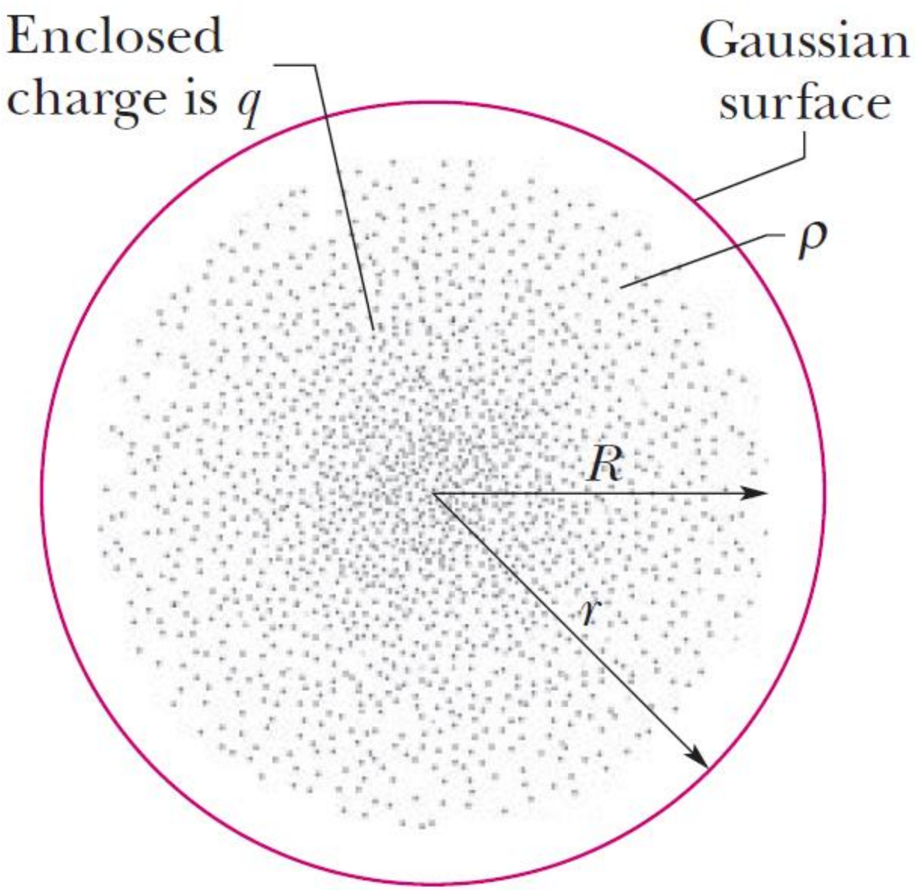
which proves the second shell theorem.



9. Applying Gauss' Law: Spherical Symmetry

We can apply Gauss' law to any **spherically symmetric** charge distribution such as that of the figure. Spherical symmetry implies that the charge density $\rho = q/V$ has a single value for each shell. Equivalently, ρ should be function of r only.

When $r > R$, the Gaussian surface encloses all the charge and the electric field is identical to the electric field of a point charge q.



9. Applying Gauss' Law: Spherical Symmetry

When $r \leq R$, the enclosed charge is q' and we get that

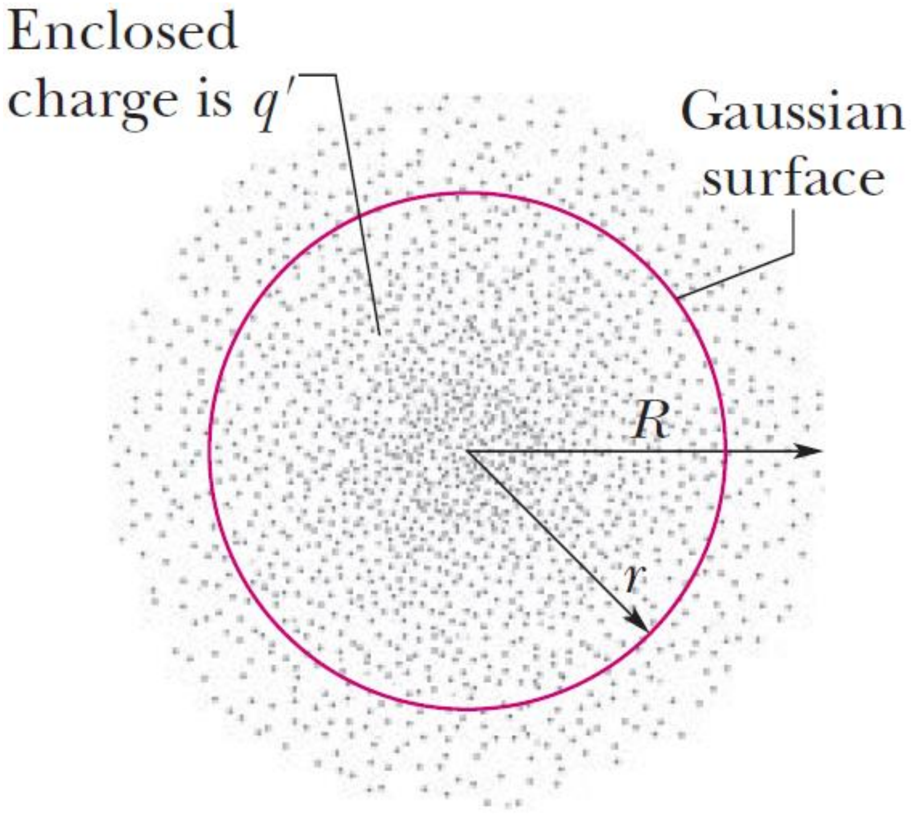
$$E = \frac{1}{4\pi\varepsilon_0}\frac{q'}{r^2}. \qquad (r \leq R).$$

If the charge q within R is uniform, or ρ is constant, then

$$q' = V_r\rho = V_r\frac{q}{V_R} = q\frac{4/3\,\pi r^3}{4/3\,\pi R^3} = q\frac{r^3}{R^3}.$$

The electric field becomes

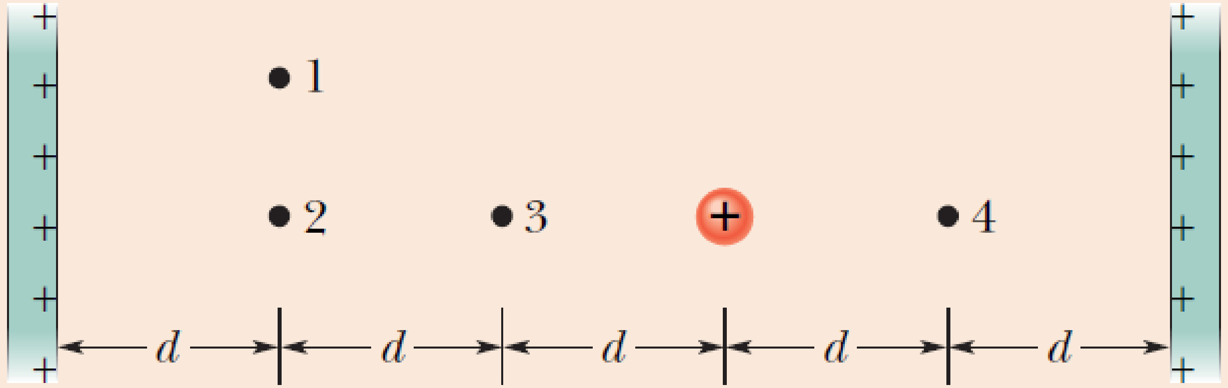
$$E = \frac{q}{4\pi\varepsilon_0 R^3}r. \qquad (r \leq R)$$



9. Applying Gauss' Law: Spherical Symmetry

CHECKPOINT 4

The figure shows two large, parallel, nonconducting sheets with identical (positive) uniform surface charge densities, and a sphere with a uniform (positive) volume charge density. Rank the four numbered points according to the magnitude of the net electric field there, greatest first.



3 & 4, 2 then 1.