Chapter 16

WAVES I

1. Types of Waves

https://youtu.be/KVc7oBKzq9U?t=3m49s

There are three main types of waves:

1. **Mechanical waves**: These are the most familiar waves. Examples include water waves, sound waves and seismic waves. Mechanical waves have two central features: (1) They are governed by Newton's laws and (2) they can exist only within a material medium, such as water, air and rock.
2. **Electromagnetic waves**: Examples include visible light, infrared light, ultraviolet light, radio waves, microwaves and x rays. Electromagnetic waves require no martial medium to exist. They travel through vacuum at the same speed $c = 299\,792\,458\ \mathrm{m/s}$.
3. **Matter waves**: They are associated with subatomic particles, atoms and molecules. Because these particles constitute matter, such waves are called matter waves.

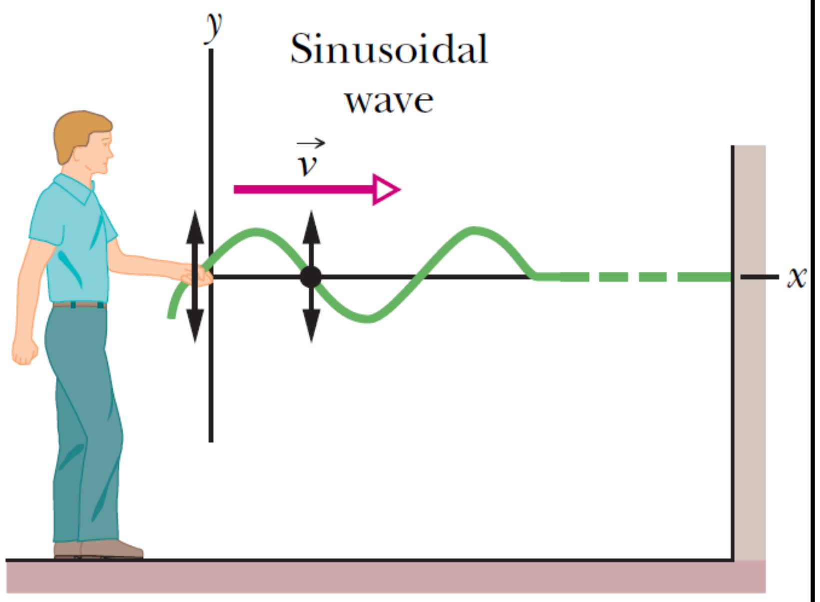
2. Transverse and Longitudinal Waves

If you move one end of a stretched wire up and down in a continuous SHM, a continuous sinusoidal wave travels along the string at velocity $\vec{v}$.

One way to study such a wave is to monitor the **wave forms** (shapes of the waves) as they move to the right.

Another way is to monitor the motion of an element of the string as the element oscillates up and down while the wave passes through it.

The displacement of every such oscillating string element is perpendicular to the direction of travel of the wave. This motion is called **transverse** and the wave is called a **transverse wave**.



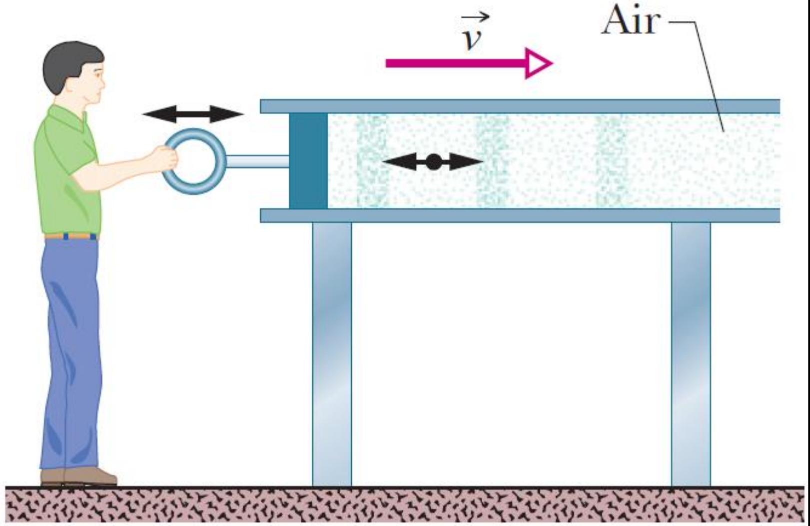
2. Transverse and Longitudinal Waves

As shown in the figure, if you push and pull on the piston in SHM, a sinusoidal wave travels along the pipe.

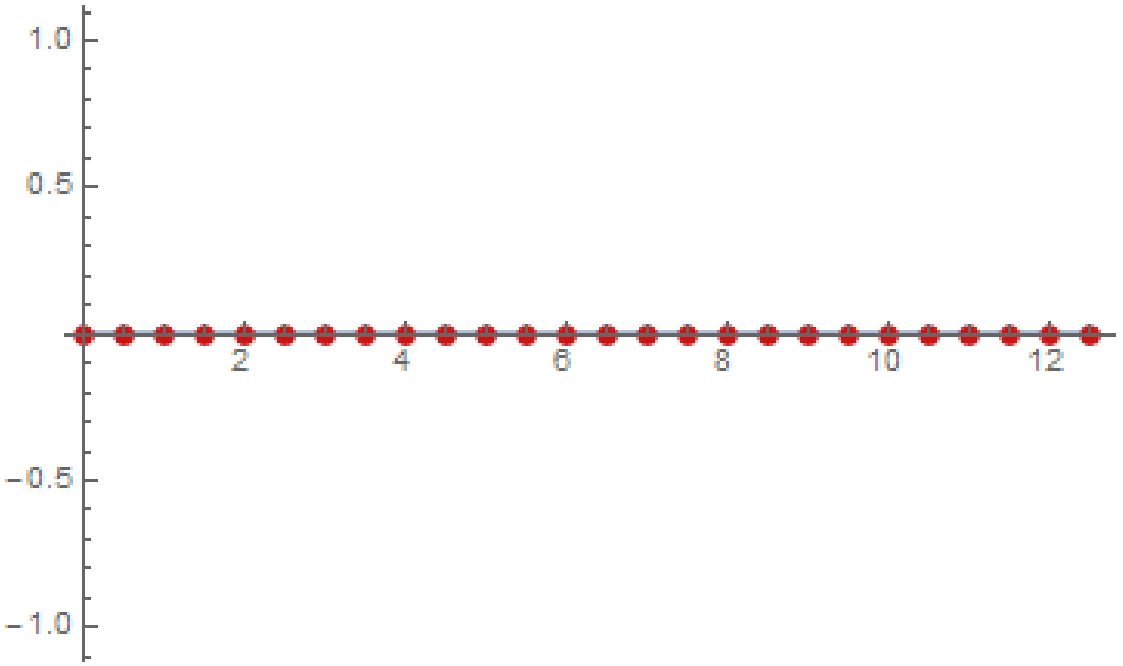
The motion of the elements of air is parallel to the direction of the wave's travel. The motion is therefore called **longitudinal** and the wave is called a **longitudinal wave**.

In this chapter we focus on transverse waves. We focus on longitudinal waves in the next chapter.

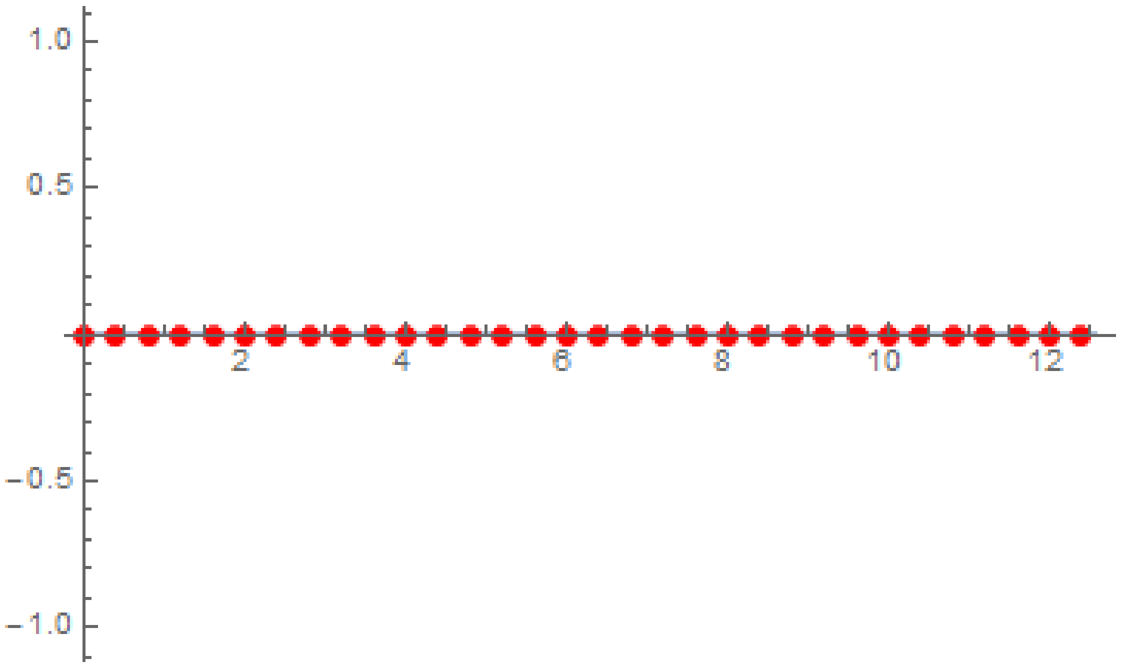
Both transverse and longitudinal waves are **travelling waves**. Note that it is the wave that travels, not the material (string or air) through which the wave moves!



2. Transverse and Longitudinal Waves



Transverse Wave



Longitudinal Wave

3. Wavelength and Frequency

To completely describe a wave on a string, we need a function that gives the shape of the wave. We need a relation of the form

$$y = h(x, t).$$

y is the transverse displacement at time t of an element of the string at position x.

In general, a sinusoidal wave can be described with h being either a sine or cosine function. We choose the sine in this chapter.

We write the displacement y of any element of the string as

$$y = y_m \sin(kx - \omega t)\,.$$

Let us define the quantities in this equation.

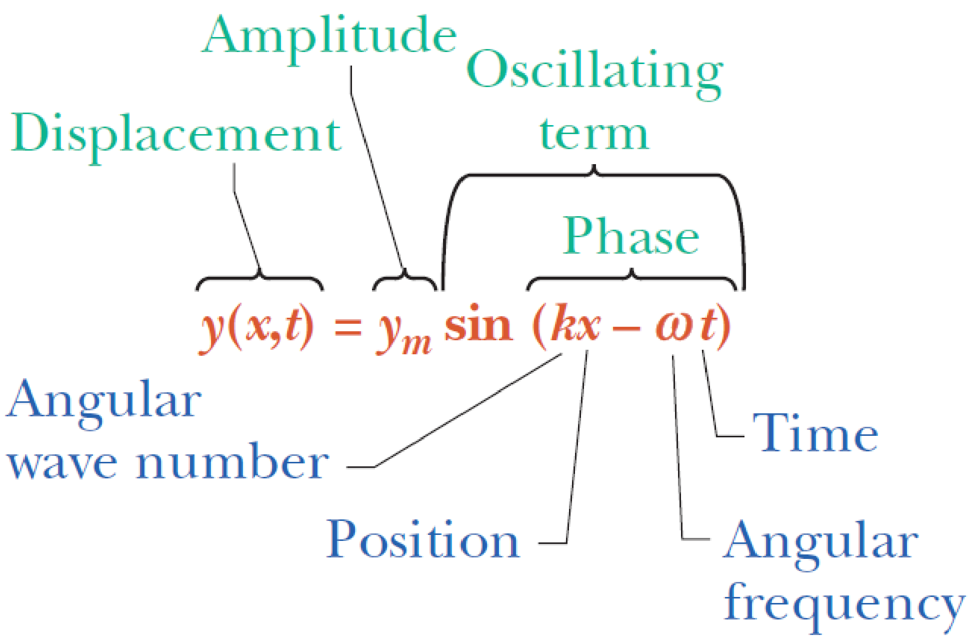
3. Wavelength and Frequency

Amplitude and Phase

The **amplitude** y_m of a wave is the magnitude of the maximum displacement of the elements from their equilibrium position. y_m is always positive.

The **phase** of the wave is the argument $kx - \omega t$ of the sine. As a wave sweeps through a string element at position x, the phase changes linearly with time.

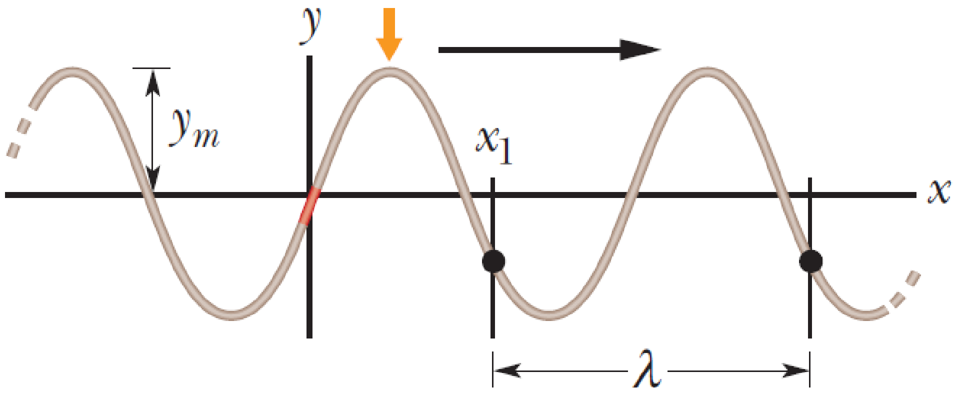
Thus, the sine also changes, oscillating between -1 and $+1$, with $+1$ corresponding to a **peak** of the wave and -1 corresponding to a **valley** of the wave.



3. Wavelength and Frequency

<u>Wavelength and Angular Wave Number</u>

The wavelength λ of a wave is the distance between repetitions of the shape of the wave.



At time $t = 0$, for example, the wave shape is given by

$$y(x, 0) = y_m \sin kx\,.$$

The displacement y is the same at both ends of a wavelength, or $y(x_1, 0) = y(x_1 + \lambda, 0)$. This gives

$$\begin{aligned} y_m \sin kx_1 &= y_m \sin k(x_1 + \lambda) \\ &= y_m \sin(kx_1 + k\lambda)\,. \end{aligned}$$

4. Wavelength and Frequency

<u>Wavelength and Angular Wave Number</u>

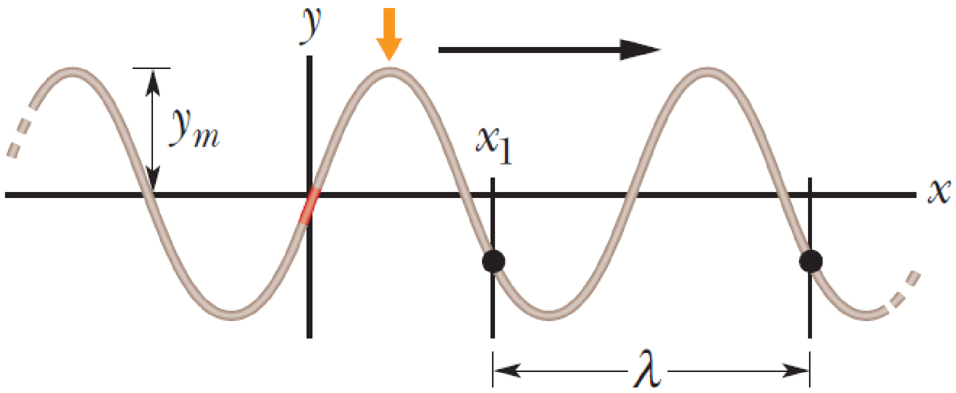
$$y_m \sin kx_1 = y_m \sin k(x_1 + \lambda)$$
$$= y_m \sin(kx_1 + k\lambda)\,.$$

This equation is true when $k\lambda = 2\pi$, or

$$k = \frac{2\pi}{\lambda}.$$

k is called the **angular wave number** of the wave. Its SI unit is $\mathrm{rad/m}$.

k can be thought of as the number of waves in a (2π) distance.

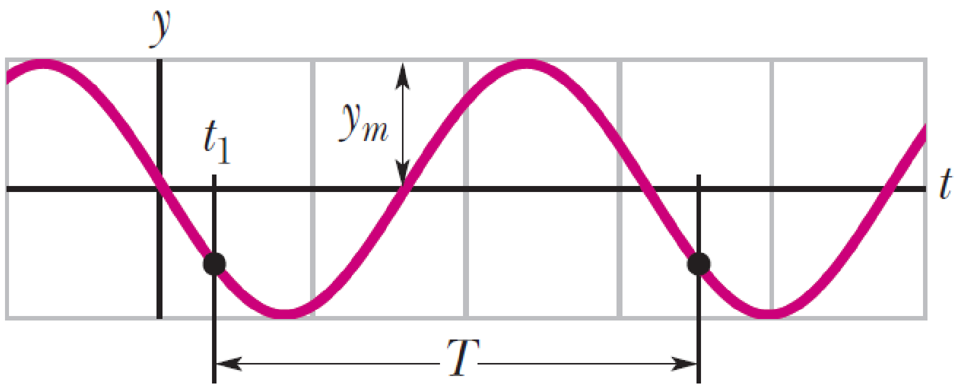


3. Wavelength and Frequency

Period, Angular Frequency, and Frequency

The element of the string at $x = 0$, for example, moves up and down in SHM given by

$$y(0,t) = y_m \sin(-\omega t)$$
$$= -y_m \sin \omega t\,.$$



We define the period of oscillation T of a wave to be the time any string element takes to move through one full oscillation.

The displacement y is the same at both ends of a period, or $y(0,t_1) = y(0,t_1+T)$. This gives

3. Wavelength and Frequency

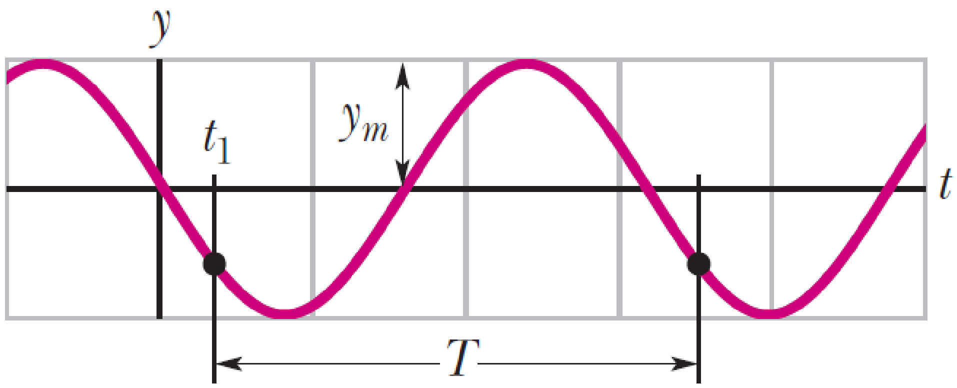
<u>Period, Angular Frequency, and Frequency</u>

$$-y_m \sin \omega t_1 = -y_m \sin \omega(t_1 + T)$$
$$= -y_m \sin(\omega t_1 + \omega T)\,.$$

This is true when $\omega T = 2\pi$, or

$$\omega = \frac{2\pi}{T}\,.$$

We call ω the **angular frequency** of the wave. Its SI unit is $\mathrm{rad/s}$.



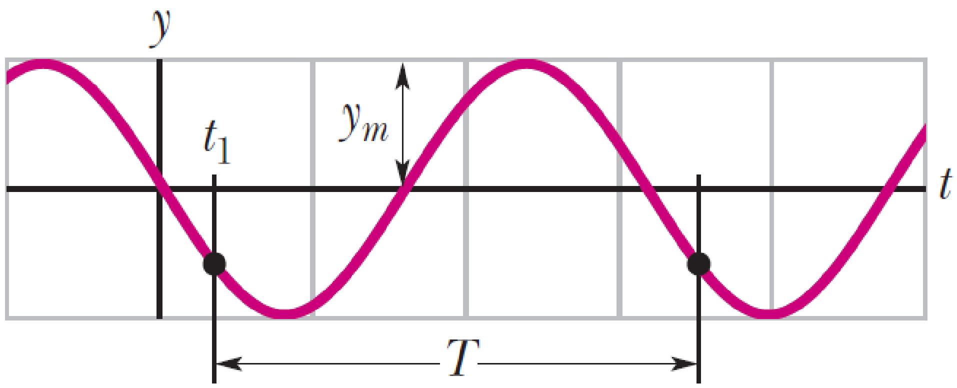
3. Wavelength and Frequency

Period, Angular Frequency, and Frequency

The **frequency** f of a wave is defined as $1/T$:

$$f = \frac{1}{T} = \frac{\omega}{2\pi}.$$

f is the number of oscillations per unit time. Its SI unit is $1/s$ or **hertz**.

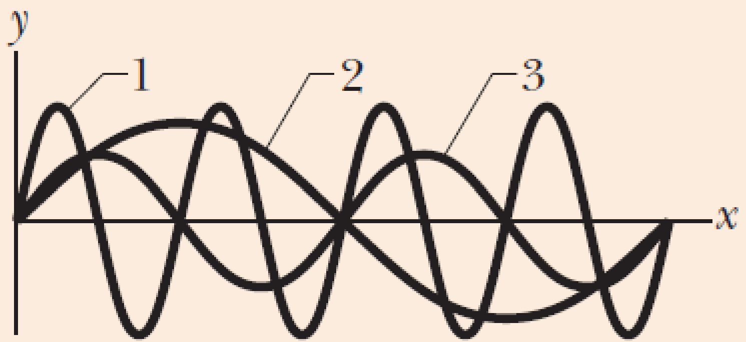


https://phet.colorado.edu/sims/wave-on-a-string/wave-on-a-string_en.html

3. Wavelength and Frequency

CHECKPOINT 1

The figure is a composite of three snapshots, each of a wave traveling along a particular string. The phases for the waves are given by (a) $2x - 4t$, (b) $4x - 8t$, and (c) $8x - 16t$. Which phase corresponds to which wave in the figure?



(a) 2

(b) 3

(c) 1

$$y = y_m \sin(kx - \omega t)$$

$$\lambda = \frac{2\pi}{k}$$

3. Wavelength and Frequency

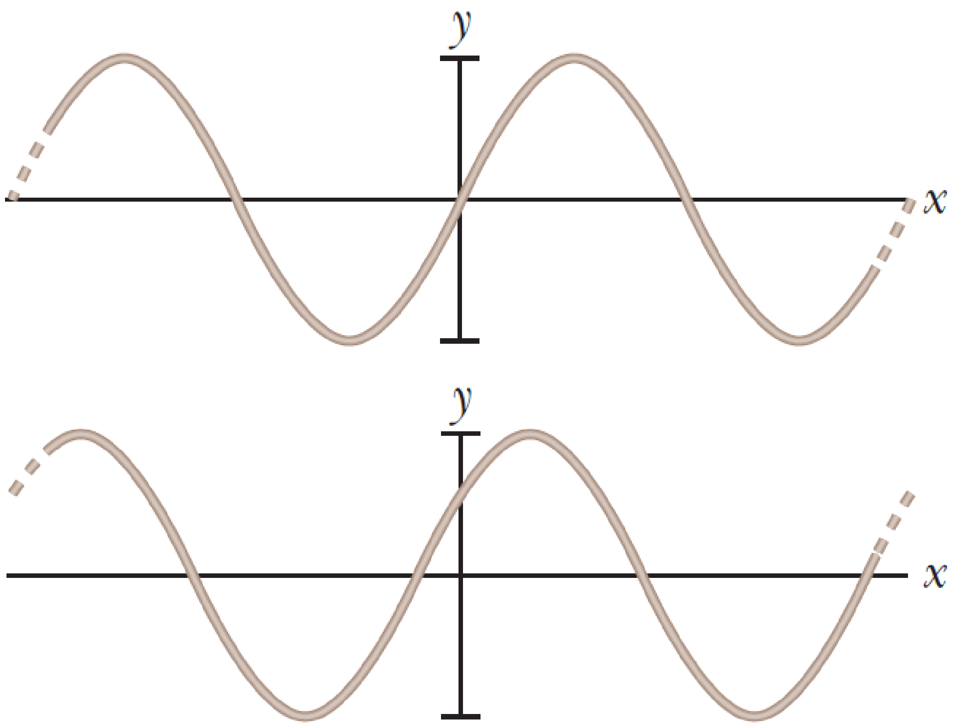
Phase Constant

Note that at $x = 0$ and $t = 0$, $y = y_m \sin 0 = 0$ and the slope is maximum. We have therefore considered a special case of sinusoidal travelling waves.

We can generalize the expression for $y(x, t)$ by inserting a **phase constant** ϕ in the wave equation:

$$y(x, t) = y_m \sin(kx - \omega t + \phi).$$

We can choose the value of ϕ so that the function gives some other displacement and slope at $x = 0$ and $t = 0$.

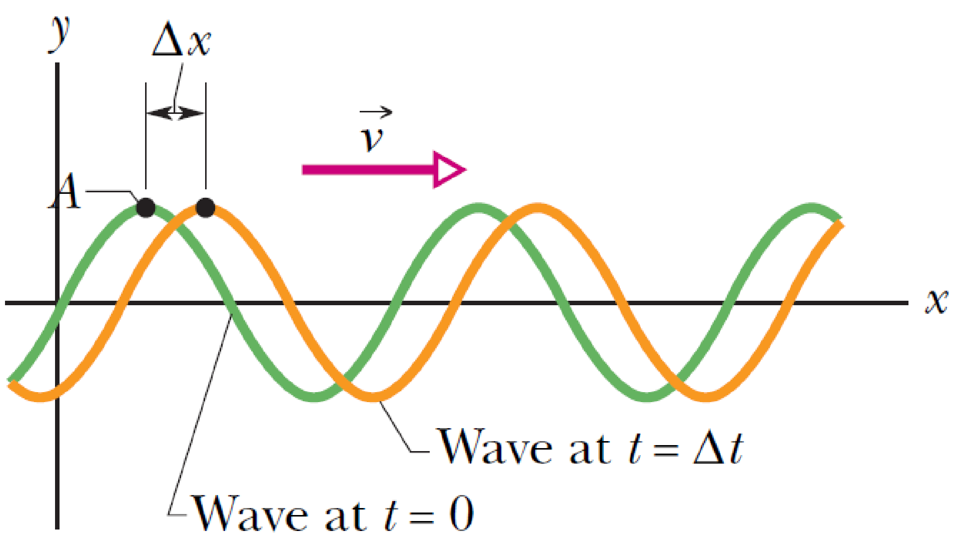


4. The Speed of a Traveling Wave

The figure shows a wave at two times, Δt apart. The entire wave moved a distance Δx. The **wave speed** v is the ratio $\Delta x/\Delta t$ (or dx/dt).

Each point on the wave form (e.g. A), retains its displacement y as the wave moves. For point A to retain its displacement y, the phase must remain constant:

$$kx - \omega t = \text{constant}.$$



4. The Speed of a Traveling Wave

Differentiating with respect to time gives

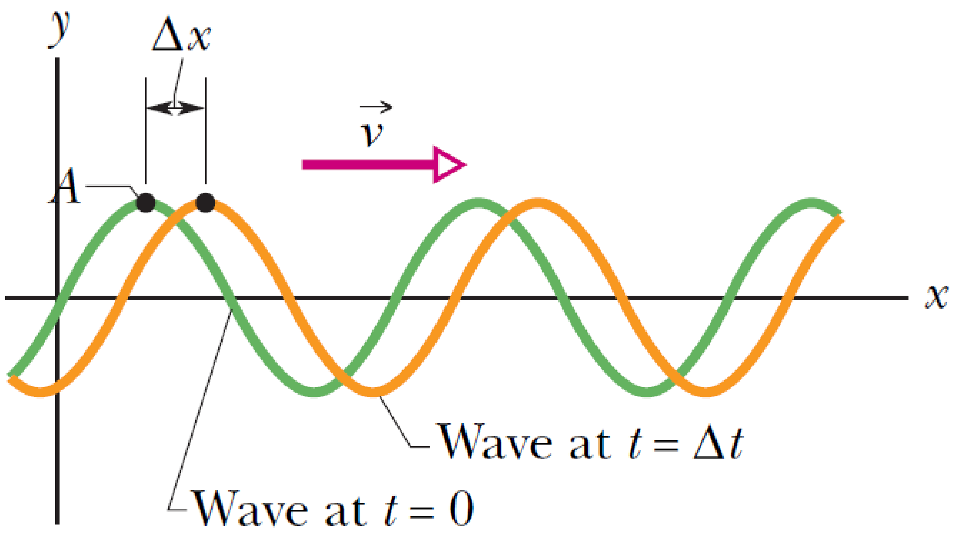
$$k\frac{dx}{dt} - \omega = 0,$$

or

$$\frac{dx}{dt} = v = \frac{\omega}{k}.$$

Using $k = 2\pi/\lambda$ and $\omega = 2\pi/T$, we can write v as

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f.$$



4. The Speed of a Traveling Wave

We can write the equation for a wave traveling in the opposite direction by replacing t with $-t$ in the phase to obtain

$$kx + \omega t = \text{constant.}$$

The equation for a wave traveling in the negative x direction is therefore

$$y(x,t) = y_m \sin(kx + \omega t + \phi)\,.$$

In general, a traveling wave of arbitrary shape is given by

$$y(x,t) = h(kx \pm \omega t),$$

where h can be any function. h represents a traveling wave since x and t enter h into the combination $kx \pm \omega t$.

4. The Speed of a Traveling Wave

CHECKPOINT 2

Here are the equations of three waves:
(1) $y(x,t) = 2\sin(4x - 2t)$, (2) $y(x,t) = \sin(3x - 4t)$, (3) $y(x,t) = 2\sin(3x - 3t)$.
Rank the waves according to their (a) wave speed and (b) maximum speed perpendicular to the wave's direction of travel (the transverse speed), greatest first.

(a) 2, 3, 1

(b) 3, 1 and 2 tie.

$$v = \frac{\omega}{k}$$

$$y(x,t) = y_m \sin(kx - \omega t + \phi)$$

$$u(x,t) = \frac{\partial y}{\partial t} = -y_m \omega \cos(kx - \omega t + \phi)$$

$$u(x,t) = -u_m \cos(kx - \omega t + \phi)$$

$$u_m = y_m \omega$$

4. The Speed of a Traveling Wave

Example 1: A wave traveling along a string is described by

$$y(x,t) = 0.00327 \sin(72.1x - 2.72t)\,,$$

in which the numerical constants are in SI units (0.00327 m, 72.1 rad/m, and 2.72 rad/s).

(a) What is the amplitude of this wave?

Comparison with the general wave equation

$$y = y_m \sin(kx - \omega t + \phi)\,,$$

reveals that $y_m = 0.00327$ m.

(b) What are the wavelength, period, and frequency of this wave?

4. The Speed of a Traveling Wave

By comparing with the general wave equation we see that $k = 72.1 \text{ rad/m}$, and $\omega = 2.72 \text{ rad/s}$.

The wavelength λ is related to k by

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{72.1 \text{ rad/m}} = 0.0871 \text{ m} = 8.71 \text{ cm}.$$

The period T is related to ω by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.72 \text{ rad/s}} = 2.31 \text{ s}.$$

The frequency is then

$$f = \frac{1}{T} = \frac{1}{2.31 \text{ s}} = 0.433 \text{ Hz}.$$

4. The Speed of a Traveling Wave

(c) What is the speed of this wave?

$$v = \frac{\omega}{k} = \frac{2.72\ \mathrm{rad/s}}{72.1\ \mathrm{rad/m}} = 0.0377\,\frac{\mathrm{m}}{\mathrm{s}} = 3.77\,\frac{\mathrm{cm}}{\mathrm{s}}.$$

(d) What is the displacement y of the string at $x = 22.5$ cm and $t = 18.9$ s?

$$\begin{aligned} y(0.225, 18.9) &= 0.00327 \sin(72.1 \times 0.225 - 2.72 \times 18.9) = 0.00192\ \mathrm{m} \\ &= 1.92\ \mathrm{mm}. \end{aligned}$$

(e) What is u, the transverse velocity of the string's element at $x = 22.5$ cm and $t = 18.9$ s?

In general,

$$u(x,t) = \frac{\partial y}{\partial t} = -y_m \omega \cos(kx - \omega t + \phi).$$

4. The Speed of a Traveling Wave

For our particular problem,

$$u(x,t) = -(0.00327)(2.72)\cos(72.1x - 2.72t)$$
$$= -0.00889\cos(72.1x - 2.72t)\,,$$

and

$$u(0.225,18.9) = -0.00889\cos(72.1 \times 0.225 - 2.72 \times 18.9)$$
$$= 0.00720\ \mathrm{m/s} = 7.20\ \mathrm{mm/s}.$$

(f) What is the transverse acceleration a_y of the same element at that time?

In general,

$$a_y(x,t) = \frac{\partial u}{\partial t} = -y_m\omega^2\sin(kx - \omega t + \phi) = -\omega^2 y(x,t).$$

4. The Speed of a Traveling Wave

Therefore,

$$a_y(0.225,18.9) = -(2.72)^2 y(0.225,18.9) = -0.0142\ \frac{\mathrm{m}}{\mathrm{s}^2} = -14.2\frac{\mathrm{mm}}{\mathrm{s}^2}.$$

5. Wave Speed on a Stretched String

The speed of a wave is set by the properties of the medium. When a wave travels through a medium, it causes the particles of that medium to oscillate. This oscillation requires both mass (for kinetic energy) and elasticity (for potential energy). Thus, the mass and elasticity of a medium determine the speed of a wave traveling though that medium.

Wave Speed Using dimensional analysis

For the mass, we use the **linear density** μ of a string which is the ratio of the string's mass m to the string's length l (or $\mu = m/l$). μ has the dimension of M/L.

We can associate the **tension** τ in a string with the stretching (elasticity) of the string. The tension has the dimension of ML/T^2.

The combination of τ and μ that has the dimension of speed (L/T) is $\sqrt{\tau/\mu}$.

5. Wave Speed on a Stretched String

Therefore, the velocity has the form $v = c\sqrt{\tau/\mu}$, where c is a constant. We can show that $c = 1$ (See your text). The speed v of the wave is then

$$v = \sqrt{\frac{\tau}{\mu}}.$$

The speed of a wave along a stretched ideal string depends only on the tension and linear density of the string and not on the frequency of the wave.

The frequency of the wave is fixed entirely by whatever generates the wave. The wavelength is then fixed by $\lambda = v/f$.

5. The Speed of a Traveling Wave

CHECKPOINT 3

You send a traveling wave along a particular string by oscillating one end. If you increase the frequency of the oscillations, do (a) the speed of the wave and (b) the wavelength of the wave increase, decrease, or remain the same? If, instead, you increase the tension in the string, do (c) the speed of the wave and (d) the wavelength of the wave increase, decrease, or remain the same?

(a) Same.

(b) Decreases.

(c) Increases.

(d) Increases.

$$v = \sqrt{\tau/\mu}$$

$$\lambda = v/f$$

6. Energy and Power of a Wave Traveling Along A String

When we generate a wave on a stretched string, we provide energy for the motion of the string. As the wave moves away, it transports that energy as both kinetic energy and potential energy.

Kinetic Energy

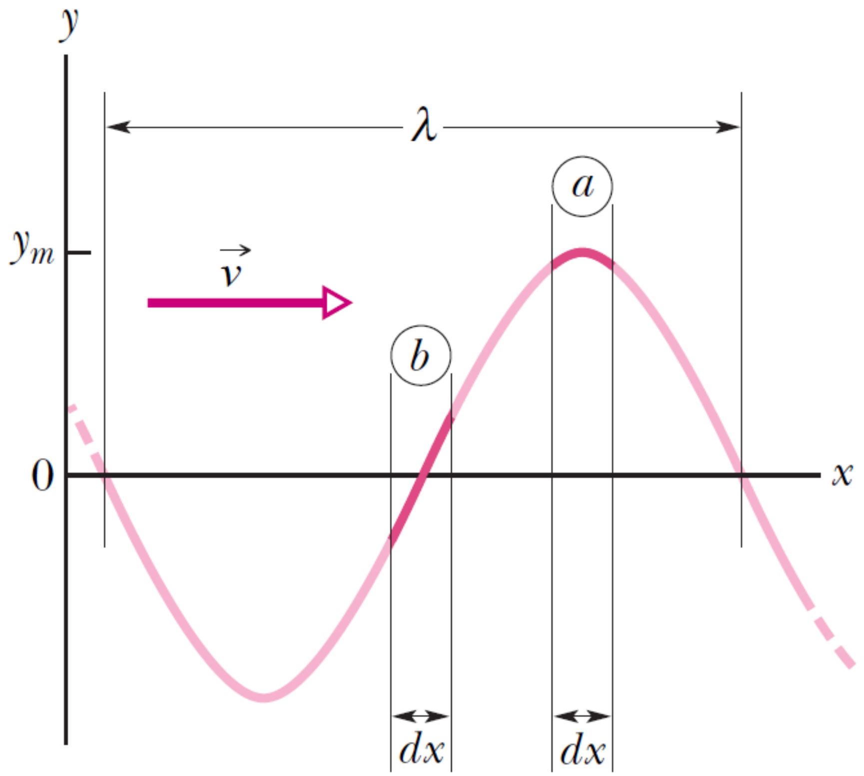
A string element of mass dm, oscillating transversely in SHM has kinetic energy associated with its transverse velocity $\vec{u}$. When the element is at $y = 0$ its speed (and hence its kinetic energy) is maximum. When the element is at one of its extreme points $y = \pm y_m$, its speed (and hence its kinetic energy) is zero.

6. Energy and Power of a Wave Traveling Along A String

Elastic Potential Energy

For a sine wave to travel along a straight string, the wave must stretch the string. To fit the wave form, a string element of length dx increases and decreases in a periodic way as it oscillates transversely.

When the element is at $y = y_m$, it has its normal undisturbed length and its elastic potential energy is zero. When the element is at $y = 0$, it has maximum stretch and thus maximum potential energy.



6. Energy and Power of a Wave Traveling Along A String

Energy Transport

The regions of the string at $y = y_m$ have zero energy (K.E. + P.E.), and the regions at $y = 0$ have maximum energy. As the wave travels along the string, forces due to the tension in the string continuously do work to transfer energy from regions with energy to regions with no energy.

Suppose that we generate a continuous wave on a stretched string. We continuously provide energy to the string. As the wave moves into sections that were previously at rest, energy is transferred into those new sections. Thus, the wave transports the energy along the string.

6. Energy and Power of a Wave Traveling Along A String

The Rate of Energy Transmission

The **average power**, or the average rate at which energy is transmitted by the wave, is

$$P_{avg} = \frac{1}{2}\mu v \omega^2 y_m^2.$$

The dependence of P_{avg} on $\omega^2 y_m^2$ is true for all types of waves.

6. Energy and Power of a Wave Traveling Along A String

Example 2: A string has linear density $\mu = 525\ \mathrm{g/m}$ and is under tension $\tau = 45\ \mathrm{N}$. We send a sinusoidal wave with frequency $f = 120\ \mathrm{Hz}$ and amplitude $y_m = 8.5\ \mathrm{mm}$ along the string. At what average rate does the wave transport energy?

To calculate P_{avg}, we need to calculate ω and v first.

$$\omega = 2\pi f = 2\pi(120\ \mathrm{Hz}) = 754\,\frac{\mathrm{rad}}{\mathrm{s}}.$$

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{45\ \mathrm{N}}{0.525\ \mathrm{kg/m}}} = 9.26\,\frac{\mathrm{m}}{\mathrm{s}}.$$

The average energy transport rate is then

$$P_{avg} = \frac{1}{2}\mu v \omega^2 y_m^2 = \frac{1}{2}\left(0.525\,\frac{\mathrm{kg}}{\mathrm{m}}\right)\left(9.26\,\frac{\mathrm{m}}{\mathrm{s}}\right)\left(754\,\frac{\mathrm{rad}}{\mathrm{s}}\right)^2 (0.0085\ \mathrm{m})^2 \approx 100\ \mathrm{W}.$$

7. The Principle of Superposition for Waves

Suppose that two waves travel simultaneously along the same string. The transverse displacements of the string due to each wave alone are $y_1(x,t)$ and $y_2(x,t)$, respectively. The displacement of the string when both waves overlap is

$$y'(x,t) = y_1(x,t) + y_2(x,t).$$

Overlapping waves algebraically add to produce a **resultant wave** or **net wave**.

Overlapping waves do not affect the travel of each other. That is, *every wave travels as if there were no other waves*.

8. Interference of Waves

Suppose we send two sinusoidal waves of the same wavelength and amplitude in the same direction along a stretched string. The resultant wave depends on the phase difference of the two waves. If the waves are exactly in phase (peaks and valleys of the two waves are aligned), they combine to double the displacement of either wave alone. If the waves are exactly out of phase (peaks of one are aligned with valleys of the other), they combine to cancel out.

This phenomenon of combining waves is called **interference**.

Let one wave be given by

$$y_1(x,t) = y_m \sin(kx - \omega t)\,,$$

and another wave, shifted from the first, be given by

$$y_2(x,t) = y_m \sin(kx - \omega t + \phi)\,.$$

8. Interference of Waves

These waves are said to be *out of phase* by ϕ or to have *phase difference* of ϕ.

From the superposition principle, the resultant wave is

$$\begin{aligned} y'(x,t) &= y_1(x,t) + y_2(x,t). \\ &= y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi)\,. \end{aligned}$$

Using the identity

$$\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)\,,$$

the resultant wave can be recast as

$$y'(x,t) = \left[2y_m \cos\frac{1}{2}\phi\right]\sin\left(kx - \omega t + \frac{1}{2}\phi\right).$$

8. Interference of Waves

Two sinusoidal waves of the same amplitude and wavelength, traveling in the same direction along a string interfere to produce a resultant sinusoidal wave traveling in the same direction.

The resultant wave differs from the interfering waves in two respects:

(1) Its phase ϕ' is $\frac{1}{2}\phi$.

(2) Its amplitude y'_m is

$$y'_m = \left|2y_m \cos\frac{1}{2}\phi\right|.$$

$$\overbrace{y'(x,t)}^{\text{Displacement}} = \underbrace{\left[2y_m \cos\tfrac{1}{2}\phi\right]}_{\substack{\text{Magnitude}\\ \text{gives}\\ \text{amplitude}}} \underbrace{\sin\left(kx - \omega t + \tfrac{1}{2}\phi\right)}_{\substack{\text{Oscillating}\\ \text{term}}}$$

8. Interference of Waves

When $\phi = 0$, the interfering waves are *exactly in phase* and $y'_m = \left|2y_m \cos\frac{1}{2}(0)\right|$ $= 2y_m$. Thus,

$$y'(x,t) = 2y_m \sin(kx - \omega t)\,.$$

The amplitude of the resultant wave is twice the amplitude of either interfering wave. The interference produces the greatest possible amplitude and thus called **fully constructive interference**.

When $\phi = \pi$ rad $(180°)$, the interfering waves are *exactly out of phase* and y'_m $= \left|2y_m \cos\frac{1}{2}(\pi)\right| = 0$. Thus,

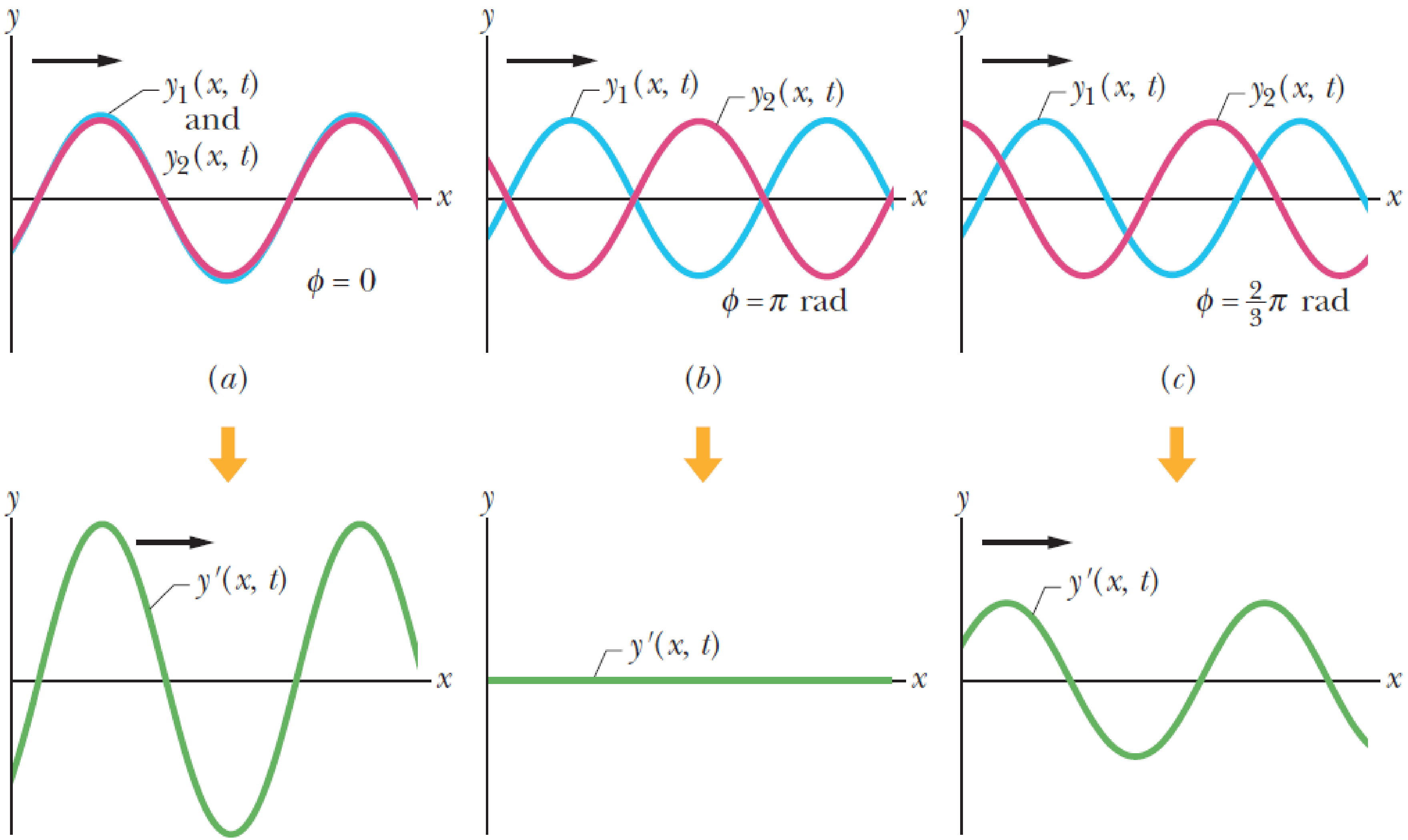
$$y'(x,t) = 0.$$

The interference eliminates the motion of the string and thus called **fully destructive interference**.

8. Interference of Waves

When the interference is neither fully constructive nor fully destructive, it is called **intermediate interference**. In this case, y'_m is between 0 and $2y_m$. For example, when $\phi = \frac{2}{3}\pi$ (or $120°$), $y'_m = \left|2y_m \cos\frac{\pi}{3}\right| = y_m$.

8. Interference of Waves



8. Interference of Waves

Phase differences can be described in terms of wavelengths as well as angles. For example, phase differences of 1 wavelength corresponds to $\phi = 2\pi$ (360°) and phase difference of 0.5 wavelengths corresponds to $\phi = \pi$ (180°).

Phase Difference, in			Amplitude of Resultant Wave	Type of Interference
Degrees	Radians	Wavelengths		
0	0	0	$2y_m$	Fully constructive
120	$\frac{2}{3}\pi$	0.33	y_m	Intermediate
180	π	0.50	0	Fully destructive
240	$\frac{4}{3}\pi$	0.67	y_m	Intermediate
360	2π	1.00	$2y_m$	Fully constructive
865	15.1	2.40	$0.60y_m$	Intermediate

8. Interference of Waves

CHECKPOINT 4

Here are four possible phase differences between two identical waves, expressed in wavelengths: 0.20, 0.45, 0.60, and 0.80. Rank them according to the amplitude of the resultant wave, greatest first.

0.20 and 0.80 tie, 0.60, 0.45.

$$\left|\cos\frac{1}{2}2\pi(0.20)\right| = 0.81$$

$$\left|\cos\frac{1}{2}2\pi(0.80)\right| = 0.81$$

$$\left|\cos\frac{1}{2}2\pi(0.45)\right| = 0.16$$

$$\left|\cos\frac{1}{2}2\pi(0.60)\right| = 0.31$$

$$y'_m = \left|2y_m \cos\frac{1}{2}\phi\right|$$

8. Interference of Waves

Example 3: Two identical sinusoidal waves, moving in the same direction along a stretched string, interfere with each other. The amplitude y_m of each wave is $9.8\ \mathrm{mm}$, and the phase difference ϕ between them is $100°$.

(a) What is the amplitude y'_m of the resultant wave due to the interference, and what is the type of this interference?

$$y'_m = 2y_m \left|\cos\frac{1}{2}\phi\right| = 2(9.8\ \mathrm{mm})\cos\frac{100°}{2} = 13\ \mathrm{mm}.$$

The interference is intermediate.

8. Interference of Waves

(b) What phase difference, in radians and wavelengths, will give the resultant wave an amplitude of 4.9 mm?

$$\phi = 2\cos^{-1}\frac{y'_m}{2y_m} = 2\cos^{-1}\frac{4.9}{2(9.8)} = \pm 2.6 \text{ rad}.$$

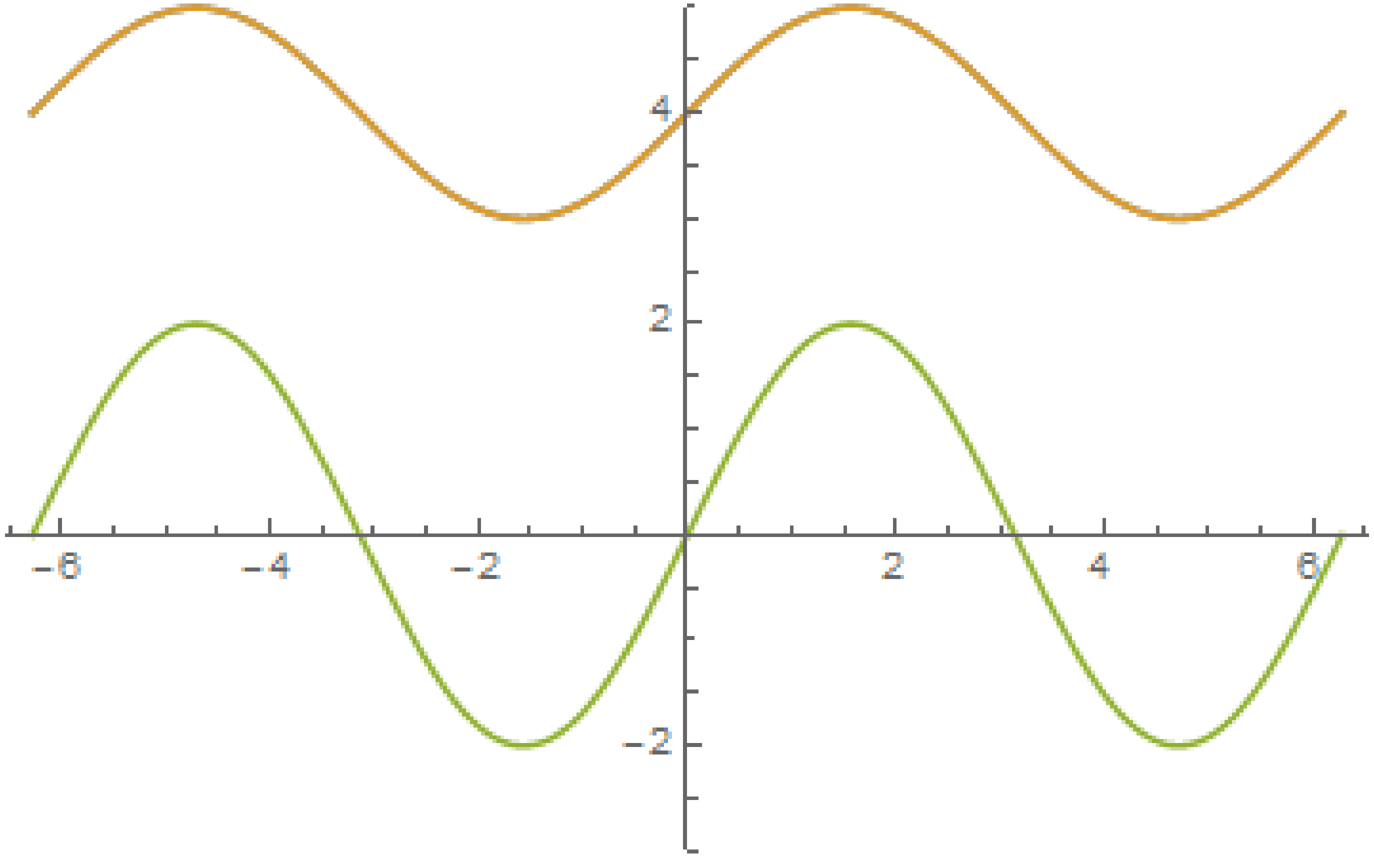
There are two solutions because we can obtain the same resultant wave by letting the first wave lead (travel ahead of) or lag (travel behind) the second wave by 2.6 rad.

The phase difference in wavelengths is

$$\phi = \pm 2.6 \text{ rad} \times \frac{1 \text{ wavelength}}{2\pi \text{ rad}}$$
$$= \pm 0.42 \text{ wavelengths}.$$

9. Standing Waves

What if the two interfering waves are traveling in opposite directions?



9. Standing Waves

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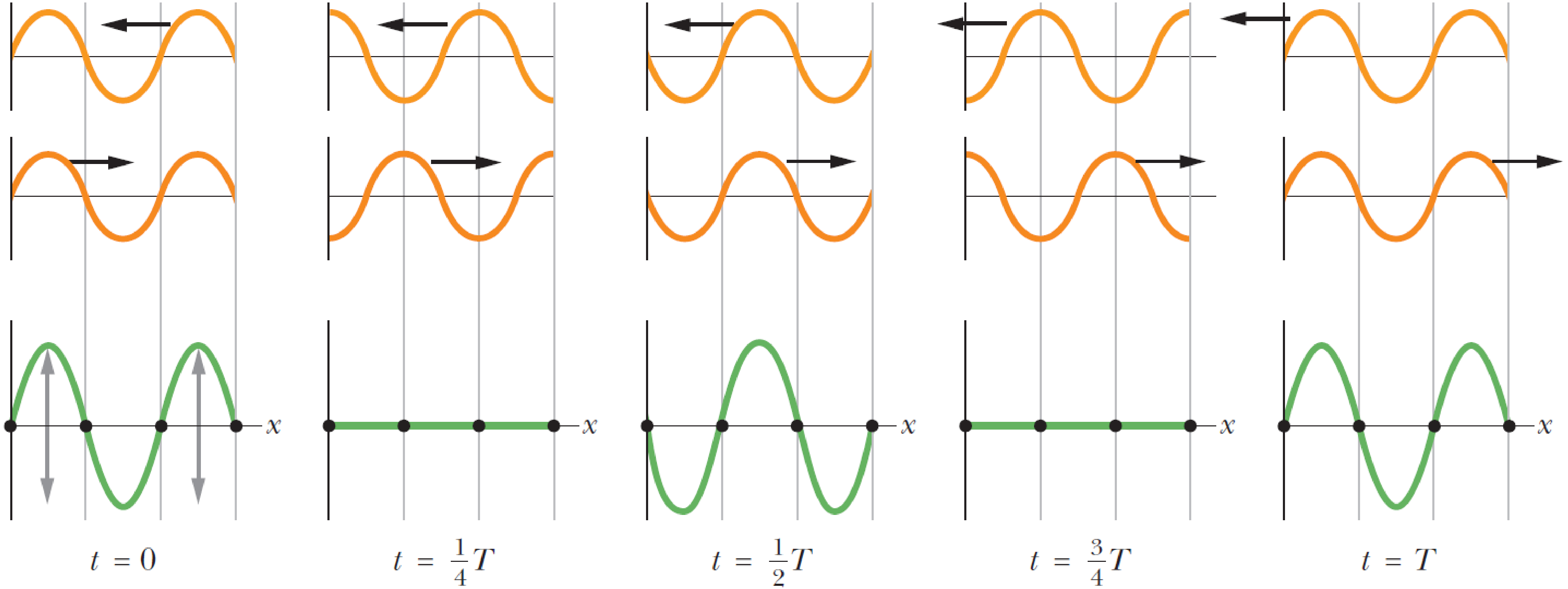


9. Standing Waves

There are places along the string, called **nodes**, where the string never moves.

Halfway between adjacent nodes are **antinodes**, where the amplitude of the resultant wave is maximum.

The resultant wave does not move left or right and thus called a **standing wave**.

Generally, if two sinusoidal waves of the same amplitude and wavelength travel in opposite directions along a string, their interference produces a standing wave.

Let the two interfering waves be represented by

$$y_1(x,t) = y_m \sin(kx - \omega t)\,,$$

and

$$y_2(x,t) = y_m \sin(kx + \omega t)\,.$$

9. Standing Waves

The resultant wave is then

$$y'(x,t) = y_1(x,t) + y_2(x,t)$$
$$= y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t)\,.$$

Using the identity

$$\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)\,,$$

we obtain

$$y'(x,t) = [2y_m \sin kx]\cos(\omega t)\,.$$

This equation does not describe a traveling wave because it is not of the form $y = h(kx \pm \omega t)$. Instead it describes a standing wave.

$$\overbrace{y'(x,t)}^{\text{Displacement}} = \underbrace{[2y_m \sin kx]}_{\text{Magnitude gives amplitude at position } x}\underbrace{\cos \omega t}_{\text{Oscillating term}}$$

9. Standing Waves

The quantity $2y_m \sin kx$ can be viewed as the amplitude of oscillation of a string element at position x.

Unlike the case of traveling waves, the amplitude of a standing wave varies with position.

The amplitude is zero for values of kx that give $\sin kx = 0$. These values are given by

$$kx = n\pi, \qquad n = 0, 1, 2, \ldots$$

Using $k = 2\pi/\lambda$ and rearranging we find

$$x = n\frac{\lambda}{2}, \qquad n = 0, 1, 2, \ldots$$

These are the positions of zero amplitude (the nodes) for the standing wave. They are separated by $\lambda/2$.

9. Standing Waves

The amplitude of the standing wave has a maximum value of $2y_m$, which occurs when $|\sin kx| = 1$. This is true when

$$kx = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$$

$$= \left(n + \frac{1}{2}\right)\pi, \qquad n = 0, 1, 2, \ldots$$

Using $k = 2\pi/\lambda$ and rearranging, we find

$$x = \frac{\lambda}{4} + n\frac{\lambda}{2}, \qquad n = 0, 1, 2, \ldots$$

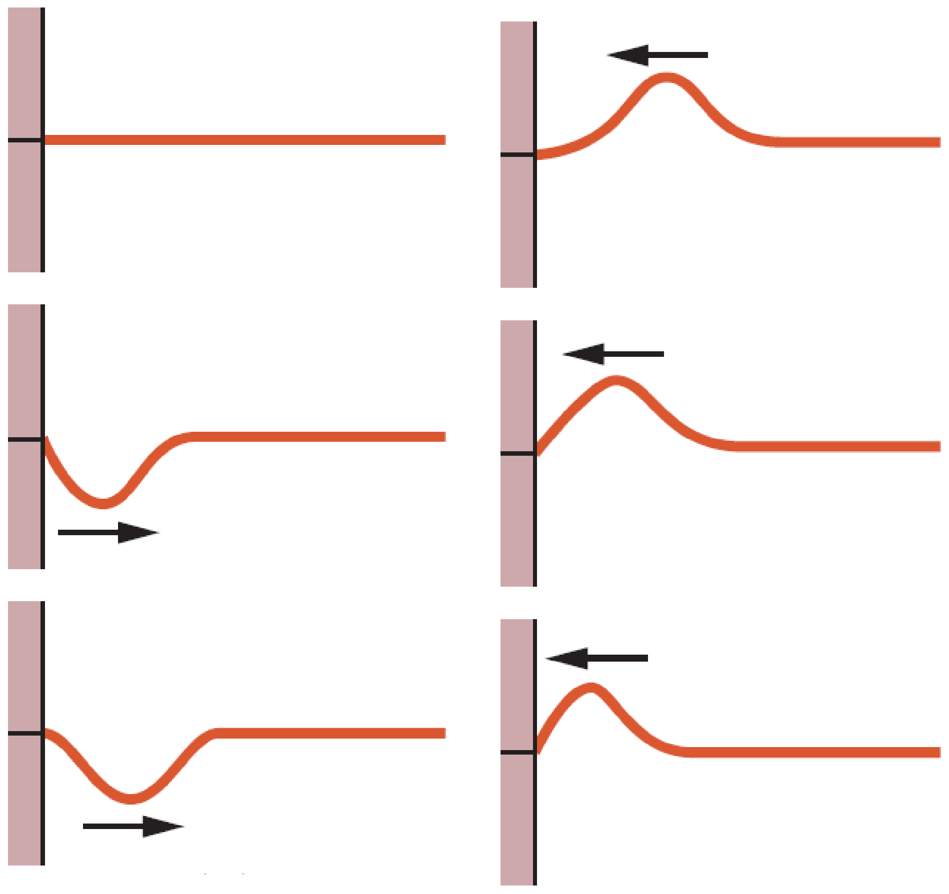
These are the positions of maximum amplitude (the antinodes) of the standing wave. They are separated by $\lambda/2$ and are located halfway between pairs of nodes.

9. Standing Waves

<u>Reflection at a Boundary</u>

We can set up a standing wave in a stretched string by letting a traveling wave get reflected at the far end of the string.

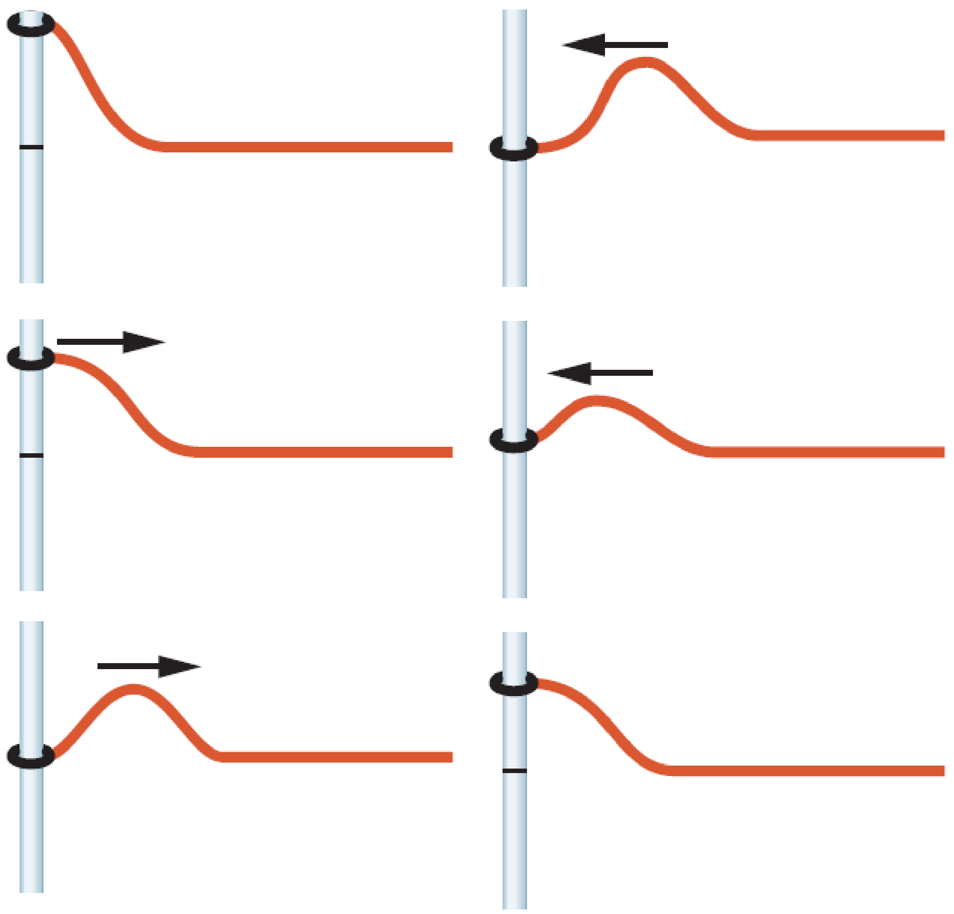
If the far end of the string is fixed, a node exits at the support point. The reflected and incident waves have the same amplitude but opposite signs. We refer to this case as **hard reflection**.



9. Standing Waves

<u>Reflection at a Boundary</u>

If the far end is free to move along the transverse direction, an antinode occurs at the free end. The incident and reflect waves have the same amplitude and signs. We refer to this case as **soft reflection**.



9. Standing Waves

CHECKPOINT 5

Two waves with the same amplitude and wavelength interfere in three different situations to produce resultant waves with the following equations:

(1) $y'(x, t) = 4\sin(5x - 4t)$

(2) $y'(x, t) = 4\sin(5x)\cos(4t)$

(3) $y'(x, t) = 4\sin(5x + 4t)$

In which situation are the two combining waves traveling (a) toward positive x, (b) toward negative x, and (c) in opposite directions?

(a) 1

(b) 3

(c) 2

https://www.youtube.com/watch?v=wvJAgrUBF4w

10. Standing Waves and Resonance

Consider a string that is stretched between two clamps. Suppose that we send a continuous sinusoidal wave along the string.

The wave gets reflected as it reaches one of the clamps and starts traveling in the other direction, interfering with the incident wave. The same process occurs repeatedly at both ends of the system.

For certain frequencies, the interference produces a standing wave pattern (or **oscillation mode**) with nodes and large antinodes. Such standing waves are said to be produced at **resonance**, and the string is said to **resonate** at these special frequencies, called **resonant frequencies**.

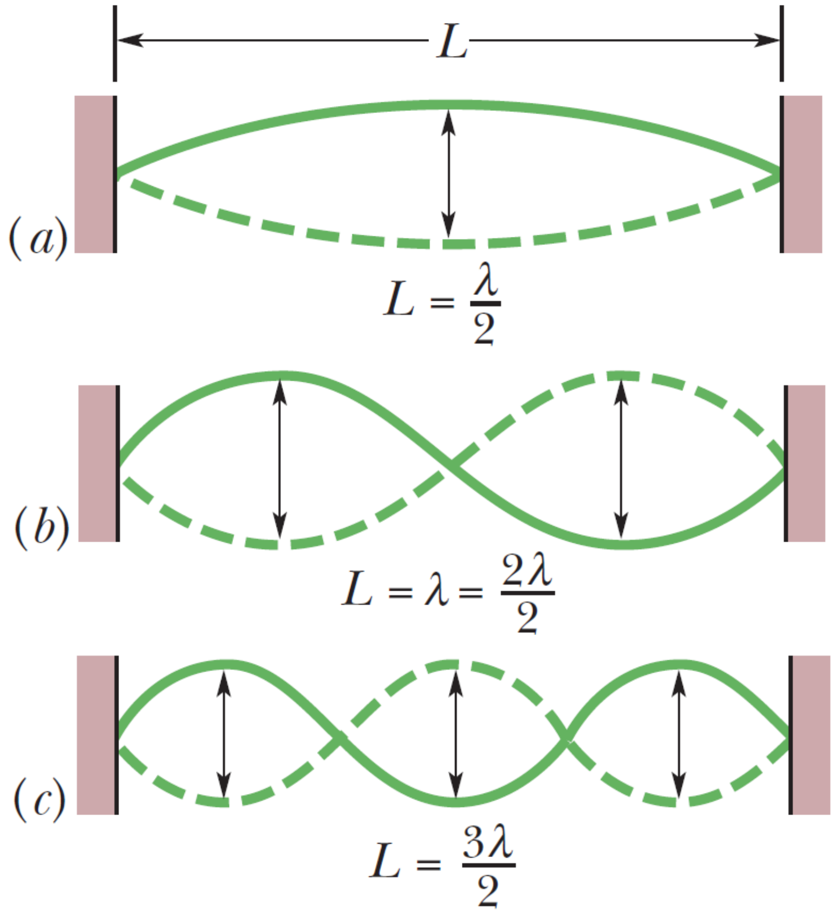
At other frequencies, the interference produces only small oscillations.

10. Standing Waves and Resonance

Let the clamps be separated by a distance L. To find an expression for resonant frequencies we use the fact that nodes must exist at the two ends of the string.

The simplest pattern that meets this requirement is shown in Fig. (*a*), where there is one antinode at the center of the string. For this pattern $L = \lambda/2$ or $\lambda = 2L$.

The second and third simplest patterns are shown in Figs. (*b*) and (*c*), respectively.



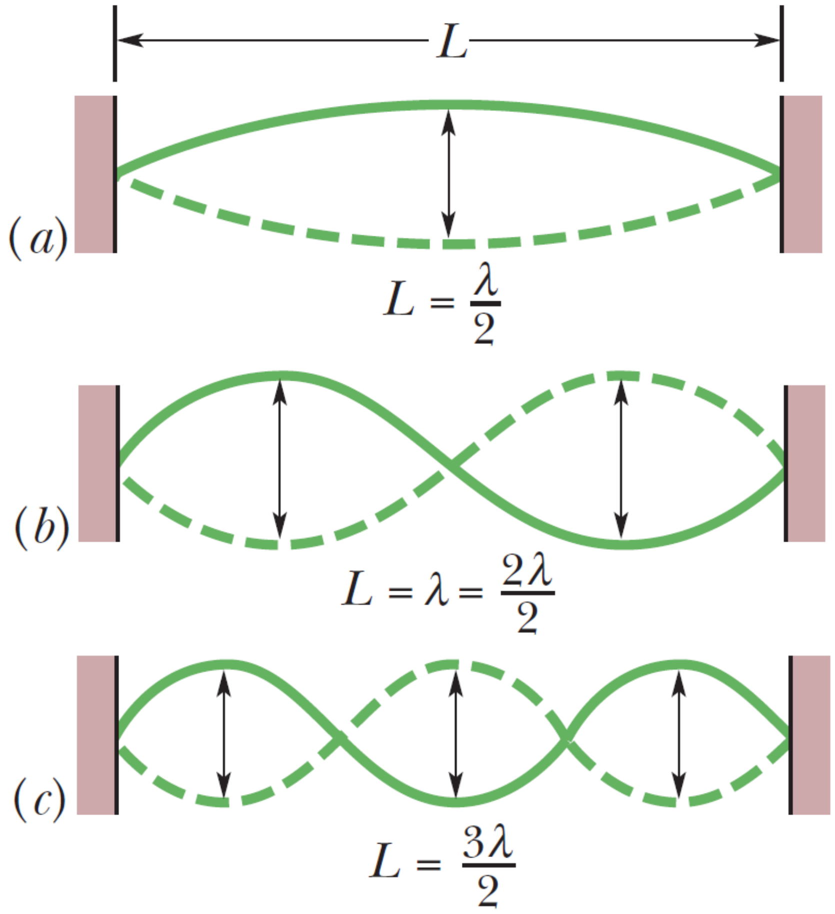
10. Standing Waves and Resonance

Generally, a standing wave can be set up on a string of length L by a wave of wavelength equal to

$$\lambda = \frac{2L}{n}, \qquad n = 1, 2, 3 \ldots$$

Using $\lambda = v/f$, the resonant frequencies that correspond to these wavelengths are

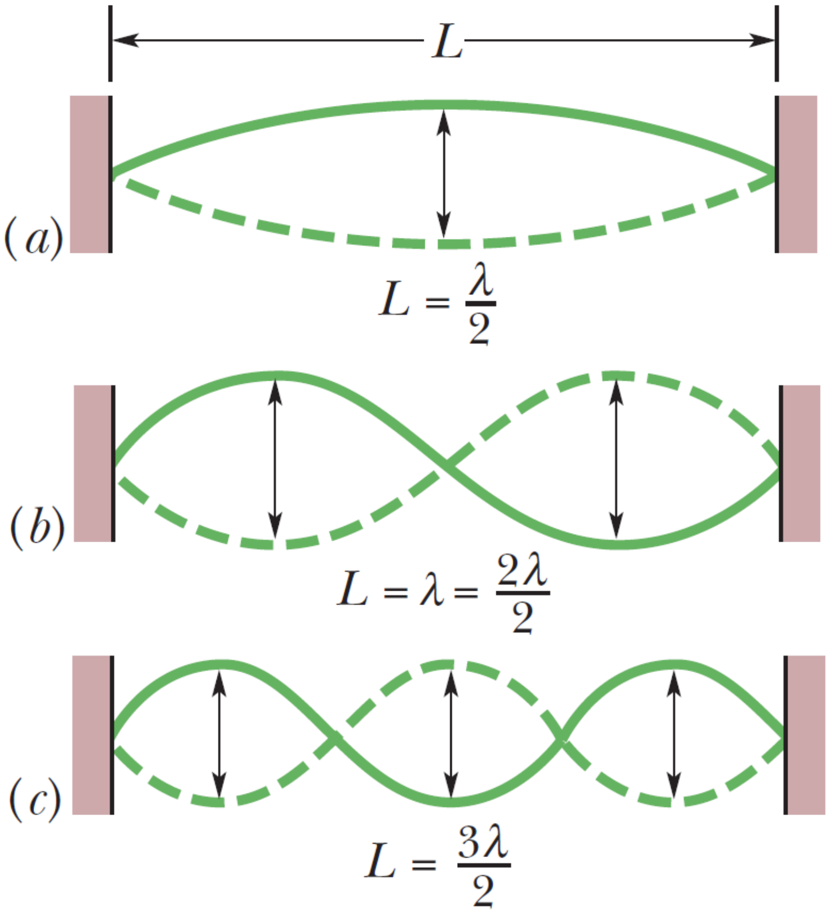
$$f = \frac{v}{\lambda} = n\frac{v}{2L}, \qquad n = 1, 2, 3 \ldots$$



10. Standing Waves and Resonance

The resonant frequencies are integer multiples of the lowest resonant frequency, $f_1 = \frac{v}{2L}\ (n = 1)$. This frequency is called the **fundamental mode** or the **first harmonic**.

The *second harmonic* mode corresponds to $n = 2$, and so on. The collection of all possible oscillations modes is called the **harmonic series** and n is called the **harmonic number** of the n^{th} harmonic.



10. Standing Waves and Resonance

CHECKPOINT 6

In the following series of resonant frequencies, one frequency (lower than 400 Hz) is missing: 150, 225, 300, 375 Hz. (a) What is the missing frequency? (b) What is the frequency of the seventh harmonic?

(a) 75 Hz.

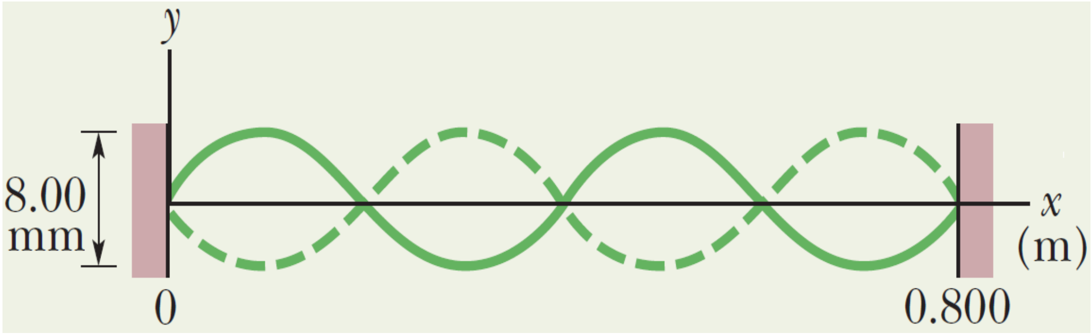
(b) $f_7 = 7f_1 = 7(75\ \text{Hz}) = 525\ \text{Hz}$.

$$f_1 = \frac{v}{2L}$$

$$f_n = nf_1$$

10. Standing Waves and Resonance

Example 4: The figure shows a pattern of resonant oscillation of a string of mass $m = 2.500\ \mathrm{g}$ and length $L = 0.800\ \mathrm{m}$ and that is under tension $\tau = 325.0\ \mathrm{N}$.



(a) What is the wavelength λ of the transverse waves producing the standing-wave pattern.

From the figure we can see that $L = 2\lambda$. Therefore,

$$\lambda = \frac{L}{2} = \frac{0.800\ \mathrm{m}}{2} = 0.400\ \mathrm{m}.$$

10. Standing Waves and Resonance

(b) What is the harmonic number n?

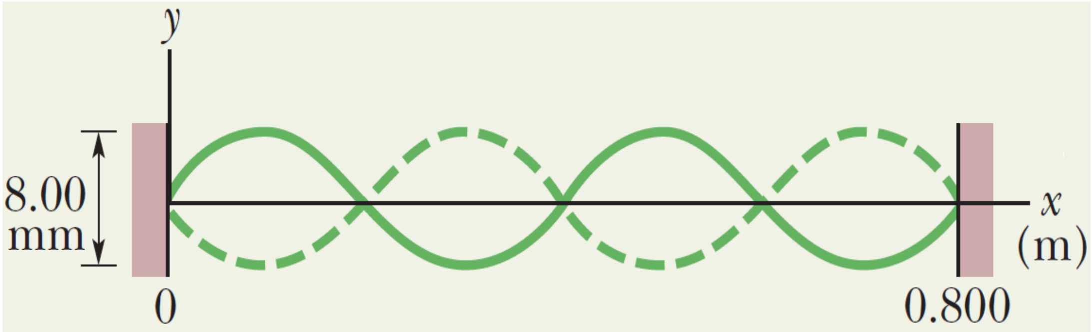
By counting the number of loops (or half-wavelengths), we see that the harmonic number is

$$n = 4.$$

We get the same answer by comparing the expressions $\lambda = L/2$ and $\lambda = 2L/n$.

(c) What is the frequency f of the transverse waves and of the oscillations of the moving string elements?

$$\mu = \frac{m}{L} = \frac{2.500 \times 10^{-3}\text{kg}}{0.800\ \text{m}} = 3.125 \times 10^{-3}\,\frac{\text{kg}}{\text{m}}.$$

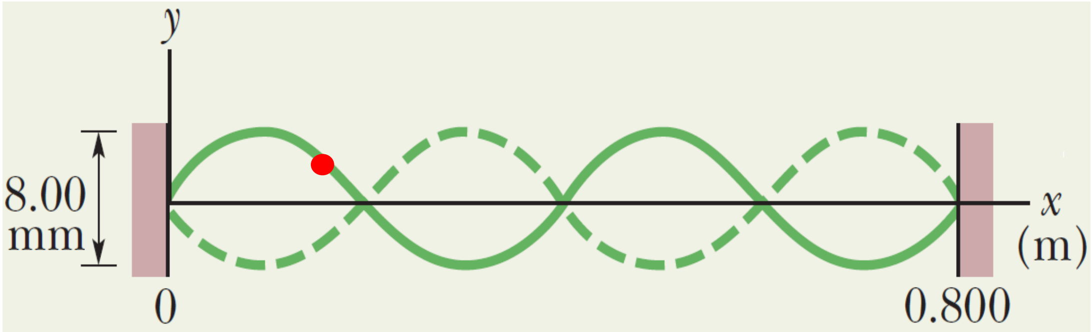


10. Standing Waves and Resonance

$$v = \left(\frac{\tau}{\mu}\right)^{1/2} = \left(\frac{325.0\ \text{N}}{3.125 \times 10^{-3}\ \frac{\text{kg}}{\text{m}}}\right)^{1/2}$$

$$= 322.5\ \frac{\text{m}}{\text{s}}.$$

$$f = \frac{v}{\lambda} = \frac{322.5\ \text{m/s}}{0.400\ \text{m}} = 806\ \text{Hz}.$$



(d) What is the maximum magnitude of the transverse velocity u_m of the element oscillating at coordinate $x = 0.180$ m?

$$u(x,t) = \frac{\partial y'}{\partial t} = \frac{\partial}{\partial t}[(2y_m \sin kx) \cos \omega t].$$

10. Standing Waves and Resonance

$$u(x,t) = (-2y_m\omega \sin kx) \sin \omega t\,.$$

The maximum transverse speed is then

$$u_m(x) = |-2y_m\omega \sin kx|.$$

At $x = 0.180\ \mathrm{m}$,

$$u_m(0.180\ \mathrm{m}) = \left|-2(0.00200\ \mathrm{m})(2\pi)(806.2\ \mathrm{Hz}) \sin\left[\frac{2\pi}{0.400\ m}(0.180\ \mathrm{m})\right]\right|$$

$$= 6.26\frac{\mathrm{m}}{\mathrm{s}}.$$

At what point during the element's oscillation is the transverse velocity maximum?

The transverse velocity is maximum then the element is at $\mathrm{y} = 0$.