3.17 Summary

The use of the α-particle as a charge probe for the measurement of the electrostatic field within the atom led to the concept of the potential barrier and to an estimate of nuclear size. It also revealed a paradox with respect to the emission of α-particles with insufficient energy to have surmounted the potential barrier. The resolution of this paradox by the abandonment of classical dynamics in favour of wave mechanics led to an explanation of the observed relationship between half-life and α-particle energy. The measurements of these two quantities for two α-emitting nuclides were used to determine R_0, the nuclear unit radius which enters the formula for the nuclear radius, namely $R = R_0 A^{\frac{1}{3}}$. The interpretation of the fine structure observed in the energy spectra of α-particles established the existence of excited states of the nucleus and led to the introduction of energy-level diagrams.

Chapter 4
Radioactivity: Beta Decay

4.1 Introduction

Beta decay, the most generally occurring mode of radioactive transformation, takes place between neighbouring isobars (i.e. without change in A and with a change of one in Z). In contrast to α-decay, which is a phenomenon limited to nuclei with medium and high A-values, β-decay has been observed for nuclei with all A-values from one upwards. Essentially in β-decay a neutron switches into a proton or vice versa. When the switch occurs a β-particle, of negative sign of electric charge if a neutron switch, of positive sign if a proton has switched, is observed to be emitted. Careful experimentation has failed to reveal any difference between the physical properties of the negative β-particle and those of the electron of atomic structure, and we assume that these particles are identical. The positive β-particle, apart from the sign of its electric charge, has the same properties as the negative β-particle. The β-particles are sometimes named *negatron* (perhaps more properly, but less usually, *negaton*) and *positron* (or *positon*), *electron* then being available to apply generically to either.

4.2 Beta decay and the conservation laws

The measurement, by Chadwick in 1914, of the energy of β-particles emitted from a source containing a single isotopic species revealed a continuous spectrum of energy ranging from zero to a finite maximum value. If it is assumed that, as in α-decay, the parent and daughter nuclei have well-defined mass values, then the conservation of mass–energy and linear momentum requires that there be at least three 'products' of the decay, that is, one product in addition to the β-particle and the recoiling daughter nucleus. Careful measurement of the energy absorbed in massive calorimeters containing strong β-sources indicated an energy per decay corresponding to the mean β-energy, not to the maximum β-energy. Thus the third 'product', if such existed, did not deposit any energy in the material of the calorimeter (Ellis and Wooster, 1927).

In addition to the difficulty thus presented in respect of energy conservation, β-decay set a problem with respect to conservation of angular momentum. The simplest β-emitter is the free neutron, which, with a half-life of about thirteen minutes, decays to a proton. We start with a neutron which has intrinsic angular momentum of $\frac{1}{2}\hbar$. If we end with only a proton and electron, each having intrinsic angular momentum of $\frac{1}{2}\hbar$ and only permitted by the rules of quantum mechanics to have an angular momentum of relative motion in multiples of $\hbar$, then clearly angular momentum cannot be conserved. The question of conservation of linear momentum was also raised by the existence of cloud-chamber photographs purporting to show the decay of ^{6}He in which the direction of the recoiling daughter nucleus was not accurately collinear with the direction of the emitted β-particle.

4.3 The neutrino hypothesis

The three conservation laws, those of energy, linear mome tum and angular momentum, can be satisfied in β-decay by the simple expedient of postulating (as was done by Pauli in 1933) the emission of a neutral particle, the *neutrino*, in addition to the β-particle. We must assign to this new particle the property of having practically no interaction with matter, in order to explain the failure to detect such a particle in the early calorimeter experiments. More recently, the interaction of the neutrino with matter has been observed in elaborate experiments with large baths of scintillating liquid in which neutrinos from the β-emitting products in a reactor undergo the reaction

$$\mathrm{p} + \bar{\nu} \rightarrow \mathrm{n} + \mathrm{e}^+,$$

where $\bar{\nu}$ represents the neutrino (Reines and Cowan, in 1959). Thus the early philosophical objections to the acceptance of the existence of a particle whose detection was virtually impossible by definition, have now been removed.

The neutron-to-proton switch, which, as mentioned above, occurs as the β-decay of free neutrons and has been observed to take place in neutron beams emerging from reactors, can be described by

$$\mathrm{n} \rightarrow \mathrm{p} + \mathrm{e}^- + \bar{\nu}. \qquad 4.1$$

Free protons on the other hand are stable against β-decay. However, in the conditions existing inside the nucleus, it is assumed that the reaction analogous to 4.1 takes place, namely

$$\mathrm{p} \rightarrow \mathrm{n} + \mathrm{e}^+ + \nu, \qquad 4.2$$

giving rise to positron emission.

It is believed that the neutrinos involved in 4.1 and 4.2 are not identical. ν, associated with positron decay, is called the neutrino; $\bar{\nu}$, associated with negatron decay, is called the antineutrino. Neutrinos, electrons and μ-mesons constitute a family of particles known as *leptons.* If we regard the electron and positron as a particle and antiparticle pair, we note that in β-decay the two leptons appearing are always in a particle and antiparticle combination.

The neutrino, for reasons which will be discussed later, is believed to have zero rest mass and always to have a velocity equal to the velocity of light. The conservation of angular momentum in β-decay requires that the neutrino should have the same spin angular momentum as the nucleons and the electrons, namely $\frac{1}{2}\hbar$.

4.4 Mass-difference conditions necessary for beta decay

In order that β-decay be energetically possible, a certain inequality must be satisfied by the masses of the parent and daughter isotopes.

4.4.1 *β⁻ decay*

Let $^A_Z\mathrm{X}$ represent a nuclide which is unstable against β^- decay and let $_{Z+1}^{\;\;A}\mathrm{Y}$ represent the isobar into which it decays. Then the process starts from X with its complete complement of orbital electrons and, it is assumed, with the electrons in their ground-state configuration. Following the emission of the β^- particle, which it is assumed does not interact with the electron system in its passage out through the atom, the daughter Y will have a nuclear charge $Z + 1$ but only the electron complement of X, namely that corresponding to a nuclear charge Z. There will be slight adjustments in orbitals necessary by virtue of the change in Z; Y will also be singly ionized. If now M_X is the mass of the X-atom and M_Y^+ the mass of the singly ionized Y-atom, the mass–energy equation of the reaction is

$$M_\mathrm{X} = M_\mathrm{Y}^+ + m_\mathrm{e} + Q, \qquad 4.3$$

where Q is the total kinetic energy available for the electron, neutrino and daughter atom, m_e is the mass of an electron, and the mass of the neutrino is assumed to be zero.

Now the (first) ionization energy of the Y-atom will be given by

$$I = M_\mathrm{Y}^+ + m_\mathrm{e} - M_\mathrm{Y},$$

where M_Y is the mass of the Y-atom in its ground state. Equation 4.3 can then be written

$$M_\mathrm{X} = M_\mathrm{Y} + I + Q.$$

I will be of the order of electronvolts and will usually be negligible compared to the reaction energy. On this assumption the condition that β^- decay be energetically possible, namely that $Q > 0$, leads to the requirement

$$M_\mathrm{X} > M_\mathrm{Y}. \qquad 4.4$$

4.4.2 *β⁺ decay*

In the case of β^+ decay the roles of X and Y are reversed. Y is now the parent nucleus. The daughter will have one electron over its full complement when it is formed. Depending on the atom concerned, this electron may remain attached, in which case a negative ion results, or it may dissociate from the atom leaving a neutral atom and a free electron. The energy difference between these two possibilities (i.e. the electron-attachment energy) is however only a few electronvolts and will usually be negligible compared to the reaction energy. In the final state, in addition to the daughter atom, we thus have one free electron and one positron. The mass–energy equation is therefore

$$M_\mathrm{Y} = M_\mathrm{X} + 2m_\mathrm{e} + Q.$$

For $Q > 0$, therefore,

$$M_Y > M_X + 2m_e. \quad 4.5$$

4.4.3 *Electron capture*

Having regard to the inequalities 4.4 and 4.5, the question arises of a possible stability of neighbouring isobars in the event of

$$M_X + 2m_e > M_Y > M_X.$$

Although β-decay, as it has been discussed above, is not energetically possible, a process equivalent to positron emission is possible and is observed to take place.

The reaction

$$p + e^- \rightarrow n + \nu \quad 4.6$$

is equivalent to reaction 4.2 in respect of the daughter nucleus formed. However, the interaction between a nucleon (i.e. proton or neutron) and a lepton is the so-called *weak interaction*, which we discuss later, and is of a magnitude which makes the reaction 4.6 experimentally unobservable in terms of a free-proton target bombarded by a beam of electrons. In the atom, the nucleus in a sense is being constantly bombarded by those orbital electrons whose wave functions have finite amplitudes within the nuclear volume. Thus, despite the weakness of the interaction process, 4.6 can be expected to occur in the course of time. This process in general terms is known as *electron capture*. The two electrons in the K-shell have, compared with electrons in other orbitals, a relatively large probability of being found within the nuclear volume and the process therefore usually proceeds by the capture of one of these electrons. In that event, it is referred to as '*K-capture*' to distinguish it from an event involving the capture of an L- or even an M-shell electron. The process of capture from shells other than the K-shell has a smaller but still finite probability and is observed.

The end product of electron capture in terms of emitted particles is solely a neutrino. It should be noted that, since the process in this case is a two-body process, as distinct from the three-body process involving β-particle emission, the neutrinos from a given nuclide are monoenergetic. It is not usually possible to detect these neutrinos. However, the disappearance of the electron leaves a vacancy in an atomic shell. The filling of this vacancy by an electron jumping in from another shell will result either in the emission of an X-ray or an Auger electron, both of which can be detected with high efficiency. For example if the electron vacancy is in the K-shell, then it can be filled by an electron from the L-shell jumping in with the emission of a K X-ray and the creation of a vacancy in the L-shell. (In this summary account we are neglecting the multiplicity of the L-shell, which has in fact three components L_I, L_{II}, L_{III}.) alternatively, the electron from the L-shell may, as before, fill the K-shell vacancy but instead of the emission of the energy difference in the K- and L-shell binding energies, namely $B_K - B_L$, in the form of an X-ray, this energy may concentrate on a second L-shell electron, ejecting it from the atom as an Auger electron having an

energy of $B_K - 2B_L$. In this event two vacancies are left in the L-shell, which are then filled by similar processes by electrons jumping from the outer shells. Thus the vacancies move out through the shells till finally an ion is left which will then neutralize itself by finding free electrons, if such are available in its environment.

As the final state immediately following electron capture consists of the daughter nucleus with one vacancy in a certain shell, but with its full electron complement, the mass-energy relation is

$$M_Y = M_X + E' + Q,$$

where E' is the energy necessary to produce the vacancy by promoting an electron to an outer orbit. In principle E' need only be a few electronvolts in magnitude if an electron in an outer shell is captured. Even when the capture is from a deeper shell, E' will in most cases be negligible in comparison with the reaction energy. Thus the condition that electron capture be energetically possible is

$$M_Y > M_X. \quad 4.7$$

It should be noted that if the inequality 4.5 is satisfied then 4.7 is also satisfied. In that event the processes of positron emission and electron capture occur in competition.

Inequality 4.7 taken with 4.3 means that one of two neighbouring isobars must in all circumstances be unstable with respect to decay into the other. In Figure 12 the three modes of β-decay are related to the mass differences.

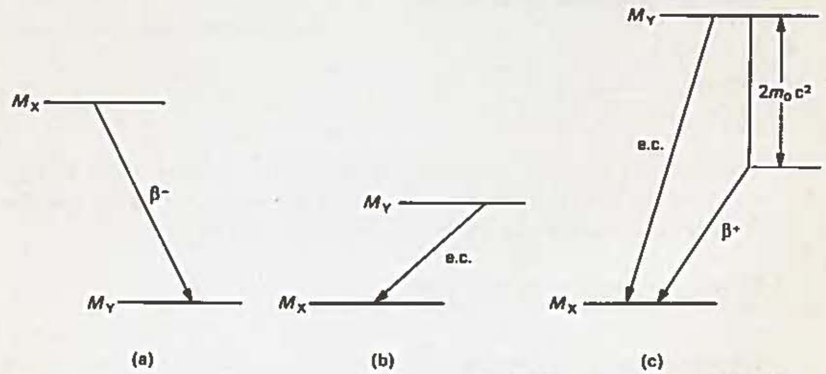

Figure 12 Relationships between mass values of neighbouring isobars giving rise to (a) β^- decay, (b) electron capture only, (c) competition between electron capture and β^+ decay

4.5 Decay energy and beta-particle energy

We define the decay energy to be the difference between the ground-state masses of the parent and daughter nuclei. If the transition is from ground state to ground state, then the whole of the decay energy is available as kinetic energy to the products of the decay in the case of β^- decay and electron capture. In the case of

β^+ decay, the decay energy less 1·022 MeV (i.e. twice the electron rest mass) is available. Very frequently, however, the transition is not to the ground state of the daughter nucleus but to an excited state which quickly decays by γ-ray emission to the ground state. In that event the energy of excitation of the excited state has to be subtracted from the decay energy to arrive at the energy available to the particles, as discussed above.

4.6 The energy distribution among products in beta decay

The energy available in a β-decay transition is shared by the daughter nucleus, the β-particle and the neutrino, the division being governed among other factors by the requirement that linear momentum be conserved. As there are three bodies involved, the division can take place in an infinite number of ways, and therefore each product has an energy lying in a continuous range from zero to a fixed maximum value for a particular transition.

The daughter nucleus will have its maximum possible recoil energy when the neutrino energy is zero, the β-particle energy has its maximum value for the transition in question and the linear momentum of the daughter nucleus balances that of the β-particle. The total energy E_β of the β-particle is given by

$$E_\beta = T_\beta + m_0 c^2,$$

where T_β is its kinetic energy; also from the special theory of relativity, the β-particle momentum p_β is given by

$$p_\beta^2 c^2 = E_\beta^2 - m_0^2 c^4.$$

It therefore follows that

$$p_\beta^2 c^2 = T_\beta^2 + 2T_\beta m_0 c^2. \qquad 4.9$$

If now p_D is the linear momentum of the daughter nucleus, whose kinetic energy will be small compared to its rest mass and whose motion can therefore be adequately described by Newtonian dynamics, then we have

$$2M_D T_D = p_D^2 = p_\beta^2 = \frac{T_\beta^2}{c^2} + 2T_\beta m_0.$$

$$\text{Therefore} \quad T_D = \frac{T_\beta^2}{2M_D c^2} + T_\beta \frac{m_0}{M_D}. \qquad 4.10$$

If this result be now applied to the particular case of ^{6}He, which has a comparatively low mass value for a β-emitter and a comparatively large maximum β-particle energy of 3·5 MeV and which should therefore have a relatively large recoil energy, we find T_D = 1·4 keV. Thus less than one part in two thousand of the available energy is taken away by the recoil. In the case of other β-emitters where a heavier recoil mass and a lower β-particle maximum energy are involved the fraction will be even smaller. Consequently one can, in most circumstances, equate the decay energy to the maximum β-particle energy, allowance having been made for any subsequent γ-ray emission as discussed above.

4.7 Range of values of $T_{\beta,\max}$

There are several hundred nuclides classified as β-emitters. They range in A-value from A = 1 for the neutron to the isobars with A = 256 (e.g. $^{256}_{99}$Es).

The largest value of decay energy in a β-transition is 18 MeV, which occurs in the case of the nuclide $^8_5$B. However, this positron emitter decays to an excited state of $^8_4$Be and the maximum particle energy is about 14 MeV. $^{12}_7$N, which has a slightly smaller decay energy of 17·6 MeV, decays to the ground state of $^{12}_6$C and consequently the maximum positron energy is 16·6 MeV. At the other end of the scale cases are known of decay energies considerably less than 10 keV. The value of $T_{\beta,\max}$ measured in the case of ^{187}Re, in what is believed to be a ground-state–ground-state transition, is about 2 keV. When the decay energy is as small as this then the assumptions made in section 4.4, concerning the effect of the orbital electrons in the atoms, have to be considered carefully. A reorganization of the extranuclear electrons may involve in the case of heavy atoms some 12 keV. In β^- decay, energy is provided by this reorganization; in β^+ (or electron capture) it has to be provided out of the nuclear energy store. A β^--active heavy atom for which, in the neutral state, the decay energy is, say, 10 keV would be stable if completely ionized. The range of particle energy in β-decay from a few thousand electronvolts to about 17 MeV is very much wider than the range of particle energies involved in α-decay, which is approximately 2 MeV to 9·2 MeV.

4.8 Range of half-lives in beta decay

In the case of β-decay, as for α-decay, the more energetic the emitted particle the shorter, as a general rule, is the half-life of the nuclide. For example, $^{12}_7$N, which emits a very-high-energy positron, has a half-life of 11 ms. On the other hand, ^{187}R which emits a 2 keV β^- particle, has a half-life of 4 x 10^{10} years. An even longer half-life, namely 1·1 x 10^{11} years, has been measured for the β^- emitter ^{138}La. This range of half-lives is not so great as the range for α-decay, where the lower limit is less than a microsecond and the upper limit, probably set by experimental technique, is at least 2 x 10^{17} years.

An early empirical attempt to relate the half-life to the maximum energy of β-particle of naturally occurring β-emitters was made by plotting log λ against log $E_{\max}$. The resulting plot is called a *Sargent diagram.* On this diagram the β-emitters were found to fall into two distinct groups. Those with the shorter half-life for a given $E_{\max}$ were said to involve 'allowed' transitions, those with the longer half-life to involve 'forbidden' transitions.

4.9 Fermi theory of beta decay

Two experimental features of β-decay present obvious challenges for theoretical interpretation, namely the shape of the measured energy spectrum of emitted β-particles and the relation between half-life and maximum β-particle energy. A theory was devised by Fermi (1934) to provide an interpretation of these features. This theory has largely stood the test of time and is still a convenient basis for the theoretical discussion of β-decay.

An analogy is made between β-decay and the transition between two states of an atom with the emission of electromagnetic radiation. The β-particle and neutrino, like the electromagnetic quantum in the atomic case, do not exist prior to the transition taking place, their creation and emission forming part of the transition process.

In the atomic case, the transition rate between initial and final states is given by perturbation theory (see, for example, R. M. Eisberg, *Fundamentals of Modern Physics*, Wiley, 1961, p. 268) and can be written as

$$\frac{2\pi}{\hbar} v_{fi}^{*} v_{fi} \rho_f,$$

where ρ_f is the number of final energy states per unit energy interval and v_{fi} is the *matrix element* of the interaction potential V, the perturbation causing the transition. In this theory, v_{fi} is defined by the equation

$$v_{fi} = \int \psi_f^* V \psi_i \, d\tau,$$

where ψ_i and ψ_f are the eigenfunctions (see section 6.4) describing the initial and final states of the system. In the atomic case, V represents the potential energy of the electric charges and magnetic moments of the atomic system in the perturbing electromagnetic field.

In the case of β-decay, a new interaction has to be assumed to provide the perturbation and initially the simple assumption is made that V can be replaced by a constant g which is called the *β-decay coupling constant*. With this assumption we now write

$$v_{fi} = g \int \psi_f^* \psi_i \, d\tau.$$

Now the initial system is simply, let us say, the β-unstable nucleus (Z, A) and therefore we will write $\psi_i = \psi_{Z,A}$.

The final system consists of the daughter nucleus $(Z \pm 1, A)$ (the sign depending on whether we have β^- or β^+ decay) together with the outgoing β-particle and neutrino. The eigenfunction ψ_f is therefore to be obtained by multiplying the eigenfunctions of the three decay products; thus

$$\psi_f = \psi_{Z\pm 1, A} \psi_\beta \psi_\nu.$$

The electron and neutrino we shall at this stage in the discussion assume to be adequately described by free-particle plane waves, that is, we assume them to emerge through a field-free region. Both of these particles have momenta in a range in which the associated de Broglie wavelength is very much larger than nuclear dimensions. Consequently as a simplification we take ψ_β and ψ_ν to be constants independent of the space coordinates throughout the nuclear volume. For the purpose of specifying boundary conditions we assume that the whole system is enclosed in a box of dimension L. Taking ψ_β, ψ_ν both equal to $L^{-\frac{3}{2}}$

normalizes the probability of finding β-particle and neutrino in the box to unity. We can therefore write

$$\psi_f = L^{-3} \psi_{Z\pm 1, A},$$

and it follows that

$$v_{fi} = gL^{-3} \int \psi_{Z\pm 1, A}^* \psi_{Z,A} \, d\tau = gL^{-3} M. \qquad 4.11$$

M is termed the *nuclear matrix element* and it is to be noted that under the assumptions made it is independent of the β-particle and neutrino momenta.

It remains to discuss ρ_f, the density of final states. We start from a result of non-relativistic quantum mechanics, that if an electron of kinetic energy T_β is confined in a box of dimension L, then $N(T_\beta)$, the density of final states, is given by

$$N(T_\beta)\, dt_\beta = \frac{m^{\frac{3}{2}} L^3 T_\beta^{\frac{1}{2}} \, dT_\beta}{2^{\frac{1}{2}} \pi^2 \hbar^3}.$$

We now proceed to express this in terms of momentum. Using the non-relativistic relation

$$\frac{p_\beta^2}{2m} = T_\beta,$$

we have $$\frac{p_\beta \, dp_\beta}{m} = dT_\beta.$$

Thus $$N(T_\beta)\, dT_\beta = \frac{L^3 p_\beta^2 \, dp_\beta}{2\pi^2 \hbar^3}. \qquad 4.12$$

It can be shown that this relation is also correct in the relativistic case and, as it now stands, it may be applied to β-particles and neutrinos. We now proceed to apply the equation 4.12 to the β-decay discussion. We consider the energy of the final state to lie between E_f and $E_f + dE_f$. If ρ_f is the density of final states, then the number of states in the energy range dE_f is $\rho_f \, dE_f$. Corresponding to dE_f there will be a range dE_β of β-particle energies and a range dE_ν of neutrino energies.

It follows from equation 4.12 that there are

$$N(E_\beta)\, dE_\beta = \frac{L^3 p_\beta^2 \, dp_\beta}{2\pi^2 \hbar^3}$$

β-particle states and

$$N(E_\nu)\, dE_\nu = \frac{L^3 p_\nu^2 \, dp_\nu}{2\pi^2 \hbar^3}$$

neutrino states. The number of states of the system is then obtained by forming the product of these two expressions relating to the individual components. Thus

$$\rho_f \, dE_f = \frac{L^6 p_\beta^2 p_\nu^2 \, dp_\beta \, dp_\nu}{4\pi^4 \hbar^6}.$$

We now consider the β-particle energy to be specified and associate dE_f solely with dE_ν. We therefore write

$$\rho_f = \frac{L^6 p_\beta^2 p_\nu^2 \, dp_\beta}{4\pi^4 \hbar^6} \left[\frac{dE_f}{dp_\nu}\right]^{-1}.$$

Assuming the neutrino rest mass to be zero we have $E_\nu = p_\nu c$. Therefore

$$\frac{dE_\nu}{dp_\nu} = c.$$

Also $E_\beta + E_\nu = E_{\beta,\max}$,

where $E_{\beta,\max} = m_0 c^2 + T_{\beta,\max}$.

In these expressions $E_{\beta,\max}$ and $T_{\beta,\max}$ are the total and kinetic energies respectively of the β-particle at the upper limit of the energy spectrum. Therefore

$$p_\nu = \frac{E_{\beta,\max} - E_\beta}{c} = \frac{T_{\beta,\max} - T_\beta}{c}.$$

We can then write

$$\rho_f = \frac{L^6 p_\beta^2 (T_{\beta,\max} - T_\beta)^2 \, dp_\beta}{c^3 4\pi^4 \hbar^6}. \tag{4.13}$$

Substitution from equations 4.11 and 4.13 into the expression for the transition rate then yields

$$P(p_\beta)\, dp_\beta = \frac{g^2 M^* M}{2\pi^3 \hbar^7 c^3} (T_{\beta,\max} - T_\beta)^2 p_\beta^2 \, dp_\beta, \tag{4.14}$$

where $P(p_\beta)\, dp_\beta$ is the probability per unit time that a β-particle of momentum in the range p_β to $p_\beta + dp_\beta$ will be emitted, i.e. $P(p_\beta)$ is the spectral function of the β-particle momentum spectrum. As it is particle momentum which is directly measured in a magnet spectrometer, and as the most accurate spectra have been measured with such an instrument, it is convenient to leave the spectral function in this form rather than to convert entirely to energy. Note that the spectral function separates into three factors, the first involving the universal constants $\hbar$, c, g, the second depending on the nuclear matrix element M, assumed independent of the lepton momenta, and the third, the *statistical factor*, giving the spectral shape.

4.10 Beta-particle momentum spectrum

The theoretical spectrum given by equation 4.14 is a bell-shaped curve of the same general shape as a typical measured spectrum. At the low-momentum end of the distribution we may obtain an approximation to the predicted shape by neglecting all terms of higher order than p_β^2 and we see that the distribution

should be proportional to p_β^2, i.e. should be parabolic. Since, up to this point, electric charge has been assigned no role in the process, the theoretical spectral shape is the same for positrons and electrons. When, however, measured spectra are compared with the theoretical spectra adjusted to fit at the maximum values, it is found that for low momentum values the measured spectra for β^- particles lie consistently above the theoretical spectra and are almost linear in shape in the neighbourhood of the origin. On the other hand, for β^+ particles the measured spectra lie consistently below the theoretical curves.

An explanation of the departure from the theoretical distributions derived above is to be looked for in the assumption in the theory that the outgoing electron can be treated as a free-particle plane wave. This assumption neglects the fact that there is an interaction between the electron's charge and that of the daughter nucleus. The Coulomb force between these particles will decelerate the outgoing particle, if it is a negatron, and thus increase the proportion of low-momentum particles in the spectrum; in the case of positrons the Coulomb force will accelerate the positrons and reduce the relative number of low-momentum particles.

A more exact treatment requires the substitution for ψ_β in section 4.9 of an eigenfunction which will take the Coulomb interaction into account, and which will have a greater amplitude in the region near the nucleus, in the case of negatrons, and a smaller amplitude in that region, in the case of positrons, than the plane-wave amplitudes. This substitution leads to the introduction into the right-hand side of equation 4.14 of a factor $F(Z, p_\beta)$, the *nuclear Coulomb factor*, which in the completely relativistic treatment is of a complicated form. It has been tabulated as a function of Z and p_β (I. Feister, *Physical Review*, 1950). A simplified non-relativistic treatment leads to the following expression for $F(Z, p_\beta)$ which for many purposes is a close enough approximation to the tabulated value.

$$F(Z, p_\beta) = \frac{2\pi\delta}{1 - e^{-2\pi\delta}}, \tag{4.15}$$

where $\delta = \pm Z\alpha \dfrac{E_{\beta,\max}}{c p_\beta}$.

α is the fine structure constant, 1/137, and the positive sign is to be taken for β^- decay, the negative sign for β^+ decay.

For β^- decay and low values of momentum p_β, δ is large and positive and therefore

$$F(Z, p_\beta) \simeq 2\pi Z\alpha \frac{E_{\beta,\max}}{c p_\beta} \propto (p_\beta)^{-1}.$$

Taken with the p^2 dependence of the remaining factors in equation 4.14, this leads to an overall dependence on p_β in agreement with the linear rise of the experimentally measured spectra.

For β^+ decay and low values of momentum, δ is large and negative. Then the denominator of equation 4.15 is dominated by the exponential term. Hence

$$F(Z, p_\beta) \simeq 2\pi|\delta|e^{-2\pi|\delta|}.$$

The presence of the exponential has the effect of severely reducing the relative number of low-momentum positrons, in accord with the experimental observations.

The nuclear Coulomb factor must of course approach unity as $Z \to 0$, its effect becoming increasingly more marked as Z increases. The full quantum-mechanical treatment indicates that it not only redistributes the particles in the spectrum, as the naïve argument based on the accelerating effect on the outgoing particle might suggest, but that it increases the probability of β-emission throughout the spectrum for negatrons, thus increasing the overall probability of β^- decay. It has the opposite effect in the case of positron emission.

4.11 The Kurie plot

$P(p_\beta)$ in equation 4.14 corresponds to the number of particles in the momentum range p_β to $p_\beta + dp_\beta$ to be found in an experimental measurement of the spectrum. A direct plot of $P(p_\beta)$ against p_β yields, as we saw above, a bell-shaped curve. When experimental errors are involved, it is not a simple matter to make a detailed comparison of observations with the theoretical predictions. Further the determination of the end point of the spectrum, from which the important quantity $T_{\beta,\mathrm{max}}$ is to be evaluated, is a difficult exercise because the approach to the momentum axis is predicted to be parabolic and of course the particle numbers in each channel of increasing moment are approaching zero. In the experimental situation therefore these points have a low statistical accuracy and are merging into a statistically fluctuating background.

If, however, use is made of the theoretically predicted shape of the spectrum and the quantity

$$\left[\frac{P(p_\beta)}{p_\beta^2\, F(Z, p_\beta)}\right]^{\frac{1}{2}}$$

is computed for each momentum range, then this quantity is predicted by equation 4.14 to be equal to a constant multiplied by $T_{\beta,\mathrm{max}} - T_\beta$. Therefore if the expression is plotted against T_β rather than against p_β, two important consequences follow. Firstly the degree of linearity of the plot establishes how well equation 4.14 predicts the spectral shape. Secondly if the plot is linear then a simple extrapolation of the plot to cut the T_β axis gives a value of $T_{\beta,\mathrm{max}}$ with a statistical accuracy determined largely by the good statistical accuracy of the points near the maximum in the bell-shaped spectrum.

This useful form of presentation, exemplified in Figure 13, is termed a *Kurie plot*. Many examples are known of spectra leading to very linear Kurie plots. Where there is a departure from linearity (after all allowance has been made for finite source thickness and scattering from materials behind the source, both effects which can cause distortion of the low-energy part of the spectrum), then it is taken as an indication that v_{fi} for the nuclide in question is not independent of p_β.

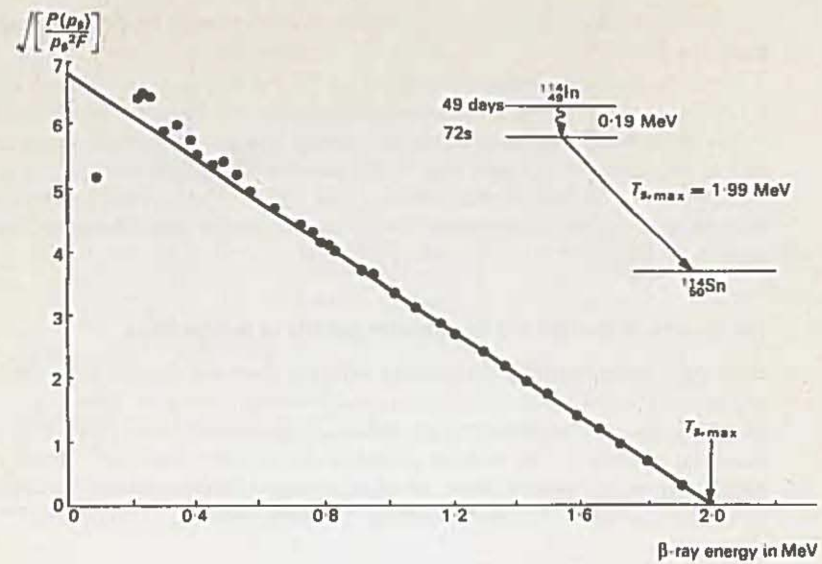

Figure 13 Kurie plot of the β^- spectrum of ^{114}In. $P(p_\beta)$ denotes the number of particles in a constant interval of momentum, while the kinetic energy plotted along the horizontal axis corresponds to the midpoint of momentum interval

4.12 Separation of complex beta spectra

There are many examples of β-transitions in which the residual nucleus may be left in its ground state or may be left in one of a set of excited states. This is analogous to fine structure in the case of α-decay. When there is, let us say, one excited state involved, there will be two β end-point energies and two superimposed continuous β-spectra. These spectra may be separated by making a Kurie plot which in the region beyond the end point of the lower-energy spectrum will be linear. By extrapolating this linear plot back to low momenta, the β-particle spectrum associated with the excited state can then be constructed by subtraction. Where there are two or more excited states involved, then the same process can be carried through in successive steps.

4.13 The mass of the neutrino

It was assumed in the derivation of equation 4.14 that the neutrino had zero rest mass. If we do not make this assumption then the full relationship

$$E_\nu^2 = p_\nu^2 c^2 + m_\nu^2 c^4$$

must be used. When we follow through the consequences of this, it is found that the effect is to modify the spectral distribution at the high energy tip, leading to a sharper cut-off to a value of $T_{\beta,\max}$ less than the value found by extrapolating the Kurie plot.

Thus the careful investigation of the Kurie plot as it approaches the energy axis is an accepted way of establishing the neutrino mass. No departure from linearity in this region has been established with certainty and the experiments therefore lead to the setting of an upper limit to the possible value of the neutrino mass. On this basis it can now be said to be less than 250 eV, i.e. less than 1/2000 of the electron mass and the experimental results are compatible with it being equal to zero.

4.14 The theoretical half-life and comparative half-life of beta emitters

$P(p_\beta)\,dp_\beta$ is the probability that β-decay will take place and that the β-particle will have a value of momentum in a particular momentum range. Thus $d\lambda = P(p_\beta)\,dp_\beta$ may be defined as the partial decay constant associated with a particular momentum requirement placed on the outgoing β-particle. To find the decay constant in the usual sense, we have to remove this momentum requirement by integrating over all values of β-particle momenta. Hence

$$\lambda = \int_0^{p_{\max}} \frac{g^2 M^*M}{2\pi^3 c^3 \hbar^7}\, p_\beta^2 (T_{\beta,\max} - T_\beta)^2 F(Z, p_\beta)\, dp_\beta$$

$$= \frac{g^2 M^*M}{2\pi^3 \hbar^7}\, m_0^5 c^4 f(Z, E_{\beta,\max}), \qquad 4.16$$

where

$$f(Z, E_{\beta,\max}) = \frac{1}{m_0^5 c^7} \int_0^{p_{\max}} F(Z, p_\beta) p_\beta^2 (T_{\beta,\max} - T_\beta)^2\, dp_\beta. \qquad 4.17$$

Since $E_\beta = m_0c^2 + T_\beta$, with a similar expression for $E_{\beta,\max}$, we may write equation 4.17 in the dimensionless form

$$f(Z, E_{\beta,\max}) = \int_0^{p_{\max}} F(Z, p_\beta) \left[\frac{E_{\beta,\max} - E_\beta}{m_0 c^2}\right]^2 \left[\frac{p_\beta}{m_0 c}\right]^2 \frac{dp_\beta}{m_0 c}.$$

An analytic expression for this integral cannot be given. It has been numerically computed and its value, for given values of Z and $E_{\beta,\max}$, tabulated and graphed (Feenberg, E. and Trigg, G., *Reviews of Modern Physics*, vol. 22, 1950, p. 399). However, the analysis can be taken further in the particular case where $F(Z, p_\beta)$ can be taken to be unity. This will correspond to β-transitions where Z has small values, strictly speaking where $Z = 0$. Then

$$f(0, E_{\beta,\max}) = \int_0^{p_{\max}} \left[\frac{E_{\beta,\max} - E_\beta}{m_0 c^2}\right]^2 \left[\frac{p_\beta}{m_0 c}\right]^2 \frac{dp_\beta}{m_0 c}$$

$$= \left[\frac{E_{\beta,\max}}{m_0 c^2}\right]^5 \int_{m_0c^2}^{E_{\beta,\max}} \left[1 - \frac{E_\beta}{E_{\beta,\max}}\right]^2 \left(\left[\frac{E_\beta}{E_{\beta,\max}}\right]^2 - \left[\frac{m_0^2 c^4}{E_{\beta,\max}^2}\right]\right)^{\frac{1}{2}} \frac{E_\beta}{E_{\beta,\max}} \frac{dE_\beta}{E_{\beta,\max}},$$

using $p_\beta^2 c^2 = E_\beta^2 - m_0^2 c^4$.

Let $\dfrac{E_\beta}{E_{\beta,\max}} = x$ and $\dfrac{m_0 c^2}{E_{\beta,\max}} = a$.

Then $f(0, E_{\beta,\max}) = \left[\dfrac{E_{\beta,\max}}{m_0 c^2}\right]^5 \displaystyle\int_a^1 (1 - x)^2 (x^2 - a^2)^{\frac{1}{2}} x\, dx.$

If now a can be assumed small (i.e. $E_{\beta,\max} \gg m_0c^2$) then

$$f(0, E_{\beta,\max}) = \left[\frac{E_{\beta,\max}}{m_0 c^2}\right]^5 \int_0^1 (1 - x)^2 x^2\, dx = \frac{1}{30} \left[\frac{E_{\beta,\max}}{m_0 c^2}\right]^5$$

Thus $\lambda = \text{constant} \times (E_{\beta,\max})^5$.

This result has been derived for low-Z nuclei emitting high-energy β-particles but its prediction that λ should be strongly dependent on the maximum energy of the emitted particle is true in the general case. In a comparison of decay constants (or equivalently in a comparison of the half-lives) of two isotopes it is very frequently convenient to remove the very strong effect of β-particle kinematics by forming the product

$f(Z, E_{\beta,\max}) T_{\frac{1}{2}}$,

which is called the *comparative half-life* and often referred to as the '*ft*' value. From equation 4.16 we see that

$$f(Z, E_{\beta,\max}) T_{\frac{1}{2}} = \frac{f(Z, E_{\beta,\max})}{\lambda} \ln 2 = \frac{\text{constant}}{M^*M}. \qquad 4.18$$

A measurement of the '*ft*' value thus gives a measure of the nuclear matrix element for the transition.

A survey of the measured '*ft*' values for a range of β-emitters reveals that, far from being constant, they vary over many orders of magnitude. As a consequence it is frequently convenient to work with log *ft*. Measuring *ft* in seconds, then log *ft* values range from about 3·5 to 23. Since the greatest possible value of M^*M will be unity and will arise when the normalized eigenfunctions $\psi_{Z\pm1,A}$

and $\psi_{Z,A}$ are identical, we assume that the eigenfunctions are closest to being identical in the cases that give rise to the lowest observed ft values. On this assumption substituting $ft = 10^{3.5}$ s and $M^*M = 1$ in equation 4.18 and using the known values of the other constants involved, we arrive at $g = 1{\cdot}4 \times 10^{-62}$ J m^3 for the value of the β-decay coupling constant as introduced above.

The cases involving ft values of the order of $10^{3.5}$ s arising from the eigenfunctions of the initial and final states being very similar, are referred to as *allowed transitions.* The so called *mirror nuclei* of which $^{17}_{8}O$ and $^{17}_{9}F$ form one example (the Z of one equals the N of the other), have ground-state eigenfunctions which are very similar, differing in fact only in respect of the Coulomb interaction. Transitions between the ground states of mirror nuclei should therefore be allowed. This is borne out in the case of the transition

$$^{17}F \rightarrow {}^{17}O + \beta^+.$$

In cases other than mirror nuclei the ground-state eigenfunctions need not have the same degree of similarity, and as a consequence M^*M will be reduced in value. This will lead to higher ft values. In cases where the transition is to an excited state of the daughter nucleus, the eigenfunctions may be markedly different and M^*M very much reduced from unity.

4.15 Beta-decay selection rules

If the two states involved in a transition are so different as to have different nuclear spins or different parities, then it follows from the results of quantum mechanics that M, and hence M^*M, are identically zero. This, on the basis of the theory developed above, would lead to an infinite half-life; in other words the transition would be forbidden. Thus the conditions that the transition be allowed are

(a) $\Delta I = 0$, where I is the nuclear spin, and
(b) nuclear parity must not change.

These constitute the *Fermi selection rules.*

It is however found that the violation of these selection rules does not lead to the transition being absolutely forbidden although it is inhibited to the extent of the ft value increasing by about three orders of magnitude. To understand why the ft value does not increase to infinity, we examine the assumption that the β-particle and neutrino wavelengths are of such a magnitude that the eigenfunctions of the outgoing leptons can be assumed constant over nuclear dimensions. This assumption is equivalent to taking only the first term, which is unity, in an expansion of factors of the form $e^{i2\pi r/\lambda}$ describing the spatial shape of the particle waves. If the second term in the expansion be included, then a term has to be added to the right-hand side of equation 4.11 which has a factor

$$\int \psi^*_{Z\pm1,A}\, r\psi_{Z,A}\, d\tau.$$

This integral is not necessarily zero when the violation of the selection rules has resulted in

$$\int \psi^*_{Z\pm1,A}\, \psi_{Z,A}\, d\tau = 0.$$

When the first term, usually the dominant term, is zero and the second term is not zero then the transition is termed *first forbidden.* The magnitude of the ft value depends on the magnitude of the second term and is about three orders of magnitude smaller than the ft value for allowed transitions. Should the second term as well as the first be zero, then the third term in the expansion has to be taken. It is again about three orders of magnitude smaller than the second term and the transition is now said to be *second forbidden.* This argument can be extended to still higher terms. There are examples (e.g. $^{115}_{49}In$) believed to be as high as fourth forbidden with log ft values of about twenty-four.

There is the further complication that there are known cases where the $\Delta I = 0$ and 'no parity change' selection rules are violated yet the ft values are as for allowed transitions. To accommodate this fact, it must be assumed that our assumption that the perturbing interaction could be expressed simply as a constant g leading to equation 4.11 is not valid. There may be coupling between the spins of the transforming nucleon and the emitted leptons contributing to the interaction energy. The more complicated matrix elements then arising lead to the *Gamow–Teller selection rules* requiring $\Delta I = 0$ or ± 1 (but *not* $0 \rightarrow 0$) with no change of parity. Very many cases of allowed transitions on the basis of ft values seem to be governed by the Gamow–Teller rules, although there are a few instances where the Fermi rules are required. For example

$$^{10}C \rightarrow {}^{10}B^* + \beta^+ \quad \text{and} \quad {}^{14}O \rightarrow {}^{14}N^* + \beta^+$$

are transitions in which the initial and final states have zero spin yet the ft values are as for allowed transitions.

4.16 The theory of electron capture

A theory to describe the process of electron capture can be set up along the same general lines as the theory discussed above for β-decay. There are however two important differences. Firstly, the initial state includes, in addition to the parent nucleus, an orbital electron. This electron will be in a well-defined energy state whose eigenfunction, derived from atomic theory, must be included in ψ_i.

Secondly, there is only one lepton, a neutrino, in the final state. Consequently the density of final states will be given by ρ_f where

$$\rho_f\, dE_f = \frac{L^3 p_\nu^2\, dp_\nu}{2\pi^2\hbar^3}.$$

Thus $$\rho_f = \frac{L^3 p_\nu^2}{2\pi^2\hbar^3 c} = \frac{L^3 E_\nu^2}{2\pi^2\hbar^3 c^3}.$$

E_ν has a fixed value, namely the decay energy if the daughter nucleus be formed in the ground state, or the decay energy less the energy of excitation if the daughter nucleus be formed in an excited state.

When allowance is made for these two considerations, it is found that

$$\lambda_{e.c.} = \frac{m_0^5 c^4}{2\pi^3 \hbar^7} g^2 M^* M f_{e.c.},$$

where $f_{e.c.} \simeq 2\pi \left[\frac{\alpha Z}{n}\right]^3 E_\nu^2,$

α being the fine-structure constant, and n the principal quantum number of the shell from which the electron is captured.

A quantity which has been studied extensively by experimental methods is the ratio of electron capture to positron emission in cases where these are competing processes. From the above expression we have in the case of K-capture ($n = 1$)

$$\frac{\lambda_K}{\lambda_{\beta^+}} = \frac{f_K}{f(Z, E_{\beta,\max})} \simeq \frac{2\pi(\alpha Z)^3 E_\nu^3}{f(Z, E_{\beta,\max})},$$

where $f(Z, E_{\beta,\max})$ is given by equation 4.17.

Note that $E_\nu = E_{\beta,\max} + m_0 c^2$ when the decay energy is great enough for positron emission to be energetically possible.

The theoretical prediction is in reasonably good agreement with the experimental results. K-capture is very much favoured over positron emission for heavy elements, being a thousand times more probably for $Z = 80$ and a decay energy of 1·5 MeV.

We also note that

$$\frac{\lambda_K}{\lambda_L} \simeq 8,$$

since $n = 2$ for the L-shell. This ratio can be altered very much in favour of L-capture when the decay energy is very small. In fact, when the decay energy is so small that atomic binding energies cannot be neglected, L-capture may be energetically possible whereas K-capture is energetically forbidden.

4.17 Double beta decay

It was shown in section 4.4 that, having regard to the mass–energy relationships governing β-instability, two neighbouring isobars cannot both be stable. There are however many instances of two stable isobars with an unstable isobar between. For example ${}^{124}_{50}Sn$ and ${}^{124}_{52}Te$ are both stable whereas ${}^{124}_{51}Sb$ is unstable. Now the mass of ${}^{124}Sn$ exceeds that of ${}^{124}Te$ and energetically the transition ${}^{124}Sn$ to ${}^{124}Te + \beta^- + \beta^-$ is possible. Theoretical estimates have been made of the decay constants for the double β-decay process on the assumption that no antineutrinos are emitted and also on the assumption that two antineutrinos, one associated with each β-particle are emitted. These calculations predicted that in the former case half-lives in the order of 10^{16} years were to be expected, in the latter case half-lives of the order of 10^{22} years.

Attempts have been made, using radiation and particle detectors, to find experimental evidence for this phenomenon, without convincing success. However a lower limit of 10^{16} years has been established for the half-life in several of the isotopes examined. Evidence has also been sought, by analysing with mass spectrometers geological specimens, to establish the extent to which the products of double β-decay may have built up. Such an investigation of tellurium-bearing ores revealed amounts of xenon consistent with a half-life of 10^{21} years for the transition

$${}^{130}Te \rightarrow {}^{130}Xe + \beta^- + \beta^-.$$

While therefore double β-decay is an important phenomenon in the theory of β-decay, the half-life is known to be so long that for all practical purposes we can consider both nuclei involved to be stable.

4.18 Summary

The theoretical explanation of β-decay transitions introduces the concept of 'weak interaction' between leptons and nucleons which takes its place with gravitation, electromagnetism and the 'strong interaction' between nucleons as one of the four basic interactions in nature. The decay constant for the β-decay process gives information concerning the degree of similarity between initial and final nuclear wave functions. The decay constant for electron capture, equivalent in a sense to positron emission but possible when positron emission is energetically impossible, yields information concerning the wave functions of orbital electrons within the nuclear volume. The β-decay energy enables mass differences to be accurately established when the transition is known to be ground state to ground state. When an excited state is also involved and the β-spectrum is therefore complex, analysis of the β-momentum distribution enables the energy of the excited state to be found.