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# The Atomic Nucleus

J. M. Reid

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# Chapter 1

## Introduction

### 1.1 Historical introduction

By the end of the nineteenth century, the successful development of chemical science had firmly established the concepts of *molecule* and *atom*. A sample of a chemical compound, it was believed, could be divided into parts, the parts having the same chemical behaviour as the original sample. Subdivision could continue without change in the chemical behaviour until the resulting specimen consisted of a single molecule, but if subdivision continued beyond this stage the chemical behaviour was no longer unaffected. Molecules were believed to be assemblies of a comparatively small number of atoms of which about ninety different types were thought to exist, each type corresponding to a chemical *element*. To take a simple example, the molecule of water was considered to be built from three atoms, two corresponding to the element hydrogen and one to the element oxygen. Its behaviour was that of water (or water vapour) unless it was dissociated into its component atoms, in which case it behaved as a mixture of the gaseous elements oxygen and hydrogen. Each type of atom was believed to be of different structure, and this structure in some way determined the chemical behaviour of the element with which the atom was associated.

Shortly before the beginning of the present century, there were indications that, just as molecules had internal structure in the sense of being constructed from atoms, so atoms themselves had structure and were built from more fundamental entities. The discovery of the electron (J. J. Thomson, 1897), which was found to be associated with a wide range of materials of different chemical behaviour, and the realization that its mass was very much less than that of the lightest atom, provided an early example of a possible 'subatomic' particle. At this time too, the study of radioactivity, a phenomenon Becquerel had stumbled upon, in 1896, in an investigation into the fluorescence of uranium salts following the then recent discovery of X-rays, provided examples of parts of atoms being ejected in some form of internal reorganization. Various models conferring internal structure on the atom were current in the first decade of this century. Rutherford in 1911, on the basis of alpha-particle scattering (which is discussed in detail in Chapter 3), proposed a new model. This model, with features added by Bohr in 1913, remains the basis of present-day theory.

## 1.2 The Rutherford-Bohr atom

According to the currently accepted picture, the atom is a comparatively diffuse empty structure, having a concentrated core or *nucleus*. Orbiting around the nucleus at distances very much greater than nuclear dimensions are electrons. The dimensions of the outermost electron orbits set the size of the atom, as strong repulsive forces operate when atoms come so close together that their electron orbits overlap. The mass of the electrons represents a very small fraction of the total mass of the atom. They are held in their orbits by the Coulomb force of attraction between their negative electric charge and a positive charge which is assumed to reside in the nucleus. At distances greater than atomic dimensions, the positive charge in the nucleus will be 'screened' by the negatively charged orbiting electrons. If the screening is not complete, that is, the charge in the nucleus is not completely compensated by the charge of the electrons, then additional electrons will be collected from the surroundings until the neutralization is exact.

Moseley's work (1913) on X-rays enabled the chemical elements to be placed precisely in order of increasing frequency of characteristic X-radiation. This order also gave the best fit of the elements into the periodic table, which had been developed as a means of revealing the pattern of chemical behaviour in the progression from lighter to heavier elements. The position of an element in this order fixes its *atomic number* (or *charge number*) which is denoted by  $Z$ . This number ranges from 1 for hydrogen to 92 for uranium, the highest value for a naturally occurring element (see Appendix E). The simple hypothesis that the number of orbiting electrons in an atom may be equated with  $Z$  turns out to be tenable. This hypothesis demands that the nuclear charge be  $+Ze$ , where  $-e$  is the electric charge on an electron. By alpha-particle-scattering experiments Chadwick (1920) was able to measure the charges on the nuclei of platinum, silver and copper. The values found experimentally were in good agreement with  $+Ze$  in each case.

## 1.3 The mass of the atom

Appeal to Avogadro's hypothesis, that 'equal volumes of all gases under the same conditions of temperature and pressure contain the same number of molecules', enables the relative weights (and hence masses) of molecules to be obtained from a comparison of gas densities. From a knowledge of molecular composition, in terms of constituent atoms, relative atomic weights may then be deduced. Standard atomic weights, as used in chemistry, are obtained from these relative values by defining the atomic weight of oxygen to be 16.

If the mass of an atom in absolute terms is required, then the total number of molecules in a given volume must be known. Usually the volume chosen is the gramme-molecular volume, which is 22.4 l. At 0 °C and one atmosphere pressure the number of molecules in this volume is known as *Avogadro's constant*. By several independent experimental methods estimates have been made of this number and the results are all in good agreement with the value  $6.023 \times 10^{23}$ .

Since 22.4 l of hydrogen at 0 °C and one atmosphere pressure weigh 2 g, and since there are two hydrogen atoms per molecule, it follows that the mass of the hydrogen atom is  $1.67 \times 10^{-24}$  g. Numerically this is the reciprocal of Avogadro's constant.

Some, but not all, chemical atomic masses are close to integral multiples of the hydrogen mass. This led to the very early suggestion, made by Prout in 1815, that the atoms of all elements are combinations of hydrogen atoms. Unfortunately the existence of elements whose chemical atomic masses fell half-way between integral multiples of the hydrogen atomic mass constituted an insuperable objection to this simple hypothesis. However, the situation was basically altered by the discovery of the existence of *isotopes*. This discovery arose in the chemical investigation (Soddy, 1906-13) of the products of radioactive decay. To take one instance, lead was produced by the decay of radium F. Lead was also produced by the decay of thorium C'. The two forms of lead were chemically indistinguishable from each other (and from ordinary lead) but one was found to have atomic mass 206, the other 208. These were then said to constitute two different isotopes of lead. The work of J. J. Thomson (1913) on positive ions revealed that there were two distinct isotopes of different mass present also in neon. It is now known that very many of the chemical elements occurring in their natural form, including hydrogen and oxygen, consist of a mixture of isotopes (see Appendix A). The chemical atomic mass, in the case of an element which has two or more stable isotopes, is an average value which depends on the relative abundance of the isotopes. If now we consider the isotopic masses instead of the chemical atomic masses, then, as will be discussed in detail later, a mass scale can be defined on which all isotopic masses have a value lying close to an integral number. For any particular isotope this integral number is called the *mass number* and is denoted by  $A$ .

## 1.4 The size of the atom

Since the atomic weight of oxygen is sixteen, the weight of a water molecule is approximately eighteen times the weight of a hydrogen atom, i.e. it is about  $30 \times 10^{-24}$  g. In 1 cm<sup>3</sup> of water, which weighs 1 g, there are thus  $10^{24}/30$  molecules. Each molecule therefore occupies  $30 \times 10^{-30}$  m<sup>3</sup>, and so the molecular linear dimension must be approximately  $3 \times 10^{-10}$  m. By simple arguments of this kind the atomic radius, assuming the atom to be spherical, is deduced to be of the order of  $10^{-10}$  m. Alpha-particle-scattering experiments show the uranium nucleus on the other hand to have a radius smaller than  $3 \times 10^{-14}$  m. We thus see the extent to which the atom is 'a diffuse empty structure'. The radius of the nucleus is about  $10^{-4}$  times the radius of the atom. If the nucleus is scaled to the size of the earth (6400 km radius) then the atom would extend to about  $64 \times 10^6$  km, that is, to a distance comparable to the sun's distance from the Earth.

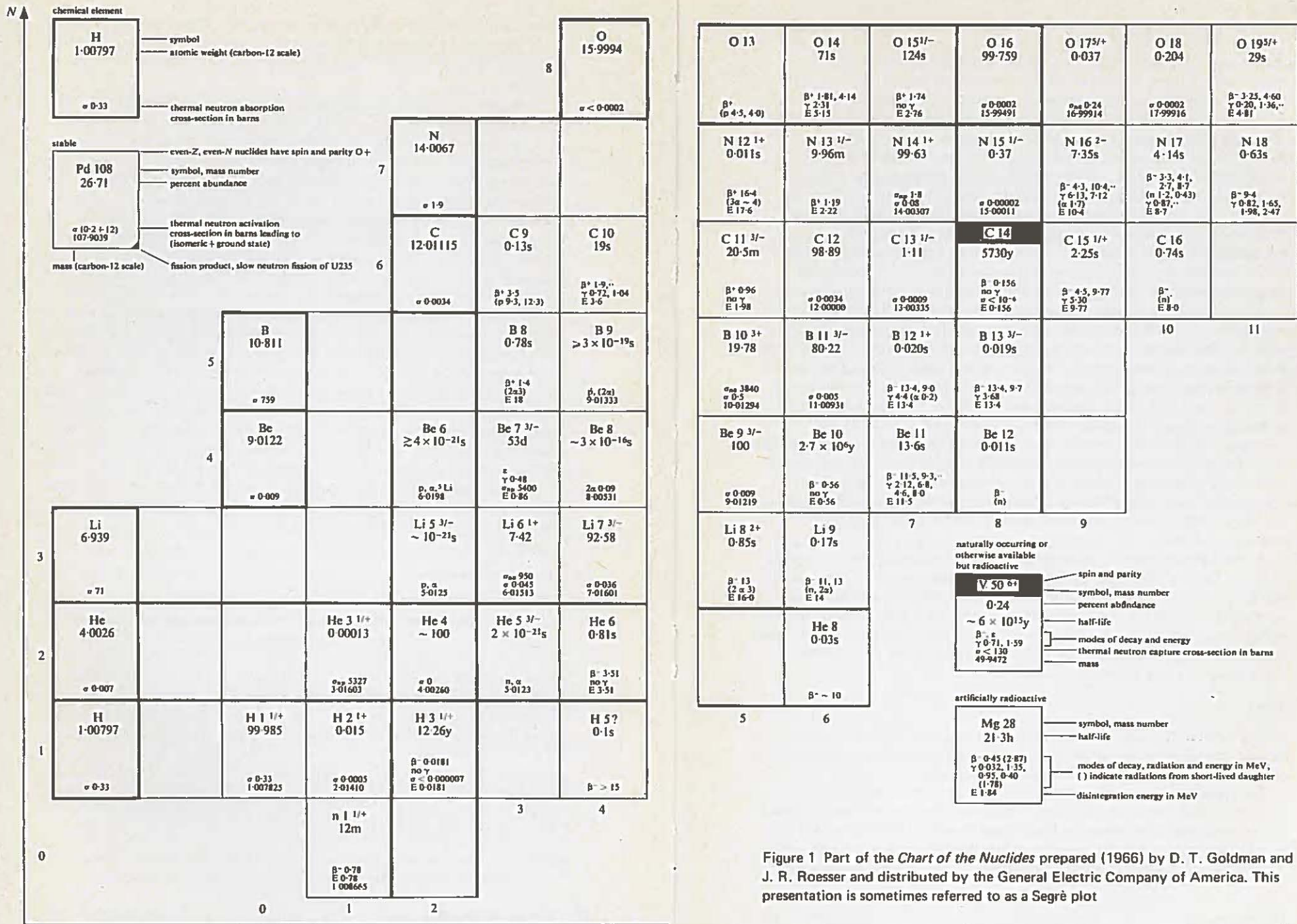


Figure 1 Part of the *Chart of the Nuclides* prepared (1966) by D. T. Goldman and J. R. Roesser and distributed by the General Electric Company of America. This presentation is sometimes referred to as a Segrè plot

## 1.5 Constitution of the nucleus

We have seen in section 1.3 that the atom of an isotope of mass number  $A$  is quite closely  $A$  times the mass of the lightest hydrogen isotope. In so far as the mass of the atomic electrons is negligible compared to the mass of the nucleus, this means that the mass of the nucleus is about  $A$  times the mass of the nucleus of the lightest hydrogen isotope. The latter nucleus, which we would expect to be the simplest of all nuclei, is given the name *proton*; it has  $A = 1$  and  $Z = 1$ . A simple hypothesis which suggests itself is that the general nucleus consists of  $A$  protons and  $A - Z$  neutrons, thus having a mass approximately  $A$  times the mass of the proton, and a net positive charge of  $Ze$ . However, as we shall see when we come to discuss intrinsic angular momentum or 'spin', such a hypothetical assembly in certain cases does not have a spin in agreement with the measured nuclear spin. There are additional objections based on the high energy that electrons would necessarily have if confined to a volume of nuclear dimensions. With the discovery of the *neutron* (Chadwick, 1932) another and, as it proved, satisfactory hypothesis could be made. The neutron has no charge and has a mass almost, but not quite, equal to that of the proton. It is convenient to introduce *nucleon* as a generic term for a particle which is either a proton or a neutron. We can then consider a nucleus to be built from  $Z$  protons and  $N$  neutrons. It will have a mass number  $A$  provided  $Z + N = A$ . In other words it contains  $A$  nucleons. It is in these terms that the nucleus is presently pictured.

Any two of the three integers  $Z, N, A$  completely determine the nuclear species or *nuclide* under discussion. A nuclide is usually denoted by its chemical symbol with a prefix indicating the  $A$ -value. A second lower prefix may be used to indicate the  $Z$ -value, though this of course is not necessary because the  $Z$ -value is already fixed by the chemical symbol. Thus the lighter of the stable helium isotopes is denoted by  ${}^3\text{He}$  or by  ${}^3_2\text{He}$ .

A useful plot in common use on which to present nuclear properties is shown in part in Figure 1. With  $N$  as abscissa and  $Z$  as ordinate, each nuclide has its own square on the chart. Since the chemical behaviour is determined by the number of orbiting electrons (i.e. the  $Z$ -value) all the isotopes of a given chemical element, having the same  $Z$ -value, lie on a horizontal line. Nuclides having the same  $N$ -value, called *isotones*, lie on a vertical line, while nuclides having the same  $A$ -value, called *isobars*, lie on a line with a backward slope of  $45^\circ$ .

## 1.6 Units

Having outlined the nature, size and mass of the structure, the atomic nucleus, which is the subject of our study, we conclude these introductory paragraphs with a few remarks concerning the units to be employed.

The praiseworthy attempt to achieve a standardized set of units in all scientific applications by the adoption of the SI (*Système International d'Unités*) units (International Conference on Weights and Measures 1960) has not yet led to the displacement of the units introduced by the pioneers in nuclear physics. The natural resistance to change, however trivial, in mental habits is supported

in some instances by the requirement for special units (for example nuclear magneton) which are not likely to have applications in outside fields. The relationships to the SI units are usually straightforward and these are included in Appendix D.

It is a happy coincidence that the unit of length, the *femtometer* ( $10^{-15}\text{m}$ ), abbreviated to fm, which is the order of size of the nucleus, allows, without causing confusion, the continuation in the use of the name 'fermi' for this unit in honour of the Italian physicist who pioneered major developments in both theoretical and experimental nuclear physics.

The continued development of experimental techniques has brought the requirement for the general use of decreasingly smaller units of time. Having left microseconds ( $10^{-6}\text{s}$ ) and nanoseconds ( $10^{-9}\text{s}$ ) behind we are passing into, and no doubt through, the era of picoseconds ( $10^{-12}\text{s}$ ). As will later be discussed, even this unit is long on the natural nuclear time scale.

The size of the useful energy unit is set by the binding energy of the nuclear components. This is about a million times greater than the binding energy of the valence atomic electrons. The unit which had been used in atomic physics was the electronvolt, defined as the change in kinetic energy of an electron moving in vacuum through a potential difference of one volt. It was therefore natural in nuclear physics to adopt a million electronvolts (MeV) equal to  $10^6\text{eV}$  as the unit of energy.

The charge on the nucleus is customarily expressed in electrostatic units (e.s.u.). This leads to the necessary introduction of the velocity of light in some symbolic electromagnetic formulae (e.g. nuclear magneton) which have become traditional in the subject. The only other unit we need mention is the gauss (G), which is used for the quantity loosely referred to as *the magnetic field*. This expression will normally refer to the magnetic flux density, the SI unit for which is the weber per metre squared which equals  $10^4$  gauss. The reader is referred to *Basic Electricity* by W. M. Gibson in the Penguin Library of Physical Sciences for a discussion of these and the other electromagnetic units.



# Chapter 2

## Radioactive Decay Laws

### 2.1 Introduction

The fact that the atoms of most materials do not spontaneously change their nuclear or chemical properties indicates that certain assemblies of nucleons with particular  $Z$ - and  $N$ -values, under the action we must assume of internal attractive forces, form stable structures. By no means all nucleon assemblies are stable. Some are so far from stability that, immediately after formation, one or more nucleons may be emitted and the grouping changed. However there exist certain assemblies that are very close to stability but are still not strictly speaking stable. They do not have enough excess energy to permit the immediate emission of even one nucleon and they exist for appreciable times before stabilizing by undergoing a transformation constituting one of the modes of radioactive decay and thereby altering their  $Z$ - and  $N$ -numbers. Before discussing in detail the various modes of radioactive decay, from which we derive considerable insight into the physical conditions within and around the nucleus, it is convenient to collect together certain mathematical results which are applicable to all modes of radioactive decay.

### 2.2 The exponential decay law

We start from the assertion that the probability that a particular nucleus undergoes radioactive decay in a time interval  $dt$  is  $\lambda dt$ , where, for a particular nuclide,  $\lambda$  is a constant called the *decay constant*. This assertion implies that the previous history of the nucleus has no bearing on its probability of decay at a particular time. For example, it means that when radioactive nuclei are being produced artificially, in, say, a reactor over a period of time, the 'older' nuclei at any instant are no more likely to decay than those more recently formed.

If now we consider a large number  $N$  of similar radioactive nuclei in a sample and if  $-dN$  of these decay in an interval  $dt$  then, providing  $N$  is very large, by definition the probability that any one nucleus decays in the interval is  $-dN/N$ . From the definition of the decay constant this probability is also  $\lambda dt$ . Therefore

$$\frac{dN}{dt} = -\lambda N.$$

The general solution of this equation is

$$\ln N = -\lambda t + \text{constant}.$$

Choosing the constant of integration to suit the initial condition that there be  $N_0$  nuclei originally in the sample, we have

$$N = N_0 e^{-\lambda t}. \quad 2.1$$

Experimentally, radioactivity is usually investigated by detecting the products of the decay. It is thus the number of decays per unit time, defined as the *activity* of the sample, which is of experimental relevance. In the present case, if we denote the activity by  $A$ , then by definition

$$A = -\frac{dN}{dt} = \lambda N_0 e^{-\lambda t},$$

from 2.1.

This may be written

$$A = A_0 e^{-\lambda t}, \quad 2.2$$

where  $A_0$  is the initial activity.

The justification for the original assertion will now lie in the accuracy with which this exponential law describes the observed decrease in activity with time. In practice it may be tested by plotting  $\ln A$  against  $t$ , when, if the assertion is valid, a straight line of negative slope  $\lambda$  should result. In the case of sources of reasonably high activity, such a linear plot is found to result and this is a recognized way of determining  $\lambda$ .

### 2.3 Half-life and mean life

Very often, instead of  $\lambda$ , an equivalent quantity, the *half-life*, denoted by  $T_{\frac{1}{2}}$  is used. It is defined as the time taken for the activity to fall to one half of its original value. From this definition and equation 2.2 we see that

$$\frac{1}{2}A_0 = A_0 e^{-\lambda T_{\frac{1}{2}}},$$

$$\text{and therefore } T_{\frac{1}{2}} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}. \quad 2.3$$

$T_{\frac{1}{2}}$  has the dimension of time;  $\lambda$  has the dimension of reciprocal time.

The *mean life*  $\tau$ , in the ordinary sense of the average life of all nuclei in a large assembly, is also of interest. The number of nuclei in the assembly having life  $t$  will be the number which decay in the interval from  $t$  to  $t + dt$ . This is  $\lambda N(t) dt$ . Forming the average in the usual way we then have

$$\tau = \frac{\int_0^{\infty} \lambda N(t) t dt}{N_0}.$$

Substituting for  $N(t)$  from equation 2.1 and carrying out the integration it follows that

$$\lambda\tau = 1 \quad 2.4$$

$$\text{and } \tau = \frac{T_{1/2}}{\ln 2}.$$

## 2.4 Series decay

It may be that the *daughter* nucleus, which is formed as the result of the decay of the *parent* nucleus, is itself radioactive with its own decay constant and half-life. Its subsequent decay may be experimentally distinguishable from that of its parent. In this circumstance, the two activities can be measured separately as functions of time.

Let  $N(t)$  be the number of parent nuclei present at a given time and  $n(t)$  the number of daughter nuclei at the same time. Let  $\lambda_P$  and  $\lambda_D$  be the respective decay constants. Then, as before, for the parent

$$dN(t) = -\lambda_P N(t) dt. \quad 2.5$$

For the daughter, however, the population is now increased by the decay of the parent as well as decreased by its own decay. Thus

$$dn(t) = -\lambda_D n(t) dt + \lambda_P N(t) dt. \quad 2.6$$

From equation 2.5, as before,

$$N(t) = N_0 e^{-\lambda_P t}.$$

Substituting this value into equation 2.6 and multiplying throughout by  $\exp(\lambda_D t)$  we have

$$\frac{d}{dt} [e^{\lambda_D t} n(t)] = \lambda_P N_0 e^{(\lambda_D - \lambda_P)t}.$$

It follows by integration that

$$e^{\lambda_D t} n(t) = \frac{\lambda_P N_0}{\lambda_D - \lambda_P} e^{(\lambda_D - \lambda_P)t} + \text{constant}.$$

If now the original conditions are that no daughter nuclei are present, i.e.  $n(0) = 0$ , then the constant of integration is such that

$$n(t) = \lambda_P N_0 \left[ \frac{e^{-\lambda_P t}}{\lambda_D - \lambda_P} + \frac{e^{-\lambda_D t}}{\lambda_P - \lambda_D} \right]. \quad 2.7$$

Since  $n(t)$  starts from zero at  $t = 0$  and must fall to zero as  $t$  approaches infinity, there must be at least one intermediate maximum value. Differentiating equation 2.7 with respect to  $t$  and equating  $dn/dt$  to zero we find

$$t_{\max} = \frac{\ln(\lambda_P/\lambda_D)}{\lambda_P - \lambda_D}. \quad 2.8$$

It is important to note that in this case the activity is *not* equal to  $dn(t)/dt$ . The number of nuclei *decaying* in time  $dt$  is not  $dn(t)$  as given in equation 2.6 but only the first term on the right-hand side of that equation. Thus

$$a(t) = \lambda_D n(t) = \lambda_D A_0 \left[ \frac{e^{-\lambda_P t}}{\lambda_D - \lambda_P} + \frac{e^{-\lambda_D t}}{\lambda_P - \lambda_D} \right]. \quad 2.9$$

The activity of the daughter, therefore, also builds up from zero to reach a maximum value at a time given by equation 2.8, then falls off as the difference between two exponentials.

Two cases of practical interest arise.

### 2.4.1 Short-lived parent

If the half-life of the parent is very short in comparison with that of the daughter nuclide, then  $\lambda_P \gg \lambda_D$ . In this case  $\exp(-\lambda_P t)$  becomes quickly negligible compared to  $\exp(-\lambda_D t)$ . If we also neglect  $\lambda_D$  in the denominator of equation 2.9 then the equation giving the long-term behaviour of the daughter activity is

$$a(t) = \frac{\lambda_D}{\lambda_P} A_0 e^{-\lambda_D t}. \quad 2.10$$

Note that to this approximation the parent nuclei transform instantaneously into the daughter nuclei.

In Figure 2 the parent and daughter activities corresponding to the case where  $\lambda_P = 10 \times \lambda_D$  are plotted as functions of time.

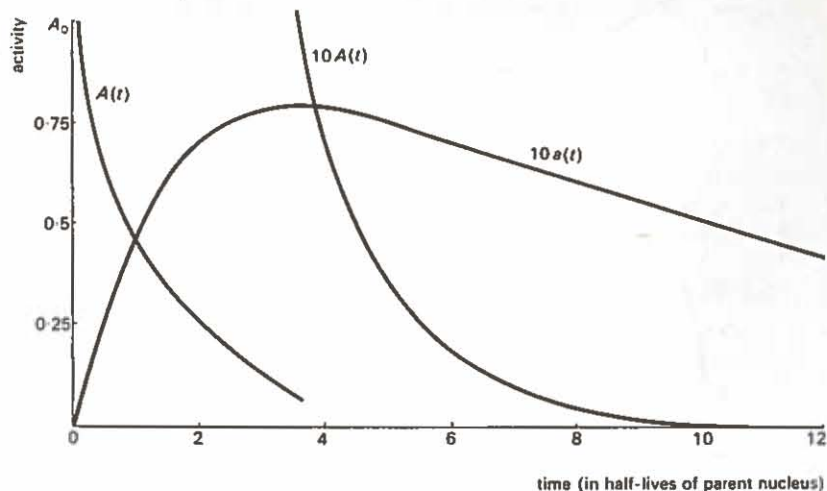


Figure 2 Plot of the variation with time of the radioactivities of short-lived parent and long-lived daughter materials in a source which initially contained only the parent material. The half-lives are taken to be in the ratio 1:10