Radiation Sources

■ Problem 1.1. Radiation Energy Spectra: Line vs. Continuous

Line (or discrete energy): a, c, d, e, f, and i.
Continuous energy: b, g, and h.

■ Problem 1.2. Conversion electron energies compared.

Since the electrons in outer shells are bound less tightly than those in closer shells, conversion electrons from outer shells will have greater emerging energies. Thus, the M shell electron will emerge with greater energy than a K or L shell electron.

■ Problem 1.3. Nuclear decay and predicted energies.

We write the conservation of energy and momentum equations and solve them for the energy of the alpha particle. Momentum is given the symbol "p", and energy is "E". For the subscripts, "al" stands for alpha, while "b" denotes the daughter nucleus.

$$p_{al} + p_b = 0 \qquad \frac{p_{al}^2}{2\,m_{al}} = E_{al} \qquad \frac{p_b^2}{2\,m_b} = E_b \qquad E_{al} + E_b = Q \quad \text{and} \quad Q = 5.5\ \text{MeV}$$

Solving our system of equations for E_{al}, E_b, p_{al}, p_b, we get the solutions shown below. Note that we have two possible sets of solutions (this does not effect the final result).

$$E_b = 5.5\left(1 - \frac{m_{al}}{m_{al} + m_b}\right) \qquad E_{al} = \frac{5.5\,m_{al}}{m_{al} + m_b}$$

$$p_{al} = \mp \frac{3.31662\sqrt{m_{al}}\sqrt{m_b}}{\sqrt{m_{al} + m_b}} \qquad p_b = \pm \frac{3.31662\sqrt{m_{al}}\sqrt{m_b}}{\sqrt{m_{al} + m_b}}$$

We are interested in finding the energy of the alpha particle in this problem, and since we know the mass of the alpha particle and the daughter nucleus, the result is easily found. By substituting our known values of $m_{al} = 4$ and $m_b = 206$ into our derived E_{al}equation we get:

$$E_{al} = 5.395\ \text{MeV}$$

Note : We can obtain solutions for all the variables by substituting $m_b = 206$ and $m_{al} = 4$ into the derived equations above :

$$E_{al} = 5.395\ \text{MeV} \qquad E_b = 0.105\ \text{MeV} \qquad p_{al} = \mp 6.570\sqrt{\text{amu} * \text{MeV}} \qquad p_b = \pm 6.570\sqrt{\text{amu} * \text{MeV}}$$

■ Problem 1.4. Calculation of Wavelength from Energy.

Since an x-ray must essentially be created by the de-excitation of a single electron, the maximum energy of an x-ray emitted in a tube operating at a potential of 195 kV must be 195 keV. Therefore, we can use the equation E=h$\vee$, which is also E=hc/λ, or λ=hc/E. Plugging in our maximum energy value into this equation gives the minimum x-ray wavelength.

$$\lambda = \frac{h \times c}{E} \qquad \text{where we substitute } h = \ 6.626 \times 10^{-34}\ J * s,\ \ c = 299\,792\,458\ m/s \text{ and } E = 195\ \text{keV}$$

$$\lambda = \frac{\textbf{1.01869 J–m}}{\textbf{KeV}} = \textbf{0.0636 Angstroms}$$

■ Problem 1.5. ^{235}U Fission Energy Release.

Using the reaction $^{235}U \rightarrow {}^{117}\text{Sn} + {}^{118}\text{Sn}$, and mass values, we calculate the mass defect of:

$$M(^{235}U) - \left[M(^{117}\text{Sn}) + M(^{118}\text{Sn})\right] = \Delta\text{M}$$

and an expected energy release of ΔMc^2.

$$Q = (235.0439 - (116.9029 + 117.9016))\,\text{AMU} \times \frac{\textbf{931.5 MeV}}{\textbf{AMU}} = 223\ \textbf{MeV}$$

This is one of the most exothermic reactions available to us. This is one reason why, of course, nuclear power from uranium fission is so attractive.

■ Problem 1.6. Specific Activity of Tritium.

Here, we use the text equation Specific Activity = (ln(2)*Av)/ ($T_{1/2}$*M), where Av is Avogadro's number, $T_{1/2}$ is the half-life of the isotope, and M is the molecular weight of the sample.

$$\textbf{Specific Activity} = \frac{\textbf{ln(2)} \times \textbf{Avogadro's Constant}}{T_{1/2}\,M}$$

We substitute $T_{1/2}$ = 12.26 years and M=$\frac{3\ \text{grams}}{\text{mole}}$to get the specific activity in disintegrations/(gram–year).

$$\textbf{Specific Activity} = \frac{\textbf{1.13492} \times \textbf{10}^{\textbf{22}}}{\textbf{gram –year}}$$

The same result expressed in terms of kCi/g is shown below

$$\textbf{Specific Activity} = \frac{\textbf{9.73 kCi}}{\textbf{gram}}$$

■ Problem 1.7. Accelerated particle energy.

The energy of a particle with charge q falling through a potential ΔV is qΔV. Since ΔV= 3 MV is our maximum potential difference, the maximum energy of an alpha particle here is q*(3 MV), where q is the charge of the alpha particle (+2). The maximum alpha particle energy expressed in MeV is thus:

$$\textbf{Energy} = \textbf{3 Mega Volts} \times \textbf{2 Electron Charges} = \textbf{6. MeV}$$

- **Problem 1.8. Photofission of deuterium. ${}^{2}_{1}D + \gamma \rightarrow {}^{1}_{0}n + {}^{1}_{1}p$ + Q (-2.226 MeV)**

The reaction of interest is ${}^{2}_{1}D + {}^{0}_{0}\gamma \rightarrow {}^{1}_{0}n + {}^{1}_{1}p$+ Q (-2.226 MeV). Thus, the γ must bring an energy of at least 2.226 MeV in order for this endothermic reaction to proceed. Interestingly, the opposite reaction will be exothermic, and one can expect to find 2.226 MeV gamma rays in the environment from stray neutrons being absorbed by hydrogen nuclei.

- **Problem 1.9. Neutron energy from D-T reaction by 150 keV deuterons.**

We write down the conservation of energy and momentum equations, and solve them for the desired energies by eliminating the momenta. In this solution, "a" represents the alpha particle, "n" represents the neutron, and "d" represents the deuteron (and, as before, "p" represents momentum, "E" represents energy, and "Q" represents the Q-value of the reaction).

$$p_a + p_n = p_d \qquad \frac{p_a^2}{2\,m_a} = E_a \qquad \frac{p_n^2}{2\,m_n} = E_n \qquad \frac{p_d^2}{2\,m_d} = E_d \qquad E_a + E_n = E_d + Q$$

Next we want to solve the above equations for the unknown energies by eliminating the momenta. (Note : Using computer software such as Mathematica is helpful for painlessly solving these equations).

We evaluate the solution by plugging in the values for particle masses (we use approximate values of "m_a," "m_n,"and "m_d" in AMU, which is okay because we are interested in obtaining an energy value at the end). We define all energies in units of MeV, namely the Q-value, and the given energy of the deuteron (both energy values are in MeV). So we substitute m_a = 4, m_n = 1, m_d = 2, Q = 17.6, E_d = 0.15 into our momenta independent equations. This yields two possible sets of solutions for the energies (in MeV). One corresponds to the neutron moving in the forward direction, which is of interest.

$$E_n = 13.340 \text{ MeV} \qquad E_a = 4.410 \text{ MeV}$$
$$E_n = 14.988 \text{ MeV} \qquad E_a = 2.762 \text{ MeV}$$

Next we solve for the momenta by eliminating the energies. When we substitute m_a = 4, m_n = 1, m_d = 2, Q = 17.6, E_d = 0.15 into these equations we get the following results.

$$p_n = \frac{p_d}{5} \mp \frac{1}{5}\sqrt{2}\,\sqrt{3\,p_d^2 + 352} \qquad p_a = \frac{1}{10}\left(8\,p_d \pm 2\sqrt{2}\,\sqrt{3\,p_d^2 + 352}\right)$$

We do know the initial momentum of the deuteron, however, since we know its energy. We can further evaluate our solutions for p_n and p_a by substituting:

$$p_d = \sqrt{2 \times 2 \times 0.15}$$

The particle momenta (in units of $\sqrt{\text{amu} * \text{MeV}}$ **)** for each set of solutions is thus:

$$p_n = -5.165 \qquad p_a = 5.940$$
$$p_n = 5.475 \qquad p_a = -4.700$$

The largest neutron momentum occurs in the forward (+) direction, so the highest neutron energy of 14.98 MeV corresponds to this direction.

Radiation Interaction Problems

■ Problem 2.1 Stopping time in silicon and hydrogen.

Here, we apply Equation 2.3 from the text.

$$T_{stop} = \frac{1.2\ \text{range}\ \sqrt{\frac{\text{mass}}{\text{energy}}}}{10^7}$$

Now we evaluate our equation for an alpha particle stopped in silicon. We obtained the value for "range" from Figure 2.8 (converting from mass thickness to distance in meters by dividing by the density of Si $\simeq$ 2330 mg/cm^3). The value for "mass" is approximated as 4 AMU for the alpha particle, and the value for "energy" is 5 MeV. We substitute range = 22 x 10^{-6}, mass = 4 and energy = 5 into Equation 2.3 to get the approximate alpha stopping time (in seconds) in silicon.

$$T_{stop} = 2.361 \times 10^{-12}\ \text{seconds}$$

Now we do the same for the same alpha particle stopped in hydrogen gas. Again, we obtain the value for "range" (in meters) from Figure 2.8 in the same manner as before (density of H $\simeq$ .08988 mg/cm^3), and, of course, the values for "mass" and "energy" are the same as before (nothing about the alpha particle has changed). We substitute range = 0.1, mass = 4 and energy =5 into Equation 2.3 to get the approximate alpha stopping time (in seconds) in hydrogen gas.

$$T_{stop} = 1.073 \times 10^{-8}\ \text{seconds}$$

The results from this problem tell us that the stopping times for alphas range from about picoseconds in solids to nanoseconds in a gas.

■ Problem 2.2. Partial energy lost in silicon for 5 MeV protons.

Clever technique: A 5 MeV proton has a range of 210 microns in silicon according to Figure 2-7. So, after passing through 100 microns, the proton has enough energy left to go another 110 microns. It takes about 3.1 MeV, according to the same figure, to go this 110 microns, so this must be the remaining energy. Thus the proton must have lost 1.9 MeV in the first 100 microns.

■ Problem 2.3. Energy loss of 1 MeV alpha in 5 microns Au.

From Figure 2.10, we find that $\frac{-1}{\rho}\frac{dE}{dx} \simeq 380\ \frac{\text{MeV}*\text{cm}^2}{g}$. Therefore, $\frac{dE}{dx} \simeq 380\ \frac{\text{MeV}*\text{cm}^2}{g} * \rho$ (ignoring the negative sign will not affect the result of this problem).

$$\text{Energy loss} = \frac{\rho\ (dE/dx)\ \Delta x}{\rho}$$

We substitute $dE/dx = \frac{380\ \text{MeV cm}^2\ \rho}{\text{gram}}$, $\rho = \frac{19.32\ \text{grams}}{\text{cm}^3}$ and $\Delta x = 5$ microns to get the energy loss of the 1 MeV α-particle in 5 μm Au (in non-SI units).

$$\text{Energy loss} = \frac{\text{36 708 MeV microns}}{\text{cm}}$$

The result in SI units is thus:

Energy loss = 3.671×10^6 eV

Since this energy loss is greater than the initial energy of the particle, all of the α-particle energy is lost before 5 μm.
Note the small range of the α , i.e. ~ μm per MeV.

■ Problem 2.4. Range of 1 MeV electrons in Al. Scaling Law.

The Bragg-Kleeman rule, or scaling law, allows us to relate the known range in one material to the range in another material. The semi-empirical rule we use is:

$$\frac{R_1}{R_0} \cong \frac{\rho_0 \sqrt{A_1}}{\rho_1 \sqrt{A_0}} \quad \text{(Equation 2.7)}$$

Here, we use Figure 2.14 to approximate the 1 MeV electron range in silicon (R_0), and since we know every other quantity in Equation 2.7, we can approximate the range of the 1 MeV electron in aluminum (R_1). Solving for R_1 we can find the estimated range of the electron in Aluminum.

$$r_{\text{Al}} = \frac{r_{\text{Si}}\, \rho_{\text{Si}} \sqrt{\frac{\text{AW}_{\text{Al}}}{\text{AW}_{\text{Si}}}}}{\rho_{\text{Al}}}$$

We substitute $r_{\text{Si}} = \frac{0.5\, g}{\text{cm}^2\, \rho_{\text{Si}}}$, $\text{AW}_{\text{Al}} = 26.9815$ amu, $\text{AW}_{\text{Si}} = 28.0855$ amu, $\rho_{\text{Al}} = \frac{2.698\, g}{\text{cm}^3}$ and $\rho_{\text{Si}} = \frac{2.329\, g}{\text{cm}^3}$ to get the approximate range of 1 MeV electrons in aluminum (in cm).

r_{Al} = 0.1816 cm

■ Problem 2.5. Compton scattering.

This problem asks for the energy of the scattered photon from a 1 MeV photon that scattered through 90 degrees. We use the Compton scattering formula (Equation 2.17). We write the Compton scattering formula, defining the scattering angle ("θ") as 90 degrees and the photon energy ("E_0") as 1 MeV.

$$\text{Energy} = \frac{E_0}{\frac{(1-\text{Cos}[\theta])\, E_0}{m_e * c^2} + 1}$$

We substitute $\theta = 90°$ and $E_0 = 1$ MeV to get the energy of the scattered photon in MeV.

Energy = 0.338 MeV

Problem 2.6. Prob of photoelectric in Si versus Ge

For a rough estimate, we can note that photoelectric probabilities vary as $\sim Z^{4.5}$ so we would expect that

$$\tau_{Si}/\tau_{Ge} = (14/32)^{4.5} = 0.0242$$

■ Problem 2.7. The dominant gamma ray interaction mechanism.

See Figure 2-20 and read off the answers (using the given gamma-ray energies and the Z-number for the given absorber in each part).
Compton scattering: a, b, and d.
Photoelectric absorption: c
Pair production: e

■ Problem 2.8. Mean free path in NaI of 1 MeV gamma-rays.

(a). The gamma-ray mean free path (λ) in NaI is $1/\mu$ (where μ is the total linear attenuation coefficient in NaI). The mass attenuation coefficient ($\frac{\mu}{\rho}$) is 0.06 cm^2/gm at 1 MeV according to Figure 2.18, and the density of NaI relative to water (ρ) is 3.67 gm/cm^3 (by the definition of specific gravity). Therefore, we have $\lambda = 1/\mu = \frac{1}{\left(\frac{\mu}{\rho}\right)*\rho}$. Here, we will denote the mass attenuation coefficient ($\frac{\mu}{\rho}$) by μ_ρ, so we have

$$\lambda = \frac{1}{(\mu_\rho \rho)}$$

We substitute $\mu_p = \frac{0.06\ cm^2}{g}$ and $\rho = \frac{3.67\ g}{cm^3}$ to get the mean free path of 1 MeV gamma-rays in NaI (in cm).

$$\lambda = 4.54\ cm$$

(b). Any photon which emerges from 1 cm cannot have undergone a photoelectric absorption. Neglecting buildup factors, the probability that a photon emerges from the slab without having an interaction is $e^{-\mu_T x}$, where μ_T is the **total** attenuation coefficient. The complement of this is the probability that a photon doesn't emerge from the slab without having had at least one interaction ($1-e^{-\mu_T x}$). The probability that the interaction is a photoelectric interaction is τ/μ_T (this is not the probability per unit path length, but the total probability that any given interaction is a photoelectric interaction). Therefore, the probability that a photon undergoes photoelectric absorption in the slab is $(\tau/\mu_T)*(1-e^{-\mu_T x})$. This equation is expressed below, along with the values for μ_T (which is just 1 divided by the previous result for λ), the attenuation distance (denoted "x" and which is 1 cm), and τ, which is just the mass attenuation coefficient for photoelectric absorption (found on Figure 2.18 to be 0.01) multiplied by the density of NaI (3.67 g/cm3).

$$\text{Probability of photoelectric absorption} = \frac{\tau\,(1 - e^{-\mu_T x})}{\mu}$$

We substitute $\mu_\tau = \frac{1}{4.54\,\text{cm}}$, $x = 1$ cm and $\tau = \frac{0.01\times3.67}{\text{cm}}$ to get the probability of photoelectric absorption for 600 keV gamma-rays in 1 cm NaI.

Probability of photoelectric absorption = 0.0329

What is interesting is that a different result is obtained using a different, although seemingly equally valid approach. We can note that the probability per unit path length of a photoelectric interaction is τ, so $1 - e^{-\tau x}$ is the probability of a photoelectric interaction in traveling a distance x.

Probability of *a* photoelectric interaction = $1 - e^{-\tau x}$

We substitute x=1cm and $\tau = \frac{0.010\times3.67}{\text{cm}}$ to get the probability of a photoelectric interaction in traveling a distance x.

Probability of *a* photoelectric interaction = 0.0360

This result is slightly (10%) larger from the previous answer because this approach does not account for the attenuation of photons through the material by other means.

■ Problem 2.9. Definitions

See text.

■ Problem 2.10. Mass attenuation coefficient for compounds.

The linear attenuation coefficient is the probability per path length of an interaction. In a compound, the total linear attenuation coefficient would be given by the sum over the i^{th} constituents multiplied by the density of the compound:

$$\mu_c = \rho_c \sum w_i \left(\frac{\mu}{\rho}\right)_i \qquad \text{(Equation 2.23)}$$

where w_i is the weight fraction of the i^{th} constituent in the compound (represented by "w_H" and "w_O," respectively, in this problem), $(\mu/\rho)_i$ is the mass attenuation coefficient of the i^{th} constituent in the compound (represented by "μ_H" and "μ_O," respectively), and ρ_c is the density of the compound (represented by "ρ_W" for the density of water). The expression below shows this sum for water attenuating 140 keV gamma rays:

$$\mu_c = \rho_W \left(\mu_H w_H + \mu_O w_O\right)$$

We substitute $\mu_H = .26\ \text{cm}^2/g$, $\mu_O = .14\ \text{cm}^2/g$, $\rho_W = 1\,g/\text{cm}^3$, $w_H = 2/18$ and $w_O = 16/18$ to get the linear attenuation coefficient for water attenuating 140 keV gamma rays.

$$\mu_c = \frac{0.153333}{\text{cm}}$$

The mean free path, λ, is just the inverse of this last calculated value, or $\frac{1}{\mu_C}$. The mean free path of 140 keV gamma rays in water is thus:

$$\lambda_{H_2O} = 6.52\ \text{cm}$$

This turns out to be an important result because Tc – 99 *m* is a radioisotope routinely used in medical diagnostics and it emits 140 keV gamma rays. Since most of the human body is made of water, this gives us an idea of how far these gamma rays can travel without an interaction through the human body and into our detectors.

■ Problem 2.11. 1 J of energy from 5 MeV depositions.

We are looking for the number of 5 MeV alpha particles that would be required to deposit 1 J of energy, which is the same as looking for how many 5 MeV energy depositions equal 1 J of energy. We expect the number to be large since 1 J is a macroscopic unit of energy. To find this number, we simply take the ratio between 1 J and 5 MeV, noting that 1 MeV = 1.6 x 10^{-13} *J*.

The number of 5 MeV alpha particles required to deposit 1 J of energy is thus:

$$n = \frac{\textbf{1 Joule}}{\textbf{5 MeV}} = \mathbf{1.25 \times 10^{12}\ alpha\ particles}$$

■ Problem 2.12. Beam energy deposition.

100 μA tells us the current, i.e., the number of Coulombs passing in unit time. If we divide this by the amount of charge per particle, it gives us the number of particles passing through the area in unit time. If we take this value, and multiply it by the energy per particle, we get the energy/time, or power. The power dissipated in the target by a beam of 1 MeV electrons with a current of 100 μA is thus:

$$\textbf{Power} = \frac{\textbf{(100 } \mu\textbf{A) (1 MeV)}}{\textbf{(Electron Charge)}} = \textbf{100 Watts}$$

This is a surprisingly low amount of power, about the same as from a light bulb, and represents the output from a small accelerating device.

■ Problem 2.13. Exposure rate 5 m from 1 Ci of ^{60}Co.

We use the following equation for exposure rate:

Exposure Rate = $\Gamma_s \frac{\alpha}{d^2}$ (Equation 2.31)

where α is the source activity, d is the distance away from the source, and Γ_s is the exposure rate constant. The exposure rate constant for Co-60 is 13.2 $R - \text{cm}^2 / \text{hr} - \text{mCi}$ (from Table 2.1). The exposure rate 5 m from a 1 Ci Co-60 source is thus:

$$\textbf{Exposure Rate} = \Gamma_s \frac{\alpha}{d^2} \quad \textit{where we substitute } \alpha = 1\ \text{Ci},\ d = 5\ m,\ \Gamma_s = 13.2\ \frac{R - \text{cm}^2}{\text{hr} - \text{mCi}}$$

$$\textbf{Exposure Rate} = \frac{\textbf{0.528 cm}^2\ \textbf{R}}{\textbf{hr mm}^2} = \frac{\textbf{52.8 mR}}{\textbf{hr}}$$

The result in SI units :

$$\textbf{Exposure Rate} = \frac{\mathbf{3.78 \times 10^{-9}\ C}}{\textbf{kg} - \textbf{s}}$$

Problem 2.14. ΔT/Δt from 10 mrad/hr.

The dose of 1 rad corresponds to an energy deposition of 100 ergs/gram. So 10 mrad/hr corresponds to 1 erg/(gram-hr). Using a specific heat of water as 1 calorie/(gram $-{}^{o}$C), we can use the equation $\Delta Q = mC_p \Delta T$. Our given 1 ergs/(gram-hr) is $\Delta Q/(m*\Delta t)$. If we divide our equation by mC_p, we are left with $\Delta T/\Delta t$ on the right side, which is the quantity of interest, i.e.

$$\Delta T/\Delta t = \frac{\Delta Q/(m\Delta t)}{C_p} = \frac{1 \text{ ergs/(gram}-\text{hr)}}{1 \text{ calorie/(gram} - {}^{o}C)}.$$

$$\frac{\Delta T}{\Delta t} = \frac{1 \text{ Erg}}{\frac{1 \text{ Gram Calorie Hour}}{\text{Gram Centigrade}}}$$

The result below is the rate of temperature rise in a sample of liquid water with an absorbed dose rate of 10 mrad/h. Note that this is virtually impossible to measure because it is so small.

$$\frac{\Delta T}{\Delta t} = \frac{2.39 \times 10^{-8} \text{ Centigrade}}{\text{Hour}}$$

There are unusual radiation detectors which actually use the temperature rise in a detecting material to detect ionizing radiation. For the curious reader, do some research on bolometers in radiation detection, and also read about the superconducting radiation detectors under development.

Problem 2.15. Fluence-dose calculations for fast neutron source.

The Cf source emits fast neutrons with the spectrum N(E) dE given in the text by Eqn. 1.6. Each of those neutrons carries a dose h(E) that depends on its energy as shown in Fig. 2.22(b). To get the total dose, we have to integrate N(E)h(E) over the energy range.

First, let's consider only the neutron dose h(E). In the MeV range, we need a linear fit to the log h-log E plot of Figure 2.22(b) by using two values read off the curve:

$$\text{Log}_{10}\left(10^{-12}\right) = b + m \, \text{Log}_{10}(0.01)$$
$$\text{Log}_{10}\left(10^{-10}\right) = b + m \, \text{Log}_{10}(1.0)$$

Solving the system of equations above yields :

$$m = 1 \quad \text{and} \quad b = -10$$

This result gives us an approximate functional form for h(E) $\left[\text{in Sv} - \text{cm}^2\right] = 10^{-10} E$ [MeV]. Recall 1 Sv = 100 Rem = 10^5 mrem. In a moment, we'll integrate N(E)h(E) to get the total dose-area, but first check the normalization of N(E):

$$\text{norm} = \int_0^\infty \sqrt{E} \; e^{-\frac{E}{1.3}} \, dE \quad = \quad 1.31359$$

We'll need this normalization factor because we will want to use N(E)/norm as the probability that a source neutron has energy E. Now, a source neutron – cm^2 arriving at the person delivers a dose (in Sv) of:

$$\text{neutron} - \text{dose} = \frac{\int_0^\infty E^{3/2} \, e^{-\frac{E}{1.3}} \, dE}{10^{10} \, \text{norm}} \quad = \quad 1.95 \times 10^{-10} \text{ Sv}$$

We now need to multiply this by the number of neutrons produced by 3 micrograms of Cf-252 (2.3 x 10^6n/sec-mg) at 5 meters over 8 hours:

$$\textbf{dose} - \textbf{equivalent} = \frac{2.3 \times 10^6\ (3\ \mu\text{g})\ (8 \times 3600\ \text{sec})}{\mu\, g\ \ \text{sec}\left(4\,\pi\,500^2\right)} \quad = \quad 63\,254.5\ \ \text{neutrons}\,/\,\text{cm}^2$$

So the total dose equivalent is given by (using 10^5mrem/Sv):

$$\textbf{dose equivalent}_{\textbf{neutrons}} = 1.23\ \textbf{mrem}$$

This is a small dose, comparable to natural background.

Aside: What about the dose from the gamma rays? They are high-energy gammas and the source emits 9.7 gammas per fission. Using a value of h_E~ 5 x 10^{-12} Sv – cm^2:

$$\textbf{dose equivalent}_{\gamma} = \frac{\left(2.3 \times 10^6\ \text{neutrons}\right)(3\ \mu\text{g})\ (8 \times 3600\ \text{sec})\ (5\ \text{Sv cm}^2)\ (100\ \text{Rad})\ (1000\ \text{mRad})}{(1\ \mu\text{g}\ \ \text{sec})\left(10^{12}\ \text{neutrons}\right)(4\,\pi\,500\ \text{cm}^2)\ (\text{Sv Rad})} = 0.0316\ \text{mRad}$$

So the gamma dose is even smaller than that from the fast neutrons, as expected. Fast neutrons have a high quality factor, i.e., they produce a heavy charged particle when they interact, and therefore do a lot more biological damage than the light electrons produced when gamma rays interact in materials.

Counting statistics problems

■ Problem 3.1. Effect of increasing number of trials on sample variance.

The relative variance of the variable x, i.e., $\frac{\sigma_x^2}{\langle x \rangle} = \frac{\langle x^2 \rangle}{\langle x \rangle^2} - 1$, is dependent only on the ratio of the means of x^2 and x. It does not depend upon the uncertainty in those quantities. Since these means are not expected to change with more samples, the relative variance (i.e., 2% of the mean) shouldn't change. Note that this conclusion is independent of the type of distribution (Poisson, Guassian, Binomial, etc.) for x. However, for any quantity that is derived from measurements, such as the mean <x>, the uncertainty in that quantity improves with additional measurements as shown by: $\sigma_{\langle x \rangle} = \sqrt{\frac{\langle x \rangle}{N}}$.

■ Problem 3.2. Probability of 8 heads occuring in 12 coin tosses.

We define the binomial distribution for n = 12 (12 trials), p = 0.5 (probability of a success is 1/2), and we give the value k = 8 (the number of successes we are interested in). We substitute the known values of n,p and k and evaluate the binomial distribution below to get the probability that exactly 8 heads (or tails for that matter) will occur in 12 tosses of a coin.

$$\textbf{Probability} = \frac{n!}{k!\,(n-k)!}\,p^k\,(1-p)^{n-k} = 0.121$$

■ Problem 3.3. Statistics of males occuring in random population samples.

The mean is well known to be (prob of success)*(number of trials) =0.75 N The probability of success of any one trial (drawing a male) is large (so Poisson statistics is not valid) and the sample size is only 15. Binomial statistics therefore apply. We substitute n = 15 and p = 0.75 in the equations below to find the mean ($\overline{x}$), variance (σ^2) and standard deviation (σ).

$$\overline{x} = n \times p = 11.25$$

$$\sigma^2 = n\,p\,(1-p) = 2.81$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{n\,p\,(1-p)} = 1.68$$

■ Problem 3.4. Probability of no sixes in ten rolls of a dice.

Here, we define our probability distribution function as a binomial distribution with n=10 (number of trials), p=1/6 (probability of a success), and evaluating it at k=0 successes, which is the same as finding the probability that no sixes (or no fives, or fours, etc.) will turn up in ten rolls of the dice. The probability that no sixes will turn up in ten rolls of a dice is thus:

$$\textbf{Probability} = \frac{n!}{k!\,(n-k)!}\,p^k\,(1-p)^{n-k} = 0.162$$

■ Problem 3.5. Statistics of errors in computer program statements.

Poisson statistics applies to this problem because the probability of success (an error) is low, but the expected number of successes is not >~20 (the expected number of successes is 250/60 ≃ 4.17, so we cannot apply Gaussian statistics).

a) Here, we simply define the expected mean and standard deviation of a Poisson distribution with an expected number of successes ($\bar{x}$ = np) of 250/60. Of course, the expected number of successes is the same as the expected mean (they are both equal to np), and in Poisson statistics, the standard deviation is the same as the square root of the expected mean. This is reflected in the results shown below:

$$\bar{x} = \frac{250}{60} = 4.17$$

$$\sigma = \sqrt{\bar{x}} = 2.04$$

b) Next, we define a Poisson distribution with $\bar{x}$= 100/60 (expected number of successes), and evaluate the Poisson probability distribution function at k=0 successes. The probability that a 100-statement program will be free of errors is thus:

$$\text{Probability} = \frac{e^{-\bar{x}}\,\bar{x}^{k}}{k!} = 0.189$$

■ Problem 3.6. When is square-root of a number an estimate of uncertainty?

The only time the square root of a number is an estimate of its uncertainty is when the number is a direct sample of a population. For example, the number cannot have units associated with it (i.e. the square root of a value is not an estimate of its uncertainty when it is not a number of counts obtained by direct measurement).

(a). Yes. This is just a number of counts.

(b). Yes. This is just a number of counts.

(c). No. This is a processed number. One must use error propagation to determine the error in the quantity derived from the measurements.

(d). No. A rate has units and involves a division of the number of counts by time. In this case, error propagation says: $\sigma^2 = \frac{N}{T^2}$.

(e). No. An average is a processed quantity derived from the measured values. Error propagation says: $\sigma = \sqrt{x/N}$ where x is the expected value and N is the number of samples in the average.

(f). Yes. Although this is a processed quantity, it would yield the same result as if we had just counted for one 5 minute period. Using error propagation: σ = Sqrt ($N_1 + N_2 + ... + N_5$), so the error in the sum is just square root of the sum. This only works for sums, and not for subtractions.

■ Problem 3.7. Source + Background -> Net counts and uncertainty

We are asked to find net = (S+B) - B and σ_{net}. Finding $\sigma_{net}$is a straight forward error propagation since all counts are taken for 1 minute. Below we define the equation for the net counts and the standard error propagation formula (Eqn. 3.37). As a shorthand notation, the error propagation equation is expressed as a dot product between the two vectors representing the squared partial derivatives and the corresponding variances. The variable "sb" refers to (S+B) and "b" refers to B in the equation for "net".

$$\text{net} = sb - b$$

$$\sigma_{\text{net}} = \sqrt{\left\{\left(\frac{\partial \text{net}}{\partial \text{sb}}\right)^2, \left(\frac{\partial \text{net}}{\partial b}\right)^2\right\}.\{\sigma_{\text{sb}}^2, \sigma_b^2\}}$$

We substitute sb=561, b=410, $\sigma_{\text{sb}}^2 = \text{sb}$ and $\sigma_b^2 = b$ to get the net number of counts and the expected uncertainty in the net number of counts (σ_{net}).

net = 151 counts

σ_{net} **= 31.16**

■ Problem 3.8. Source + Background -> Net count rate and uncertainty

In this problem, the source plus background count rate (S+B) is 846 counts in 10 min, and the background count rate (B) is 73 counts in 10 min. We want the net count rate and its associated standard deviation. The net count rate is simply (S+B)-B, which is calculated below.

$$\text{net count rate} = \frac{846 - 73}{10} = \frac{77.3 \text{ counts}}{\text{minute}}$$

The expected error, or the standard deviation, in the count rates for (S+B) and B are calculated using the error propagation formula for division by a constant (Eq. 3.40), and then the standard deviation in the net count rate is calculated using error propagtion for differences of counts (Eq. 3.38), where σ_{S+B} and σ_B are substituted for σ_x and σ_y. The expected error, or standard deviation, in the net count rate (counts/min) is thus:

$$\sigma_{\text{net}} = \frac{\sqrt{846 + 73}}{10} = \frac{3.03 \text{ counts}}{\text{minute}}$$

■ Problem 3.9. Results using optimal counting times for Problem 3.8.

We first solve the equation giving the optimal division of time (Eq. 3.54). This is done below, where we have again defined the variables "sb" to denote the total counts of (S+B), and "b" to denote the background B in the equation. First we solve the system of equations for T_{sb} and T_b, and then we substitute the measured values for the count rates (84.6 and 7.3 counts/min, respectively) and the total amount of time allowed for the measurements to be done (20 min) to find the numerical values of $T_{\text{sb}}$and T_b.

$$T_{\text{sb}} = \sqrt{\frac{\text{sb}}{b}}\, T_b \qquad T_b + T_{\text{sb}} = T_{\text{tot}}$$

The resulting solutions for T_{sb} and T_b solved from the equations above is thus:.

$$T_{\text{sb}} = \frac{\sqrt{\frac{\text{sb}}{b}}\, T_{\text{tot}}}{\sqrt{\frac{\text{sb}}{b}} + 1} \qquad T_b = \frac{T_{\text{tot}}}{\sqrt{\frac{\text{sb}}{b}} + 1}$$

We substitute sb = 84.6, b = 7.3 and $T_{\text{tot}} = 20$ to get the numerical values of T_{sb} and T_b in minutes (i.e. the optimal times for measuring (S+B) and B, respectively).

T_{sb} = 15.46 minutes T_b = 4.54 minutes

We now calculate the uncertainty in the net count rate using error propagation. When defining the net count rate, we denote the number of counts over the new optimal time intervals as "n_{sb}" and "n_b," respectively, and the optimal counting times as "t_{sb}" and "t_b," respectively. We then use the basic formula for error propagation with the appropriate variables (as in problem 3.7, the dot between the two bracketed quantities signifies the dot product between the two, just as if we thought of them as two vectors). Next we substitute in known values to get the expected error in the net count rate.

$$\text{net} = \frac{n_{sb}}{t_{sb}} - \frac{n_b}{t_b}$$

$$\sigma_{\text{net count rate}} = \sqrt{\left\{\left(\frac{\partial \text{net}}{\partial n_{sb}}\right)^2, \left(\frac{\partial \text{net}}{\partial n_b}\right)^2\right\} \cdot \left\{\sigma_{n_{sb}}{}^2, \sigma_{n_b}{}^2\right\}}$$

We substitute $\sigma_{n_{sb}}{}^2 = n_{sb} = 84.6\ t_{sb}$, $\sigma_{n_b}{}^2 = n_b = 7.3\ t_b$, $t_{sb} = 15.5$ and $t_b = 4.5$ to get the expected uncertainty in the net count rate when the optimal time intervals are used.

$\sigma_{\text{net count rate}}$ = 2.66

The improvement factor is thus 3.03/2.66.

■ Problem 3.10. Counting time versus uncertainty.

We are looking for the improvement in the relative uncertainty of a measurement by longer counting. We know:

$\frac{\sigma_N}{N} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{\text{RT}}}$, where N is the number of counts, R is the count rate, and T is the measurement time interval.

Since we have only one measurement, N is assumed to be our experimental mean (see Eq. 3.29). Since the count rate remains constant ($R_1 = R_0$, and $\frac{1}{R_1} = \frac{1}{R_0}$), we have :

$$\left[\left(\frac{\sigma_N}{N}\right)^2 T\right]_1 = \left[\left(\frac{\sigma_N}{N}\right)^2 T\right]_0 \qquad \text{or} \qquad T_1 = T_0 \left[\frac{\left(\frac{\sigma_N}{N}\right)_0}{\left(\frac{\sigma_N}{N}\right)_1}\right]^2$$

We already know that $\left[\frac{\left(\frac{\sigma_N}{N}\right)_0}{\left(\frac{\sigma_N}{N}\right)_1}\right] = \left(\frac{2.8}{1.0}\right)$, and that T_0 = 10 min. From here, calculating T_1 is very simple. The new counting time in minutes is thus:

$$\text{counting time} = 10\left(\frac{2.8}{1.0}\right)^2 = 78.4 \text{ minutes}$$

Since the original counting time was 10 minutes, 68.4 minutes must be added to reduce the statistical uncertainty from 2.8% to 1.0%.

■ Problem 3.11. Better to increase source or decrease background?

Recall our relationship that the relative uncertainty (fractional standard deviation squared, or ϵ^2) in the source is given by:

$$\frac{1}{T}\frac{\left(\sqrt{S+B}+\sqrt{B}\right)^2}{S^2} \quad \text{(from Eqn. 3.55)}$$

(a). If the S >> B, then this becomes $\frac{1}{ST}$. Doubling the source strength is the best choice since the background is nearly irrelevant.

(b). If S << B, then this becomes $\frac{4B}{S^2T}$. Doubling source improves the ratio by 4 times, whereas halving background only improves ratio by 2 times. Doubling the source strength is again the best choice.

■ Problem 3.12. Probabilities of getting desired counts.

The expected number of counts in two minutes is (2 min)*(2.87 counts/min) = 5.74 counts. This is too small to apply Gaussian statistics, so we assume Poisson statistics to be accurate.

a) Here, we define a Poisson distribution with a mean of 5.74 (or 2.87*2) counts (since this is the only measurement made), and we use the Poisson probability density function (PDF):

$$\textbf{Probability} = \frac{e^{-\mu}\mu^k}{k!}$$

We substitute μ=5.74 and k=5 successes (i.e., counts) to get the probability that a given 2 minute count will contain exactly 5 counts.

Probability = 0.167

b) In order to determine the probability that at least one count will be recorded, we simply subtract the probability of 0 counts being recorded from 1 (since the integral of the probability distribution function from 0 to infinity is normalized to 1).

$$\textbf{Probability} = 1 - \frac{e^{-\mu}\mu^k}{k!}$$

We substitute μ=5.74 and k=0 to get the probability that a given 2 minute count will contain at least 1 count:

Probability = 0.997

For the last part, we want to know how many counts are needed to ensure at least one count with probability >99%. Note that this is the same as looking for the measurement time "t" required to acheive this number of counts (i.e. number of counts = (2.87 counts/min)*t). The key to this problem is to note that the probability of 0 counts must be less than 1% to satisfy this condition. We again define our Poisson Distribution, but this time with a mean value of 2.87*t. We define the probability of observing no counts to be 1%, and we solve for the resulting value for "t" that satisfies this condition:

$$\frac{e^{-\mu}\mu^k}{k!} = .01$$

We substitute μ=2.87t and k=0 then solve for t to get the minimum counting time required (in minutes) to ensure with > 99% probability that at least one count is recorded:

t = 1.61 minutes

■ Problem 3.13. Percent standard deviation between the activity ratio of two sources.

The ratio of the activity of Source B to Source A is simply given by:

$$\text{Ratio} = \frac{[(\text{Source } B + \text{background count rate}) - (\text{background count rate})]}{[(\text{Source } A + \text{background count rate}) - (\text{background count rate})]}$$

This is represented below where we denote the number of counts by "c," measurement time by "t," background by "b," Source B + background by "bb," and Source A + background by "ab."

$$\textbf{Ratio } R = \frac{\frac{c_{bb}}{t_{bb}} - \frac{c_b}{t_b}}{\frac{c_{ab}}{t_{ab}} - \frac{c_b}{t_b}}$$

We substitute $c_{ab} = 251$, $t_{ab} = 5$, $c_{bb} = 717$, $t_{bb} = 2$, $c_b = 51$ and $t_b = 10$ to get the ratio of the activity of Source B to Source A.

Ratio R = 7.84

Next, we define the explicit error propagation formula with the appropriate variables (as in previous problems, using the dot product notation under the square root) and give the known values in counts and minutes, respectively.

$$\sigma = \sqrt{\left\{\left(\frac{\partial R}{\partial c_{ab}}\right)^2, \left(\frac{\partial R}{\partial c_{bb}}\right)^2, \left(\frac{\partial R}{\partial c_b}\right)^2\right\} . \{c_{ab}, c_{bb}, c_b\}}$$

We substitute $c_{ab} = 251$, $t_{ab} = 5$, $c_{bb} = 717$, $t_{bb} = 2$, $c_b = 51$ and $t_b = 10$ to get the percent standard deviation in the ratio of the activity of Source B to Source A.

$\sigma_{B/A}$ = 0.635

■ Problem 3.14. Estimating source measurement time based on desired error.

This problem is asking us to find the minimum measurement time interval necessary for the (source + background) counting rate measurement such that the fractional standard deviation of the net source counting rate (source alone) is at most 3%. To do this, we simply write the equation for the fractional standard deviation, set it equal to .03, and solve for the unknown time interval for the source + background.

Below, we define the net counting rate as "R," "c" denotes a number of counts, "t" denotes a measurement time interval, "sb" indicates a measurement for source + background, and "b" indicates a measurement for background. Next we define the fractional standard deviation for "R" (which is $\frac{\sigma_R}{R}$) using the explicit form of the error propagation formula in the numerator in dot product notation, as in previous problems. We then set $\frac{\sigma_R}{R}$ equal to .03, substitute in known values and solve for t_{ab}.

$$R = \frac{c_{sb}}{t_{sb}} - \frac{c_b}{t_b}$$

$$\frac{\sigma_R}{R} = \frac{\sqrt{\left\{\left(\frac{\partial R}{\partial c_{sb}}\right)^2, \left(\frac{\partial R}{\partial c_b}\right)^2\right\}.\{c_{sb}, c_b\}}}{R} = 0.03$$

We substitute $c_{sb} = 80\, t_{ab}$, $c_b = 845$ and $t_b = 30$ then solve the above equation to get the time interval the source should be counted for (with background) to determine the counting rate due to the source alone to within a fractional standard deviation of 3% (in minutes).

t_{ab} = 54.1 minutes

■ Problem 3.15. Uncertainty in groups of measurements.

(a). The data fluctuations are expected to be statistical if the standard deviation of the sample population is the square root of the mean. This seems true, but we check this with a Chi-squared distribution to be sure.

Here, we define the student's data set and calculate the standard deviation of that data set.

data = {25, 35, 30, 23, 27}

σ = 4.69

Next, we take the square root of the mean (which is 28).

$$\sqrt{\overline{x}} = 5.29$$

Here, we define chi-squared (χ^2) for the data set (using Eqn. 3.36).

$$\chi^2 = \frac{(N-1)\,\sigma^2}{\overline{x}}$$

We substitute N=5, σ^2=22 and $\overline{x}$=28 to get the value of $\chi^2$for our data set.

$$\chi^2 = \frac{22}{7}$$

This is our "measured" value of χ^2. One can also calculate the expected value of χ^2 from data values drawn from a predicted distribution. (The χ^2 distribution is defined as the distribution of the quantity $\sum_{i=1}^{n} x_i^2$, where the x_i are random variables following a normal distribution that has a unit variance and a mean value of zero).

To be consistent with the approach taken in the textbook, we actually want to calculate 1 minus the χ^2 "**C**umulative **D**istribution **F**unction" (CDF). The χ^2 CDF is the integral from zero up to some argument, which in this case is $\frac{22}{7}$. The function CDF(x) gives the probability that the expected value of χ^2 ranges between 0 and the value x, assuming the data follow the normal distribution. The complement of this is then the probability that the expected value of χ^2 is larger than this value, which is what the textbook uses. One can look these values up in statitics tables, but we use *Mathematica* to do this calculation for us:

$$1 - \text{CDF}\left[\chi^2\left(4\ \text{degrees}-\text{of}-\text{freedom},\ \frac{22}{7}\right)\right] = 0.534$$

Because this value is close to 0.5, the data are random (i.e. a true Poisson distribution would have a χ^2 probability of 0.5). As a side note, this probability could also have been estimated using the χ^2 distribution table.

b) This question is asking that since the data appears to be random, what is the EXPECTED standard deviation of the MEAN of 5 single measurements using just these data. For this data set, given a mean , $\sigma_{\bar{x}} = \sqrt{\frac{\bar{X}}{N}}$ (Eqn. 3.44).

Using the provided data, the expected standard deviation in the mean of the data set is:

$$\sigma_x = \sqrt{\frac{\overline{x}}{N}} = 2.37$$

(c). Now, suppose we have the 30 measurements of the mean from the 30 students. What would we expect to measure for the variance of the set of these mean values $\{x_1, x_2, x_3,.., x_{30}\}$?

The variance estimated in (b) above IS the expected fluctuation when samples are drawn identically from that population. So we expect that the 30 mean values WILL show this variance (i.e. we would expect the sample variance in this situation to be the result above squared, or $s^2 = \sigma^2_{\bar{\bar{x}}}$; this is calculated to be $\simeq 5.59$).

(d). Suppose we now average these 30 values to get a better estimate of the mean. What is the standard deviation for the mean?

One way to look at this is to see it as 5 x 30 data points and calculate $\sigma_{\bar{\bar{x}}}$. The standard deviation goes down by $\sqrt{30}$, so the expected standard deviation of the mean when we use all 30 mean values will be 0.432049.

Another way is to calculate the standard deviation of the average of the averages. If we define $<\bar{x}>$ as the average of the 30 students individual means, then:

$$\sigma_{<\bar{x}>} = \sqrt{\frac{<\bar{x}>}{N}}$$

which, of course, turns out to be exactly the same formula.

■ Problem 3.16. Chi-squared test on a data set of counting measurements.

Define the data set we are applying the chi-squared (χ^2) test to as the variable "data":

data = {3626, 3731, 3572, 3689, 3625, 3711, 3617, 3572, 3578, 3569, 3677, 3630, 3615, 3605, 3591, 3678, 3624, 3652, 3595, 3636, 3465, 3574, 3601, 3540, 3629}

We determine the χ^2 value of the data by using the data values to calculate the sample variance (s^2) and the sample mean $\bar{x}$, and then calculate:

$$\chi^2_{\text{data}} = \frac{(\text{No. of data values} - 1)\ \text{Variance}(\text{data})}{\text{Mean}(\text{data})}$$

Using a calculator to determine the mean and variance of the data set, one finds the mean is 3616.08 and χ^2 is 21.2191.

In problem 3.15(a) above, we described how to determine the probability that a true Poisson distribution would have fluctuations larger than the data set by using the complement to the cumulative probability function of the χ^2 distribution.

We used *Mathematica* to calculate this probability, but it can also be found in your textbook or in standard tables of probabilities.

1 - CDF[ChiSquareDistribution(No. of data points - 1, χ^2_{data})] = 0.626

Since this $\chi 2$ probability is close to 0.5, the fluctuations are consistent with random fluctuations (relatively close to a true Poisson distribution). In general, one looks for values that are very small (~5%) or very large (~95%) to indicate that the data is not following the expected statistical model. If this is the case, then one looks at whether there is a problem with the measurement system.

■ **Problem 3.17. Uncertainty in the difference between two measurements.**

Suppose we are given that a set of I counts $\{N_i\}$, each taken over a period of time $t_I$results in an average rate $< r>$. The uncertainty in the average rate can be determined from error propagation once we write the average rate in terms of the measured counts:

$<r> = (1/I)(N_1/t_I + N_2/t_I + ... N_I/t_I) = (1/It_I)\,(N_1 + N_2 + ... N_I) = N_{total}/t_{total}$

so

$\sigma_{<r>}^2 = (1/It_I)^2\,\{N_1 + N_2 + \ ... \ + N_I\} = N_{total}/t_{total}^2$

We know that the total number of counts $N_{total} = \ < r > \ t_{total}$. Since for this problem, we are given <r> and $t_{total}$for each group, we can find $N_{total}$for each group.

Wtih N_A and N_B (total counts from Group A and Group B), we want to find out if the difference between $< r >_A$and $< r >_B$ is significant. Define this difference as:

$\Delta = \frac{N_A}{T_A} - \frac{N_B}{T_B}$.

where $T_A$is the total time for Group A (i.e., I * t_I), and similarly for Group B.

Using error propagation on these independent measurements,

$\sigma_\Delta^2 = \left(\frac{\sigma_{N_A}}{T_A}\right)^2 + \left(\frac{\sigma_B}{T_B}\right)^2 = (< r_A > /T_A + < r_B > /T_B)$, since $\sigma_{N_A}^2 = N_A$ (i.e., the standard deviation in a number of counts is equal to the square root of that number).

If both groups are making identical measurements, we expect the probability of observing a given value of Δ, i.e., P(Δ), to be Gaussian with <Δ>=0 and $\sigma_\Delta = \sqrt{< r > \left(\frac{1}{T_A} + \frac{1}{T_B}\right)}$

We look at our measured value of Δ, and see if it lies within $\pm\sigma_\Delta$ of 0. We will look at $\frac{\Delta_{meas}}{\sigma_{\Delta_{theory}}}$and if this value is >> 1, then the difference is significant since the probability of observing our value of Δ would be small.

Below we define $\frac{\Delta}{\sigma_\Delta}$, and substitute the known values for "N_A," "N_B," "T_A," and "T_B"

$$\frac{\Delta}{\sigma_\Delta} = \frac{\frac{N_A}{T_A} - \frac{N_B}{T_B}}{\sqrt{\frac{N_A}{T_A^2} + \frac{N_B}{T_B^2}}}$$

We substitute $N_A = 2162.4 \times 10,\ N_B = 2081.5 \times 20,\ T_A = 10$ and $T_B = 20$ to get the value of $\frac{\Delta}{\sigma_\Delta}$.

$$\frac{\Delta}{\sigma_\Delta} = 4.52$$

Since the difference between the two measured averages is 4.5 standard deviations from the expected mean (of zero), the difference is significant (i.e. there is a very small probability of observing such a value). We conclude that the premise that both groups were making the same measurement is highly unlikely.

Problem 3.18. Optimal total counting time in source activity calibration.

This problem is solved based on the assumption that if both sources are counted for the same amount of time, then $\frac{A_u}{A_s} = \frac{N_u}{N_s} = R$, or $A_u = A_s\ R$, where "A" represents the activity of a source, "N" represents the number of counts measured by the detector over the counting period for either source (not the total counting period), "u" stands for the unknown source, and "s" stands for the reference source. From this and error propagation for multiplication of counts (Eqn. 3.41), we have:

$$\left(\frac{\sigma_{A_u}}{A_u}\right)^2 = \left(\frac{\sigma_{A_s}}{A_s}\right)^2 + \left(\frac{\sigma_R}{R}\right)^2$$

We already know that $\frac{\sigma_{A_u}}{A_u} = .02$, since we want an expected standard deviation in the unknown activity to be 2%. We also know that $\frac{\sigma_{A_s}}{A_s} = \frac{0.05}{3.50}$, since we are given the quoted reference source activity of 3.50±0.05 μCi. Therefore:

$$(.02)^2 = \left(\frac{.05}{3.50}\right)^2 + \left(\frac{\sigma R}{R}\right)^2$$

Then, through further error propagation for division of counts and the assumption that $\sigma_{N^2} = N$, we have:

$$\left(\frac{\sigma_R}{R}\right)^2 = \left(\frac{\sigma_{N_u}}{N_u}\right)^2 + \left(\frac{\sigma_{N_s}}{N_s}\right)^2 = \frac{1}{N_u} + \frac{1}{N_s}$$

$$= \frac{1}{(1000\ \text{counts/sec})*(t/2)} + \frac{1}{(1000\ \text{counts/sec})*(t/2)} = \frac{4}{(1000\ \text{counts/sec})*(t)}$$

where "t" is the **total** counting time for both measurements (i.e. each measurement occupies an equal amount of time, t/2). Therefore, our final equation that we have to solve for t is:

$$\mathbf{.02^2 = \left(\frac{0.05}{3.5}\right)^2 + \frac{4}{1000\ t}}$$

We solve the above equation for t to get the total counting time, t, in seconds.

t = 20.4 seconds

■ Problem 3.19. Measuring half-life and its expected standard deviation of a source.

a) To solve this, we write the expression for the half life from the data measured, two measured counts taken over two measurement times that are separated by time t. The background is assumed to have no uncertainty. This expression is shown below, where the time separation between the measurements is denoted by "t," number of measured counts by n_1 and n_2, the measurement times by t_1 and t_2. We denote the background count rate by b_{rate}, and then the calculated half-life t_{half} is given by:

$$\mathbf{t_{half} = -\frac{\ln(2)\ t}{\ln\left(\frac{\frac{n_2}{t_2} - b_{rate}}{\frac{n_1}{t_1} - b_{rate}}\right)}}$$

We substitute $t = 24\,\text{hours}$, $n_2 = 914$, $t_2 = 10\,\text{minutes}$, $n_1 = 1683$, $t_1 = 10\,\text{minutes}$ and $b_{\text{rate}} = 50\,\text{minute}^{-1}$ to get the calculated half-life of the source:

$$\mathbf{t_{half} = 15.8\ hours}$$

b) Now to determine the uncertainty in this value, we apply the error propagation formula. This is shown below (as in previous problems, we use the shorthand dot product notation here for the error propagation formula). Note that we are applying the fact that σ_{n^2} = n (which is acceptable because "n" is a number of counts).

$$\sigma_{t_{\text{half}}} = \sqrt{\left\{\left(\frac{\partial t_{\text{half}}}{\partial n_1}\right)^2, \left(\frac{\partial t_{\text{half}}}{\partial n_2}\right)^2\right\}.\{n_1, n_2\}} = \sqrt{\frac{n_1 t^2 \ln^2(2)}{\left(\frac{n_1}{t_1} - b_{\text{rate}}\right)^2 t_1^2 \ln^4\left(\frac{\frac{n_2}{t_2} - b_{\text{rate}}}{\frac{n_1}{t_1} - b_{\text{rate}}}\right)} + \frac{n_2 t^2 \ln^2(2)}{\left(\frac{n_2}{t_2} - b_{\text{rate}}\right)^2 t_2^2 \ln^4\left(\frac{\frac{n_2}{t_2} - b_{\text{rate}}}{\frac{n_1}{t_1} - b_{\text{rate}}}\right)}}$$

We substitute $t = 24\,\text{hours}$, $n_2 = 914$, $t_2 = 10\,\text{minutes}$, $n_1 = 1683$, $t_1 = 10\,\text{minutes}$ and $b_{\text{rate}} = 50\,\text{minute}^{-1}$ to get the expected standard deviation of the half-life

$$\sigma_{t_{\text{half}}} = \mathbf{1.22\ hours}$$

■ Problem 3.20. Calculating the weight of radioactive piston ring particles and the standard deviation.

This problem is solved based on the assumption that $m_{\text{samp}} = m_{\text{std}}\left(\frac{R_{\text{samp}}}{R_{\text{std}}}\right)$, where "m" is the weight of a sample, "R" is the measured net count rate from a sample (minus background), "std" represents a quantity for the standard sample, and "samp" represents a quantity for the unknown sample. This equation is expressed below, where "R" for either sample is represented by "($\frac{n}{t}$ - rb)" ("rb" is the background count rate).

$$m_{\text{samp}} = \frac{\left(\frac{n_{\text{samp}}}{t_{\text{samp}}} - \text{rb}\right) m_{\text{std}}}{\frac{n_{\text{std}}}{t_{\text{std}}} - \text{rb}}$$

We substitute $n_{\text{samp}} = 13\,834$, $t_{\text{samp}} = 3\,\text{minutes}$, $n_{\text{std}} = 91\,396$, $t_{\text{std}} = 10\,\text{minutes}$, $\text{rb} = \frac{281}{\text{minute}}$ and $m_{\text{std}} = 100\,\mu\text{g}$ to get the weight of the particles in the unknown sample.

$$m_{\text{samp}} = \mathbf{48.9\ \mu g}$$

We are also asked to find the expected fractional standard deviation in this weight calculation. To do this, we just apply the error propagation formula. This is expressed below (in partial derivative and dot product notation, noting that $\sigma n^2 = n$).

$$\sigma = \sqrt{\left\{\left(\frac{\partial m_{\text{samp}}}{\partial n_{\text{std}}}\right)^2, \left(\frac{\partial m_{\text{samp}}}{\partial n_{\text{samp}}}\right)^2\right\}.\{n_{\text{std}}, n_{\text{samp}}\}} = \sqrt{\frac{m_{\text{std}}^2 n_{\text{samp}}}{t_{\text{samp}}^2\left(\frac{n_{\text{std}}}{t_{\text{std}}} - \text{rb}\right)^2} + \frac{m_{\text{std}}^2 n_{\text{std}}\left(\frac{n_{\text{samp}}}{t_{\text{samp}}} - \text{rb}\right)^2}{t_{\text{std}}^2\left(\frac{n_{\text{std}}}{t_{\text{std}}} - \text{rb}\right)^4}}$$

We substitute $n_{\text{samp}} = 13\,834$, $t_{\text{samp}} = 3\,\text{minutes}$, $n_{\text{std}} = 91\,396$, $t_{\text{std}} = 10\,\text{minutes}$, $\text{rb} = \frac{281}{\text{minute}}$ and $m_{\text{std}} = 100\,\mu\text{g}$ to get the expected standard deviation in the calculated weight of the particles in the unknown sample.

$$\sigma = \mathbf{0.473\ \mu g}$$

However, we are asked for the expected **fractional** standard deviation, so we take our previous result and divide it by the calculated weight of the sample. The expected fractional standard deviation in the calculated weight of the sample is thus:

$$\textbf{fractional standard deviation} = \frac{0.472973\ \mu g}{48.8828\ \mu g} = 0.968\ \%$$

■ Problem 3.21. Uncertainty in decay constant measurement.

In this problem, we wish to adjust the waiting time to minimize the predicted uncertainty in the decay constant from two measurements. We have an approximate value of the decay constant, λ_p, which will enable us to predict the number of counts expected at any value of the waiting time. Begin by defining the decay constant and its uncertainty in terms of the measured counts, noting that we expect $n(t_1) = n(t_0)\, e^{-\lambda_p \Delta t}$.

$$\lambda = \frac{\ln\left(\frac{n_1}{n_0}\right)}{\Delta t}$$

$$\sigma_\lambda = \sqrt{n_0 \left(\frac{\partial \lambda}{\partial n_0}\right)^2 + n_1 \left(\frac{\partial \lambda}{\partial n_1}\right)^2}$$

We substitute $n_1 = n_0\, e^{-\lambda \Delta t}$ to get the variance in the decay constant.

$$\sigma_\lambda = \sqrt{\frac{e^{\lambda_p \Delta t}}{\Delta t^2\, n_0} + \frac{1}{\Delta t^2\, n_0}}$$

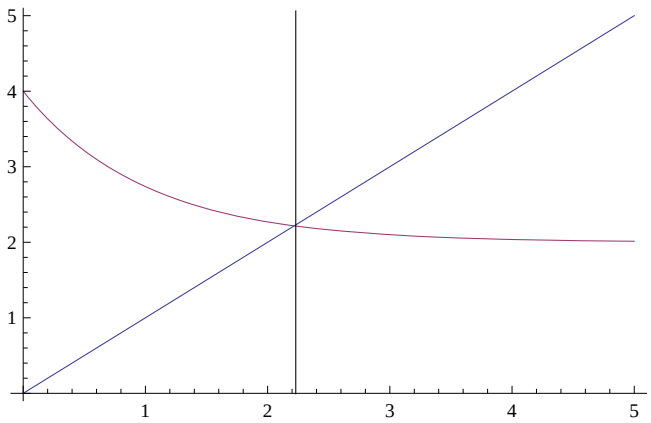
To find the minimum in the variance with respect to the wait time Δt, we set the derivative equal to zero:

$$\frac{\partial \sigma_\lambda}{\partial \Delta t} = \frac{e^{\lambda_p \Delta t}\left(\lambda_p \Delta t - 2\right) - 2}{2\, \Delta t^3\, n_0 \sqrt{\frac{e^{\lambda_p \Delta t}+1}{\Delta t^2\, n_0}}} = 0$$

We recognize that the numerator must be zero, so we solve for Δt:

$$e^{\Delta t \lambda_p}\left(\Delta t\, \lambda_p - 2\right) - 2 = 0$$

This is an equation of the form $x = 2(1 + e^{-x})$. We plot the left-hand-side and the right-hand-side of the equation, and see that they are equal at about $x = \lambda_p \Delta t = 2.2$.



We can also use a nonlinear root finder to find the numeric value for the numerator, which yields a more accurate answer for the optimal wait time Δt:

$$\Delta t = \frac{2.22}{\lambda_p}$$

- **Problem 3.22. Optimal choice of attenuation coefficient to minimize error in transmission thickness measurement.**

a) First, we define the expression for the measured material thickness x_m (different than the true thickness, x_t) in terms of the problem parameters. In our derived equation, n_x is the measured number of counts at a true absorber thickness x_t , n_0 is the measured number of counts with no absorber, and μ is the linear attenuation coefficient. Next, we define the expected standard deviation of x_m , "σ_{xm} ," which was derived using the error propagation formula. Note that we use "$n_0 e^{-\mu x_t}$" for "${\sigma_{nx}}^2$" because ${\sigma_{nx}}^2 = n_x$, and n_x can be assumed to be equal to $n_0 e^{-\mu x_t}$ from Eqn. 2.20. Also, we do not replace "${\sigma_{n_0}}^2$" with n_0 because we are told that n_0 is measured over a 'long time,' so its uncertainty can be assumed to be negligible. At this point, we leave it in the equation so that we may test the case when its uncertainty is not negligible. Finally, we replace n_x with $n_0 e^{-\mu x_t}$, which comes from Eqn. 2.20, as before.

$$x_m = -\frac{\ln\left(\frac{n_x}{n_0}\right)}{\mu}$$

$$\sigma_{xm} = \sqrt{{\sigma_{n_0}}^2 \left(\frac{\partial x_m}{\partial n_0}\right)^2 + n_0 e^{-\mu x_t}\left(\frac{\partial x_m}{\partial n_0}\right)^2} = \sqrt{\frac{n_0 e^{-\mu x_t}}{\mu^2 {n_x}^2} + \frac{{\sigma_{n_0}}^2}{\mu^2 {n_0}^2}}$$

We substitute $n_x = n_0 e^{-\mu x_t}$ to get the formula for σ_{xm} with n_x in terms of n_0

$$\sigma_{xm} = \sqrt{\frac{e^{\mu x_t}}{\mu^2 n_0} + \frac{{\sigma_{n_0}}^2}{\mu^2 {n_0}^2}} = \frac{\sqrt{n_0 e^{\mu x_t} + {\sigma_{n_0}}^2}}{\mu n_0}$$

As previously stated, we are given that σ_{n0} =0. Next, determine the partial derivative of σ_{xm} with respect to μ.

$$\frac{\partial \sigma_{xm}}{\partial \mu} = \frac{\frac{e^{\mu x_t} x_t}{\mu^2 n_0} - \frac{2 e^{\mu x_t}}{\mu^3 n_0}}{2\sqrt{\frac{e^{\mu x_t}}{\mu^2 n_0}}}$$

Recognizing the numerator of this partial derivative of σ_{xm} with respect to μ is zero at the minimum, we solve for μ.

$$\frac{e^{\mu x_t} x_t}{\mu^2 n_0} - \frac{2 e^{\mu x_t}}{\mu^3 n_0} = 0$$

$$\mu = -\frac{2}{x_t}$$

Thus, the optimum value of the linear attenuation coefficient that will minimize the uncertainty in the derived sheet thickness, x_m, is $\frac{2}{x_t}$, so $\mu = \frac{2}{1\text{ cm}} = 2\text{ cm}^{-1}$.

b) To find the minimum fractional standard deviation of the measured thickness, we take the minimal σ_{xm} (i.e. σ_{xm} when $\mu = \frac{2}{x_t}$) and divide it by x_t (since x_t is the known value of the sheet thickness). This is accomplished below, where we define our fractional standard deviation equation, giving the known values of n_0, x_t, and μ.

$$\textbf{fractional standard deviation} = \frac{\sigma_{xm}}{x_t}$$

We substitute $n_0 = 10^4$, $x_t = 1$ and $\mu = \frac{2}{x_t}$ to get the minimum fractional standard deviation in the measured absorber thickness.

fractional standard deviation = 0.0136

- What if σ_{n_0} had not been zero? We would expect σ_{n_0} to be $\sqrt{n_0}$, so we substitute $\sigma_{n_0} = \sqrt{n_0}$ into the simplified formula for σ_{xm}. The new formula for σ_{xm} is thus:

$$\sigma_{xm} = \sqrt{\frac{e^{\mu x_t}}{\mu^2 n_0} + \frac{1}{\mu^2 n_0}} = \frac{\sqrt{e^{\mu x_t} + 1}}{\mu\sqrt{n_0}}$$

We take the derivative again with respect to μ, as before.

$$\frac{\partial \sigma_{xm}}{\partial \mu} = \frac{\frac{x_t e^{\mu x_t}}{\mu^2 n_0} - \frac{2 e^{\mu x_t}}{\mu^3 n_0} - \frac{2}{\mu^3 n_0}}{2\sqrt{\frac{e^{\mu x_t}}{\mu^2 n_0} + \frac{1}{\mu^2 n_0}}}$$

We then look at the numerator, after rearranging and defining $x = \mu x_t$. We multiply each term of the numerator by $\frac{\mu^3 n_0 e^{-\mu x_t}}{2}$ and substitute $x = \mu x_t$ to get our new equation. Our new equation with the substitutions and eliminations made is thus:

$$-e^{-x} + \frac{x}{2} - 1$$

We set the above equation to 0 and solve to get the approximate value of x (= μx_t) when the partial derivative of σ_{xm} with respect to μ is set equal to zero.

$$x = 2.22$$

This is an interesting result. It says that the minimum σ_{xm} occurs at μx_t = 2.22, which is only slightly larger than our earlier value of 2. For this problem, x_t = 1 cm, so μ=2.22 cm^{-1}. If we assume n_0=10,000 counts, then our uncertainty in the counting measurement without absorption contributes little to the overall uncertainty in the derivation of the absorber thickness. Here we calculate the lowest value of σ_{xm} with optimal μ under the circumstances in which we do have uncertainty in n_0 (which is actually the same as the fractional standard deviation in $x_m$because $\sigma_{xm}/x_m = \sigma_{xm}/1 = \sigma_{xm}$).

$$\sigma_{xm} = \frac{\sqrt{e^{\mu x_t} + 1}}{\mu \sqrt{n_0}}$$

We substitute $x_t = 1$, $\mu = 2.21771$ and $n_0 = 10\,000$ to get the fractional standard deviation in $x_m$when the uncertainty in the counting measurement without absorption is not negligible.

$$\sigma_{xm} = 0.0144$$

Note that this is only slightly larger than the previous value of 1.36% when we assumed that the uncertainty in the counting measurement without absorption was negligible.

■ Problem 3.23. Calculating fraction of intervals less than Δt with a known average count rate.

The probability of the next event taking place in an interval dt after a delay Δt is given by (r*$e^{-r\Delta t}$) dt. (cf. Eqn.3.71). The probability that the time between two events is less than Δt is given by the integral of this interval distribution from 0 to Δt. Of course, this must be the same as the the complement of the probability that no events occur during the time interval Δt, or $1 - e^{-rt}$. We evaluate this function:

$$\text{Fraction} = 1 - e^{-r\,\Delta t}$$

We substitute $\Delta t = 10^{-2}$ sec and $r = 100$ sec^{-1} to get the fraction of the intervals that are less than 10 ms when the average event rate is 100 sec^{-1}.

$$\text{Fraction} = 0.632$$

■ Problem 3.24. Minimum Detectable Activity.

The concept of Minimum Detectable Activity (MDA) is one of most misunderstood concepts in radiation measurements because it is somewhat of a misnomer. In literal terms, there is no such thing as a minimum detectable activity because no matter how small the source strength, if one counts long enough, the source can be distinguished above the background. So the "MDA" that is used in operational settings is understood to be a value resulting from a defined procedure of counting the source and the background for the same length of time, which as you have seen, may not be optimal. Nevertheless, it is a well accepted figure-of-merit.

(a). L_C is set to ensure that S=T-B gives only 5% false positives when S=0, We define "S" as the number of counts from the source alone, "T" is the number of counts including the source and background, and "B" is the number of background counts, and all counts are taken over the same time interval. This corresponds to L_C=2.326 $\sqrt{B}$ (= 2.326 σ_B). We substitute B=100×30 below to get the appropriate value of L_C in units of counts per 30 minute measurement.

$$L_c = 2.326\sqrt{B} = 127.4 \text{ counts}$$

(b). The MDA is set to ensure that if S > N_D it will be detected with a 95% confidence, or only 5% false negatives. First, we find N_D and then translate this to MDA. We use the "Currie Equation" (Eqn. 3.67) to calculate $N_D$below, where we substitute B=100×30. The value of N_D in units of counts per 30 minute measurement is thus:

$$N_D = 4.653\sqrt{B} + 2.706 = 257.5 \text{ counts}$$

Now we convert $N_D$to MDA (or α) using Eqn. 3.68. This is expressed below, where ϵ is the detection efficiency (counting efficiency), T is the count time in seconds, and f is the branching ratio for the observed radiation (the result is expressed in Bq).

$$\alpha = \frac{N_D}{\epsilon T f}$$

We substitute ϵ = 0.15, T = 30×60 and f = 0.85 to get the minimum detectable amount (MDA) of Cs-137 on the filter.

$$\alpha = 1.12 \text{ Bq}$$

■ Problem 3.25. Intervals between events.

(a). For a fixed frequency f, the period is T=1/f, and the mean wait time is (1/2)T. For an analogy, think of blindly putting your pencil down on a ruler. The intervals between tick marks is constant. Since all points are equally probable, the average pencil mark is at 1/2 of the scale. Therefore, the average (mean) waiting time until the next bus arrives is (1/2)*(30 minutes) = 15 minutes.

(b). If the interval distribution is not a constant, then the mean wait time is the integral of the probability of arriving during an interval of Δt between buses (P(Δt) dt), times the mean waiting time for this interval (which is Δt/2) from part (a) above. The way to realize this is to consider 5 intervals of length 1, and 1 interval of length 5. The probability of an interval of length 1 is 5/6, but the probability of arriving during an interval of length 1 is actually 1/2 (5 of the 10 length scale). So the mean distance (time) to the next tick (bus) for this contrived example would be: 1/2*(1/2) + 1/2*(5/2) = 1.5 .

This concept is expressed below. The probability of an interval t when the mean rate is r is given by Poisson statistics as $e^{-rt}\,r\,dt$. If the average wait for this interval is t/2, then the overall mean wait time is the integral over all possible intervals t. We have noted that the average time between Poisson distributed events (i.e., interval) is the inverse of the rate.

$$\text{average wait time} = \frac{\int_0^\infty \frac{1}{2} r t e^{-rt} t\,dt}{\int_0^\infty r t e^{-rt}\,dt}$$

We substitute $r = \frac{1}{T}$and evaluate to find the average wait time.

$$\text{average wait time} = T$$

The average waiting time for randomized buses is thus T, twice as long as for buses which are periodic. This makes sense intuitively because for a Poisson process, the average time between events is 1/r = 1/f = T.

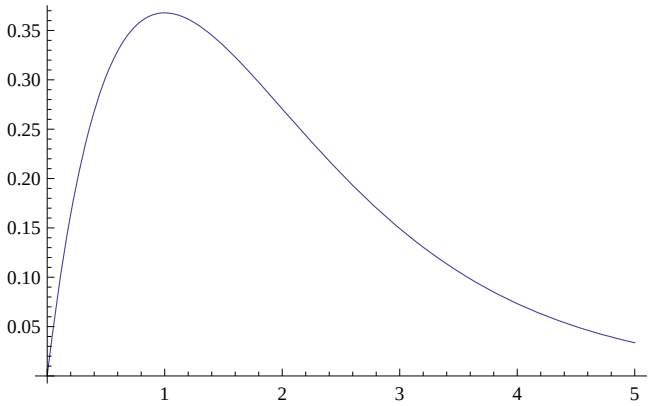
Supplemental Problem: Professor Kerr's Tricky Problem.

A detector detects gamma rays from a radioactive source with the detected events occurring at a rate such that the average interval between events is a time T. Begin observing at time zero. At what time t will the probability of having detected exactly one event be 0.5?

Start by assuming Poisson statistics are valid. Now we want to look at the probability distribution function. Here, we define our distribution function, "P," as a function of two variables, "n" and "t." We measure time t with respect to T, so our variable "t" actually corresponds to t/T, and "n" is simply the number of counts recorded at time t..

$$P = \frac{t^n}{e^t\, n!}$$

We can plot the probability of observing exactly 1 count at time t by giving "n" that value of 1 and plotting "t" from 0 to 5.



Note that the maximum probability of observing 1 count occurs near t=T (where t/T=1 on our graph) and takes on a value of only about 0.36. We can find this exact maximum by taking the derivative of the distribution with respect to t, setting it equal to zero, solving for t, and then substituting this value for t back into the distribution.

$$\frac{\partial P}{\partial t} = \frac{e^{-t}\, n\, t^{n-1}}{n!} - \frac{e^{-t}\, t^n}{n!} = 0$$

We solve the above equation for "t" to find the value of "t" for when the probability distribution function reaches its maximum (or when the derivative equals zero).

$$t = 1$$

Now we substitute t=1 and n=1 back into our probability distribution function "P" to get the exact value of the maximum probability of detecting only one event.

$$P = \frac{t^n}{e^t\, n!} = \frac{1}{e} = 0.368$$

So, to answer the question posed by Professor Kerr, there is no point at which the probability of detecting exactly one event will be 0.5, and, as expected, the maximum probability of this detecting only one event occurs at t=T, and has a value of 1/e. It's a tricky question because the answer is "no time" rather than a specific time.

Supplemental Problem: Uncertainty in signal to noise ratio.

The signal to noise ratio, "R," is defined by the net number of counts divided by the noise level (i.e. the statistical uncertainty) of the background. If "T" is the total number of counts from the source, "S," and background, "B," (i.e. S = T-B) then $R = \frac{(T-B)}{\sqrt{B}}$. Find the uncertainty and relative uncertainty in the signal to noise ratio, and evaluate this for S=B.

Solution : First we define the signal to noise ratio. Then we define the uncertainty in R using the error propagation formula (using partial derivative and dot product notation). Then we define the relative uncertainty (fractional uncertainty) in R.

$$\boldsymbol{R = \frac{T - B}{\sqrt{B}}}$$

$$\boldsymbol{\sigma_R = \sqrt{\left\{\left(\frac{\partial R}{\partial T}\right)^2, \left(\frac{\partial R}{\partial B}\right)^2\right\}.\{T,\ B\}} = \frac{1}{2}\sqrt{\frac{B^2 + 6\,T\,B + T^2}{B^2}}}$$

$$\textbf{relative uncertainty} = \boldsymbol{\frac{\sigma_R}{R} = \frac{\sqrt{B}\sqrt{\frac{B^2+6\,T\,B+T^2}{B^2}}}{2\,(T - B)}}$$

Next we substitute T=2B (since S=B and S=T-B) into the above equations to get R, the uncertainty in R (σ_R), and the relative uncertainty in R $\left(\frac{\sigma_R}{R}\right)$.

$$\boldsymbol{R = \sqrt{B}} \qquad \boldsymbol{\sigma_R = \frac{\sqrt{17}}{2}} \qquad \textbf{and} \qquad \boldsymbol{\frac{\sigma_R}{R} = \frac{\sqrt{17}}{2\sqrt{B}}}$$

Supplemental problem: Professor Fleming's Attenuation Coefficient and Count Time Statistics

Suppose we have a constant source of monoenergetic particles with strength I_0 particles per unit time. We have a detector with constant efficiency and no background. Find the optimum thickness of the sample which minimizes the relative error in the measured value of μ given that the measurement must be carried out in a fixed measurement time T.

There are two variables at work here: the material thickness (which affects the number of counts obtainable when the sample is in the beam) and the observation times (divided between sample-in T_s and sample-out T_0 times, where $T_s + T_0 = T$).

Let $y = \mu t = \ln\left(\frac{I_0}{I_s}\right) = \ln\left(\frac{N_0}{N_s}\right) - \ln\left(\frac{T_0}{T_s}\right)$, where y is the thickness in mfp units.

Then:

$$\sigma_y{}^2 = \left(\frac{1}{N_0} + \frac{1}{N_s}\right) = \frac{1}{I_0\,T_o}\left(1 + e^y\,\frac{T_0}{T_s}\right) = \frac{1}{I_0\,T}\,\frac{1}{f}\left(1 + e^y\,\frac{f}{1-f}\right)$$

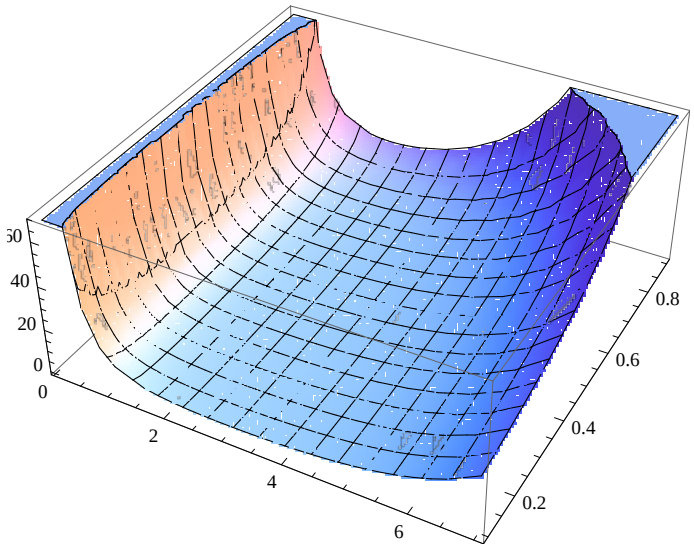
where $f = \frac{T_0}{T}$. Define the relative variance to be r, where:

$$\mathrm{r} = \frac{1}{I_0\,T}\,\frac{1}{\mathrm{f}y^2}\left(1 + e^y\,\frac{f}{1-f}\right)$$

Choosing $I_0\,T = 1$ for convenience:

$$r(y,\ f) = \frac{\frac{f\, e^{y}}{1-f} + 1}{y^2\, f}$$

Let's plot the shape of the variance surface as a function of sample thickness y and fraction of time spent observing the open beam f:



The minimum relative variance occurs at r=3.22404, y=2.55693 and f=0.217811. Note that these can be found by simultaneously solving $\frac{\delta r}{\delta y} = \frac{\delta r}{\delta f} = 0$

- **Supplemental Problem: What is the probability of observing the same (or different by Δ) subsequent number of counts?**

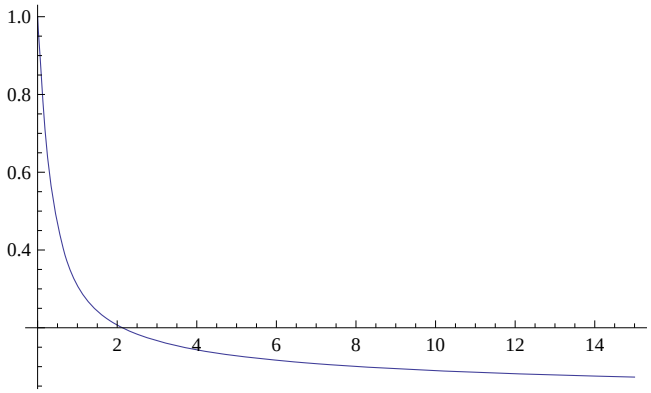
The probability distribution function for the Poisson is given by:

$$\text{pdf} = \frac{e^{-\mu}\, \mu^{x}}{x!}$$

We calculate, for every observable value of counts x, the probability of observing two subsequent counts that have the same value. So we sum pdf[x, μ]^2 for all possible values of x given the mean μ:

$$\text{Probability} = \sum_{x=0}^{\infty} \left(\frac{e^{-\mu}\, \mu^{x}}{x!} \right)^2$$

A plot of this function is shown below where μ goes from 0 to 15.

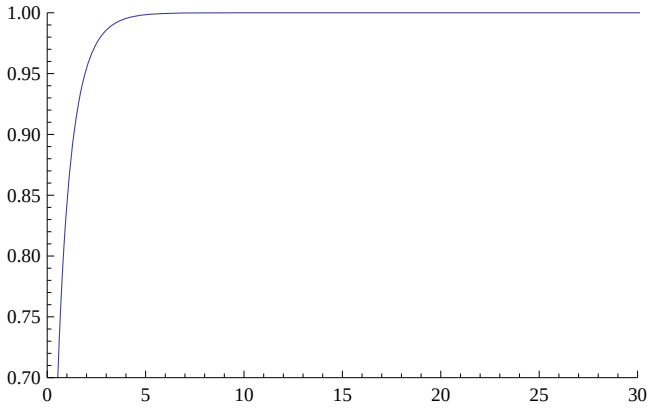


This makes sense. If the mean is small, the probability of two subsequent counts having the same number is higher. Indeed, if the mean of the background is 0, then two subsequent counts have to have the same value (namely, zero!).

Suppose we look at the same question using Normal distribution: m is the mean now. In order to use the normal distribution, we must now assume that the mean is large. We integrate the probability of observing two values of x, and integrate over all possible values of x:

$$\textbf{Probability} = \frac{1}{2\pi m}\int_0^\infty e^{\frac{-(x-m)^2}{m}}\,dx = \frac{\text{erf}\left(\sqrt{m}\right)+1}{4\sqrt{\pi}\,\sqrt{m}}$$

Below is a plot of y=Erf($\sqrt{m}$) where m goes from 0 to 30.



Note that Erf[m] → 1 for m large, so our result becomes $\frac{1}{2\sqrt{\pi m}}$.

Another way to look at this problem is to calculate the probability of observing 0 net counts. That is the same as assuming two subsequent counts that have the same value. We assume two Normal distributions, the difference is a Normal distribution with mean of 0 and standard deviation of $\sqrt{2m}$. We could use this same approach to calculate that two subsequent values differ by any amount, not just 0.

$$\textbf{Probability} = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$$

We substitute x=0, μ=0 and σ=$\sqrt{2m}$ to find the probability as thus:

$$\text{Probability} = \frac{1}{2\sqrt{\pi}\ \sqrt{m}}$$

We, of course, get the same answer, but with a much simpler approach.

■ A related problem

We ask the question what is the probability of observing a difference of delta between two measurements from a single distribution with mean $n_{1\,\text{av}}$.

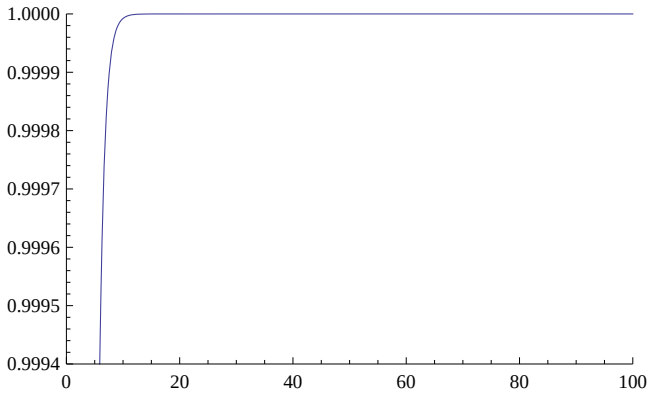
$$\text{Probability} = \frac{1}{2\pi n_{1\,\text{av}}} \int_0^\infty \int_0^\infty \frac{\delta(-\delta - n_1 + n_2)}{e^{\frac{(n_1 - n_{1\,\text{av}})^2}{2 n_{1\,\text{av}}}}\ e^{\frac{(n_2 - n_{1\,\text{av}})^2}{2 n_{1\,\text{av}}}}} \, dn_1\, dn_2$$

$$\text{Probability} = \frac{e^{-\frac{\delta^2}{4 n_{\text{av}}}} \left(\theta(-\delta)\left(\text{erf}\left(\frac{\delta - 2 n_{\text{av}}}{2\sqrt{n_{\text{av}}}}\right) + \text{erf}\left(\frac{2 n_{\text{av}} + \delta}{2\sqrt{n_{\text{av}}}}\right)\right) + \text{erfc}\left(\frac{\delta - 2 n_{\text{av}}}{2\sqrt{n_{\text{av}}}}\right)\right)}{4\sqrt{\pi}\ \sqrt{n_{\text{av}}}}$$

Note that the answer is a Gaussian distribution with mean of zero, as before, with a new variance $\sqrt{2}$ larger. We now substitute δ=0 to get the probability.

$$\text{Probability} = \frac{\text{erf}\left(\sqrt{n_{\text{av}}}\right) + 1}{4\sqrt{\pi}\ \sqrt{n_{\text{av}}}}$$

A plot of y=Erf$\left(\sqrt{x}\right)$ is shown below where x goes from 0 to 100.



which is the same answer we had before (the problem above has δ=0 too) if we let Erf[...] -> 1.

General Properties of Radiation Detectors problems

■ Problem 4.1. Voltage from collected charge Q on capacitance C.

Since we know that 10^6 electrons are collected, and we know that the charge of an electron is approximately 1.6 x 10^{-19} C, the total charge collected is easily computed. We also know that the amplitude of the voltage pulse is V = $\frac{Q}{C}$. The following expression defines the equation for the voltage, giving the appropriate value for Q in terms of the number of electrons collected and the charge of a single electron and giving the known capacitance:

$$V = \frac{Q}{C}$$

We substitute Q = $\frac{10^6 \text{ electrons } 1.6 \text{ Coulombs}}{10^{19} \text{ electrons}}$ and C = $\frac{100 \text{ Farads}}{10^{12}}$ to get the amplitude of the voltage pulse.

$$V = 0.0016 \text{ Volts}$$

■ Problem 4.2. Comparison of pulse, MSV, and current mode operation.

See text for the description of these modes of operation. Note that pulse mode allows spectroscopy information to be measured, current mode allows average rate information but not spectroscopy, and MSV gives average rate information but heavily weighted to the type of radiation that gives more charge per pulse (e.g., neutrons in a mixed gamma-neutron field).

■ Problem 4.3. Derive expression for mean squared fluctuation in voltage.

Combine [Eq'n 4.2] $I_0 = r\,Q$ with [Eq'n 4.7] $\overline{\sigma_I^2(t)} = \frac{I_0^2}{rT}$ to solve for $\overline{\sigma_I^2(t)}$ by eliminating I_0:

$$I_0 = r\,Q,$$
$$\overline{\sigma_I^2(t)} = \frac{I_0^2}{r\,T}$$

Substituting for I_0^2 in the latter expression yields the answer we seek:

$$\overline{\sigma_I^2[t]} = \frac{Q^2\,r}{T}$$

■ Problem 4.4. RC collection circuit time constant.

For this problem, τ= RC= 3 μs >> t_c= 150 ns, so this is a ***large time constant*** problem. ($t_c$is the charge collection time). This is true in most pulsed spectroscopy applications since we want to collect all of the charge, and have some time to measure this maximum voltage.

Although this is beyond the question being asked, let's solve the differential equation which describes the circuit of a capacitor and resistor in parallel, being fed a constant current of Q/t_C during the collection time t_C, from the detector. This is done below, where "v(t)" is the voltage-time profile for the circuit, "τ" is the circuit's time-constant ("RC"), "v'(t)" is the derivative of the voltage-time profile with respect to "t," "q" is the charge collected over a time "t_C," and "R" is the resistance. We also give the intial value of v(0) = 0.

$$\frac{v(t)}{\tau} + v'(t) = \frac{q\,R}{t_C\,\tau} \qquad \text{where } v(0) = 0$$

The equation for the voltage-time profile of the circuit is thus:

$$v(t) = \frac{e^{t/\tau}\,q\,R - q\,R}{e^{t/\tau}\,t_C}$$

which can be put into a simplified form of v(t).

$$v(t) = \frac{\left(1 - e^{-\frac{t}{\tau}}\right) q\,R}{t_C}$$

Note that for our case τ= RC= 3 μs >> t_c= 150 ns, so this is a ***large time constant*** problem. But let's look at the behavior for small times. We will expand the small value of the exponential after first substituting x= $\frac{t}{\tau}$ to simplify the equatin:

$$v(t) = \frac{\left(1 - e^{-x}\right) q\,R}{t_C}$$

We then expand this in a Taylor Series expansion around x=0 to Order 1, then substitute back in for x and τ, yielding:

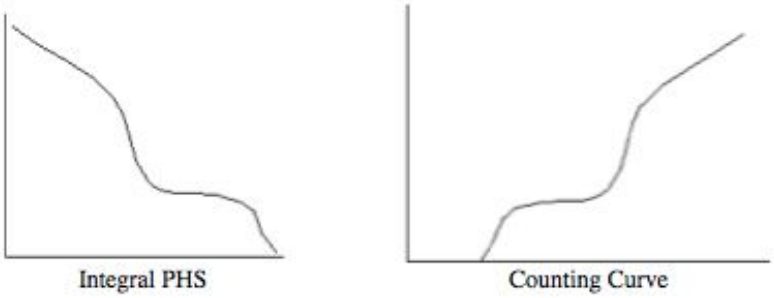
$$v(t) = \frac{q\,t}{C\,t_C} \qquad \text{after we substitute } \tau = R\,C \text{ and } x = \frac{t}{\tau}$$

Expanding the exponential thus shows an initial linear rise in the voltage:

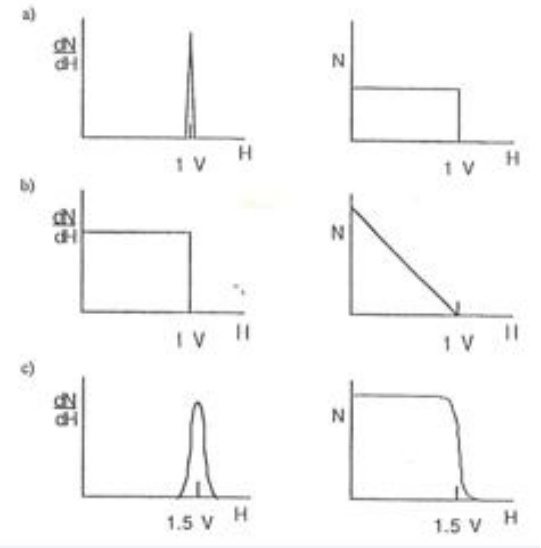
$$\mathbf{v(t)} = \frac{Q}{C}\left(\frac{t}{t_c}\right) \qquad \text{for } t < t_C$$

Looking at the v(t) expression for longer times, the voltage builds up to the maximum value of $\frac{Q}{C}$during the collection time t_C. After t_c, the differential equation is homogeneous and leads to an exponential fall off of the voltage. This is the normal condition we want -- the maximum amplitude of the pulse is proportional to the total charge formed in the detector, and the total charge formed is proportional to the energy deposited by the radiation interaction.

- **Problem 4.5. DPHS -> IPHS and Counting Curve**



- **Problem 4.6. DPH and IPH Spectra**



- **Problem 4.7. Energy resolution.**

The two peaks are separated by 55 keV. We want the peaks separated by at least one FWHM (see the discussion in the text-book). First, we calculate the energy resolution of the 435 keV peak. This is done using $R = \frac{FWHM}{H_0} = \frac{55\,keV}{435\,keV}$. The energy resolution for the 435 keV peak assuming a FWHM of 55 keV is thus:

$$R_{435} = \frac{55}{435} = .126437 = 12.6\,\%$$

Next, we do the same calculation for the 490 keV peak. The energy resolution for the 490 keV peak assuming a FWHM of 55 keV is thus:

$$R_{490} = \frac{55}{490} = .112245 = 11.2\,\%$$

In order for the peaks to be separated by at least 1 FWHM, we take the smaller energy resolution as the value of the system energy resolution. Therefore, the required system energy resolution is about 11.2 % (or less).

Problem 4.8. Fano factor role in energy resolution

The energy resolution is given by $\frac{\Delta E}{E} = 2.355\,\frac{\sigma_E}{E} = 2.355\,\frac{\sqrt{F}\,\sigma_n}{n} = 2.355\sqrt{\frac{F}{n}}$. Given a resolution of 0.5%, F=0.1, we solve for n.

$$R = 2.355\sqrt{\frac{F}{n}}$$

$$n = \frac{5.5456\,F}{R^2}$$

We then substitute F = 0.1 and R=0.005 in the last result to find the number of carriers n:

$$n = 22\,184$$

■ Problem 4.9. Electronic noise added to intrinsic resolution.

The key idea here is that electronic noise adds in quadrature with intrinsic resolution to determine the system's overall resolution, i.e. $\Delta E_{tot}^{\,2} = \Delta E_{int}^{\,2} + \Delta E_{noise}^{\,2}$

Using this relationship, we want to evaluate ΔE_{tot}. This is done below, where we solve for ΔE_{tot} and give the appropriate energy resolution values (in fractions, not percents).

$$\Delta E_{tot} = \sqrt{\Delta E_{int}^2 + \Delta E_{noise}^2}$$

We substitute $\Delta E_{int} = 0.04$, $\Delta E_{noise} = 0.02$ to get the expected overal pulse height resolution (expressed as a fraction, not a percent).

$$\Delta E_{tot} = 0.0447$$

■ Problem 4.10. Solid angle.

We use the correct and approximate expressions for a right circular cylinder's solid angle to solve this problem (note that we should get a relatively accurate result using the approximate calculation because, using the same variable meanings as in the textbook, since d=4a then. d>>a). The formula for the correct solid angle calculation is:

$$\Omega_{correct} = 2\,\pi\left(1 - \frac{d}{\sqrt{a^2 + d^2}}\right)$$

and the formula for the approximate calculation is:

$$\Omega_{approx} = \frac{\pi\,a^2}{d^2}$$

Next, we plug our known values d = 20 cm and a = 5 cm into $\Omega_{correct}$ to get the correct value of the solid angle (in steradians).

$\Omega_{correct} = 0.188$ steradians

Now, we plug our known values d=20cm and a=5cm into Ω_{approx} to get the approximate value of the solid angle (in steradians).

$\Omega_{approx} = 0.196$ steradians

The approximate value is not far off from the correct value, indicating that using the approximate calculation is appropriate in this case.

■ Problem 4.11. Solid Angle.

We use the integral over $\frac{d\Omega}{4\pi} = \frac{\sin\theta\, d\theta\, d\phi}{4\pi}$ for $\phi \in \{0, 2\pi\}$, $\theta \in \{0, 0.25°\}$ to find $\frac{\Omega}{4\pi}$, which is the probability that the laser beam will strike the moon. This integral is expressed below (when $d\phi$ is integrated from 0 to 2π, we get a factor of 2π in the overall integral).

$$\text{Probability} = \int_0^{\theta} \frac{2\pi (\sin\theta)}{4\pi}\, d\theta$$

The equation for $\frac{\Omega}{4\pi}$ is dependant on the upper limit of θ.

$$\text{Probability} = \frac{1}{2} - \frac{\cos\theta}{2}$$

Next, we substitute $\theta = 0.25°$ to get the probability that the laser will strike the moon.

$\text{Probability} = 4.75964 \times 10^{-6}$

Just for fun, we perform the Taylor series expansion for Cos[θ ~ small] up to order 2 on our equation for $\frac{\Omega}{4\pi}$.

The Taylor series expansion of the equation for $\frac{\Omega}{4\pi}$ is thus:

$$\text{Probability} = \frac{\theta^2}{4}$$

Again, we substitute $\theta = 0.25°$ to get an approximation of the probability that the laser will strike the moon.

$\text{Probability} = 4.75965 \times 10^{-6}$

This is obviously a good approximation and doesn't require looking up the value of the cosine.

We could also use the approximate expression for Ω where d >> a, $\frac{\Omega}{4\pi} = \frac{\pi a^2}{4\pi d^2}$ and since tan[θ]=a/d, $\frac{\Omega}{4\pi} = \frac{\tan^2\theta}{4}$ This is expressed below, where we give the value for θ.

$\text{Probability} = 0.25 \tan^2(0.25°)$

Another approximation of the probability that the laser will strike the moon is thus:

$\text{Probability} = 4.75971 \times 10^{-6}$

Again, this is a good approximation.

■ Problem 4.12. Expected counts under photopeak given ϵ_{int}, Ω, yield, S_0, and t_{count}.

The total number of counts under the full-energy peak will be $S_0 \frac{\Omega}{4\pi} \epsilon_{int} y_\gamma t_{count}$, where all values are given in the problem. To clarify, "S_0" is the **activity** of the source, explaining the necessity of a factor of "t_{count}," and "y_γ" is the **yield** of the source, or the fraction of the decays that result in 1 MeV gamma rays (which is the energy of our full-energy peak). Below, we express this equation, substituting in our known values.

$$\textbf{counts} = \frac{S_0\, \Omega\, \epsilon_{int}\, y_\gamma\, t_{count}}{4\,\pi}$$

We substitute $S_0 = 20\,000$, $\Omega = 0.1876$, $\epsilon_{int} = 0.12$, $y_\gamma = 0.8$ and $t_{count} = 100$ to get the expected number of counts that will appear under the full-energy peak over the 100 sec measurement period.

```
counts = 2866
```

■ Problem 4.13. Dead time models for decaying source.

To solve this, we find the analytic solution to the two governing equations for the paralyzable model, the most appropriate model for the GM tube. We take the natural log of both sides of the paralyzable model expression.

$$\ln(m_0) = \ln(n_0) - n_0\,\tau$$

In the next line, we express Eqn. 4.36 (we make the distinction between "m_0" and "m_1" because one is a measured count rate at time 0 and one is measured count rate after an elapsed amount of time "t").

$$m_1 \ln = -n_0\,\tau\, e^{t(-\lambda)} + n_0 \ln - t\,\lambda$$

Next, we solve our equations for $\ln(n_0)$ to give the analytical solution :

$$\ln(n_0) = \frac{t\, e^{t\,\lambda}\,\lambda - \ln(m_0) + e^{t\,\lambda} \ln(m_1)}{-1 + e^{t\,\lambda}}$$

A simplified form of $\ln(n_0)$ is thus:

$$\ln(n_0) = \frac{e^{t\,\lambda}\,(t\,\lambda + \ln(m_1)) - \ln(m_0)}{-1 + e^{t\,\lambda}}$$

Using this solution, we eliminate the ln on the left hand side by raising "e" to the power of both sides, thus solving for n_0, and giving the appropriate known values.

$$n_0 = e^{\frac{t\lambda - \frac{\ln(m_0)}{e^{t\lambda}} + \ln(m_1)}{1 - e^{-t\lambda}}}$$

We substitute $m_0 = 131\,340$, $m_1 = 93\,384$, $\lambda = \frac{\ln(2)}{54.3}$ and $t = 40$ to get the calculated true interaction rate in the G-M tube at 12:00, n_0(in min^{-1}) for the paralyzable case.

$$n_0 = 200\,691 \text{ min}^{-1}$$

Now we do the same thing for the non-paralyzable case, solving the appropriate governing equations (Eqn. 4.23 and Eqn. 4.35, slightly rearranged).

$$n_0 - m_0 = n_0 m_0 \tau$$
$$n_0 e^{-t\lambda} - m_1 = n_0 e^{-t\lambda} m_1 \tau$$

The solution for n_0.

$$n_0 = \frac{\left(-1 + e^{\lambda t}\right) m_0 m_1}{m_0 - m_1}$$

We substitute $m_0 = 131\,340$, $m_1 = 93\,384$, $\lambda = \frac{\log(2)}{54.3}$ and $t = 40$ to get the calculated true interaction rate in the G-M tube at 12:00, n_0(in min^{-1}) for the non-paralyzable case.

$$n_0 = 215\,307 \text{ min}^{-1}$$

Note that the two results are about 8% different depending on which model we chose to be most representative of our true detection system.

■ Problem 4.14. Dead time models applied to two detectors.

To solve this problem, we first write the governing equations for detector A and detector B. Since they are both non-paralyzable, dead time losses for either detector is represented by $m\tau$, and also note that $m_B \tau_B = 2 m_A \tau_A$. These equations are expressed below, where we solve them similtaneously for "n" by eliminating "m_A."

$$n - m_A = n m_A \tau_A,$$
$$n - \frac{2 \tau_A m_A}{\tau_B} = 2 n m_A \tau_A$$

As you can see, we get two solutions (of course, we are not interested in the trivial solution n=0).

$$n = 0 \quad \text{or} \quad n = \frac{\tau_B - 2\tau_A}{\tau_A \tau_B}$$

Now we substitute $\tau_A = \frac{30}{10^6}$ and $\tau_B = \frac{100}{10^6}$ into the non–trivial solution for n to get the value of n we are interested in, namely, the true event rate that will give dead time losses in counter B that are twice as great as those for counter A (in sec^{-1}):

$$n = \frac{40\,000}{3} = 13\,333 \text{ sec}^{-1}$$

■ Problem 4.15. Dead time.

The key to this problem is to recognize that two measured rates are given and the ratio of the true source rates are known (if one source is in place, we have a true source rate of n, but if two identical sources are in place, we have a true source rate of 2n). The value of τ is desired, and background is negligible. Let's assume a non-paralyzable model, i.e. $n = \frac{m}{1-m\tau}$. We could just use equation 4.32, but let's solve the two equations directly. The two equations for the different measured rate are shown below, where we solve for "τ" by eliminating "n."

$$n = \frac{10\,000}{1 - 10\,000\,\tau}$$
$$2n = \frac{19\,000}{1 - 19\,000\,\tau}$$

The counter dead time in seconds for the non-paralyzable model is thus:

$$\tau = 5.26 \times 10^{-6}$$

This is the same result as if equation 4.32 was used, of course. We can do the same approach for the paralyzable model, i.e. $m = n\,e^{-n\tau}$. The corresponding equations for our problem are shown below.

$$10\,000 = n\,x,$$
$$19\,000 = 2\,n\,x^2$$
$$x = e^{-n\,\tau}$$

Our solution for τ in fraction form.

$$\tau = \frac{19 \ln\left(\frac{20}{19}\right)}{200\,000}$$

The counter dead time in seconds for the paralyzable model is thus:

$$\tau = 4.87 \times 10^{-6}$$

Note that the two results are slightly different by about 6%.

■ Problem 4.16. Paralyzable model: true count rate.

For the paralyzable model, we know that the true event rate n is related to the measured event rate m by the relationship: $m = ne^{-n\tau}$. Given $m = 10^5\,s^{-1}$ and τ=1.5 μs, we find n using that equation. This is expressed below, where we plug in our known values and solve for "n."

$$10^5 = n\,e^{-n\,\tau}$$

We substitute $\tau = \frac{1.5}{10^6}$ to get two possible values for the true interaction rate (in sec^{-1}).

$$n = 119\,661 \quad \text{and} \quad n = 1.996 \times 10^6$$

Note the two possible values for n are vastly different; either low or very high -- a disconcerting (and perhaps dangerous) ambiguity.

Problem 4.17. Finding τ given the maximum observed count rate of a paralyzable detector.

We know the maximum value of m is 50000 counts/sec, and we also know that $m = ne^{-n\tau}$. We just need to find the value of n that maximizes m. This calculation is shown below, where we differentiate the equation for m with respect to n, set it equal to zero, and solve for n.

$$\frac{\partial (n\, e^{-n\tau})}{\partial n} = 0$$

The value of n that maximizes m in terms of τ is thus:

$$n = \frac{1}{\tau}$$

Indeed, this tells us that the maximum observed count rate occurs when the true event rate is $1/\tau$. Next, we substitute this value for n and solve for τ, plugging in the maximum value of m.

$$m = n\, e^{-n\tau}$$

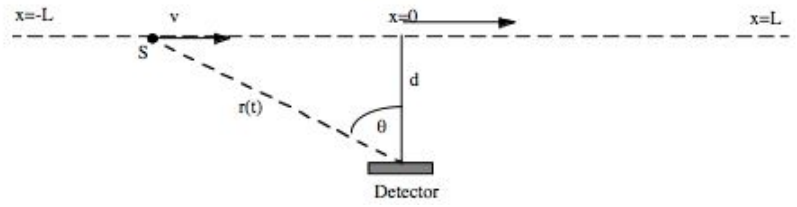
We substitute $n = \frac{1}{\tau}$ and m = 50 000 to get the dead time of the detector in seconds.

$$\tau = 7.358 \times 10^{-6}$$

- **Supplemental Problem on Moving Sources**
- **Moving Sources**

A cylindrical detector of area A =100 cm2 and intrinsic efficiency =100% is placed a distance d = 3 m from a road. The detector alarms if any passing vehicle causes 100 counts above background. The vehicles move at a constant speed v=3 m/s and are in view of the detector when they pass from x = -L until x = +L. Estimate the minimum activity for the alarm level if L is chosen to be a large value. [Neglect background and attenuation].

Assume $\Omega = \frac{A\cos\theta}{r^2}$ so the number of counts when the source is at x(t) is $dC(t) = \frac{S\Omega(t)\,\epsilon_i}{4\pi}\, dt$



We let $r^2 = x^2 + d^2$, so

$dC(x) = S\, A\, d\, dx \Big/ \left(4\pi \left(x^2 + d^2\right)^{(3/2)}\right) v.$

where we substituted dt = dx/v, and used from geometry, $\cos(\theta) = d/r$.To find the total counts accumulated, we integrate dC(x) from - L to L :

$$C = \int_{-\infty}^{\infty} \frac{S\,A\,d}{4\,\pi\,v\,(d^2 + x^2)^{3/2}}\,dx \qquad \text{assuming } d > 0$$

$$C = \frac{A\,S}{2\,d\,\pi\,v}$$

Solving the above equation for S and plugging in known values for C, d,v and A gives us:

$$S = \frac{2\,C\,d\,\pi\,v}{A} \qquad \text{where we substitute } C = \ 100,\ d = 3\,m,\ v = 3\,m/s,\ \text{and } A = 0.01\,m^2$$

$$S = 565\,487\ \text{sec}^{-1}$$

With *a* spherical detector, the cos(θ) term disappears, making the integral :

$$C = \int_{-\infty}^{\infty} \frac{S\,A}{4\,\pi v(x^2 + d^2)^{2/2}}\,dx \qquad \text{assuming } d > 0$$

$$C = \frac{A\,S}{4\,d\,v}$$

Solving the above equation for S and plugging in known values for C, d,v and A gives us:

$$S = \frac{4\,C\,d\,v}{A} \qquad \text{where we substitute } C = \ 100,\ d = 3\,m,\ v = 3\,m/s,\ \text{and } A = 0.01\,m^2$$

$$S = 360\,000\ \text{sec}^{-1}$$

This makes sense. The system can detect a smaller source because the detector is more efficient by virtue of the full area of the detector being exposed to the source at all times. For finite viewing of the source from -L to L:

$$C = \int_{-L}^{L} \frac{S\,A\,d}{4\,\pi v(d^2 + x^2)^{3/2}}\,dx \qquad \text{assuming } d > 0 \text{ and } L > 0$$

$$C = \frac{A\,L\,S}{2\,d\,\sqrt{d^2 + L^2}\ \pi v}$$

An interesting exercise left to the reader is to plot S as L is varied from small to large.

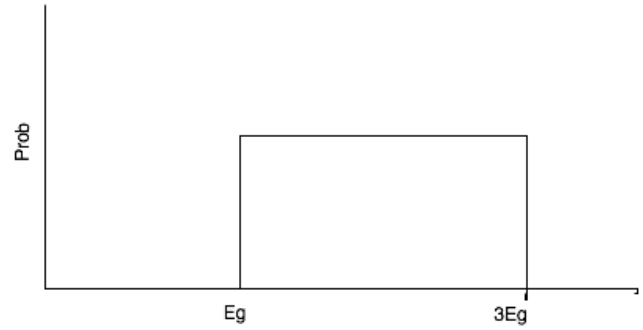
More on the Fano Factor

The Fano factor, F, is defined as the variance of the pulse height (or energy) distribution divided by its mean. If Poisson or Gaussian statistics applied, the Fano factor should be 1. And yet we note that F is about 0.1 for semiconductor detectors. This note looks at the rationale for this large difference.

- **Charge formation**

The primary electron (perhaps a photoelectron) moves through the detector material and interacts with the atoms along its path. Suppose N of these interactions result in an electron-hole pair being formed. Each of these interactions will result in a different amount of energy being lost by the primary electron. (We add any non-charge-producing energy losses from other interactions to this energy loss so that the total energy lost equals the primary electron energy loss).

Suppose the distribution of the amount of energy lost in an interaction that produces a signal carrier is given by g(E). One simple example of g(E) is shown below where E_g is the band gap energy.

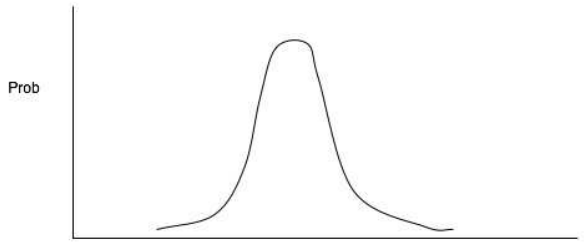


- **Combination of multiple interactions drawn from g(E)**

We can view our problem as either fixing the number of interactions, N, and calculating the probability distribution of the total energy lost E_{tot} , or fixing the energy lost and calculating the probability distribution of the number of interactions. We'll choose the former for now.

If we look at all of the interactions leading to a carrier, we have a set of N random variables $\{\Delta E_1, \Delta E_2, \ldots, \Delta E_N\}$. where each of these is drawn from the distribution g(E). But we seek the distribution for the total energy deposition, i.e., of the sum $E_{tot} = \Delta E_1 + \Delta E_2 + \ldots + \Delta E_N$.

The Central Limit Theorem immediately gives us the distribution for E_{tot} assuming N is large. This fundamental theorem of statistics states that the combination of independent and identically-distributed random variables will follow a normal distribution which has a mean of N<E> and a variance of N $\sigma_{g(E)}^2$. The reason for this is that we have N independent variables which can be combined in multiple ways to total E_{tot}. The interested student should consult any statistics textbook (or just see the example at http://wiki.stat.ucla.edu/socr/index.php/SOCR_EduMaterials_Activities_GeneralCentralLimitTheorem) for further elaboration.



Using this result, we can immediately write down the Fano factor:

$$F = \frac{\mathrm{var}(E_{\mathrm{tot}})}{<E_{\mathrm{tot}}>} = \frac{N\,\sigma_g{}^2}{N<g(E)>} = \frac{\sigma_g{}^2}{<g(E)>}$$

This tells us that the variance of the pulse height is actually due to the underlying distribution g(E). If g(E) happens to be Gaussian or Poisson, then F=1. For the case where the total energy is fixed, and N varies, a very similar argument can be made, and the same result for F will emerge. (cf. Jordan, D.V., et al., NIMA 585 (2008) 146-154).

- **Extra Problem**

1. Calculate the Fano factor using g(E) shown in the figure above except allow g(E) to be uniform over the variable range from E_g to $N * E_g$. Although this is a crude model, compare your results for N=5 to the Fano factor for typical semiconductor (Knoll: Table 11.1, 2nd Ed) and gas-filled detectors (Knoll: Table 6.2, 2nd Ed.). Plot the Fano factor for N ranging from 1 to 10. Your plot shows generally how the Fano factor changes as g(E) broadens.

(As an aside, Fano factors are usually not applied to scintillators, although this is a current topic of discussion. If we follow the arguments above, the primary electron produces excited atoms, each of which may produce a photon. If we could count these photons directly, then the Fano factor may have some relevance. But each photon must be collected and converted to a photoelectron. One might expect that the net result of these additional random processes is to ultimately produce a g(E) distribution for the final signal carriers emerging from the PMT that is Gaussian in shape, which would imply F~1).

Ionization Chamber problems

■ Problem 5.1. Charge created by alpha particles in He.

The W-value (energy dissipation/ion-pair) for α particles in He is 42.7 eV. We simply divide the energy of one α particle (5.5 MeV) by the W-value to find the number of charge carriers created when the α particle is stopped in He:

$$N = \frac{5.5\ \text{MeV}}{\frac{42.7\ \text{eV}}{\text{Carrier}}}$$

The number of charge carriers created (electrons or positive ions) is thus:

$$N = 128.8\ \text{Kilo Carriers}$$

Next, we simply convert the number of carriers created to charge by multiplying by the charge of one electron.

$$Q = \frac{128.8\ \text{Kilo Carriers} \times \text{ElectronCharge}}{\text{Carriers}}$$

The charge created when one 5.5 MeV α particle deposits all its energy in He is thus:

$$Q = 2.06 \times 10^{-14}\ \text{Coulombs}$$

To find the corresponding saturated current when 300 α particles per second enter the He filled ion chamber, we simply take the number of charge carriers created by one α particle, and multiply that by the charge of one electron and the factor 300/second:

$$I = \frac{(128.8\ \text{Kilo Carriers} \times \text{ElectronCharge})\ 300}{\text{Carriers Second}}$$

The saturated current created when 300 α particles per second (each with energy 5.5 MeV) enter the He filled ion chamber and deposit their full energy (in pA) is thus:

$$I = 6.19\ \text{pA}$$

Note that this is an extremely small current to be measured.

■ Problem 5.2. Finding the saturated ionization current given voltage and current data.

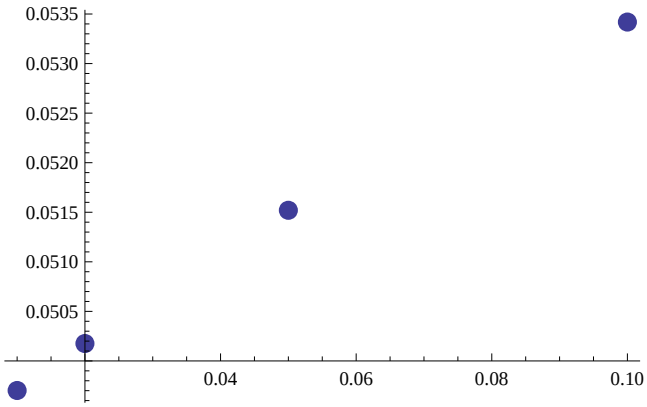
In the first two lines below, we define our lists of voltage values and current values, and on the third line, we create a set of points consisting of corresponding {1/V, 1/I} values. The idea is that we can extrapolate 1/V to zero (which would be infinite applied voltage) to find the saturated current.

```
volt = {10., 20., 50., 100.}

current = {18.72, 19.41, 19.93, 20.12}

data = {1/volt, 1/current}
```

The set of 1/V and 1/I data points we have created are thus:

```
{0.1, 0.0534188}, {0.05, 0.0515198}, {0.02, 0.0501756}, {0.01, 0.0497018}
```

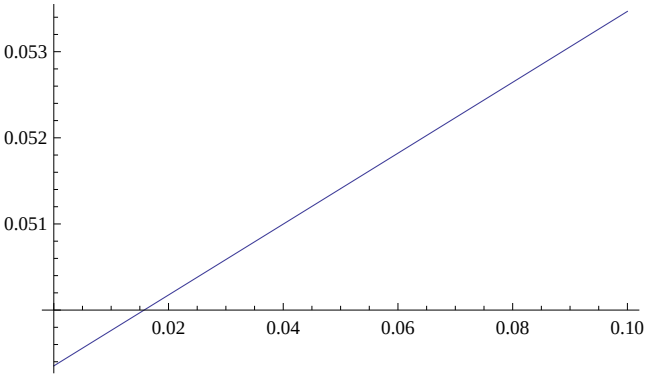
A plot of our data points (1/I versus 1/V):



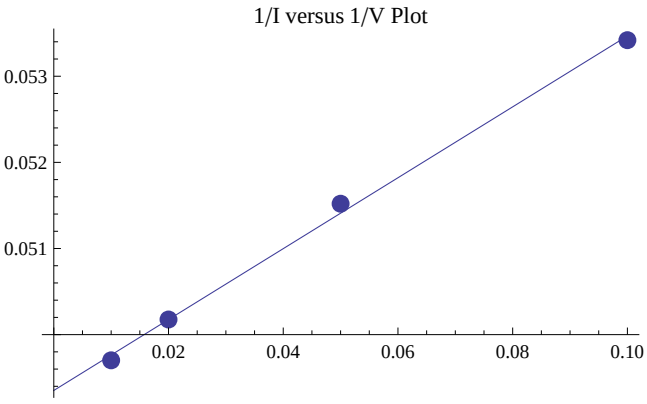
A linear equation that best fits our data (which is an approximate equation for 1/I in terms of 1/V, or "x") is determined from least-squares fitting:

$$y = 0.0411593\,x + 0.0493518$$

Next we plot our linear fit equation for the data set for x (or 1/V) from 0 to 0.1.



Next we plot our data set and the linear fit equation on the same graph to see how well our line fits the data:



The saturated current occurs at the y-intercept, so to make an approximation of the saturated current ("i_{sat}"), we calculate 1 divided by the value of our linear fit equation's y-intercept:

$$i_{sat} = \frac{1}{0.0493518} = 20.3 \text{ pA}$$

which is an approximated value of the saturated ionization current (in pA).

We can solve the equation which fits (1/I versus 1/V) to find the value of V which corresponds to .995 i_{sat} (i.e. the minimum applied voltage needed to reach a current within 0.5% of saturation). This is shown below along with the numerical value for 1/x (or V).

$$0.0411593\, x + 0.0493518 = \frac{1}{0.995\, i_{sat}}$$

The minimum applied voltage needed to reach a current within 0.5% of saturation (in volts) is thus:

$$V = \frac{1}{x} = 166 \text{ volts}$$

- **Problem 5.3. Finding the gamma-ray exposure necessary to reduce ion chamber voltage.**

Exposure is found by determining the charge formed (and lost from the capacitor) per volume of air at STP, and then converting that into charge/mass air, which is exposure. We define our equation for the change in charge per volume air with the voltage drop, i.e. $\frac{\Delta Q}{\text{Volume Air}} = \frac{C\Delta V}{\text{Volume Air}}$ (for Volume Air, we use the factor $\rho_{air} = 1.293 \times 10^{-3}\, g/\text{cm}^3$ to find the mass of the air, which we need for units to cancel), and convert to roentgen (R) using the factor $R = 2.58 \times 10^{-7}$ C/g. We then take our result and plug in the factor 1 Farad-V = 1 C.

$$\frac{C(V_0 - V_1)R}{\frac{(50 \text{ cm}^3)(1.293 \text{ grams})(2.58 \text{ Coulombs})}{(10^3 \text{ cm}^3)(10^7 \text{ grams})}}$$

We substitute $C = \frac{75 \text{ Farads}}{10^{12}}$, $V_0 = 25$ Volts and $V_1 = 20$ Volts to getthe gamma-ray exposure in roentgen (R):

$$\text{Exposure} = 0.0225\, R$$

- **Problem 5.4. Finding maximum e^- collection time given voltage, pressure in methane, and distance between electrodes.**

First, we must find the value of $\frac{E}{p}$ (electric field strength over gas pressure), which is easily calculated since voltage, distance between electrodes, and gas pressure are all given. This calculation is shown, where we use "E_0" to represent $\frac{E}{p}$.

$$E_0 = \frac{1000 \text{ Volts}}{\frac{(5 \text{ meters})\text{ atm}}{10^2}}$$

The value of $\frac{E}{p}$ to be used in the textbook figure is thus:

$$E = \frac{\textbf{20 000 volts}}{\textbf{atm–meter}}$$

We approximate the drift velocity from textbook figure as v = .36 x 10^5 m/s. Thus, the **maximum** electron collection time can easily be computed by assuming that the free electrons are formed at the farthest possible distance from where they are to be collected, which is the given distance between electrodes. We calculate this collection time below:

$$t_c = \frac{\textbf{5 meters}}{\frac{\textbf{10}^{\textbf{2}}\left(\textbf{.36} \times \textbf{10}^{\textbf{5}}\ \textbf{meters}\right)}{\textbf{second}}}$$

The maximum electron collection time is thus:

$$t_c = 1.39 \times 10^{-6}\ \textbf{seconds}$$

■ Problem 5.5. Finding average current over gamma-ray exposure period.

This is similar to problem 5.3. We find the charge generated corresponding to the voltage drop on the chamber, and dividing this by the exposure time, gives the average current. We define the average current as $\frac{\Delta Q}{\Delta T} = \frac{C\Delta V}{\Delta T}$ (denoted by "i" below) and give the appropriate unit conversions so that we get our result in amperes.:

$$i = \frac{c\left(v_0 - v_1\right)}{\textbf{30} \times \textbf{60 seconds}}$$

We substitute c = $\frac{250\ \text{Farad}}{10^{12}}$, v_0 = 1000 Volts and v_1 = 850 Volts to get the average current over the exposure period:

$$i = 2.083 \times 10^{-11}\ \textbf{Amperes}$$

■ Problem 5.6. Range of 0.5 MeV electron in air with and without using scaling laws.

Referring to Fig. 2.14 in the textbook, Range x Density = 0.25 gm/cm^2 (in silicon)-- however, this is approximately a constant for a wide variety of materials. Using a density of air of 0.001293 g/cm^3 gives a range (in cm) of 0.25/ρ. This is shown below.

$$\textbf{Range} = \frac{\textbf{0.25 gm/cm}^{\textbf{2}}}{\textbf{0.001293 } g/\textbf{cm}^{\textbf{3}}}$$

The approximate range in cm is thus:

$$\textbf{Range} = \textbf{193.349 cm or roughly 2 } m$$

A more accurate approach is to use the Bragg-Kleeman rule which accounts for variations in atomic masses somewhat. Using an atomic weight (in amu) for air of 28.97 and for Silicon using 28.09 gives a range in air (in cm) of $(0.25/\rho)\left(\sqrt{\frac{A_{air}}{A_{si}}}\right)$, or:

$$\textbf{Range} = \frac{\textbf{0.25}\sqrt{\frac{\textbf{28.97}}{\textbf{28.09}}}}{\textbf{0.001293}} = \textbf{196 cm}$$

which should be a more accurate approximation of the range. This result is not much different from that above because of the similarities in the atomic numbers between silicon and air.

■ Problem 5.7. Compensation of free air chambers for high energy gamma-rays.

The key idea of the 1-D Free Air ion chamber is that the range of the secondary electron is less than twice the electrode separation distance. Under the worst case, a 5 MeV gamma will lead to a 5 MeV electron through a photoelectric absorption. Using the textbook Fig. 2.14, we estimate $\rho R \sim 3$ gm$/$cm^2 for 5 MeV electrons. Thus, we estimate the range of these electrons using $\rho_{air} = .001293$ g/cm^3 as in Problem 5.6. The estimated electron range, and hence roughly half the spacing needed between electrodes (in cm) is thus:

$$\textbf{Range} = \frac{3.0}{0.001293} = 2320 \text{ cm}$$

This implies a separation distance of ~ 50 meters! Not very attractive. Fortunately, a photoelectric absorption is not very likely at these energies. Nevertheless, pair production will still yield electron positron pairs with ~ 2 MeV of energy each. This would still require large separations of several meters between electrodes.

■ Problem 5.8. Saturated ion current predicted from source activity.

We first convert the source activity to Ci for the sake of discussion after the problem is solved. The source activity in Ci.

$$150 \text{ kBq} = 4.054 \times 10^{-6} \text{ Ci}$$

To calculate the saturated ion current, we first note that since we have 150 kBq source activity, this also means that 150 x 10^3 incident electrons are hitting the argon gas per second. The ion current that is produced comes from the secondary electrons produced, thus for every incident electron hitting the argon gas, we have a factor of $\frac{\text{incident electron energy}}{W-\text{value of argon}} = \frac{49 \, x \, 10^3 \text{ eV/electron}}{26.4 \text{ eV/electron}}$, which determines the number of secondary electrons produced for each incident electron. Finally, we must multiply this out by the charge of a single electron (1.6 x 10^{-19}C), since current is in C/s. This calculation is shown below:

$$i = \frac{(150 \times 10^3 \text{ electrons})(49 \times 10^3 \text{ eV})(1.6 \text{ Coulombs})}{\frac{(\text{sec})(\text{electron})(26.4 \text{ eV})(10^{19} \text{ electrons})}{(\text{electron})}} = 4.45 \times 10^{-11} \text{ Amps}$$

Therefore, roughly 1 microcurie of C-14 gives a current of ~ 10 pA -- very small to measure.

■ Problem 5.9. Current in an ion chamber at low dose rate.

We need to find the current produced in 1 liter of STP air if the dose rate is 0.5 mR/h (35.8 pC/kg-s). First, we find the mass of 1 liter of air using its density, which is shown below.

$$m = \frac{(0.001293 \text{ kg})(1000 \text{ cm}^3) \text{ liter}}{(10^3 \text{ cm}^3) \text{ liter}}$$

The mass of 1 liter of air is thus:

$$m = 0.00129 \text{ kg}$$

Then we multiply this mass of air by the dose rate in pC/kg-s to get the minimum current that must be measured.:

$$i = \quad 0.001293 \text{ kg} \left(\frac{35.8 \text{ Coulombs}}{10^{12} \text{ (kg– sec)}} \right) = 4.63 \times 10^{-14} \text{ Amps}$$

The answer tells us that this is a very small current -- probably too small to measure reliably. We need much bigger signals (x 10^6 bigger) to measure this level of dose rate.

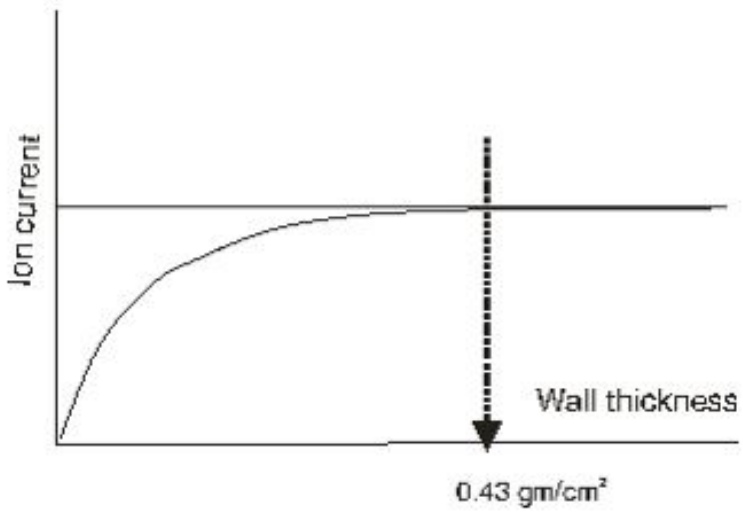
- **Problem 5.10. Electron sensitive mode ion chamber pulse size.**

We know V=Q/C (x/d) for an ion chamber in electron sensitive mode. Therefore, we simply multiply the number of ion pairs formed (or electrons formed) by the charge of a single electron (1.6 x 10^{-19}C) to obtain Q. Also, we are given x, since x is the distance from where the ion pairs are formed to the anode (x = 2cm), and we are also given d, which is the total spacing between the plates (d = 5cm). Of course, we are given the capacitance, (C = 100 pF). The calculation for V is shown below.

$$V = \frac{(1000 \times 1.6 \text{ Coulombs}) \, (2 \text{ cm})}{\left(10^{19}\right) (150 \text{ pF}) \, (5 \text{ cm})} = 4.27 \times 10^{-7} \text{ volts}$$

- **Problem 5.11. Electronic equilibrium**

a) According to Table 5-2, electronic equilibrium is reached for a wall thickness of 0.43 g/cm^2.



b) In a vacuum, no secondary electrons can enter the chamber, so there is inadequate compensation just using thin walls. This reduces the measured ion current compared with air surroundings.

- **Problem 5.12. Electronic equilibrium for 10 MeV gamma-rays.**

For 10 MeV gamma-rays, we need a wall thickness of 4.9 g/cm^2(from Table 5.2). Dividing this by the density of aluminum (2.7 g/cm^3) will give the desired wall thickness, t:

$$t = \frac{4.9\ g/cm^2}{2.7\ g/cm^3} = 1.81\ cm$$

which is the minimum aluminum wall thickness for compensation to be maintained.

■ Problem 5.13. Ion current predicted from exposure rate.

We use the ideal gas law to find the mass of air in the chamber. Using the relation that pressure is proportional to density times temperature, p $\propto \rho$*T:

$$\rho = \left(\frac{p}{p_0}\right)\left(\frac{T_0}{T}\right)\rho_0$$

we are able to find the air density at the specified conditions (the subscript "0" refers to STP conditions, and "p" and "T" refer to pressure and temperature, respectively). From this, we can find the mass of the air in the ion chamber, which we multiply by the gamma-ray exposure rate to find the saturated ion current.
First, we define the value of the exposure rate:

$$\text{exposure rate} = \frac{100\ pC}{kg - sec}$$

And we calculate the mass of the air using the method outlined above:

$$\text{mass}_{air} = \frac{p\ t_0\ \rho_0 \times \text{volume}}{p_0\ t}$$

We substitute p = 3 Atm, $p_0 = 1$ Atm, $T_0 = 273\,K$, $T = 373\,K$, $\rho_0 = \frac{1.293\ mg}{cm^3}$ and volume = 2500 cm^3 to get the mass of air in the ion chamber:

$$\text{mass}_{air} = 7.098\ g$$

Now we calculate the saturated ion current by multiplying the exposure rate by the mass of air in the ion chamber:

$$i_{saturated} = \text{exposure rate} \times \text{mass}_{air} = 0.7098\ pA$$

■ Problem 5.14. Pocket dosimeter minimum sensitivity.

Exposure = $\frac{\Delta Q}{M} = \frac{\Delta VC}{M}$, where M is the mass of air in the pocket chamber. First, we calculate the mass of air in the pocket chamber (denoted by "M"). Here, we use $\rho_{air} = .001293\ g/cm^3$:

$$M = \frac{(10\ cm^3)\ (.001293\ grams)}{cm^3} = 0.0129\ grams$$

Using the ΔV and C values provided:

$$\Delta V = \frac{50\ Volt}{10^3} \text{ and } C = \frac{20\ Farad}{10^{12}}$$

We calculate the exposure, using the value of "M" (mass of air in the chamber) we found above (using the conversion factor of 1 mR = 2.58 x10^{-10}C/g)

$$\text{Exposure} = \frac{C\,\Delta V}{\frac{(2.58\ \text{Coulombs})\,m}{10^{10}\ \text{grams}}} = 0.2998\ \text{mR}$$

■ Problem 5.15. Rise time slopes for electrons and ions.

We need the slope of the curves for electron + ion travel, $V_{ei}(t)$ (cf. textbook Eqn. 5.15), and for just ion travel, $V_i(t)$ (cf. textbook Eqn. 5.16). Recall that a particle's velocity is given by $\frac{\mu E}{p}$ (Eqn. 5.3), where μ is the mobility and $\frac{E}{p}$ is the electric field strength divided by the gas pressure. In the third line, we calculate the ratio of the derivative (with respect to t) of $V_{ei}(t)$ to that of $V_i(t)$, giving the appropriate relationships for v^+ and v^-.

We express Eqn. 5.15

$$V_{ei} = \frac{n_0\, e\, (v^+ + v^-)\, t}{d\, c}$$

We express Eqn. 5.16 as

$$V_i = \frac{n_0\, e\, (t\, v^+ + x)}{d\, c}$$

Calculate the ratio of the derivative (with respect to t) of $V_{ei}(t)$ to that of $V_i(t)$, giving the appropriate relationships for v^+ and v^-.

$$\frac{\left(\frac{\partial V_{ei}}{\partial t}\right)}{\left(\frac{\partial V_i}{\partial t}\right)}$$

where we substitute $v^+ = \mu^+ E$ and $v^- = \mu^- E$ to get the equation for the ratio of the slopes of the V_{ei} and V_i portions of Fig. 5.16.

$$\frac{\mu^- + \mu^+}{\mu^+}$$

Now we take our previous result and plug in estimated values for the mobilities (read pg. 135), $\mu^- = 1000\,\mu^+$ and $\mu^+ = \frac{1.25\ \text{m}^2\ \text{Atm}}{10^4\ (\text{Volt–Second})}$, to get the approximate ratio of the slopes of the V_{ei} and V_i portions of Fig. 5.16.

$$\text{Slope Ratio} = 1001$$

The rise time is proportional to the velocity, which is proportional to the carrier mobility. So the ratio of the fast (electron-dominated) rise to the slow (ion-only) rise is about the ratio of the mobilities, which is about 10^3. Note that we have assumed that the ions aren't moving much during the first part of the pulse. Otherwise, we would need to include them, and the ratio would more precisely be given by $1+\frac{\mu^-}{\mu^+}$, which is what we have found in our result.

- **Supplemental Problem: Minimum pulse rise time**

Assume for simplicity a parallel plate geometry so that the electric field is constant in the detector volume. Assume that electrons and ions move with saturated velocities v_e and v_i. The distance between the negatively charged cathode and positively charged anode is d.

Show that the minimum rise time of the resulting pulse occurs when $\frac{x}{d} = \frac{v_e}{v_e+v_i}$ and is $t_{\text{rise}}^{\text{min}} = \frac{d}{v_{\text{total}}}$ where $v_{\text{total}} = v_e + v_i$.

Solution: Plot t_{rise}(= distance to collection/velocity) as a function of interaction position x for both the electrons and ions. The plot will form an "X", with the upper "v" of the "X" shape representing the total charge collection time as a function of x. The point of intersection of the two curves is $\left(x_{\text{rise}}^{\text{min}},\ t_{\text{rise}}^{\text{min}}\right)$. The reader can extend this problem to include other electric field distributions (e.g., cylindrical, spherical).

Proportional counter problems

■ Problem 6.1. Variance in charge carriers without multiplication.

We expect the mean number of ion pairs formed to be equal to the energy of the radiation divided by the W-value of argon, and we expect the standard deviation in the number of ion pairs to be the square root of the Fano factor times the calculated mean number of ion pairs. Also, the relative standard deviation is simply the standard deviation divided by the mean. In the following calculation Fano = 0.17. We express this below.

$$\text{mean number of ion pairs} = \frac{10^6\ \text{eV}}{26.2\ \text{eV}} = 38\,170$$

$$\text{Expected standard deviation } (\sigma) = \sqrt{\text{Fano} \times \text{mean}} = 80.6$$

$$\text{Relative std deviation} = \frac{\sigma}{\text{mean}} = 0.00211$$

Now we look at what happens if the Fano factor equals 1.

$$\text{mean number of ion pairs} = \frac{10^6\ \text{eV}}{26.2\ \text{eV}} = 38\,170$$

$$\text{Expected standard deviation } (\sigma) = \sqrt{\text{Fano} \times \text{mean}} = 195$$

$$\text{Relative std deviation} = \frac{\sigma}{\text{mean}} = 0.00512$$

Observe that the Fano factor makes a **large** difference in reconciling experiment to theory in this case, primarily because the non-Poisson recombination along the tracks reduces the fluctuation (standard deviation) in the number of ion pairs formed. (For the inquiring reader, earlier in this solutions manual is a simple explanation of the basis for the Fano factor).

■ Problem 6.2 Using air in gas-filled chambers

Because oxygen has a large electron attachment coefficient, the electrons produced from a radiation interaction would be unlikely to reach the small multiplication volume around the anode wire. On the other hand, air is useful in ionization chambers since either electrons, or the negative ions formed when the electron becomes attached, can be collected. An air-filled chamber also provides a direct reading of exposure.

■ Diethorn Model.

The Diethorn model plays a role in the following problems. We define it here as "multiplication", and use it throughout the following problems.

$$\text{multiplication}(V, b, a, \Delta V, K, p) = e^{\frac{V \ln(2) \ln\left(\frac{V}{K\, p\, a \ln\left(\frac{b}{a}\right)}\right)}{\ln\left(\frac{b}{a}\right) \Delta V}}$$

■ Problem 6.3. Effect of anode/cathode radius on multiplication.

a) We solve this part by simply giving the known values of our variables in our function "multiplication," setting "multiplication" equal to 1000, and solving for "V". Our known values in this problem are: b=1, a=.003, ΔV=23.6, K=4.8×10^4, and p=1.

The operating voltage (in Volts) required to achieve a gas multiplication factor of 1000 is thus:

$$V = 1790 \text{ volts}$$

b) Here, we look for the effect of doubling the anode radius at this operating voltage on the multiplication factor. We calculate the factor that the multiplication changes by taking the value of "multiplication" at double the anode radius (.006 cm) and divide it by the value of "multiplication" at the previous anode radius (.003 cm). The factor the multiplication changes by (or is "increased" by) is thus:

$$\frac{\text{multiplication}(V,\ 1,\ .006,\ 23.6,\ 4.8\times10^4,\ 1)}{\text{multiplication}(V,\ 1,\ .003,\ 23.6,\ 4.8\times10^4,\ 1)} = 0.00752$$

We just take the inverse of the previous result to get the factor the multiplication is decreased by when changing the anode radius from .003 to .006 cm.

$$0.007522^{-1} = 133$$

Note that increasing the anode radius by just 30 μm decreases the multiplication by more than 100!
c) Here, we do the same as in part b, but this time we are looking at the effect of changing the cathode radius from 1 to 2 cm (we put the anode radius back to the original .003 cm). The factor the multiplication changes by is thus:

$$\frac{\text{multiplication}(V,\ 2,\ .003,\ 23.6,\ 4.8\times10^4,\ 1)}{\text{multiplication}(V,\ 1,\ .003,\ 23.6,\ 4.8\times10^4,\ 1)} = 0.192$$

Again, we take the inverse of the previous result to get the factor the multiplication is decreased by when changing the cathode radius from 1 to 2 cm.

$$0.192226^{-1} = 5.202$$

The effect is only a factor of 5 as opposed to more than 100, showing how the anode radius is quite critical in determining the multiplication.

■ Problem 6.4. Pulse size from proportional counter.

To solve this problem, we calculate the multiplication expected in the tube under the given operating conditions, and from this we obtain the charge formed in the tube by multiplying the initial number of ion pairs (energy of radiation divided by W-value) formed by the multiplication factor and the charge of a single electron. Then, we divide this charge by the capacitance of the tube to obtain the maximum voltage pulse (which is a good estimate for our purposes here). First, we show the calculation of the multiplication factor, which will be used in further discussion.

$$\text{multiplication}(2000,\ 5,\ .005,\ 23.6,\ 4.8\times10^4) = 4.93$$

Then, we calculate the voltage pulse as discussed above.

$$\frac{\text{multiplication}\left(2000,\ 5,\ .005,\ 23.6,\ 4.8\times10^{4}\right)\left(5\times10^{6}\ \text{eV}\right)\left(1.602\ x\ 10^{-19}\ \text{Coulombs}\right)}{(26\ \text{eV})\ (500\ \text{pF})} = .304\ \text{mV}$$

This is a small, but measurable voltage because we deposited 5 MeV and had a multiplication factor of 5. Larger values of the multiplication would yield a more reasonably sized pulse for spectroscopy.

■ Problem 6.5. Testing of proportionality of tube.

Using different, but known, energy depositions, measure the resulting pulse amplitudes. If the ratio of pulse amplitudes is consistent with the ratio of energy depositions, the tube is operating in proportional mode.

■ Problem 6.6. Pulse height versus energy for alpha, beta

Alpha particles deposit all of the their energy in a small region. Although incident energy may increase, so does deposited energy since the particle range is typically small compared to the tube dimensions. Hence pulse height increases. The range of a beta particle is long in a gas, usually longer than the tube dimensions. As incident energy increases, the deposited energy decreases (Bethe's law) and hence the number of carriers and pulse height also decreases.

■ Problem 6.7. Gas multiplication needed for given pulse height desired.

This is the same type of problem as Problem 6.4, except here we need to find the gas multiplication which yields a pulse amplitude of 50 mV for 50 keV input. This is expressed below, where we solve the equation V= Q/C for the multiplication, "M."

$$10\ \text{Milli Volts} = \frac{M\left(5\times10^{4}\ \text{eV}\right)1.602\ x\ 10^{-19}\ \text{Coulombs}}{(26\ \text{eV})\ (200\ \text{pF})}$$

Solving for the multiplication, M, yields:

$$M = 6490$$

■ Problem 6.8. Fraction of active volume for multiplication.

For a cylindrical wire, the electric field is given by $E(r) = \frac{V}{r\ln(b/a)}$. In this problem, we want to know what fraction of the tube has a field larger than a given value. Find the r for this electric field strength, then the fraction is just the ratio of the radii (r and b) squared. First, we solve for "r" in the above equation.

$$E = \frac{V}{r\ln\left(\frac{b}{a}\right)}$$

The equation for r:

$$r = \frac{V}{E\ln\left(\frac{b}{a}\right)}$$

Now we take the ratio of $r^2$and b^2, substituting the known values V=2000Volts, a=0.003 cm, b=2cm and V=$\frac{10^6 \text{ Volts}}{\text{meter}}$ to get the percentage of the tube volume corresponding to the multiplication region.

$$\frac{r^2}{b^2} = 0.0237\ \%$$

This is a negligibly small fraction of the tube volume, which is a characteristic of the proportional tube.

■ Problem 6.9. Field tube function.

The function of the field tube is to reduce the electric field near the ends of the tube where distortions in the electric field would occur (particularly near corners).

■ Problem 6.10. Photoelectric cross section discontinuities at shell energies

The discontinuities are absorption edges corresponding to the binding energies of electrons in K, L, M, etc. shells in the absorber atoms (with K being the tightest bound, and so it occurs at the highest energy). If the gamma energy is above an absorption edge, photoelectric absorption can occur with higher probability the closer the gamma energy is to an electron's binding energy. But once the gamma energy drops below the edge, the photoelectric process is not energetically possible for that shell, and the total interaction probability drops abruptly.

■ Problem 6.11. Gas scintillation detector voltage.

In gas scintillators, our signal carriers are the photons generated from the electrons moving in the high field region of the tube (around the anode). If we operated the detector in the proportional regime, we would increase the number of electrons (and hence photons), but would suffer from the loss of energy resolution associated with gas multiplication. In order to avoid this energy resolution loss, the tube is normally operated below the threshold for multiplication.

■ Supplemental Problem: Electric Field Distributions for Cylinders and Spheres

For the Townsend coefficient α to be positive, the electric field must exceed $\sim 10^6$ V/m. Above this value, the electron carriers will undergo multiplication. Given that a fixed voltage V_{appl} is to be placed on a central anode of radius r_i, with a cathode at radius r_o, determine the fraction of the detector volume that will undergo multiplication for spherical and cylindrical geometries. Comment on which geometry will require the least voltage to achieve multiplication. Use reasonable values for any parameters needed, such as $V_{\text{appl}} = 10^3\ V,\ r_i = 10^{-5}\ m,\ r_0 = 10^{-2}\ m$.

[The motivated reader will solve the homogeneous Gauss's Law for the electric field distribution using the appropriate coordinate system. The result will show that the radius at which the electric field is at the critical value, r_{crit}, is similar for both geometries, but the fraction of volume that is active is much smaller for the spherical geometry].

GM counter problems

■ Problem 7.1. GM quench gas ionization potential

The purpose of the quench gas in a GM tube is solely to absorb the charge from a major fill gas ion to prevent an electron from being liberated when that ion hits the cathode wall. For the quench gas to give up an electron to the positive ion, it must have a lower ionization potential than the ion.

■ Problem 7.2. Parameter changes affecting GM tube operation.

Operating an ionization chamber in the GM region requires a high degree of multiplication, M, described by the Diethorn model of Eqn. 6-10. If we change a basic parameter and the multiplication M decreases, then the voltage has to be increased to compensate. So, if we:
(a) double the anode radius, M decreases, and we must increase V to operate in the GM region.
(b) double P, M decreases slightly, and we must increase V again.
(c) to see the effect of the concentration of the quench gas, look at the values of K and ΔV for P-5 and P-10 gas. By increasing the concentration, both variables increase, leading to a decrease in M and thus an increase in V.

■ Problem 7.3. GM Pulse height increases with V.

It is the space charge buildup around the anode wire that limits the size of the GM pulse. This limit is achieved later (higher multiplication) with increased electric fields in the detector. Since increased applied voltage implies increased electric field, the pulse height will be larger with higher applied voltages.

■ Problem 7.4. Voltage for Geiger discharge in tube.

To operate in the Geiger region, $(n_0)'p \geq 1$ (let's just assume that $(n_0)'p = 1$ for this problem, since we are interested in finding the minimum voltage for Geiger operation). Since $(n_0)'$ is the number of excited atoms created in an avalanche, and there are 3 excited atoms per ion pair created in an avalanche, then $(n_0)' = 3M$, where M is the gas multiplication factor (an avalanche is initiated by a single electron, thus M is the number of ion pairs created in an avalanche). Also, p is the probability of photoelectric absorption of any given photon, which is the same as the probability that a de-excitation photon from an excited atom will create another avalanche. Thus, $M = \frac{1}{3p} = \frac{1}{3 \times 10^{-5}} = 3.3 \times 10^4 = 33{,}000$. To find the operating voltage from this, we simply plug M and all the other parameters from Problem 6.3 into the Diethorn function and solve for V, i.e., we need to solve:

$$33\,000 = e^{\frac{V \ln(2) \ln\left(\frac{V}{K p a \ln\left(\frac{b}{a}\right)}\right)}{\ln\left(\frac{b}{a}\right) \Delta V}}$$

In the above equation, substitute b=1, a=.003, ΔV=23.6, $K = 4.8 \times 10^4$ and p=1 to get the voltage necessary to operate in the Geiger region (in volts), and solve for V:

$$V = 2164 \text{ volts}$$

■ Problem 7.5. Proportional versus Geiger tubes.

a) Proportional: The pulse height varies as the avalanche amplitude which, in turn, depends on voltage in an approximately exponential manner.

Geiger: The pulse amplitude corresponds to the number of ion pairs at the point at which the accumulated positive space charge is sufficient to reduce the electric field below its critical value. This number will increase in approximate proportion to the original electric field or linearly with the applied voltage.

b) Proportional: The quench gas must absorb UV photons.

Geiger: The quench gas must pick up positive charges from the original positive ions through charge transfer collisions.

c) Proportional: Because heavy charged particles tend to deposit all of the energy, and electrons only part of theirs, the two radiations can be separated by their different pulse heights.

Geiger: No differentiation can be achieved, because pulse height is independent of particle type and energy.

d) Proportional: The maximum counting rate is often set by pulse pile-up. The minimum pulse shaping time (that will minimize pile-up) is limited by the finite rise time of the pulses.

Geiger: The maximum counting rate is limited by the long dead time of the tube itself.

e) Proportional: Gamma rays produce very small amplitude pulses and are often below the discrimination level.

Geiger: Counting efficiency is a few percent due primarily to the liberation of secondary electrons from the detector walls.

■ Problem 7.6. Geiger tube efficiency with varying wall thickness

For the detection of gamma rays, the GM tube relies upon the wall of the tube to induce a (γ, e^-) reaction. While such reactions are possible in the gas itself, they are unlikely because of gas's low density. Recall that it is this electron which, as it moves through the gas, causes the ionizations that produce the electron-ion pairs that ultimately lead to avalanching. So, as the thickness of the wall increases, the greater the probability that the gamma ray will undergo a reaction that liberates an electron that will escape from the wall and lead to an avalanche. This efficiency begins to decrease for thicker walls because the probability that the electrons escape from the wall and into the gas decreases.

Since the electron must escape from the wall, we need to find the range of a characteristic electron produced in the (γ, e^-) reaction. As a crude approximation, assume that electron is produced by a photoelectric interaction and thus has 1 MeV of energy. As noted in the textbook in Chapter 2, the range of a 1MeV electron is roughly ~1-2 mm/MeV in most solids. Therefore, it makes little sense to make the walls thicker than this.

■ Problem 7.7 GM Tubes paralyzable or non-paralyzable?

The GM tube has historically been treated as a paralyzable detector. After an avalance, the positive ions produced around the anode wire must be cleared to the cathode to re-establish the full electric field in the tube. As this positive sheath moves to the cathode and the electric field begins to be restored, another interaction in tube can initiate a smaller avalanche and require the clearing process to begin again. If the counting system requires a full discharge to register a count, then the deadtime continues to be extended with each additional interaction in the tube. Extendable deadtimes are the hallmark of the paralyzable model.

■ Problem 7.8. Electron drift time.

First, we define the necessary parameters to be used throughout the problem.

$$p = 0.5\ \text{Atm} \qquad \mu = \frac{0.05\ m^2\ \text{Atm}}{(\text{Volt}-\text{sec})} \qquad E_{thresh} = \frac{2\times10^6\ \text{Volts}}{\text{meter}}$$

$$a = .005\ \text{cm} \qquad b = 2\ \text{cm} \qquad V = 1500\ \text{Volts}$$

Next, we calculate the radius (from the anode center) at which avalanche formation begins ("r_{crit}") using Eqn. 6.3 solved for r.

$$r_{crit} = \frac{V}{E_{thresh}\ \ln\left(\frac{b}{a}\right)}$$

The radius (from the anode) at which avalanche formation begins is thus:

$$r_{crit} = 0.000125\ m$$

Using Eqn 6.35 for the charge carrier drift time from the cathode to this point:

$$t_{drift} = \frac{p\ \ln\left(\frac{b}{a}\right)\left(b^2 - r_{crit}^{\ 2}\right)}{2\,\mu\,V}$$

and subsituting in the known values yields the electron drift time from cathode to multiplying region:

$$t_{drift} = 7.99\ \text{micro seconds}$$

In this problem we used Eqn 6.35 to arrive at the solution quickly. Without that expression at hand, we could still solve this because we would recognize that the drift time is the integral of $\frac{dr}{v(r)}$ from r_{crit} to b. We would write v(r) = $\frac{\mu E\ (r)}{p}$ (Eqn. 5.3), and E(r) = $\frac{V}{r\ln(b/a)}$ and do the integral, substituting in the parameters as shown below:

$$t_{drift} = \int_{r_{crit}}^{b}\left(\frac{1}{v}\right)dr$$

We substitute v = $\frac{\mu E}{p}$ and E = $\frac{V}{r\ln\left(\frac{b}{a}\right)}$ to get an equation for the electron drift time without the known values entered in.

$$\frac{\left(\frac{b^2}{2} - 7.83478\times10^{-9}\ m^2\right)p\ln\left(\frac{b}{a}\right)}{V\,\mu}$$

Now we plug in our given parameters to get the electron drift time from cathode to multiplying region. Of course this is the same answer as before.

$$t_{drift} = 7.99\ \text{micro seconds}$$

This is a LONG time on the scale of nuclear events. By making the cathode radius smaller, it can be made faster. And of course, not all particles are formed at the cathode.

Problem 7.9. Dead time losses in GM tubes

We are given a GM tube with a resolving time of 350 μs. We want to find the true rate for which the measured count rate is 1/2 the true rate, using both paralyzable and non-paralyzable cases.

Recall the resolving time is the time required for the GM tube to rest after a measured event before another pulse can be recognized (above some finite threshold). The dead time is the time during which no pulse of any finite amplitude can be generated (i.e. shorter than the resolving time), and the recovery time is the time from pulse onset until another full pulse can be generated (i.e. much longer than the resolving time).

First, we look at the non-paralyzable case. To find the true rate, n, we solve Eqn. 4.23 similtaneously with m = $\frac{n}{2}$(given) for n by eliminating m. Our two expressions are thus:

$$n - m = n\,m\,\tau \qquad m = \frac{n}{2}$$

We get the trivial solution n = 0 and the solution we are interested in, n = $\frac{1}{\tau}$.

We take the second solution from above and plug in our known value τ = 0.000350 to get the true rate for the non-paralyzable case in sec^{-1} of:

$$\boldsymbol{n = 2857\ \text{sec}^{-1}}$$

Now, we look at the paralyzable case, doing the same thing as before, except using Eqn. 4.27. The two expressions are now:

$$m = n\,e^{-n\tau} \qquad m = \frac{n}{2}$$

Again, we get the trivial solution n = 0 and the solution we are interested in.

$$n = 0 \qquad \text{and} \qquad n = \frac{\ln(2)}{\tau}$$

We take the second solution from above and plug in our known value τ = 0.000350 to get the true rate for the paralyzable case in sec^{-1}:

$$\boldsymbol{n = 1980\ \text{sec}^{-1}}$$

Note that the limit is reached sooner for the paralyzable case, but not by that much. Also note that these detectors will be useful only up to a couple of thousand of counts/sec.

■ Problem 7.10. GM tubes time-to-first count.

The time-to-first count method is limited by the finite switching time of the voltage. In this problem, this time is 0.3 μs, and the tube's quiet time is 900 μs, while the resolving time is 350 μs.

a). We want to find the true event rate having a 5% probability of one or more pulses occurring during 0.3 μs. We will call this our limiting rate, n. From Eqn. 3.70, we know that the probability of **zero** events occurring during a time interval t is e^{-nt}, where n is the true event rate we are interested in. So the probability that **one or more** events will occur during t is 1 - e^{-nt}. Below, we set this equation equal to our desired probability of one or pulses occurring during 0.3 μs (5%, or 0.05), and we solve it for n (our limiting rate), giving the appropriate value of the time interval t (in seconds).

$$\mathbf{1 - e^{-nt} = 0.05}$$

We substitute t = $\frac{0.3}{10^6}$ to get the limiting event rate in sec^{-1}.

$$\mathbf{n = 170\,978\ sec^{-1}}$$

Note that this rate is substantially higher than that in problem 7.9 for the equivalent tube (~2000 cps) -- we have gained two orders of magnitude. The key idea here is that *the limiting time period between pulses is now 0.3 μs as compared to 350 μs*, leading to the ability to tolerate much higher count rates.

b). At this limiting rate, what is the mean time to first count? This is analogous to the problem 3.25 earlier. The voltage is increased at some time which is random relative to the true-event pulse train. If the intervals have the Poisson distribution, the mean wait time is 1/n (see Fig. 7.10), or

$$\mathbf{t_{mean} = \frac{1}{n}}$$

We substitute n=170978 $sec^{-1}$and seconds = 10^6micro seconds to get the mean time–to–first count following switching of the voltage.

$$\mathbf{t_{wait} = 5.85\ micro\ seconds}$$

c). In this experiment, we count for 2-seconds during which the true event rate is the limiting rate in part (a) above. For each measured event, there is 900.3 μs when the tube is offline. Since the mean time between events is only about 6 μs (from part (b)), we estimate a measured count rate, m, of 1 count per 906.3 μs. Thus, the expected measured number of counts in 2 s is given by:

$$\mathbf{expected\ number\ of\ measured\ counts\ in\ 2\ s = \frac{2}{\frac{906.3}{10^6}}}$$

or

$$\mathbf{expected\ number\ of\ measured\ counts\ in\ 2\ s = 2207}$$

This is expected since there is a dead time of ~1 ms per count and the events are arriving at high frequency. Thus, in 2 seconds, we would expect about 2 s/(1 ms/count) or ~2000 counts.

Another way to attack this problem is to use our non-paralyzable model with τ=900.3 μs and n = part (a) answer over 2-seconds. This calculation for the measured counts over 2-seconds is shown below.

$$\boldsymbol{M = \frac{2n}{n\tau + 1}}$$

We substitute $\tau = \frac{900.3}{10^6}$ and n = 170 978 to get the expected number of measured counts in 2 s using the non-paralyzable model:

$$\boldsymbol{M = 2207.14}$$

Finally, to answer the question being asked, the percent relative uncertainty in the true count rate will be obtained from an error analysis on the time-to-first count relationship: $n = \frac{1}{<t>} = \frac{M}{\Sigma t_i}$ so that the relative uncertainty in the true count rate n is given by $\sigma_n / n = \sigma_M / M = 1 \Big/ \sqrt{M}$ which is :

$$\boldsymbol{\sigma_n / n = \frac{1}{\sqrt{M}} = 0.0213}$$

Scintillation detector problems.

■ Problem 8.1. Scintillation efficiency.

We calculate the efficiency by multiplying the number of photons created by the energy per photon, and divide this by the total energy deposited.

Using Photon Energy = Planck's Constant x Speed of Light / Wavelength:

$$\epsilon = \frac{20\,300 \times \text{Planck's Constant} \times c}{(447\text{ nm})\left(10^6\text{ eV}\right)}$$

The scintillation efficiency expressed as a percent is thus:

$$\epsilon = 5.63\,\%$$

This is actually a fairly inefficient process in converting incident energy into light energy. More importantly is the number of carriers created -- ~ 20,000 for 1 MeV. Next, we approximate the number of carriers created per eV of deposited energy:

$$\frac{20\,000\text{ carriers}}{10^6\text{ eV}} = \frac{0.02\text{ carriers}}{\text{eV}}$$

We'd like to relate this to the W-value in gas-filled detectors, so we take the inverse of the previous result to get the approximate amount of energy in eV required to create one carrier.

$$= \frac{50\text{ eV}}{\text{carrier}}$$

This is even worse than for a gas-filled detector (~ 30 eV/carrier). Fortunately, sodium iodide, the workhorse of scintillators, is much better than anthracene.

■ Problem 8.2. Time at which 99% of light is yielded in NaI(Tl).

We know that I(t) = $I_0\, e^{-t/\tau}$, where I(t) is the light intensity with respect to time, t. We will solve this by solving this equation for t and setting I(t) = 0.01 I_0. First, we define this equation and take the natural log of both sides. Next, we subtract the natural log from both sides. Then, we multiply both sides by -τ so that we have solved for t, and we assign the given values for I(t) and τ.

Our equation with the natural log taken on both sides.

$$\ln(I(t)) = \ln(I_0) - \frac{t}{\tau}$$

The previous equation with the ln (I_0) subtracted from both sides.

$$\ln(I(t)) - \ln(I_0) = -\frac{t}{\tau}$$

The previous equation with both sides multiplied by -τ, and the appropriate values of I(t) and τ substituted in.

$$-\frac{23\,(\ln(0.01 \times I_0) - \ln(I_0))}{100\,000\,000} = t$$

This is the solution for t, but in a rather inconvenient equation format. We would like it in a format that we can plug in values and evaluate. To do this we reverse the order of the equation to get t on the left hand side and cancel the I_0 in the numerator and get a numerical result for our answer. The time required for a NaI(Tl) event to emit 99% of the total light yield (in seconds) is thus:

$$\boldsymbol{t = 1.059 \times 10^{-6} \text{ seconds}}$$

■ Problem 8.3. Maximum brightness of scintillator.

The brightness is the rate of photon emission. We assume that the excited states are formed instantaneously, followed by scintillation governed by a single time constant τ, so that $\frac{\mathrm{dn}}{\mathrm{dt}} = \frac{-1}{\tau} n = I(t) = \frac{-n_0}{\tau} e^{\frac{-t}{\tau}}$. Since I(t=0) is the largest value, the ratio of maximum brightness between NaI(Tl) and anthracene will be given by $\left(\frac{n_{0,\,\mathrm{NaI(Tl)}}}{n_{0,\,\mathrm{anthra}}}\right) * \left(\frac{\tau_{\mathrm{anthra}}}{\tau_{\mathrm{NaI(Tl)}}}\right)$. This is shown in the following expression using values from Tables 8.1 and 8.3:

$$\frac{\boldsymbol{n_{0,\,\mathrm{NaI(Tl)}}}\ \ \boldsymbol{\tau_{\mathrm{anthra}}}}{\boldsymbol{n_{0,\,\mathrm{anthra}}}\ \ \boldsymbol{\tau_{\mathrm{NaI(Tl)}}}}$$

We substitute $n_{0,\mathrm{NaI(Tl)}} = 230$, $n_{0,\mathrm{anthra}} = 100$, $\tau_{\mathrm{anthra}} = 0.030$ and $\tau_{\mathrm{NaI(Tl)}} = 0.23$ to get the ratio of maximum brightness between NaI(Tl) and anthracene:

$$\boldsymbol{\text{maximum brightness NaI(Tl) / Anthracene} = 0.3}$$

Note that while the light output is 2.3 times larger for NaI, the brightness also depends on how fast the light comes out, which is much faster for anthracene. Thus, anthracene is brighter (maximum light emission rate) by a factor of ~3.

■ Problem 8.4. Comparing Organic and Inorganic Scintillators.

a. For speed of response, organic scintillators are much faster than most inorganic scintillators. This is because of the scintillation mechanisms involved.
b. For light output, the inorganic scintillators are much brighter than the organic scintillators.
c. Linearity of light output. The inorganic scintillators tend to have a more linear response to the amount of energy deposited, and they have become even more so in recent years.
d. Detection efficiency, for the same volume of detector material, is much better for the higher-Z detectors that are associated with inorganic materials.
e. The cost of plastic or liquid inorganic scintillators is generally much less than that of inorganic detectors. These inorganic detectors have to be carefully grown as single crystals and are thus relatively expensive.

■ Problem 8.5. The role of activators.

The activator in an inorganic scintillator is the atom impurity that actually produces the detectable scintillation light. Its energy structure lies in the forbidden energy gap of the host material. When an electron is excited to the conduction band in the host material, it migrates to an activator site and non-radiatively loses energy so that it occupies the lowest level of the activator's excited state. The subsequent de-excitation is at a wavelength that is too long to cause ionization in the bulk material, and this scintillation photon flows freely to a photodetector for detection.

In an inorganic material, the excitation and de-excitation occur with the molecule itself.

■ Problem 8.6. Highest Scintillation Efficiency.

Asking for the highest scintillation efficiency material is like asking for the highest batting average in baseball. Over time, new records are made, so the reader is wise to keep track of the current literature on this subject. Check the textbook for the material with the largest number of photons/MeV. Currently, $LaBr_3(Ce)$ holds this record.

■ Problem 8.7. Scintillation Mechanisms.

This problem is similar to problem 8-5. The key is to recognize that in an inorganic scintillator, a crystal lattice is required to establish the energy levels and the activator (a deliberate impurity) is chosen to have energy levels that lie within the forbidden energy gap of the host material. Without the crystal structure present, there would be no scintillation possible.

In contrast, inorganic scintillators function using the energy levels of the molecules themselves. Thus, a single molecule can function as a scintillator, and does not require a crystalline structure.

■ Problem 8.8. Fraction of escaping light from a plastic slab into air.

Here, we use Eqn. 8.15 which gives the fraction of light **trapped** in the slab. We are interested in finding the fraction of light that **escapes** from the slab, so we just take 1 minus Eqn. 8.15 to find this. We can assume that the slab is surrounded by air, so $n_1 \simeq 1$. We find the refractive index for plastic scintillators from Table 8.1 (which is $n_0$in our equation, or denoted as "n" below) to be 1.58. Thus, the equation for the fraction of light that escapes is:

$$f_{\text{escape}} = 1 - \sqrt{1 - \left(\frac{1}{n}\right)^2}$$

We substitute n = 1.58 to get the fraction of light that escapes from either slab surface from a plastic scintillator.

$$f_{\text{escape}} = 0.226$$

■ Problem 8.9. Photons generated with 1 MeV deposition in NaI.

Using data from p. 237, we calculate that 40,000 photons are created per MeV deposited in NaI(Tl) emitted isotropically from the following relation: $\frac{(\text{Scint.eff .for 1 MeV } \beta)\,(1\text{ MeV})}{(hc/\lambda)} = \frac{12\,\% * 1\text{ MeV}}{3\text{ eV}}$ which is close to the value of 38,000 in Table 8.3. The solid angle subtended by the pupil is $\Omega \sim \frac{A}{d^2}$ (look at Eqn. 4.22; we assume d>>a), and probability of detection (or the probability that an emitted photon hits the pupil in this case) is $\frac{\Omega}{4\pi}$. Therefore, the expected number of photons that will hit the pupil per event are (43,300 photons)*($\frac{\Omega}{4\pi}$). This is expressed below:

$$\textbf{Number of photons we expect to hit the pupil per event} = \frac{40\,000\,\pi\,(1.5\text{ mm})^2}{4\,\pi\,(10\text{ cm})^2} = 2.25$$

This is not sufficient for detection by the human eye (we need at least 10 photons to hit the pupil per event).

■ Problem 8.10. Light generation and transmission in a scintillating fiber.

(a). A 1 MeV electron passes across a 0.3 mm scintillating plastic fiber. Using data from Chapter 2 (Figure 2.14), the range (assuming a density of ~1 g/cm^3) is about 7 mm. Thus $\Delta E \sim (dE/dx)\,\Delta x$.
We note that for minimum ionizing particles such as fast electrons, the specific energy loss is about 2 MeV/cm (see Figure 2.1 and note that the density of air is 1.293 g/cm^3 at STP and that of plastic is about 1 g/cm^3). Therefore, the energy loss in .03 cm is about 60 keV.

(b). Using data from Table 8.5, we estimate about 10 photons/keV. Thus, the total number of scintillation photons is only ~600 photons.

(c). The fraction of captured light which is passed to one end (without attenuation) is calculated using the ratio of the indices of refraction of the core and clad materials. The equation that gives this fraction is expressed below (from pg. 266), where we will plug in the respective values of the indices of refraction.

$$F = \frac{1}{2}\left(1 - \frac{n_1}{n_0}\right)$$

We substitute $n_1 = 1.49$ and $n_0 = 1.58$ to get the fraction of captured light which is passed to one end (without attenuation).

$$F = 0.0285$$

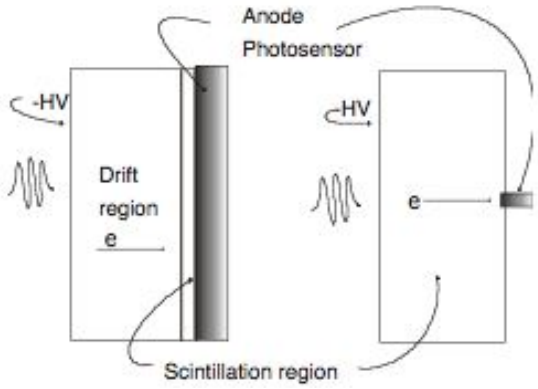
The light is attenuated exponentially along its travel direction, with an attenuation length of 2 m. We can estimate the number of photons emerging from one end after traveling 1 m using the equation $\frac{I}{I_0} = e^{-x/L}$ (from pg. 267), where x = 1 m and L = 2 m, and multiplying the result of this by the fraction calculated above and the number of photons originally created:

$$\textbf{scintillation photons arriving 1 } m \textbf{ from the point of interaction} = 600\,F\,e^{-\frac{1\,m}{2\,m}} = 10.4$$

Thus, our total light output arriving at our light sensor is only ~10 photons! Clearly, low-noise, high-efficiency photon detectors are needed to recognize such a small signal.

Supplemental Problem: Grid-less GPSC using single carrier approach

The gas proportional scintillation counter (GPSC) is commonly used in the detection of X-rays. In a conventional GPSC, the x-rays interact in a drift/absorption region having an electric field below the scintillation and ionization thresholds.



They drift past a grid into a thin higher electric field region where they then cause scintillation, but not ionization, which is detected by the anode photosensor. In the new gridless detectors, the entire volume is operated at a voltage where electrons cause scintillation during their entire travel distance, but the photosensor now is chosen to be small enough to subtend a small solid angle until the electron-induced scintillations occur close to the anode.

For both detectors, derive expressions for and sketch the shape of the observed anode signal as a function of the x-ray interaction position z_0 and time t. For simplicity you may assume: the electrons move directly to the anode with constant velocity v_0 , a constant light generation per electron path length k in the scintillation region, and a cylindrical photosensor of area A. For simplicity, you may assume $\Omega = \frac{A}{z(t)^2}$ for z > A.

The electrons move with velocity v after being formed at position z away from the photosensor. They isotropically produce scintillation light at a rate of k photons/path-length, which is detected by a cylindrical photosensor of area A and radius a. The sensed light per path length of electron travel is given by:

$S(z) = \frac{k\Omega}{4\pi}$

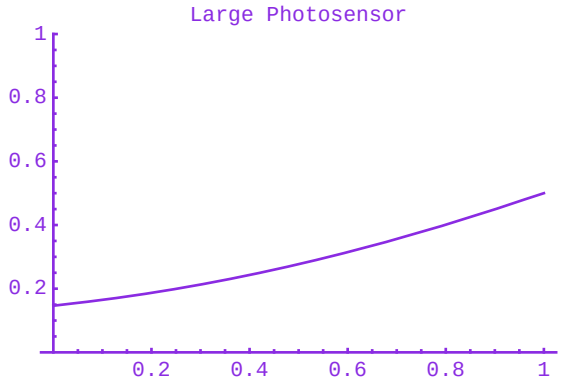
Assume that the electron is formed on the axis of the photosensor and moves directly to it, then

$$\frac{\Omega}{4\pi} = \frac{1}{2}\left(1 - \frac{d}{\sqrt{d^2+a^2}}\right) = \frac{1}{2}\left(1 - \frac{z-vt}{\sqrt{(z-vt)^2+a^2}}\right)$$

Since S(z)dz=S(t)dt, S(t)= v S(z). Let's assume some unit dimensions (k = v = z = a = 1), and write

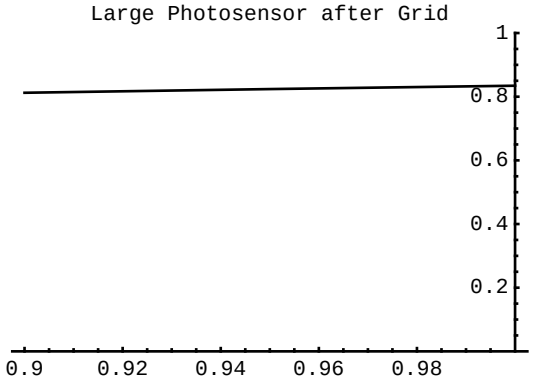
$$\boldsymbol{S(t) = \frac{1}{2}(k\,v)\left(1 - \frac{z-t\,v}{\sqrt{a^2+(z-t\,v)^2}}\right)}$$

First, ignore the grid and assume that the electron scintillates all the way to the photosensor (i.e., choose z~a). A plot of S(t) yields:



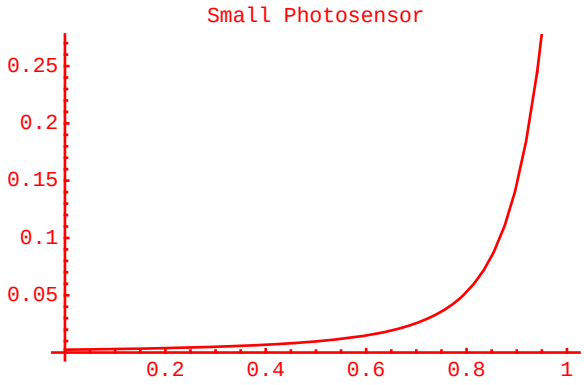
We have a pulse that increases from z_0 to the end. The pulse will be dependent upon the electron's formation in the chamber.

Next, assume that the electron only radiates only after passing through the grid, so z<<a. A plot of the output for distance greater than 90% is:

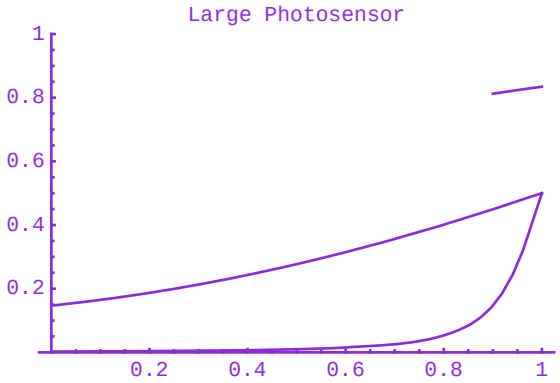


This is like a Frisch grid ionization chamber. Any electron first travels to the grid, then produces its signal in the small region between grid and photosensor. Therefore, each electron, no matter where it is formed, creates the same size pulse. This is what is needed for spectroscopy. The disadvantage of a grid is that it is typically noisy, and there may be some electrons that are initially formed in between the grid and photosensor.

Finally, assume a small photosensor without a grid. We assume a=0.1, and the full travel distance. The S(t) output looks like:



This is analogous to the "single carrier approach" that is popular today. The electron produces most of its signal only when it is very close to being collected, therefore the signal is largely independent of the formation position of the electron. Combining our results on one plot:



□ Further exercises:

1. For the small photosensor, let a=.01 and see how your results change.
2. From the results, which of these 3 approaches will have better:

a. energy resolution (Consider the size of the signal and whether the signal is dependent on the position of interaction).
b. timing resolution (shorter and larger pulses have better timing)
c. spatial resolution (in the z-direction)

PM Tube and PD Problems

■ Problem 9.1. Cutoff wavelength due to photocathode work function.

There will be no photoelectric effect if the incoming photon energy is less than the work function of the material, i.e. the energy of the photon must be at least equal to the work function (in this case, 1.5 eV). To solve this problem, we set the energy of the incoming photon equal to the work function and solve our equation for the photon wavelength (denoted "λ").

$$\frac{\textbf{(Planck's Constant)} \times c}{\lambda} = \textbf{work function}$$

We solve the above equation for the photon wavelength (λ) and substitute work function =1.5 eV to get the maximum photon wavelength (long-wavelength limit) in angstroms.

$$\lambda = 8270 \text{ angstroms}$$

■ Problem 9.2. Interdynode transit time calculation.

Recall that for a constant acceleration, the position of a particle with respect to time is:
$x(t) = \frac{1}{2}\, a\, t^2 + v_0\, t + x_0$.

We solve this for t assuming $a = \frac{F}{m_{\text{electron}}} = \frac{(q_{\text{electron}})(E)}{m_{\text{electron}}} = \frac{(q_{\text{electron}})(V)}{(m_{\text{electron}})(\Delta x)}$.

We assume that x_0 and v_0 both are zero, so $t = \sqrt{\frac{2\,\Delta x}{a}}$.

This is expressed below, where we substitute $a = \frac{(\text{Electron Charge})\ 150 \text{ volts}}{\Delta x\ (\text{Electron Mass})}$ and Δx = 12 mm to get the electron transit time between dynodes in seconds:

$$t = \sqrt{\frac{2\,\Delta x}{a}} = 3.304 \times 10^{-9} \text{ seconds} = 3.304 \text{ ns}$$

This is quick -- only a few nanoseconds between dynodes.

■ Problem 9.3. Multistage gain.

We are given that $\delta^N = 10^6 = \delta^6$ (because we know that N = 6), which means that $\delta = 10$. From Fig. 9.3, we find that the primary electron energy must be ~ 200 eV. Therefore, the voltage between dynodes (or the voltage per stage) must be 200 V, which means that we would need 1200 V across the entire tube.

■ Problem 9.4. Dark current.

We calculate the electron emission rate from:

I_{dark}= [(electron emission rate)*(PM tube gain)*(charge of single electron)]

Below, this equation has been solved for the electron emission rate.

$$\textbf{electron emission rate} = \frac{\textbf{current}}{\textbf{gain (Electron Charge)}}$$

We substitute gain = $10^6$and current = 2×10^{-9} amperes to get the electron emission rate in $\sec^{-1}$:

electron emission rate = 12 480 $\sec^{-1}$

■ Problem 9.5. Gain change with voltage fluctuation.

A 10-stage tube with each stage having a multiplication $\delta = V_S^{0.6} = \left(\frac{V}{N}\right)^{0.6}$ operates with V= 1000 Volts ($V_S$is the interdynode voltage). Since N=10, we have $\delta = \left(\frac{V}{10}\right)^{0.6}$. Therefore, the total gain of the tube is $G = \delta^N = \left[\left(\frac{V}{10}\right)^{0.6}\right]^N = \left[\left(\frac{V}{10}\right)^{0.6}\right]^{10} = 10^{-6}\, V^6$. The gain fluctuation (expressed as "ΔG") depends on voltage fluctuation (expressed as "ΔV") by taking the differential of the total gain with respect to the voltage (i.e. we assume that $\frac{\Delta G}{\Delta V} = \frac{dG}{dV}$ for the small change in gain we are considering). This is shown below, where we have solved for ΔG.

$$\Delta G = \frac{\partial\left(\frac{V^6}{10^6}\right)}{\partial V}\Delta V = 6 \times 10^{-6}\, V^5\, \Delta V$$

We are interested in finding ΔG/G for V=1000 volts. (Note that ΔG = 6 x 10^9ΔV --- so tiny changes in voltage yield large changes in gain).

Looking at the relative change in gain for V=1000 V:

$$\frac{\Delta G}{G} = \frac{\Delta G}{\left(\frac{V^6}{10^6}\right)} = \frac{6\, x\, 10^{-6}\, V^5\, \Delta V}{\left(\frac{V^6}{10^6}\right)} = 6\, x\, 10^{-3}\, \Delta V$$

We want to solve this equation for the ΔV which yields a relative uncertainty in the gain of 1% (i.e. ΔG/G = 0.01). The voltage fluctuation (in volts) that can be tolerated if the gain is not to change by more that 1% is thus:

$$6\, x\, 10^{-3}\, \Delta V = 0.01$$

or solving for ΔV

ΔV = 1.67 volts

This implies that the voltage must be held constant to 1 part in 10^3 to keep the gain constant to within 1%.

Problem 9.6. Find max voltage for NaI pulse with various RC.

In order to get an equation for V(t), we must solve the differential equation shown below (Eqn. 9.13). To avoid any confusion, $\theta = \frac{1}{RC}$. Below, we solve the differential equation for "v(t)" taking "t" as the independent variable and giving the initial value of v(0) = 0.

$$v'(t) + \theta\, v(t) = \frac{\lambda q}{c}\, e^{-\lambda t} \qquad \text{where } v(0) = 0$$

which has a solution of:

$$v(t) = \frac{\left(-e^{\lambda t} + e^{\theta t}\right) \lambda q}{e^{t(\lambda+\theta)}\,(c\theta - c\lambda)}$$

We need to find the time when the pulse voltage reaches a maximum, so we differentiate v with respect to t and set it equal to zero.

$$\frac{\partial v}{\partial t} = \frac{\lambda q \left(e^{\theta t}\,\theta - e^{\lambda t}\,\lambda\right)}{e^{t(\lambda+\theta)}\,(c\,\theta - c\,\lambda)} + \frac{\left(-e^{\lambda t} + e^{\theta t}\right) \lambda q\,(-\lambda - \theta)}{e^{t(\lambda+\theta)}\,(c\,\theta - c\,\lambda)} = 0$$

This solution is not in a very convenient form for t, so we rearrange the equation in the following several steps. First, we multiply both sides by $e^{t(\lambda+\theta)}$ and simplify to cancel out the $e^{t(\lambda+\theta)}$terms . Our new equation is thus:

$$\frac{\lambda q \left(e^{\lambda t}\,\theta - e^{\theta t}\,\lambda\right)}{c\,\theta - c\,\lambda} = 0$$

Next, we multiply both sides by the factor $\frac{(c\,\theta - c\,\lambda)}{\lambda q}$ to get our equation with (c θ - c λ) canceled out of the denominator and λq canceled out of the numerator. This yields:

$$e^{\lambda t}\,\theta - e^{\theta t}\,\lambda = 0$$

Now, we add $e^{\theta t}\,\lambda$ to both sides and take the natural log of both sides:

$$\ln\left(e^{\lambda t}\,\theta\right) = \ln\left(e^{\theta t}\,\lambda\right)$$

A simplified version of the previous result is thus:

$$\lambda t + \ln(\theta) = \theta t + \ln(\lambda)$$

Now we solve for t to obtain an equation for the time when the pulse reaches its maximum.

$$t_{max} = -\frac{\ln(\theta) - \ln(\lambda)}{\lambda - \theta}$$

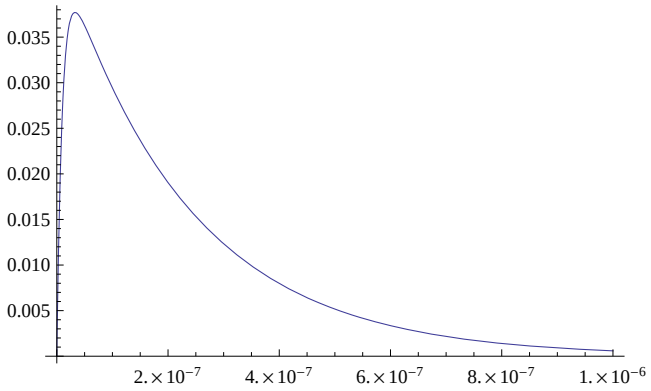
We now evaluate the voltage at this time (i.e. the time at which v(t) is maximum) for the cases given in the problem. Note that we have assumed Q=C=1 so this is the voltage per unit Q/C (this is the same as saying that the result will be the ratio of $v_{max}$to the maximum that would be observed for an infinite time constant, which is what is asked for). First, we evaluate this for $\frac{1}{\theta}$= RC = 10 ns. Then, we plot v(t) from time 0 to 10^{-6}seconds.

$$v(t) = \frac{\left(-e^{\lambda t} + e^{\theta t}\right)\lambda q}{e^{t(\lambda+\theta)}(c\theta - c\lambda)}$$

We substitute $t = -\frac{\ln(\theta)-\ln(\lambda)}{\lambda-\theta}$, $\lambda = \frac{1}{230}$,and θ = .1 to get the maximum of v(t) (in volts per unit Q/C) for RC = 10 ns.

$$v = 0.037703 \text{ volts per unit } Q/C$$

A plot of v(t) for $0 < t < 10^{-6}$seconds (RC = 10 ns) is thus:



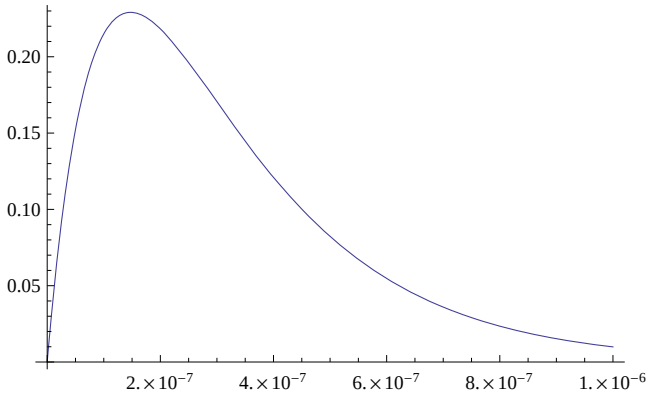
Now we do the same for RC = 100 ns. Using our equation for v(t):

$$v(t) = \frac{\left(-e^{\lambda t} + e^{\theta t}\right)\lambda q}{e^{t(\lambda+\theta)}(c\theta - c\lambda)}$$

we substitute $t = -\frac{\ln(\theta)-\ln(\lambda)}{\lambda-\theta}$, $\lambda = \frac{1}{230}$,and θ = .01 to get the maximum of v(t) (in volts per unit Q/C) for RC = 100 ns.

$$v = 0.229 \text{ volts per unit } Q/C$$

A plot of v(t) for $0 < t < 10^{-6}$seconds (RC = 100 ns) is thus:



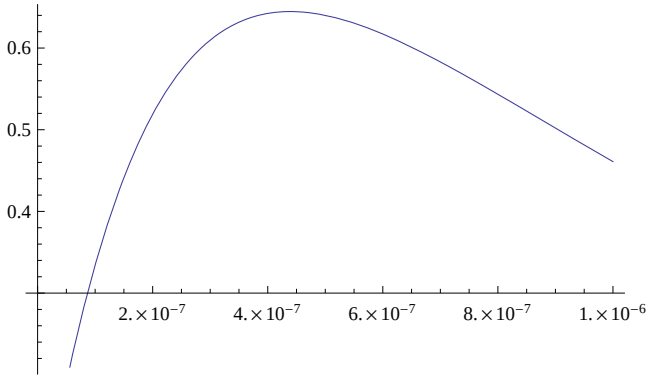
And now we do the same for RC = 1000 ns. Using our equation for v(t) again:

$$v(t) = \frac{\left(-e^{\lambda t} + e^{\theta t}\right)\lambda q}{e^{t(\lambda+\theta)}(c\theta - c\lambda)}$$

We substitute $t = -\frac{\ln(\theta)-\ln(\lambda)}{\lambda-\theta}$, $\lambda = \frac{1}{230}$,and θ = .001 to get the maximum of v(t) (in volts per unit Q/C) for RC = 1000 ns.

v = 0.645 volts per unit Q/C

A plot of v(t) for $0 < t < 10^{-6}$seconds (RC = 1000 ns) is thus:



Now we answer the question: for what value of θ will V = 0.9 $\left(\frac{Q}{C}\right)$? Here, we set v equal to 0.9 (since v is the ratio between the voltage and $\frac{Q}{C}$), give the condition that t is equal to the time at which maximum voltage is achieved, and plug in the value $\lambda = \frac{1}{230}$.

$$\frac{\lambda q \left(e^{\lambda t} \theta - e^{\theta t} \lambda\right)}{c\theta - c\lambda} = 0.9$$

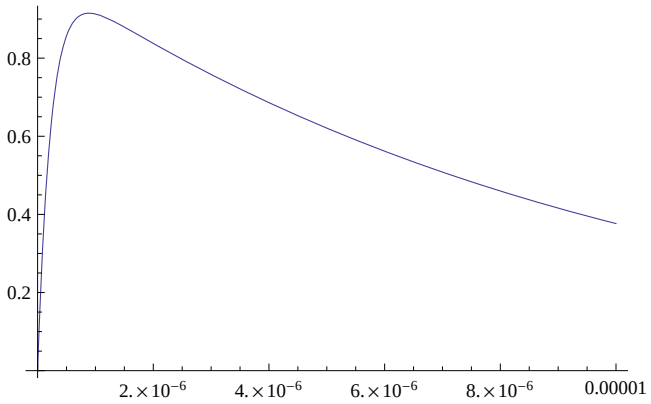
We substitute known values into the above equation and solve for θ to get the value of θ (in ns^{-1}) that gives the ratio $\frac{V}{Q/C}$= 0.9.

θ = 0.000125 ns^{-1}

We are interested in the time constant, RC, so we calculate the inverse of θ. The value of RC (in ns) that gives the ratio $\frac{V}{Q/C}$= 0.9 is thus:

$$\mathbf{RC} = \frac{1}{\theta} = 7969.25 \text{ ns} = 7.97\ \mu\text{s}$$

Note what this problem shows us. As RC increases, the maximum voltage increases because the charge integration time increases. When RC = 7967 ns >> τ = 230 ns (the decay time for scintillations of NaI(Tl)), the maximum pulse amplitude is 90% of the maximum value Q/C. For fun, let's look at the shape of v(t) (for RC= 10 microseconds). In this plot, t is in seconds, so the abscissa runs from 0 to 10 microseconds.



The key is to note that as RC increases, $\frac{V_{max}}{Q/C}$ --> 1, but the time to the maximum pulse height also increases. So we frequently have to make a trade off when we want fast response for timing or high count rates (i.e. small RC), yet also wanting large pulses (i.e. large RC, long integration time) to minimize the effects of ballistic deficit and yield high accuracy spectroscopy.

■ Problem 9.7. Voltage response of a finite step current into parallel RC circuit.

We must first solve the differential circuit equation (Eqn. 9.12) for v(t). This is expressed below, where we make "t" the independent variable and we give the initial value v(0) = 0 (note that we wrote "I_0" instead of "I(t)" because I is a constant until it abruptly changes to 0 at t=T). (In this problem v_1 = v(t) from 0 < t < T, and v_2 = v(t) from T < t)

$$\frac{v_1(t)}{\tau} + v_1'(t) = \frac{I_0}{c} \qquad \text{where } v_1(0) = 0$$

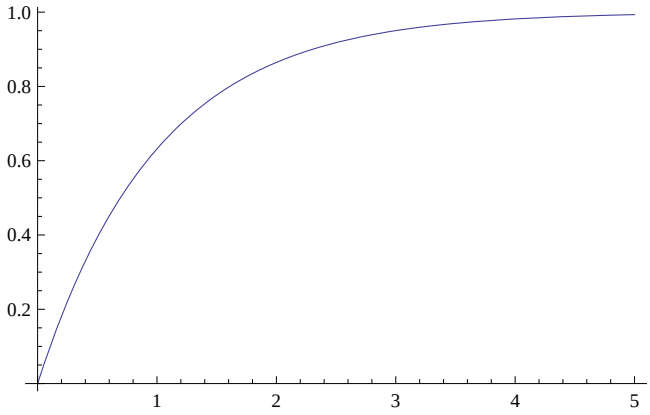
The solution for v(t) is thus:

$$v_1(t) = \frac{I_0\,\tau}{c} - \frac{I_0\,\tau}{c\,e^{t/\tau}}$$

We factor the right hand side of v(t) so we have one term instead of two.

$$v_1(t) = \frac{(-1 + e^{t/\tau})\,I_0\,\tau}{c\,e^{t/\tau}}$$

Now, we plot v(t) for 0 < t < T (we set T = 5 and τ = RC = 1).



After the current stops flowing at T=5, the differential equation has i(t) =0, and the initial condition that the voltage matches at t=5 (i.e. $v_1(t = 5) = v_2(t=5)$ is our initial voltage value for t = T). Below, we solve the differential equation given these conditions.

$$\frac{v_2(t)}{\tau} + v_2'(t) = 0 \qquad \text{where } v_2(5) = v_1(5)$$

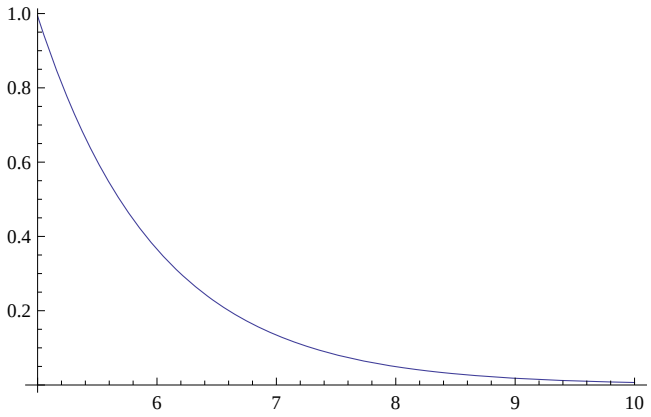
The solution for v(t) for t >T is thus:

$$v_2(t) = e^{-\frac{t}{\tau} + \frac{5}{\tau} - 5}\left(-1 + e^5\right)$$

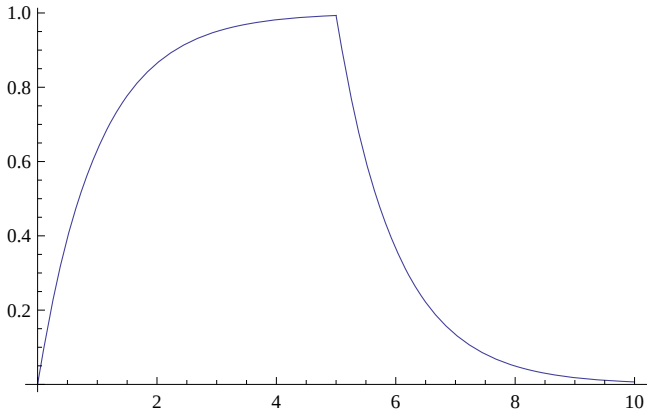
The solution for $v_2(t)$ in terms of t with τ = 1 (t > T).

$$v_2(t) = \frac{-1 + e^5}{e^t}$$

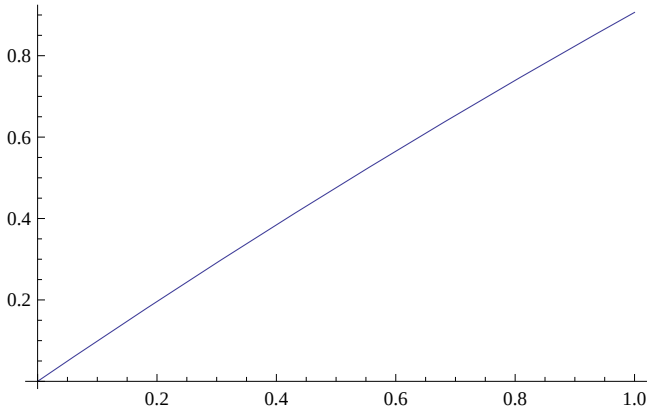
Now we plot v(t) for t > T (T = 5 and τ = RC = 1).



Next, we put the two solutions together on one plot to give the time profile of v(t) with τ = RC = 1 and T = 5 (and C = 1, I = 1 for 0 < t < T, and I = 0 for t > T).



Now, we switch the values of τ = RC and T (i.e. RC = 5, T=1). We plot $v_1(t)$ for 0 < t < T, where T = 1.



As before, we solve the differential equation for v(t) for t > T, giving the initial value of v(T) = v(1) from the previous plot. In this part of the problem T = 1, I_0 = 1and c = 1.

$$\frac{v_2(t)}{\tau} + v_2'(t) = 0 \qquad \text{where } v_2(1) = v_1(1)$$

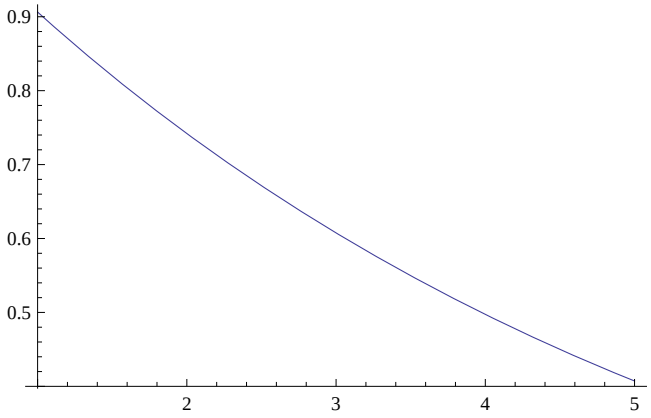
The solution for v(t) for t >T is thus:

$$v_2(t) = 5\left(-1 + \sqrt[5]{e}\right) e^{-\frac{t}{\tau} - \frac{1}{5} + \frac{1}{\tau}} = 5\left(1 - e^{-0.2}\right) e^{-(t-1)/\tau}$$

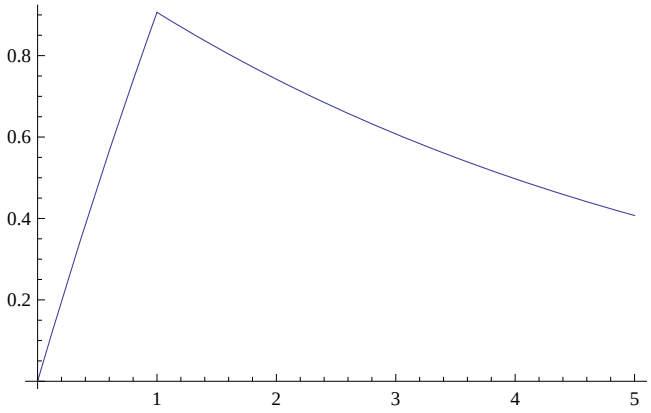
The solution for v(t) for t > T in terms of t with τ = RC = 5.

$$v_2(t) = \frac{5\left(-1 + \sqrt[5]{e}\right)}{e^{t/5}}$$

We now plot v(t) for t > T (τ = RC = 5 and T = 1).



Again, we put the two sections of the voltage-time profile together on the same plot. A plot of the voltage-time profile for τ = RC = 5 and T = 1 (this is actually what part (a) was asking for, i.e. RC >> T) is thus:



What we see here is a direct demonstration of the output pulse waveform for RC >> $t_{collection}$ and RC << $t_{collection}$. In practice, the actual wave forms are smoothed out, but have this general form. It is clear that we usually would want RC >> $t_{collection}$ so that we can accurately measure V_{max}.

Problem 9.8. Expected pulse height from a NaI(Tl)/PM tube combination.

The expected pulse height is just V = Q/C (if RC >> 1/λ), where Q = n_0e, and n_0= # electrons collected at anode, and e = electron charge.

Remember that:

n_0= (# photons created)*(light collection eff.)*(photocathode quantum eff.)*(PM gain)

(# photons created) = $\frac{(\text{energy deposited})*(\text{scintillation eff.})}{\text{light energy per photon}}$, where the scintillation efficiency = light energy created per energy deposited,
so we have

(# photons created) = (energy deposited)*(# photons created per MeV deposited in NaI(Tl))
= (energy deposited)*(38,500 photons created per MeV deposited in NaI(Tl))

Now, we know that RC = (10^5 Ω)*(100 x 10^{-12}F) = 10 μs >> τ= 1/λ = μs, so our approximation is valid. We simply plug our values into the expressions above and write the answers in sentence form below.

$$\textbf{photons} = E_{\text{dep}} \times 38\,500 \text{ photons / MeV}$$

$$n_0 = \text{photons} \times \text{light collection efficiency} \times \text{quantum efficiency} \times \text{PMgain}$$

$$\text{pulseheight} = \frac{n_0(\text{ElectronCharge})}{C}$$

We substitute E_{dep} = 1.2, light collection efficiency = 0.7, quantum efficiency = 0.2, PMgain =$10^5$and C= 100 pF into the above equations to find that:

The energy deposited yields **46 200 photons** and **6.47 x10^8 electrons** and yields a pulse height of **1.036 Volts**.

Problem 9.9. Microchannel plate primary advantage

Microchannel plates (MCPs) are very popular readouts for a number of reasons. They are compact and less sensitive to magnetic fields than traditional PMTs, but their primary advantage lies in their timing resolution. Because they are quite thin, the electron transit time is much shorter than in a traditional PMT and therefore the spread in arrival times at the anode, the factor which governs timing resolution, is much narrower.

■ Problem 9.10. A longer wavelength cutoff for silicon.

This is a repeat of Problem 9.1. Substituting the workfunction = 1.11 eV in that expression yields:

λ= 11,271 Angstroms = 11, 271 x 10^{-10} m = 1,127 nm

What do we learn from this problem? Silicon won't see radiation with longer wavelengths than 11,271 Angstroms or 1,127 nm. This is well within the emission band of most scintillators (see Figure 8.7 page 237 of the text). Thus, silicon is a logical candidate for our light converter. And, comparing the result to the answer to Problem 9.1, silicon has a larger cutoff, thus being able to see longer wavelength radiations, such as from CsI. Since CsI emits more photons per energy deposited, this implies potentially better energy resolution if the light converter can convert these extra photons to electrons.

■ Problem 9.11. Scintillator-photodiode current calculation.

To solve this, we first calculate the number of photons generated per time (alpha particle energy multiplied by a scintillation efficiency of 3%, and divided by average photon energy, which is then multiplied by the alpha particle flux, or "rate"). We then multiply that by the fraction of photons collected (light collection efficiency, or "ϵ_L") and the fraction of photons generating photoelectrons (quantum efficiency, or "QE"). Finally, we take this result and multiply it by the charge of a single photoelectron to obtain the photodiode current. This is shown below.

$$\textbf{signal} = \frac{\textbf{(3 \%) (5 MeV)}}{\textbf{(Planck's Constant)}\left(\frac{c}{\lambda}\right)} \times \textbf{rate} \times \epsilon_L \times \textbf{QE} \times \textbf{(Electron Charge)}$$

We substitute $\epsilon_L = 80\ \%$, $\lambda = 420$ nm, QE = 75 % and rate = 10^6 $\text{sec}^{-1}$to get the expected signal (or photodiode current) when the photodiode is operated in current mode.

$$\textbf{signal} = 4.88 \times 10^{-9}\ \textbf{amperes} = \textbf{4.88 nA}$$

Note that we can expect about a nanoamp per MeV deposited in the detector-PD combination. This is a very small current to measure, but can be done.

■ Problem 9.12. Hybrid photodiode voltages.

We wish to estimate the acceleration voltage required so that each electron from the photocathode will have enough energy for a gain of 5000. From the text (p. 312), we know that a 10 keV electron yields a gain of 2800. From this, we can calculate the photoelectron energy required per additional electron produced (in eV/electron). This is done below.

$$\textbf{Energy} = \frac{\textbf{10 keV}}{\textbf{2800 electrons}} = \frac{\textbf{3.57143 eV}}{\textbf{electron}}$$

So to get 5000 electrons from a single photoelectron, we simply multiply our ratio from above by 5000. An estimate of the single photoelectron energy required to produce a gain of 5000 for a hybrid photomultiplier tube using a silicon diode is thus:

$$\textbf{Single Photoelectron Energy} = \frac{\textbf{3.57143 eV}}{\textbf{electron}} \times \textbf{5000 electrons} = \textbf{17 857 eV} = \textbf{17.9 keV}$$

From this, we know that we must apply about 18 kV to the electron in order to get a gain of 5000. This is a very large voltage that needs to be sustained over a relatively small distance.

Spectroscopy with Scintillator problems

■ Compton scattering relations.

The energy relations for Compton scattering play a role in some of the following problems. We have defined them here for future use. The scattered photon energy hν' is denoted as "E_p," the energy of the Compton electron is denoted as "E_e," and the incident photon energy hν is denoted as "E_0." Using these definitions, our key relationships for Compton scattering are given by:

$$E_p(E_0,\,\theta) = \frac{E_0}{\frac{E_0\,(1-\cos(\theta))}{m_e\,c^2} + 1}$$

$$E_e(E_0,\,\theta) = E_0 - E_p(E_0,\,\theta)$$

■ Problem 10.1. Find E_0 given hν' and θ in Compton scattering.

We are given the energy of the scattered photon hν' (or "E_p" from the relationships above), so all we need to do is solve for E_0using our known values.

$$E_p = \frac{E_0}{\frac{E_0\,(1-\cos(\theta))}{m_e\,c^2} + 1}$$

We substitute θ=90° and E_p=0.5 MeV into the above equation and solve for $E_0$to get the value of E_0(= hν) in eV.

$$E_0 = 2.32\times10^7\ \text{eV}$$

■ Problem 10.2. Two Compton scatterings: is order irrelevant?

To find the energy of a photon scattered twice, we simply apply the relationship for the scattered photon energy ($E_p(E_0, \theta)$) twice. The scattered photon energy in eV (for a scattering order of 30 degrees, then 60 degrees) is thus:

$$E_p\left(E_p(2\,\text{MeV},\,30\,°),\,60\,°\right) = 0.574\,\text{MeV}$$

The deposited energy in the material is just the initial energy minus the final energy of the gamma ray. The total energy deposited in the detector is thus:

$$E_{\text{dep}} = 2\,\text{MeV} - 0.574496\,\text{MeV} = 1.43\,\text{MeV}$$

To see if the result changes when the order of scattering is reversed, we simply apply the same analysis in reversed order. The scattered photon energy in eV (for a scattering order of 60 degrees, then 30 degrees) is thus:

$$E_p\left(E_p(2\,\text{MeV},\,60\,°),\,30\,°\right) = 0.574\,\text{MeV}$$

Since the final energy of the gamma ray is unchanged by the order, there is no difference in the energy deposited.

Is this a coincidence? We try a different combination of angles, 10 and 90 degrees, below.

$$E_p\left(E_p(2\,\text{MeV},\,10\,°),\,90\,°\right) = 0.402\,\text{MeV}$$

$$E_p\left(E_p(2\,\text{MeV},\ 90\,°),\ 10\,°\right) = 0.402\,\text{MeV}$$

We can see that the deposited energy doesn't change for scattering angles of 90 degrees and 10 degrees when the order is changed.

The key idea here is that the order of the scatterings is irrelevant for the total deposited energy. Can we prove this analytically? Using $E_{dep} = E_0 - E_p$, one can analytically prove that the energy deposited is given by the relationship below. Note that the order of θ_1 and θ_2 has no influence on the answer.

$$E_{\text{dep}} = \frac{E_0^{\,2}\,(-\cos(\theta_1) - \cos(\theta_2) + 2)}{(m_e * c^2)\left(\frac{E_0\,(-\cos(\theta_1)-\cos(\theta_2)+2)}{m_e * c^2} + 1\right)}$$

- **Problem 10.3. Maximum energy deposited by two Compton scatterings.**

The largest deposited energy occurs for a 180 degree scattering. Here, we calculate the final photon energy for two 180 degree scatterings.

$$E_p\left(E_p(1\,\text{MeV},\ 180\,°),\ 180\,°\right) = 0.113\,\text{MeV}$$

The deposited energy is given by the initial photon energy minus the final photon energy. The maximum deposited energy by a 1 MeV gamma ray from two successive Compton scatterings is thus:

$$E_{\text{dep}} = 1\,\text{MeV} - 0.113278\,\text{MeV} = 0.887\,\text{MeV}$$

Note that the photon has lost nearly **90%** of its energy with just these two scatterings.

- **Problem 10.4. Scattered photon travel time compared to scintillation decay time.**

The question we are really asking here is whether we can identify two separate interactions in the scintillator by virtue of the fact that they occur at different times. The time difference between events is the scattered photon travel time to the second event (i.e. the distance between events divided by the gamma-ray velocity in NaI(Tl)). Is this time large compared to the individual scintillation times? If so, we could resolve them independently in time. The calculation for the time between scatterings is shown here (where we have included the fact that $v_{\text{gamma in NaI (Tl)}} = \frac{\text{speed of light}}{\text{index of refraction}}$, where the index of refraction for NaI(Tl) is 1.85). The time between two gamma-ray scattering interactions that are 3 cm apart is thus:

$$t = \frac{3\,\text{cm}}{\left(\frac{c}{1.85}\right)} = .185\,\text{nano seconds}$$

This travel time is very short compared to the characteristic scintillation decay time in Sodium Iodide of 230 ns. *The two events cannot be separated in time by virtue of the travel time between the scatterings.* This also tells us that, on the time scale of the scintillation process, the photons appear almost instantaneously at the photocathode/photodiode.

- **Problem 10.5. Predicted peak-to-total ratio based on attenuation coefficients.**

The peak-to-total ratio will be larger than the ratio of attenuation coefficients because of histories that may begin with Compton scattering or pair production, but for which the full energy is eventually absorbed. For small detectors, it will be closer to the attenuation coefficient ratios, whereas it will be larger for larger detectors.

Problem 10.6. Resolution extrapolated to other energies.

Recall that energy resolution, R(E), is limited by the statistical fluctuation in the number of carriers -- i.e., the physics will never let us do better than this resolution. We also recall that R(E) is proportional to $\frac{1}{\sqrt{E}}$, so $\frac{R_2}{R_1} = \sqrt{\frac{E_1}{E_2}}$, or:

$$R_2 = R_1 \sqrt{\frac{E_1}{E_2}}$$

We substitute E_1=0.662 MeV, E_2=1.28 MeV and R_1=0.07 to get the estimated energy resolution (expressed as a fraction, not a percent) of a particular NaI(Tl) scintillator for the 1.28 MeV Na^{22} gamma-rays:

$$R_2 = 0.0503$$

■ Problem 10.7. Intrinsic total efficiency of NaI(Tl) at 0.5 MeV.

Since any interaction will give some signal, the total efficiency counts every gamma-ray which has had some interaction in the detector. Since we know that $\frac{I}{I_0}$ is the percentage of photons that don't interact in an absorber (i.e. the detector) , we know that 1-$\frac{I}{I_0}$ is the percentage that do interact in the detector, which is the same as the intrinsic total efficiency. We are given the mass attenuation coefficient $\left(\frac{\mu}{\rho}\right)$ and the thickness of the detector, and the density of NaI(Tl) is found from the specific gravity in Table 8.3 to be 3.67 g/cm^3, so the intrinsic total efficiency can be calculated as:

$$\text{efficiency}_{\text{total}} = 1 - e^{-\frac{\mu \rho t}{\rho}}$$

We substitute μ = 0.955 ρ, ρ = 3.67, and t = 0.5 to get the intrinsic total efficiency of a 0.5 cm thick NaI(Tl) slab detector at a gamma-ray energy of 0.5 MeV (expressed as a fraction, not a percent).

$$\text{efficiency}_{\text{total}} = 0.827$$

Since the photofraction is the fraction of all counts in the spectrum which lie under the peak, the intrinsic peak efficiency is the intrinsic total efficiency multiplied by the photofraction. The intrinsic peak efficiency of the same detector under the same conditions (expressed as a fraction, not a percent) is thus:

$$\text{efficiency}_{\text{peak}} = 0.826647 \times 0.4 = 0.331$$

This is rather informative. It says that for 500 keV, nearly 80% of the incident gammas will have some interaction in the 0.5 cm thick detector, and nearly half of these will deposit all of their energy in the scintillator. As a result, nearly 1/3 of incident gammas at 500 keV will deposit all their energy in the scintillator. This is a high efficiency detector.

■ Problem 10.8. Compton edge location.

(a). The Compton edge occurs at the energy corresponding to when the incident gamma-ray backscatters in the material (i.e. scatters at an angle, θ, of 180 degrees, thereby depositing the most energy possible through the Compton interaction). Using the Compton scattering relations defined previously, we calculate this maximum energy deposition below.

$$E_c = E_0 - \frac{E_0}{\frac{E_0\,(1-\cos(\theta))}{m_e\,c^2} + 1}$$

We substitute E_0= 1.17 MeV and θ = 180° to get the energy of the Compton edge in MeV.

$$\boldsymbol{E_c = 0.960\ \text{MeV}}$$

(b). The backscatter peak is due to backscatter from surrounding materials and usually occurs roughly around 0.25 MeV. We want to see how this varies for initial energies of 1, 2, and 3 MeV. To do this, we again use the equation for the scattered photon (E_p), using a scattering angle of 180 degrees. This calculation is shown below, where we show the results for 1, 2, and 3 MeV, respectively. Using

$$E_p(E_0,\,\theta) = \frac{E_0}{\frac{E_0\,(1-\cos(\theta))}{m_e\,c^2} + 1}$$

We substitute θ = 180° and evaluate to get the backscatter peak energies for incident gamma-ray energies E_0 of 1, 2, and 3 MeV, respectively;

$$\boldsymbol{E_p(1\ \text{MeV}) = 0.204\ \text{MeV}}$$
$$\boldsymbol{E_p(2\ \text{MeV}) = 0.227\ \text{MeV}}$$
$$\boldsymbol{E_p(3\ \text{MeV}) = 0.235\ \text{MeV}}$$

Note that indeed the backscatter peak doesn't vary much as the incident energy changes from 1 to 3 MeV.

■ Problem 10.9. Factors relating to peak efficiency and energy resolution.

Factors which influence intrinsic peak efficiency are those which affect the capture of the the full energy of the radiation in the detector: (a) density of detector, (c) atomic number.

Factors which influence energy resolution are those which affect the number of information carriers: (b) kinetic energy required to create an information carrier (i.e. photon), (e) gain of PM tube (multiplication statistics degrade energy resolution slightly, see text pg. 344), (f) quantum efficiency of photocathode, (h) light collection efficiency.

The amplifier gain has no effect on either. It may introduce a small amount of noise.

The source-detector geometry has no first-order effect on the *intrinsic* peak efficiency.

■ Problem 10.10. Expected pulse height from 1 MeV deposition for NaI(Tl)/PM tube combination.

The problem is nearly identical to Problem 9.8 except that we now include the 1st dynode efficiency since it is given as part of the problem. Also note that τ=RC is identical to what it was in Problem 9.8, and since we concluded that V=Q/C was valid there, it is still valid here. **(**As in Problem 9.8, we expect the answer to be slightly different if using a scintillation efficiency from pg. 237 and energy/photon instead of our 38,500 photons/MeV).

$$\textbf{Number of photons} = E_{\text{dep}} \times (38\,500\ \text{photons/MeV}) = 38\,500\ \text{photons}$$

so the number of electrons generated is:

$$n_0 = \text{(collection efficiency) (efficiency quantum) (gain PM) (dynode efficiency first) (number of photons)} = 2.93732 \times 10^7 \text{ electrons}$$

and the pulse height will be:

$$\text{pulseheight} = \frac{n_0 \text{ (Charge Electron)}}{C} = 0.047060 \text{ Volts}$$

We substituted E_{dep}=1.0 MeV, collection efficiency = 0.50, quantum efficiency = 0.2, first dynode efficiency = 0.8, PM gain = $2.5^{10}$and C=100 pF into the above equations to find that the energy deposited yields 38500 photons, 2.94×10^7electrons, and a pulse height of 0.0471 volts.

■ Problem 10.11. True and chance peak count coincidences.

This problem looks at the area under the sum peak compared to a single peak for a source emitting two uncorrelated gamma-rays. We first recall our expression for solid angle from Chapter 4 below:

$$\Omega[d, a] = 2\pi\left(1 - \frac{d}{\sqrt{a^2 + d^2}}\right)$$

(a). The ratio we want is the number of true coincident full-energy (i.e. sum) counts divided by the number of counts under the γ_1 peak. Note that the number of counts under the γ_1 peak is reduced by the fact that some of the γ_1's are lost due to the coincident summing. Hence, the ratio we want is:

$$\frac{N_{12}}{N_1|_{\text{with summation}}} = \frac{N_{12}}{N_1 - N_{12}} = \frac{S\epsilon_1\,\epsilon_2\,y_1\,y_2\,{\Omega_f}^2}{S\epsilon_1\,y_1\,\Omega_f\left(1-\epsilon_2\,y_2\,\Omega_f\right)} \text{(from Eqns. 10.11 and 10.12)} = \frac{\epsilon_2\,y_2\,\Omega_f}{\left(1-\epsilon_2\,y_2\,\Omega_f\right)},$$

where $\epsilon_2$is the intrinsic peak efficiency of γ_2, $y_2$is the yield of γ_2 ($y_1$and $y_2$are both 100% in this case, so we need not worry about them in computations), and $\Omega_f$is the fractional solid angle (i.e. $\frac{\Omega}{4\pi}$). This calculation is shown below, where $\epsilon_{2,abs}$= $\epsilon_2\,y_2\,\Omega_f$(absolute peak efficiency).

$$\text{ratio} = \frac{\epsilon_{2,abs}}{1 - \epsilon_{2,abs}}$$

We substitute $\epsilon_{2,abs} = \frac{\Omega(10,5)\,\epsilon_2}{4\pi}$ and $\epsilon_2 = 0.3$to get the ratio of counts under the sum peak (N_{12})to the counts under the $\gamma_1$peak ($N_1|_{\text{with summation}}$, or $N_1 - N_{12}$).

ratio = 0.0161

(b). We first want to find the sum peak count rate given S = 100 kBq. To do this, we simply use Eqn. 10.11 (noting, again, that the yields, $y_1$and y_2, both equal 100%). This calculation is shown below:

$$\text{rate} = S * \epsilon_{2,abs} * \epsilon_{1,abs}$$

We substitute $\epsilon_{2,abs} = \frac{\Omega(10,5)\,\epsilon_2}{4\pi}$, $\epsilon_2 = 0.3$, $\epsilon_{1,abs} = \frac{\Omega(10,5)\,\epsilon_1}{4\pi}$, $\epsilon_1 = 0.5$ and $S = 100\times10^3$to get the rate at which events are recorded in the sum peak in sec^{-1}:

rate = 41.8 $\sec^{-1}$

What is the additional chance coincidence rate in the sum peak?

This is tricky. To get a chance count in the sum peak, we must get a full-energy peak detection from one gamma-ray and NO true coincidence from the other gamma-ray, followed by another decay which yields only the full-energy peak of the other gamma-ray (again with no true coincidence allowed). These conditions are required because we want the chance coincidence count to lie in the sum peak. Thus, the rate we want is:
$2\,(r_1 - r_{12})(r_2 - r_{12})\,\tau = 2\left(S\,\epsilon_{1,\text{abs}}\right)\left(1-\epsilon_{2,\text{abs}}\right)\left(S\,\epsilon_{2,\text{abs}}\right)\left(1-\epsilon_{1,\text{abs}}\right)\tau$ where τ is the detector resolving time. This calculation is shown below.

additional rate = $2\left(S\,\epsilon_{1,\text{abs}}\right)\left(1-\epsilon_{2,\text{abs}}\right)\left(S\,\epsilon_{2,\text{abs}}\right)\left(1-\epsilon_{1,\text{abs}}\right)\tau$

We substitute $\epsilon_{2,\text{abs}} = \frac{\Omega(10,5)\,\epsilon_2}{4\pi}$, $\epsilon_2 = 0.3$, $\epsilon_{1,\text{abs}} = \frac{\Omega(10,5)\,\epsilon_1}{4\pi}$, $\epsilon_1 = 0.5$, $S = 100\times10^3$ and $\tau = 3\,x\,10^{-6}$to get the additional rate expected from chance coincidences (in $\sec^{-1}$) for a detector with resolving time of 3μs:

additional rate = 24.0 $\sec^{-1}$

Another way to get counts under the sum peak is to have two true coincidences which sum to less than the sum peak, and then a chance coincidence which happens to sum the total deposition up into the peak. This is highly unlikely.

■ Problem 10.12. Fraction of NaI volume where electrons might escape.

We calculate the volume of the crystal where electron escape is possible. From Figure 2.14, range x density $\simeq 0.6\ g/\text{cm}^2$, and we know that the density of NaI = $3.67\ g/\text{cm}^3$, so the range is (0.6/3.67) cm. We look the the volume of the cylinder inside this range from the surface and calculate what fraction this is of the total crystal volume. This is expressed below, where we calculate the fraction of the volume that will fully stop the electrons, and subtract that from 1 to find the fraction of the volume where electrons will not be stopped. Denoting the range is denoted by "range," the cylinder diameter by "d," and the cylinder height by "h", we can write this as:

$$f = 1 - \frac{\pi\left(\frac{d-2\,\text{range}}{2}\right)^2 (h - 2\,\text{range})}{\left(\pi\left(\frac{d}{2}\right)^2 h\right)}$$

We substitute for range = $\frac{.6}{3.67}$, $d = 5.08$ and $h = 5.08$ to getthe fraction of the total NaI cylindrical crystal that lies near enough to the surface so that electron escape is possible:

f = 0.181

Note that this is a rather substantial fraction. It grossly overestimates the potential loss because few electrons will actually be heading directly for the surface. A few percent is probably more realistic.

■ Problem 10.13. Source of 511 keV gamma-rays.

The source might produce (1) positrons which produce annihilation photons, or (2) gamma-rays of sufficiently high energy to undergo pair production followed by annihilation photons in surrounding materials.

Problem 10.14. Relation between resolution, FWHM, and σ.

We obtain the relationship between energy resolution (R), photopeak centroid energy (E), and the standard deviation in the energy (σ_E) from Eqn. 4.13. This relationship is:

$$R = \frac{FWHM}{H_0} = \frac{2.35\,\sigma_E}{E} \text{ (for a Gaussian distribution, FWHM = 2.35}\sigma\text{), or } \sigma_E = \frac{R*E}{2.35}$$

Here, we simply plug our known values into this equation to solve for σ_E.

$$\sigma_E = \frac{\textbf{resolution} \times \textbf{energy}}{\textbf{2.35}}$$

We substitute resolution = .085 and energy = 662 keV to get the standard deviation in keV of the Gaussian curve that would be a fit to the photopeak at 662 keV:

σ_E **= 23.9 keV**

Problem 10.15 Low versus high Z scintillators

For spectroscopy, one wants the full energy of the incident radiation deposited within the detector material. In order to promote interactions with gamma rays, a high Z material is needed since all of the interaction mechanism probabilities increase with Z, particularly a photoelectric absorption which is required for a full energy deposition. For electrons, high Z materials tend to backscatter the electrons and promote bremshralung, both of which are undesirable.

Problem 10.16. Liquid scintillation counting advantage for low E beta particles

The primary advantage of liquid scintillators is that the radioactive source can be dissolved into the liquid with the scintillator. For low energy β particles that are emitted inside the liquid scintillating material, they do not have to travel a distance to the scintillator and then make it through the detector casing. Furthermore, by being dissolved in the liquid, they are not subject to self-absorption in the sample itself. Therefore, the efficiency of detection is much higher for liquid scintillation counting.

Semiconductor Diode problems

■ Problem 11.1. Intrinsic and doped concentrations.

First, from Table 11.1 we take the necessary constants for Si and Ge (i.e. the intrinsic carrier densities, or "n_i," the atomic weights, or "A," and the densities, or "ρ"). These constants are shown below.

$$n_{i\text{Si}} = \frac{1.5\times10^{10}}{\text{cm}^3} \qquad A_{\text{Si}} = \frac{28.09\ \text{grams}}{\text{mole}} \qquad \rho_{\text{Si}} = \frac{2.33\ \text{grams}}{\text{cm}^3}$$

$$n_{i_{\text{Ge}}} = \frac{2.4\times10^{13}}{\text{cm}^3} \qquad A_{\text{Ge}} = \frac{72.6\ \text{grams}}{\text{mole}} \qquad \rho_{\text{Ge}} = \frac{5.32\ \text{grams}}{\text{cm}^3}$$

We know that the material is "doped" if $n_d >> n_i$ (n_d is the dopant concentration). For Si or Ge, this begins at, say, $n_d \sim n_i$. What atom fraction is this in parts per billion? First, we answer this question for Si. To calculate this, we take n_i and divide it by ρ to give units of $\frac{\text{carriers}}{\text{gram}}$, then multiply that by A to give units of $\frac{\text{carriers}}{\text{mole}}$, then divide that by Avagadro's number to give units of $\frac{\text{carriers}}{\text{atom}}$, or $\frac{\text{impurity atoms}}{\text{Si atoms}}$, which is the atom fraction we are interested in. This calculation is shown below. The approximate atom fraction (in parts per billion) of impurity levels in Si such that we consider it to be "doped" is thus:

$$\text{atom fraction}_{\text{Si}} = \frac{n_{i\text{Si}}\,A_{\text{Si}}}{\rho_{\text{Si}}(\text{Avogadro's Constant})} = 0.0003003\ \text{parts per billiion}$$

Now, we do exactly the same calculation for Ge. The approximate atom fraction (in parts per billion) of impurity levels in Ge such that we consider it to be "doped" is thus:

$$\text{atom fraction}_{\text{Ge}} = \frac{n_{i\text{Ge}}\,A_{\text{Ge}}}{\rho_{\text{Ge}}(\text{Avogadro's Constant})} = 0.544\ \text{parts per billion}$$

Note what this says. Because Ge has a much higher intrinsic carrier density (x 1000), we require a doping concentration in Ge 1000 times larger than in Si in order to have a "doped" material.

■ Problem 11.2. Charge carriers produced in air versus Si.

The energy required to produce a carrier is about 35 eV in air and about 3.62 eV in Si. Thus there will be about 10 times more carriers for the same energy deposition in Si. This is, of course, a main reason for the improved energy resolution.

■ Problem 11.3. Expected electron-hole pairs in Si by 100 keV energy deposition.

The mean value of the number of electron-hole pairs (carriers) produced is the energy deposition divided by the energy required to produce a single electron-hole pair. This calculation is shown below. The mean number of electron-hole pairs produced is thus:

$$<\text{carriers}> = \frac{100\ \text{keV}}{\frac{3.76\ \text{eV}}{\text{carrier}}} = 26\,600\ \text{carriers}$$

The variance is given by the statistical variance (i.e. the mean) times the Fano factor (we use an approximate value F=0.1). This calculation is shown below:

$$\text{variance} = 0.1 \times 26\,595.7 \text{ carriers} = 2660 \text{ carriers}$$

■ Problem 11.4. Thermal carrier generation rates.

This question is asking how the thermal generation of carriers in Ge is reduced by cooling the detector to liquid nitrogen temperature from room temperature. We recall the generation probability per unit time (Eqn. 11.1) below (neglecting the proportionality constant):

$$p(T, E_g) = T^{3/2}\, e^{-\frac{E_g}{2\,(\text{Boltzmann's Constant})\,T}}$$

The energy gap at 77K is found by interpolation (from Table 11.1) to be 0.725 eV. Reading off from Table 11.1 at room temperature (300K), we find the energy gap there to be 0.665 eV. We want to find the factor the thermal charge generation is reduced by, so we find the ratio of thermal charge at the two temperatures (i.e. the generation rate at 77 K divided by that at 300 K). The factor that the thermal charge generation is reduced by in cooling to liquid nitrogen temperature versus room temperature for Ge (expressed as a percent) is thus:

$$\text{factor} = \frac{p(77,\ 0.725)}{p(300,\ 0.665)} = 9.41 \times 10^{-18}\ \%$$

This is a **significant** difference. The thermal generation is a factor of ~ 10^{-19} of that occurring at room temperature!

■ This is a brief review of the key relationships dealing with depleted semiconductors:

Depletion depth : $d(V, \epsilon, N) = \sqrt{\frac{2\,\epsilon\,V}{(\text{Electron Charge})\,N}}$ **which can also be written as :** $d(V, \epsilon, \mu, \rho_d) = \sqrt{2\,\epsilon\,V\,\mu\,\rho_d}$

Capacitance per unit area : $C(\epsilon, N, V) = \sqrt{\frac{(\text{Electron Charge})\,\epsilon\,N}{2\,V}}$

Maximum Electric Field Strength at Junction : $E_{\max}(V, d) = \frac{2\,V}{d}$

■ Problem 11.5. Which bias on which contact?

For a reverse bias, one wants to extract the majority carrier by applying the opposite polarity. So, the positive voltage is applied to the n+ contact. This removes the excess electrons, and leave the fixed positive charge at the positive voltage. This positive voltage pulls on the electrons in the opposite p+ contact where they are only a minority carrier.

■ Problem 11.6. How does voltage affect depletion depth, capacitance, and maximum electric field strength?

For a partially depleted detector, these relationships are derived in the textbook as:

depletion depth = $\sqrt{\frac{2\epsilon V}{eN}}$; capacitance = $\sqrt{\frac{e\epsilon N}{2V}}$; and $E_{max} = \sqrt{\frac{2VNe}{\epsilon}}$

■ Problem 11.7. Largest depletion depth per unit voltage

The depletion depth is given by $\sqrt{\frac{2\epsilon V}{eN}}$ where N is the lowest dopant concentration. By making N as low as possible, i.e., by starting with the purest of materials, one achieves the highest depletion depth for any given voltage.

■ Problem 11.8. Bias voltage for given depletion depth

Using the nomogram, one finds a bias voltage of about 35 V.

■ Problem 11.9. Energy loss measurement in dead layer.

To solve this, we look at the two equations that describe the two situations we have. Each equation describes the energy loss of the alpha particles in silicon when the alpha particles are either perpendicular or at an angle to the surface of the detector. We know that the energy lost is $\Delta E = E_i - E_f = E_a - hc$, where E_a is the incident alpha particle energy, h is the channel number of the alpha peak (depends on the situation), and c is the energy per channel number of the MCA (we assume a zero offset, so this is valid). For the case of the alpha particles perpendicular to the detector, we have $\Delta E_0 = t\frac{dE_0}{dx}$ (Eqn. 11.22), so then we can say $E_a = t\frac{dE_0}{dx} + h_0 c$ for the case of perpendicular alphas (h_0 is the MCA channel number corresponding to this situation). For the case of the alpha particles at an angle to the detector, we have $\Delta E(\theta) = \frac{\Delta E_0}{\cos(\theta)}$ (Eqn. 11.23) = $\frac{t\frac{dE_0}{dx}}{\cos(\theta)}$, so then we can say $E_a = \frac{t\frac{dE_0}{dx}}{\cos(\theta)} + h_p c$ for the case of alphas at an angle (h_p is the MCA channel number corresponding to this situation). These equations for E_a are the equations that we would like to solve simultaneously for the dead layer thickness, t. We express these two equations below, where we denote $\frac{dE_0}{dx}$ with "c_1" and c with "c_2," and we solve for t by eliminating c_2, which is unknown.

$$E_a = t\,c_1 + h_0\,c_2 \qquad E_a = \frac{t\,c_1}{\cos(\theta)} + h_p\,c_2$$

The solution for t in terms of our known variables.

$$t = -\frac{E_a\,h_p - E_a\,h_0}{h_0\,c_1\sec(\theta) - h_p\,c_1}$$

Here, we just plug our known values into the solution for t. Note that we get E_a to be roughly 5.486 MeV (from Table 1.3 for the E_α of Am^{241} in the dominant decay branch), and also, we approximate $\frac{dE}{dx}$ (i.e. $\frac{dE_0}{dx} = c_1$) in Si at this energy to be 120 keV/micron (from Figure 2.9). We substitute $E_a = 5.486\,\text{MeV}$, $h_p = 455$, $h_0 = 461$, $c_1 = \frac{120\,\text{keV}}{\mu\text{m}}$ and $\theta = 35°$ to get the dead layer thickness in units of microns.

$$t = 2.55\,\mu\text{m}$$

We are asked for the dead layer thickness in units of alpha particle energy loss, i.e. how much alpha particle energy is lost when the alphas are perpendicular to the detector surface? This is simply calculated by $t\frac{dE_0}{dx} = t * c_1$, which is shown in the below. The dead layer thickness of the silicon junction detector in units of (perpendicular) alpha particle energy loss (in keV) is thus:

$$t = 2.55\,\mu\text{m} \times \frac{120\,\text{keV}}{\mu\text{m}} = 305\,\text{keV}$$

■ Problem 11.10 Surface barrier detector noise versus area.

As the area of the detector increases, so does the capacitance. Since capacitance serves as a major noise source at the input of the preamplifier, larger areas imply reduced energy resolution.

■ Problem 11.11. Electron and hole collection times in Si.

The maximum collection time is given by $t_c = \frac{d}{v}$, where we assume the particles must drift across the entire wafer (so d = 0.1 mm = 0.01 cm). From Figure 11.2 (a and b), the saturated electron and hole velocities (v) both appear to be about 10^7 cm/s. The calculation for t_c is shown below.

$$t_c = \frac{d}{v}$$

We substitute d=0.01 and $v=10^7$ to get the approximate maximum collection time for both electrons and holes in seconds.

$$t_c = 1 \times 10^{-9}\,\text{seconds}$$

This very fast collection time is characteristic of thin semiconductor detectors.

■ Problem 11.12. Pulse amplitude variation with bias voltage fluctuation.

The pulse amplitude H changes due to the capacitance changing with applied voltage V.
The pulse amplitude H is given by $\frac{Q}{C}$, but C= $\frac{k}{V^{0.5}}$ (from Eqn. 11.20, where k is a constant which will have no bearing on this problem).
Therefore, since Q is constant, H $\propto V^{0.5}$.

Taking the differential and dividing yields:
$\left(\frac{\delta \mathrm{H}}{H}\right) = \frac{1}{2}\left(\frac{\delta \mathrm{V}}{V}\right)$

Substituting in $\left(\frac{\delta \mathrm{V}}{V}\right)$=5% for this problem yields: $\left(\frac{\delta \mathrm{H}}{H}\right) = \frac{1}{2}(5\,\%) = 2.5\,\%$.

■ Problem 11.13. Radiation damage to Si.

The text notes that serious damage occurs by $10^{11}\ \alpha/\mathrm{cm}^2$. Given a 10 MBq α source at 10 cm from the detector, the flux (in $\frac{\alpha}{\mathrm{cm}^2 * s}$) is $S\,\frac{\Omega}{4\pi}\frac{}{A}$, where $\Omega \sim A/d^2$. Hence, to find the length of exposure time required for damage to occur, we simply take the flux multiplied by the unknown time, set it equal to $10^{11}\ \alpha/\mathrm{cm}^2$, and solve for the time. This calculation is expressed below.

$$\frac{S\,t}{4\,\pi\,d^2} = 10^{11}\ \mathrm{cm}^{-2}$$

We substitute S=10 MBq, and d=10 cm to get the length of exposure time required for radiation damage to occur to the Si surface barrier detector.

$$t = 4\,000\,000\,\pi\ \text{seconds} = 145\ \text{days}$$

This says that the detector will be seriously damaged if left in this position for ~5 months.

■ Problem 11.14. Pulse height defect for heavy ions.

We assume the response of the system is linear over the entire MCA channel range. Then, we expect the 21.0 MeV heavy ion to be placed in channel $(\frac{116}{5.486\ \mathrm{MeV}})$*(21.0 MeV) (because we expect the plot of channel number vs. energy to have a slope of $\frac{\Delta \mathrm{Channel}}{\Delta \mathrm{Energy}} = \frac{(116-0)\ \mathrm{Channels}}{(5.486-0)\ \mathrm{MeV}}$). The pulse height defect (in channel units) for the 21.0 MeV heavy ion is the expected channel number minus the recorded channel number for that energy (which is 402). This calculation is shown below:

$$\text{pulse height defect} = \frac{116 \times 21.0}{5.486} - 402 = 42.0\ \text{channel units}$$

Now, we convert to energy units (in MeV) by multiplying by the inverse of our slope from before (i.e. MeV/Channel). The pulse height defect for the 21.0 MeV heavy ion (in MeV) is thus:

$$\text{pulse height defect} = \frac{42.0394 \times 5.486}{116} = 1.99\ \text{MeV}$$

Problem 11.15. Pulse height spectra for alphas for different depletion depths.

(a). In this case, all of the energy is deposited and the pulse height is of corresponding amplitude.
(b). Since alphas lose most of their energy at the end of their path, less than half of the energy is deposited in the detector, so the pulse amplitude is less than half of that in part (a).
(c). From (b) above, the energy deposited must be more than half of that in part (a).

■ Problem 11.16. Spatial broadening of diffusing electron cloud in Ge.

We want to know how much lateral motion occurs when the electrons drift a given distance in a Ge detector. The electric field is 1000 V/cm, the detector is at liquid nitrogen's 77 degrees K, and the drift distance is 1 cm. To estimate this spatial broadening, we use Eqn. 11.6 and the given values. This calculation is shown below, where we convert the result to μm.

$$\sigma = \sqrt{\frac{2\,(\text{Boltzmann's Constant}) \times T \times \text{drift}}{(\text{Electron Charge})\,(E\ \text{field})}}$$

We substitute T=77 K, E field=$\frac{1000\ \text{volts}}{\text{cm}}$ and drift =1 cm to get the estimated spatial broadening in μm.

σ = 36.4 microns

Note that 100 microns (or on that order, like our result) isn't very much, unless the spacing between contacts on a position-sensitive detector is on this order. Then, the electrons can be collected by more than one electrode and lead to some ambiguity, along with signal processing complications that must be anticipated. (The astute student may note that the problem asks us to use the mobility values in Table 11.1 when we don't need them actually).

Germanium gamma-ray detector problems.

■ Problem 12.1. Fano factor role.

Decreasing the Fano factor F plays a major role in adjusting the FWHM and, hence, the energy resolution downward. Recall from Eqn. 4.15 that

FWHM $\propto$ 2.35 $\sqrt{N}$ $\sqrt{F}$, and the energy resolution is given by: $2.35\sqrt{\frac{F}{N}}$. If the Fano factor decreases by a factor of 2, both the FWHM and energy resolution are decreased by $\sqrt{2}$.

■ Problem 12.2. Voltage required for v_{sat} and carrier lifetime required for 0.1% losses.

The first part of the problem is straight forward. We know that $v_{sat} = \mu E = \mu V/d$ (when the voltage V is sufficient to saturate the carrier velocity v), so we write this equation and solve for V. Note that we are interested in **both** carrier velocities to be saturated, so we must find the V that would accomplish this (i.e. we should use the smaller mobility between that of electrons and holes in Ge to find the sufficient applied voltage). Below, we solve our equation (here, we write two equations) for V by eliminating E.

$$v_{sat} = \mu E \qquad E = \frac{V}{d}$$

The solution for V is thus:

$$V = \frac{d\, v_{sat}}{\mu}$$

Using Fig. 11.2, we find that v_{sat} is roughly 10^7cm/s for both electrons and holes, and we find from Table 11.1 that the smaller of the two mobilities is that of electrons (which is somewhat surprising), which is 3.6 x 10^4 $\frac{cm^2}{V*s}$. We substitute μ=3.6 x 10^4 $\frac{cm^2}{V*s}$, d=1cm and v_{sat}=10^7cm/s into our equation for V above to get the minimum applied voltage necessary to saturate both carrier velocities.

V = 278 Volts

The second part looks for the minimum carrier lifetime to ensure that 99.9% of the carriers live to be collected. This is an exponential process, where the fraction of collected carriers (=.999) = $e^{\frac{-t}{\tau}}$. The maximum collection time t = $\frac{d}{v_{sat}} = \frac{1\,cm}{10^7\,cm/s}$, and τ is the minimum carrier lifetime. We express this equation below:

$$0.999 = e^{-\frac{t}{\tau}}$$

Now we can solve for τ. The solution for τ in terms of t is thus:

$$\tau = 999.5\, t$$

We substitute t = 10^{-7} seconds to get the minimum carrier lifetime such that no more than 0.1% of either holes or electrons are to be lost from any pulse:

$$\tau = 99.95\ \mu s$$

While this seems quick when looking at the time scales involved with moving ions in gas-filled detectors, this is slower than what we would ideally like and will limit the count rates that we can accept to roughly $1/\tau$.

■ Problem 12.3. Resolution contributions: statistics, electronics, incomplete charge collection negligible.

We look at the component representing the statistical fluctuation in the number of charge carriers from Eqn. 12.12, W_D, which is calculated by

FWHM = $2.35K\sqrt{N}\ \sqrt{F} = 2.35\ \epsilon \sqrt{F * \frac{E}{\epsilon}}$ ($=2.35\sqrt{F\epsilon E}$, from Eqn. 12.13; either equation works just as well), where ϵ is the energy/charge carrier (ionization energy), and E is the total energy deposited in the detector (140 keV). We use Fano= 0.08 and ϵ = 2.96 eV/carrier. The calculation for W_D is shown below.

$$W_D = 2.35 \sqrt{\text{Fano} \times 2.96\ \text{eV} \times 140\ \text{keV}} = 428\ \text{eV}$$

Note that this says that the statistical component, .427 keV, is much smaller than the electronic noise contribution of 1.2 keV (which will dominate). This is a realistic problem, and one is always looking for low noise electronic components which make the carrier statistics the dominant factor. Since the electronic noise is constant but the carrier statistics vary with $\sqrt{N}$, the electronic component is especially critical for low energy applications, such as in nuclear medicine where 140 keV is used extensively. Frequently, people will quote the electronic noise in electrons -- this is just a factor of 2.96 eV (about 3 eV) to convert from energy to electrons -- about 140 electrons in our example above.

Below, we use Eqn. 12.12 to calculate $W_T$as a percentage of the total energy deposited (note that we have assumed W_X= 0, i.e. we are assuming charge collection is complete).

$$\frac{\sqrt{(1.2\ \text{keV})^2 + {W_D}^2}}{140\ \text{keV}} = 0.910\ \%$$

This is a typical (i.e., remarkably good) value for the energy resolution of a semiconductor detector.

■ Problem 12.4. Incident gamma-ray energy and location of Compton edge.

First, we set up the equation for the Compton edge, which is the incident gamma ray energy minus the energy of the scattered photon at a scattering angle of 180 degrees (the angle at which maximum energy is transferred to an electron in the detector). Below we express this equation, set it equal to 1.16 MeV, and solve for E_0(the energy of the incident gamma rays).

$$E_0 - \frac{E_0}{\frac{E_0\,(1-\cos(180\,^\circ))}{m_e\ c^2} + 1} = 1.16\ \text{MeV}$$

The incident gamma-ray energy in MeV is thus:

$$E_0 = 1.38\ \text{MeV}$$

To answer the second part of the problem, the location of the Compton edge (1.16 MeV) is detector independent because it is due to interactions with free electrons, so the Compton edge will also occur at 1.16 MeV using a sodium iodide detector.

Problem 12.5. Escape peaks in Ge versus NaI.

Ge detectors are typically smaller in size and have lower value of Z. Both of these factors lead to increased probability of escape for the 511 keV annihilation photons.

■ Problem 12.6. Count rate for Ge detector from point source.

The standard value of the absolute peak efficiency for NaI(Tl) (gamma-ray energy of 1.333 MeV) for this size crystal is 1.2 x 10^{-3} at d = 25 cm. We need to adjust this value of $\epsilon_{abs,p}$ by accounting for the different source-detector distance that we are given (we just multiply by the ratio of the solid angle at d = 40 cm to that of d = 25 cm). Recalling that the solid angle is given by:

$$\Omega = 2\pi\left(1 - \frac{d}{\sqrt{a^2 + d^2}}\right)$$

then our value for the absolute peak efficiency for this NaI(Tl) detector for 1.333 MeV gamma rays from a point source at 25 cm will be given by:

$$\epsilon_{abs,p} = \frac{1.2\ x\ 10^{-3}\left(1 - \frac{d}{\sqrt{a^2+d^2}}\right)}{\left(1 - \frac{d_2}{\sqrt{a^2+d_2{}^2}}\right)}$$

where we substitute d = 40, d_2 = 25 and a = $\frac{7.62}{2}$ to get :

$$\epsilon_{abs,p} = 0.000473679$$

We will now use this value of $\epsilon_{abs,p}$ to calculate the expected peak count rate (counts per second) for Ge (which has an ϵ_{ip} of 40% of that for NaI(Tl)) for 150 kBq of Co-60. To do this, we simply multiply our value of $\epsilon_{abs,p}$ for NaI(Tl) (our answer above) by 0.4 (or 40%), which gives $\epsilon_{abs,p}$ for Ge under the given conditions, and then multiply by the source activity of 150 kBq (or 150,000 disintegrations per second). This calculation is shown below. The estimated counting rate (in counts per second) for Ge with a 150 kBq Co-60 source at a distance of 40 cm is thus:

$$\text{counting rate} = 0.000473679 \times 0.40 \times 150\,000 = 28.4207 \text{ counts per second}$$

■ Problem 12.7. Expected energy resolution for Ge at 662 keV.

Since we can assume charge collection to be complete and electronic noise negligible, then the energy resolution is only a function of charge carrier statistics. Recall that the FWHM from charge carrier statistics (or W_D) is given by $2.35\sqrt{F\epsilon E}$, where ϵ = 2.96 eV/carrier-pair, F=0.08 (both from pg. 368), and E = 662 keV. This calculation for the FWHM is shown below.

$$\text{FWHM} = 2.35\sqrt{F\,\epsilon\,E}$$

We substitute ϵ = 2.96 eV, F = 0.08, E = 662 keV to get the FWHM in eV.

$$\text{FWHM} = 930 \text{ eV}$$

Now, to calculate the energy resolution, we simply divide the FWHM by the gamma-ray energy. This is shown below as a percent.

$$\frac{\mathbf{930.439\ eV}}{\mathbf{662\ keV}} = \mathbf{0.141\ \%}$$

■ Problem 12.8. Location of peak in pair spectrometer.

The pair spectrometer records the energy of the double escape peak, which is $E_0 - 1.02$ MeV. For 2.10 MeV incident gamma rays, the remaining 1.08 MeV will be recorded in the central Ge detector (i.e. the peak will occur at 1.08 MeV).

Other Semiconductor Devices.

■ Problem 13.1. X-ray escape peak smaller for Si than Ge.

Two reasons the x-ray escape peak from Si is smaller are:
(1). Greater penetration of the incident radiation into Si rather than Ge (smaller photoelectric cross section in Si) leads to fewer events occurring near the surface in Si, which is where the x-ray has the greatest probability of escape.
(2). A greater x-ray energy for Ge (11 keV) as opposed to Si (1.8 keV) allows the Ge x-rays to escape more easily (higher probability of escape).

■ Problem 13.2. Si(Li) 4 mm detector at 2000 V. Charge collection time.

See Problem 11.11 above. The maximum collection time is given by $t_c = \frac{d}{v}$, where we assume the particles must drift across the entire wafer (so d = 4 mm = 0.4 cm). From Figure 11.2 (a and b), the saturated electron and hole velocities (v) both appear to be about 10^7 cm/s. The calculation for t_c (assuming that the carrier velocities are saturated, which is not stated in the problem) is shown below. We substitute d = 0.4 cm and v = 10^7 cm/s:

$$t_c = \frac{d}{v} = 4 \times 10^{-8} \text{ seconds}$$

to get the maximum collection time in seconds for saturated carrier velocities using this rough value for the saturated velocity:

Now, we solve the same problem using the given voltage. We know that $v = \mu E = \frac{\mu V}{d}$. We know that the maximum collection time will be dominated by the lowest carrier mobility between electrons and holes, so we calculate the velocity of interest, v, using the hole mobility at 300 K (from Table 11.1).

$$v = \frac{\mu V}{d}$$

We substitute $\mu = 1.1 \times 10^4$, d = 0.4 and V = 2000 to get the calculated hole velocity using the given voltage in cm/s.

$$v = 5.5 \times 10^7 \text{ cm/s}$$

But this is not close to the saturated velocity value of 1.5 x 10^6cm/s (knowing that E = V/d = 5 x 10^3V/cm, and assuming T = 300 K) from Figure 11.2 (b). Since this is well above the saturated velocity, we discard the calculated hole velocity and use the value from Figure 11.2 (b) to calculate the maximum collection time. This calculation is shown below.

$$t_c = \frac{d}{1.5 \times 10^6}$$

We substitute d=0.4 to get the estimated maximum collection time.

$$t_c = \frac{d}{1.5 \times 10^6} = 2.67 \times 10^{-7} \text{ seconds}$$

Thus, the maximum charge collection time is about 270 ns (from the holes). In actuality, this estimate is way too big and is likely due to the fact that the saturated velocity is for the <111> direction in Si ... this is much lower than in the common <100> orientation. A value of ~10 ns might be more representative of what is seen in practice using the higher saturated velocity for that direction.

■ Problem 13.3. Required resolution to separate Cu and Zn K_α X-rays.

The closest x-rays of interest are Cu $K_{\alpha 1}$at 8.048 keV and Zn $K_{\alpha 2}$at 8.616 keV (using these x-ray energies will give us the resolution necessary to resolve separately any of the K-characteristic x-rays from Cu and Zn). We want the FWHM to be (at least) equal to the difference between these two energies, ΔE, to resolve separately the X-rays. We know that R = $\frac{\text{FWHM}}{E}$, or here, R = $\frac{\Delta E}{E_{avg}}$. The calculation for the required energy resolution is shown below.

$$R = \frac{\Delta E}{E_{avg}} = \frac{8.616 - 8.048}{\frac{8.616+8.048}{2}}$$

The energy resolution expressed as a fraction, not a percent is thus:

$$R = 0.0682$$

So, we need an energy resolution of about 6.8% to resolve separately the K-characteristic x-rays from copper and zinc.

■ Problem 13.4. Causes of decreased Si(Li) efficiency at low X-ray energies (below 5 keV).

The primary reason for the decreased efficiency is that the x-rays are unable to reach the active region (the x-rays fail to penetrate the window materials and the dead layer at the surface of the detector).

■ Problem 13.5. Efficiency of planar Si(Li) detector versus NaI well-counter.

This problem is solved using the equation N = $S\,\epsilon_{abs,p}$= S ϵ_{ip} $\frac{\Omega}{4\pi}$, and the intensity ratio (X/γ from Table 13.2, which is the same as $\frac{S_X}{S_\gamma}$) for Co-57 at an X-ray energy of 6.4 keV (or 6.397 keV), which is given as 0.5863. In the end, we are interested in finding $\epsilon_{ip}$for this x-ray energy in Si(Li) (denoted as $\epsilon_{ip,X}$). We start by noting that $S_\gamma = \frac{N_\gamma}{\epsilon_{abs,p,\gamma}}$($\epsilon_{abs,p}$ and N_γ are both given for the gamma-rays in NaI(Tl)), and then we note that $S_X = (\frac{S_X}{S_\gamma})S_\gamma$ = .5863 S_γ = .5863 $\left(\frac{N_\gamma}{\epsilon_{abs,p,\gamma}}\right)$. From this, we can calculate $\epsilon_{abs,p}$ for the x-rays in Si(Li) by $\epsilon_{abs,p,X} = \frac{N_X}{S_X} = \frac{N_X}{.5863\left(\frac{N_\gamma}{\epsilon_{abs,p,\gamma}}\right)} = \epsilon_{abs,p,\gamma}\left(\frac{N_X}{.5863\,N_\gamma}\right)$. The calculation for $\epsilon_{abs,p,X}$is shown below.

$$\epsilon_{abs,p,X} = .83\left(\frac{730}{60}\right)\left(\frac{15}{.5863 \times 146\,835}\right)$$

The absolute peak efficiency for the Co-57 X-rays (expressed as a fraction, not a percent) is thus:

$$\epsilon_{abs,p,X} = 0.00176$$

From this, we calculate $\epsilon_{ip,X}$ from $\epsilon_{ip} = \frac{\epsilon_{abs,p}}{\left(\frac{\Omega}{4\pi}\right)}$, and we approximate Ω as $\frac{A}{d^2}$ (A is expressed in cm^2 since d is in cm). This calculation is expressed below.

$$\epsilon_{ip,X} = \frac{\epsilon_{abs,p,X}}{\frac{\Omega}{4\pi}}$$

We substitute $\Omega = \frac{A}{d^2}$, A = 3 and d = 10 to get the intrinsic peak efficiency of the Si(Li) detector for the 6.4 keV X-rays from Co-57 (expressed as a fraction, not a percent).

$$\epsilon_{ip,X} = 0.737$$

This high intrinsic peak efficiency is comparable to the NaI(Tl) well counter. Note how the geometric factor of moving the source just 10 cm away reduces the efficiency by about 10^3!

■ Problem 13.6. Statistical limit for energy resolution: Si(Li) versus HPGe.

The problem asks for the statistical limiting energy resolution for Si versus Ge (stated as Si(Li) vs. HPGe). For Si, we use a Fano factor of F=0.11 and ϵ = 3.76 eV (for T = 77K). For Ge, we use F = 0.08 and ϵ = 2.96 eV (for T = 77K) (note that we use T = 77K for both detectors because Ge must be cooled to this temperature, and we need to keep conditions constant for comparison). We know that the statistical limiting FWHM is $W_D = 2.35\sqrt{F\epsilon E}$, and the statistical limiting energy resolution is $\frac{W_D}{E}$. We substitute in the values for Ge to give its energy resolution.

$$W_D = \frac{2.35\sqrt{F\,\epsilon\,\text{Energy}}}{\text{Energy}}$$

We substitute F = 0.08, ϵ = 2.96 and Energy = 59.5×10^3 to get the statistical limiting energy resolution of a HPGe detector for the 59.5 keV gamma-rays from Am-241:

$$W_{D,Ge} = 0.469\ \%$$

Now, we do the same for Si. We substitute F = 0.11, ϵ = 3.76 and Energy = 59.5×10^3 into the above equation to get the statistical limiting energy resolution of a Si(Li) detector for the 59.5 keV gamma-rays from Am-241.

$$W_{D,Si} = 0.620\ \%$$

For a silicon drift detector, the statistical limit will be the same as for any silicon-based detector since F and ϵ are functions only of the detector material. Note that the lower value of ϵ helps germanium beat silicon in energy resolution, yet also is the factor that requires Ge to be cooled during normal operations. (Large Si(Li) detectors, being Lithium-drifted, also require cooling).

■ Problem 13.7. Detector thicknesses needed: Si, Ge, CdTe, and HgI_2 are compared.

This problem shows the advantages of higher Z (i.e. higher attenuation coefficients) in requiring smaller quantities of material for equivalent detection efficiencies. The probability of having at least one interaction in thickness t is given by $1 - e^{-\mu t}$ (since the probability of having zero interactions in thickness t is $e^{-\mu t}$, same as $\frac{I}{I_0}$). We want this probability to be 50%, and we are interested in finding t, so we set this equation equal to 0.5 and solve for t in terms of μ:

$$1 - e^{-\mu t} = \frac{1}{2}$$

The solution for t in terms of μ.

$$t = \frac{\ln(2)}{\mu}$$

The only data needed are the attenuation coefficients for {Si, Ge, CdTe, HgI_2}, found in the textbook for incident 662 keV gamma-rays. These are expressed below in cm^{-1}, where we have one set for photoelectric absorption, one set for Compton scattering, and one set that sums the two to give the total attenuation coefficients (662 keV is too low of an energy for pair production, so the attenuation coefficients for pair production would be zero).

$$\mu_{pe} = \left\{\frac{2.13}{10^4}, \frac{9.48}{10^3}, \frac{5.08}{10^2}, \frac{1.50}{10}\right\}$$

$$\mu_C = \left\{\frac{1.78}{10}, \frac{3.60}{10}, \frac{3.86}{10}, \frac{3.99}{10}\right\}$$

$$\mu_T = \mu_C + \mu_{pe}$$

Adding the photoelectric and Compton attenuation coefficients gives the total attenuation coefficient, μ_T. We list the values of μ_T, (in cm^{-1}) for {Si, Ge, CdTe, HgI_2}:

$$\mu_T = \{0.178,\ 0.369,\ 0.437,\ 0.549\}$$

We plug these attenuation coefficients into our solution for t to give the required thicknesses for each detector.

$$t = \frac{\ln(2)}{\mu_T}$$

The required thicknesses (in cm) for Si, Ge, CdTe, and HgI_2, respectively, such that the probability of a 662 keV gamma-ray having at least one interaction is 50% is thus:

$$t = \{3.89,\ 1.87,\ 1.59,\ 1.26\}\ \text{cm}$$

Note the remarkable advantage of high Z, high density materials like HgI_2. One gains a factor of 2-3 (in each dimension), so the required volumes are substantially reduced.

The other question asks for the fraction of interactions at 662 keV that are photoelectric events (desired for high photopeak efficiencies). This is simply the photoelectric attenuation coefficients divided by the total attenuation coefficients:

$$f_{pe} = \frac{\mu_{pe}}{\mu_T}$$

The fraction of interactions at 662 keV that are photoelectric events for Si, Ge, CdTe, and HgI_2, respectively:

$$f_{pe} = \frac{\mu_{pe}}{\mu_T} = \{0.0012,\ 0.0257,\ 0.116,\ 0.273\}$$

Again, note the distinct advantage of the high Z materials -- silicon and germanium have a very low percentage of single interaction photoelectric events, while the higher Z materials of CdTe and HgI_2do dramatically better. This also implies much smaller volumes required to acquire full energy depositions.

■ Problem 13.8. CCDs as x-ray detectors.

In this problem, we look at the limitations of using CCDs as x-ray detectors. At 10 keV, 300μm of Si is sufficient to fully absorb most of the incident x-rays. The pixel readout frequency is 100,000 pixels/second (100 kHz), and the detector has 256x256 pixels. So the readout time (in seconds) is simply calculated by dividing the total number of pixels by the readout frequency:

$$t_{readout} = \frac{256 \times 256}{100\,000}$$

The readout time (in seconds) is thus:

$$t_{readout} = 0.655 \text{ seconds}$$

The exposure time must be much longer to ensure that interactions which occur during the readout process don't significantly alter the individual pixel electron populations as the electron packages are moved from pixel to pixel. The problem states that the exposure time is 20 times larger than the readout time, so we calculate the exposure time as:

$$t_{exposure} = 20 \times t_{readout}$$

The exposure time (in seconds) is thus:

$$t_{exposure} = 13.1 \text{ seconds}$$

This is a long exposure time, primarily due to the fact that the readout time for CCDs is notoriously slow on our time scales.

(a). First we want to know what the maximum interaction rate will be over the full device so that, given there is an interaction in a pixel, the probability of more than one interaction per pixel is only 0.05. This brings out an interesting concept -- given that an event has occurred, what is the probability that another event won't occur? Since the events are totally independent of each other, we are really asking at what rate will ONE OR MORE interactions occur in the pixel with a probability of 5%. This is given by:

$$P_{>1} = 1 - P_0 = 1 - e^{-r\,t_{exposure}} = 0.05$$

We solve this equation for the x-ray interaction rate per pixel, r (in terms of $t_{exposure}$).

$$1 - e^{-r t_{exposure}} = 0.05$$

The equation for r in terms of $t_{exposure}$.

$$r = \frac{0.0512933}{t_{exposure}}$$

Now, we plug in $t_{exposure}$= 13.1072 seconds to get the tolerable x–ray interaction rate per pixel (in interactions per second per pixel):

$$r = 0.00391$$

We are interested in finding the acceptable interaction rate over the full array, and we know that we have 256 x 256 pixels, so we calculate the interaction rate over the full array by multiplying r from above by 256 x 256. This is shown below.

$$x-\text{ray interaction rate over the full array} = r \times 256 \times 256$$

We substitute r=0.00391 to get the tolerable x-ray interaction rate over the full array (in interactions per second).

$$x-\text{ray interaction rate over the full array} = 256$$

(b). The minimum storage requirement per pixel will be the total charge generated in a single photoelectric absorption in silicon, or expressed in terms of numbers of electrons. This is calculated by $n_{\text{signal}} = \frac{(\text{Photon Energy})}{\epsilon}$, and for silicon ϵ = 3.62 eV/ charge-pair (for T = 300 K):

$$n_{\text{signal}} = \frac{10\,000}{3.62} = \frac{2762 \text{ electrons}}{\text{pixel}}$$

which is the minimum required storage capacity of a typical pixel (in electrons per pixel).

(c). The charge due to leakage should be kept small since this is a noise term. In this problem, we estimate the leakage current that will produce 10% of the signal charge in part (b) above. Note that when we estimate the total allowable leakage current here, we are assuming it is spread evenly into each pixel. The time over which the leakage current is present during one cycle is $t_{\text{exposure}} + t_{\text{readout}}$. We calculate the leakage current by multiplying the number of electrons per pixel from above by 10%, then multiplying this by the charge of a single electron and the total number of pixels in the array (256 x 256). Then, we take this total charge due to leakage and divide it by the time over which leakage current is present, which is discussed above. This leakage current calculation is carried out below:

$$\text{current} = \frac{10\,\% \times n_{\text{signal}} \times \text{ElectronCharge} \times 256^2}{(t_{\text{exposure}} + t_{\text{readout}})\,\text{second}} = 0.211 \text{ pA}$$

which is the estimated maximum leakage current across the entire area of the CCD that can be tolerated (in pA).

Slow neutron detectors

■ Problem 14.1. Operation in Proportional Region

We choose to operate gas-filled neutron detectors in the proportional region. If we operated in the ionization region, the individual pulses would be inconveniently small to handle, as noted in problem 14.2 below. In the Geiger region, any interaction in the detector would yield the same size pulse, so we would be unable to discriminate based on pulse height against the small pulses produced by gamma-ray interactions, background, or noise sources.

■ Problem 14.2. Pulse height in a ^{3}He tube from a thermal neutron.

The pulse height is given by V=Q/C, where Q = n_0eM=(E/W)eM (e is the electron charge, M is the gas multiplication factor, E is the Q-value for the ^{3}He(n,p) reaction, which will also be conventionally denoted as Q below, and W is the energy/ion pair, which is found in the textbook to be 42.7 eV/ion pair for alpha particles (the proton is another heavy charged particle)). We calculate the pulse height using:

$$\text{pulse ht} = \frac{Q\,e\,M}{W\,C}$$

We substitute Q = 0.764 MeV, M = 1000, W = 42.7 eV and C = 100 pF to get the estimated pulse height produced by a thermal neutron in a ^{3}He tube of a 100 pF capacitance:

pulse ht = 28.7 mV

If we operated the detector in the ionization regime, M=1, and the pulse height would be only 28.7 microvolts -- too small.

■ Problem 14.3. Gain in efficiency using enriched ^{10}B.

Natural boron is composed of 19.8% ^{10}B. ^9B has a negligible cross section for (n, α). First, using Eqn. 14.4, we find solve for f = $\Sigma_a(E)\,L$ when the detection efficiency is 1% (i.e. ^{10}B enrichment = 19.8%). This is done below, where we set Eqn. 14.4 equal to .01 (with the appropriate "f" substitution made) and solve for f.

$$1 - e^{-f} = .01$$

The quantity f = $\Sigma_a(E)\,L$ for ^{10}B enrichment = 19.8% (efficiency = 1%) is thus:

$$f = 0.0101$$

Recall that $\Sigma_a(E) = N_{B-10}\,\sigma_{(n,\alpha)}$, and so is linearly proportional to the ^{10}B enrichment through the number density N_{B-10}. Now, to find the gain in efficiency when the ^{10}B enrichment is raised to 96%, we must divide the efficiency at this enrichment with the efficiency at a ^{10}B enrichment of 19.8% (efficiency of 1%). To find the new efficiency, we note that $\frac{f_{\text{new}}}{f_{\text{old}}} = \frac{\Sigma_a(E)_{\text{new}}}{\Sigma_a(E)_{\text{old}}}$ (since L does not change) = $\frac{\text{new } ^{10}B \text{ enrichment}}{\text{old } ^{10}B \text{ enrichment}}$ (since the neutron energy does not change). Therefore, the quantity "f" that we use for the new efficiency is simply f calculated above (or f_{old}) multiplied by the ratio of the new ^{10}B enrichment to the old ^{10}B enrichment. The calculation of this efficiency gain is expressed below.

$$\text{efficiency gain factor} = \frac{1 - e^{-\frac{f}{.198}.96}}{.01}$$

We substitute f=0.0101 to get the detection efficiency gain for 10 eV neutrons when natural boron is replaced with boron enriched to 96% ^{10}B in a BF_3 tube:

$$\text{efficiency gain factor} = 4.76$$

Note that this is close to the ratio of the abundances ($\frac{96\,\%}{19.8\,\%} = 4.85$) because of the relatively low attenuation (i.e., in this case, $e^{-f} \sim$ 1-f).

■ Problem 14.4. Thermal efficiency of BF_3 tube at 600 torr.

We first must find the atom density of the boron-10 atoms in the gas in order to find the macroscopic cross section, $\Sigma_a(E)$ by Eqn. 2.26. Using the ideal gas law: PV=nRT

$$\text{atom density} = \frac{0.96\ (\text{Avogadro's Constant})\ \text{Pressure}}{(\text{Molar Gas Constant})\ \text{Temp}}$$

We substitute Pressure = 80 kPa and Temp = 300 K to get the boron-10 atom density in cm^{-3}.

$$\text{atom density} = 1.85421 \times 10^{19}\ \text{cm}^{-3}$$

We next calculate the efficiency along the path length t, using the thermal neutron cross section, σ, of boron-10 under the given conditions, which is 3840 barns (given in the textbook), and using Eqn. 2.26 to find the macroscopic cross section and Eqn. 14.4 to finally calculate the detection efficiency:

$$\text{efficiency} = 1 - e^{-\text{atomdensity} * \sigma * t}$$

We substitute σ = 3840 Barns and t = 10 cm to get the detection efficiency of a BF_3tube (96% enriched in B-10) filled to 80 kPa for thermal neutrons with a 10 cm path length (expressed as a fraction, not a percent):

$$\text{efficiency} = 0.509$$

Note that this is different than the text value of 91.5% for the same pressure and enrichment (we can also assume the temperature to be the same), but for a longer length. This implies that our detection efficiency increases dramatically by increasing the length of the tube. Now, we will run the same calculation for a thickness of 30 cm below.

$$\text{efficiency} = 1 - e^{-\text{atomdensity} * \sigma * t}$$

We substitute σ = 3840 Barns and t = 30 cm to get the detection efficiency for thermal neutrons of BF_3tube under the same conditions but with a length of 30 cm.

$$\text{efficiency} = 0.882$$

This is certainly closer to the textbook value, although still slightly less.

■ Problem 14.5. BF_3 versus ^3He

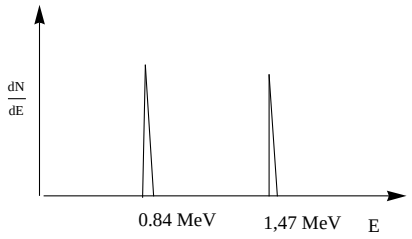
He-3 has the advantages of being easy to work with, being nontoxic and noncorrosive, and has the ability to be used at quite high pressures. Therefore, when efficiency is the primary concern, He-3 is preferred. In general, it is the preferred gas of choice for slow neutron detection, and most manufacturers choose this material in modern thermal neutron detectors. Unfortunately, because the source of He-3 comes from the diminishing weapons program, there is a forecast shortage of this isotope, and researchers are looking at viable alternatives or nontradiational sources to supply the demand for He-3.

■ Problem 14.6. Slope in spectrum.

Recall that the spectrum is showing the energy deposition in the BF_3 tube from the 1.47 MeV α particle and the 0.84 MeV Li nucleus. The steps originate from having one of these particles being directed towards the wall and therefore depositing only a fraction of their full energy in the gas. The rectangular shapes assumed a planr detector wall surface. When the wall has a curvature, as in the tube design, some particles that would have deposited their full energy in the gas with a flat wall will now reach the wall and deposit only part of their energy. This will introduce a slope at the transitions in the spectra. Because the alpha particles have a greater range than the Li nucclei, the effect is more noticeable for this particle, and thus the 1.47 MeV edge, corresponding to where the alpha has deposited all of its energy and the Li has deposited none, will slope more dramatically.

■ Problem 14.7. Ultra-thin-wall boron-lined detector spectrum.

If there is no energy loss in the wall, then the two possibilities are either a full energy deposition of the alpha particle (1.47 MeV) if the alpha particle is ejected into the gas or a full energy deposition of the Li particle (0.84 MeV). Thus, we would expect to isolated peaks at these two energies.



■ Problem 14.8. Scintillation photons from ^{6}LiI(Eu) and Li glass from a thermal neutron.

With Q as the energy released in the ^{6}Li(n,α) reaction, where Q=4.78 MeV, and the values for absolute light yield [photons generated/(MeV deposited)] from Table 8.3 of the text given as:

Q = 4.78 MeV
LiI = 11 000 photons/(MeV)
Liglass = 3500 photons/(MeV)

then the total number of photons produced is Q * light yield, or

$$Q \times \text{LiI} = 52\,580 \text{ photons}$$
$$Q \times \text{Liglass} = 16\,730 \text{ photons}$$

Note that LiI produces about 3 times the number of photons as Li glass, which implies better discrimination against noise or low amplitude pulses.

■ Problem 14.9. Fission chamber efficiency limit.

Increasing the thickness of the fissile material deposit increases the probability that an incidnet neutron will undergo fission, but for detection, at least one of the fission fragments must reach the active gas to cause an ionization. Thus, any fissile material located further from the inner wall than the largest fission product range will not contribute to detection efficiency.

■ Problem 14.10. Self-powered neutron detector (SPND) with nA signal current.

1 nA is the signal current (which is amazingly small), which mean 10^{-9}Coulomb/second is flowing. Since each electron carries 1.6 x 10^{-19} Coulombs/electron, the number of electrons (or beta particles) flowing per second is easily calculated by dividing the signal current by the electron charge. This calculation is shown below.

$$n_e = \frac{10^{-9}\ \text{Coulomb/second}}{\frac{1.6 \times 10^{-19}\ \text{Coulomb}}{\text{electron}}} = \frac{6.24 \times 10^{9}\ \text{electrons}}{\text{second}}$$

■ Problem 14.11. Fractional burnup of Rhodium SPN detector.

The number of reactions that have occurred after time t (number of target atoms lost) is approximated by $N_0\sigma\phi$t (where N_0 is the original number of target atoms ... we ignore the slight loss due to burnup), so the fractional burnup is just $\sigma\phi$t. The thermal absorption cross section is given as σ_{th} = 150 barns (we sum the Table 14.1 two values of 139 and 11 barns), the flux is given as 3 x 10^{13} $/\left(\text{cm}^2 * s\right)$, and the exposure time is given as 6 months. We use this equation to approximate the fractional burn–up after 6 months of exposure:

$$\text{fractional burn–up} = \frac{(150\ \text{barns})\left(3 \times 10^{13}\right)(6\ \text{months})}{\text{cm}^2\ \text{second}} = 0.0710$$

A more rigorous approach is to write the number of target atoms as N and note $\frac{\text{dN}}{\text{dt}} = -\text{N}\sigma\phi$ (by assuming a steady-state current), so that $\frac{N(t)}{N_0} = e^{-\sigma\phi \text{t}}$. The fractional burn-up is given by $\frac{(N_0 - N(t))}{N_0} = 1 - \frac{N(t)}{N_o} = 1 - e^{-\sigma\phi \text{t}}$ (which we had approximated as $\sigma\phi$t above). We calculate the fractional burn-up using this more accurate equation below:

$$\text{fractional burn–up} = 1 - e^{-\frac{(150\ \text{barns})\left(3 \times 10^{13}\right)(6\ \text{months})}{\text{cm}^2\ \text{second}}} = 0.0685$$

where the unit conversions of barns-to-cm^2 and months-to-seconds are not shown in the equation above. Thus, the sensitivity of the detector drops by about 7% over six months. In the case where one is monitoring reactor power using the SPND, this must be accounted for to avoid underestimating the true neutron flux in the core.

Fast Neutron detectors

■ Problem 15.1. Efficiency of ^{6}LiI scintillator for 1 MeV neutrons.

We know that ϵ(E) = 1-exp[$-\Sigma_a(E)\,L$] = 1-exp[$-N_{\mathrm{LiI}}\,\sigma_a(E)\,L$]. For simplicity, we assume that 100% enrichment of ^{6}Li and we neglect absorption of neutrons in I. From Fig. 14.1, we find that σ_a(1 MeV)= .25 barns, and from pg. 522 we know that σ_a(.025 eV)= 940 barns. We are also given that t (=L) = 4 mm. Therefore, the first task is to calculate the atom density of LiI, N_{Li}. This is done by noting that ρ_{LiI}= 4.08 g/cm^3(cf. Table 8.3), and the molar mass of LiI is 133 g/mole (=6+127). From this, it is easy to calculate the atom density:

$$N_{\mathrm{Li}} = \frac{\frac{4.08\,g}{\mathrm{mole}} \times (\mathrm{Avogadro's\ Constant})}{\frac{133\,g}{\mathrm{mole}}} = \frac{1.84739 \times 10^{22}}{\mathrm{cm}^3}$$

Given N_{Li}, we calculate ϵ(E) below.

$$\epsilon(E) = 1 - e^{N_{\mathrm{Li}}\sigma_a t}$$

For fast neutrons, we substitute σ_a = .25 barns and t = 4 mm and the calculated value of $\epsilon(E)$ is :

$$\epsilon_{\mathrm{fast}}(E) = 0.00185$$

For thermal neutrons, we substitute σ_a = 940 barns and t = 4 mm and the calculated value of $\epsilon(E)$ is :

$$\epsilon_{\mathrm{thermal}}(E) = 0.999$$

We learn that the efficiency for 1 MeV neutrons is ~500 times smaller than the efficiency for thermal neutrons. This means that the detector will usually have a peak, i.e. an epithermal peak, even when we are trying to measure predominately fast neutrons since thermal neutrons are ubiquitous. Note that even a 4 mm thick detector is essentially black (i.e., efficiency ~ 100%) to thermal neutrons.

■ Problem 15.2. Moderating Sphere Thickness

With 5 MeV neutrons, it will take several cm of hydrogenated material to thermalize them. As the thicknes of the sphere increases, there is a greater probability that the neutron will become thermalized and diffuse into the detector region. This probability continues to increase with moderator thickness until the loss of the thermal neutrons due to absorption in the bulk and surface leakage outweights any gains in neutron moderation. Increases the thickness beyond this point results in reduced thermal detection efficiency.

■ Problem 15.3. LiI detector

At the center of the moderating sphere is a LiI detector. Since these are now thermal neutrons that are to be detected, the incident neutrons bring a neglibly small incident energy. The most likely ensuing reaction is thus $^6_3\mathrm{Li}(n,\,\alpha)\ ^3_1H$ with a Q value of 4.78 MeV. Thus, we would expect to see a single photopeak at 4.78 MeV since both reaction products will likely be contained in the detector.

■ Problem 15.4. Detection time.

The question here is how long does it take for the fast neutron to induce a pulse? By far the slowest time is the time to moderation followed by diffusion (~ milliseconds). To travel 10 cm at 2200 m/sec (using t=d/v) takes:

$$t = \frac{0.1\ m}{2200\ m/s} = 45.5 \text{ micro seconds}$$

■ Problem 15.5. Find E_{alpha} and E_{triton} in Li sandwich spectrometer. E_n = 3 MeV.

This is similar to Problem 1.3. We must satisfy conservation of momentum and energy. The variables are self-explanatory, and the conservation of momentum and energy are written as:

$$p_n = p_\alpha + p_t \qquad KE_n + Q = KE_\alpha + KE_t \qquad KE_n = \frac{p_n^2}{2\,m_n}$$

$$KE_\alpha = \frac{p_\alpha^2}{2\,m_\alpha} \qquad KE_t = \frac{p_t^2}{2\,m_t} \qquad m_t = 3\,m_n \qquad m_\alpha = 4\,m_n$$

We solve these quadratic equations for KE_α and KE_t (all energies are in MeV) by eliminating momenta, and find the reaction product kinetic energies:

$$KE_\alpha = \frac{1}{49}\left(-4\sqrt{3}\ \sqrt{7\,Q\,KE_n + 6\,KE_n^2} + 22\,KE_n + 21\,Q\right)$$

$$KE_t = \frac{1}{49}\left(4\sqrt{3}\ \sqrt{7\,Q\,KE_n + 6\,KE_n^2} + 27\,KE_n + 28\,Q\right)$$

and

$$KE_\alpha = \frac{1}{49}\sqrt{3}\ 4\sqrt{7\,Q\,KE_n + 6\,KE_n^2} + \frac{22\,KE_n}{49} + \frac{3\,Q}{7}$$

$$KE_t = \frac{1}{49}\left(-4\sqrt{3}\ \sqrt{7\,Q\,KE_n + 6\,KE_n^2} + 27\,KE_n + 28\,Q\right)$$

We note that there are two possible solutions for KE_α and two solutions for KE_t. We substitute Q=4.78 and KE_n = 3 to evaluate both of them.

1) $KE_\alpha = 6.14$ $KE_t = 1.64$

2) $KE_\alpha = 2.63$ $KE_t = 5.15$

The two solutions correspond to the cases where the alpha goes in the forward direction or backward direction, respectively. The former is the case for this problem. The fact that both particles have > 1 MeV worth of energy is encouraging. We could take these energy values, and look at the particles' ranges in Li to get an idea of the maximum thickness of Li we can have and still have the reaction products emerge into the Si diode detector for detection.

■ Problem 15.6. Maximum proton energy from ^{3}He(n,p)^{3}t reaction from 1.5 MeV neutrons. Q= 0.764 MeV

We write down our conservation of momentum and energy equations and solve for the desired energies of the reaction products:

$$p_n = p_p + p_t \qquad \mathrm{KE}_n + Q = \mathrm{KE}_p + \mathrm{KE}_t \qquad \mathrm{KE}_n = \frac{{p_n}^2}{2\,m_n}$$

$$\mathrm{KE}_p = \frac{{p_p}^2}{2\,m_p} \qquad \mathrm{KE}_t = \frac{{p_t}^2}{2\,m_t} \qquad m_t = 3\,m_n \qquad m_p = m_n$$

We solve for the solution of these equations for KE_α and KE_t (all energies in MeV), the reaction products:

$$\mathrm{KE}_t = \frac{1}{8}\left(-\sqrt{3}\,\sqrt{4\,Q\,\mathrm{KE}_n + 3\,\mathrm{KE}_n^2} + 3\,\mathrm{KE}_n + 2\,Q\right)$$

$$\mathrm{KE}_p = \frac{1}{8}\left(\sqrt{3}\,\sqrt{4\,Q\,\mathrm{KE}_n + 3\,\mathrm{KE}_n^2} + 5\,\mathrm{KE}_n + 6\,Q\right)$$

and

$$\mathrm{KE}_t = \frac{1}{8}\sqrt{3}\,\sqrt{4\,Q\,\mathrm{KE}_n + 3\,\mathrm{KE}_n^2} + \frac{3\,\mathrm{KE}_n}{8} + \frac{Q}{4}$$

$$\mathrm{KE}_p = \frac{1}{8}\left(-\sqrt{3}\,\sqrt{4\,Q\,\mathrm{KE}_n + 3\,\mathrm{KE}_n^2} + 5\,\mathrm{KE}_n + 6\,Q\right)$$

We evaluate the first solution for the given initial conditions: Q=0.764 MeV and KE_n = 1.5 MeV

$$\mathrm{KE}_t = 0.0246\ \mathrm{MeV}$$
$$\mathrm{KE}_p = 2.24\ \mathrm{MeV}$$

The second solution evaluated given the same initial conditions yields:

$$\mathrm{KE}_t = 1.48\ \mathrm{MeV}$$
$$\mathrm{KE}_p = 0.782\ \mathrm{MeV}$$

Clearly the first case has the proton with the most energy -- i.e. forward directed, with 2.24 MeV of energy.

■ Problem 15.7. Thermal peak for fast neutrons

Although we may be trying to measure the fast neutron spectrum, in practice there will always be thermal neutrons also present. This is inevitable since the surrounding environment, such as the walls, floor, and/or people, will serve as a moderating material that will return some thermal neutrons to the detector. These will give rise to a detected peak at the Q value of the detection reaction, which, for He-3, will be 764 keV.

■ Problem 15.8. Detection efficiency of proton recoil proportional counter.

100 keV neutrons are incident on a detector having path length t=5 cm and p=1 atm of methane (CH_4). Find the number density of molecules in the gas first using the ideal gas law PV=nRT.

$$N_{CH_4} = \frac{(\text{Avogadro's Constant}) \times \text{Pressure}}{(\text{Molar Gas Constant}) \times \text{Temperature}}$$

We substitute Pressure = 1 Atm and Temperature = 300K to get the number density of the gas.

$$N_{CH_4} = 2.44631 \times 10^{19}\ \text{cm}^{-3}$$

Note that this density, typical of a gas, is low and will lead to a low detection efficiency unless the interaction cross section is abnormally large. The cross section for H(100 keV) is σ_H =15 barns, and for carbon, σ_C= 4.5 barns from Figure 15-16. The efficiency is given by eqn. 15-9 and the results are calculated below:

$$\text{efficiency} = \frac{(N_H \sigma_H)\left(1 - e^{-(N_C \sigma_C + N_H \sigma_H)t}\right)}{N_C \sigma_C + N_H \sigma_H}$$

We substitute σ_H = 15 Barns, σ_C = 4.5 Barns, t = 5 cm, $N_H = 4\, N_{CH_4}$ and $N_C = N_{CH_4}$ to get the efficiency :

efficiency = 0.00731

Note that this is a very low efficiency, typical of gas filled detectors relying on fast neutron elastic scattering.

■ Problem 15.9. Fast Neutron Scatterings separated by 3 cm.

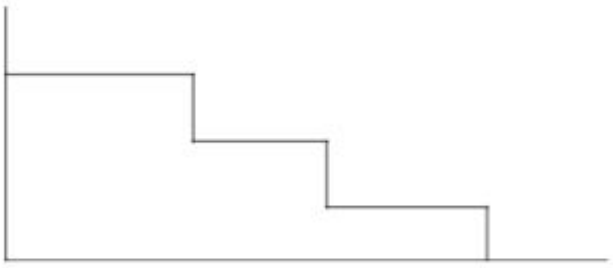
We recall that, in the laboratory frame of reference, $E_n' = E_n \cos^2 \theta_n$. (You should be able to prove this using the conservation of momentum and energy equations). The time between the two scatterings is given by $t = \Delta x \Big/ \sqrt{2\, E_n'/m}$] so:

$$t = \frac{\Delta x}{\sqrt{\frac{2\,\text{KE}_n \cos^2(\theta)}{m_n}}}$$ where we substitute θ = 40 °, KE_n = MeV and Δx = 3 cm to yield :

t = 2.83 nano seconds

This time is significantly less than the anode time constant, so the two events cannot be resolved in time.

■ Problem 15.10. Ideal DPHS from proton recoil detector from 3 neutron energies.



Since each neutron energy gives rise to a shoe-box shape response function, the combination of the three incident energies yields a pulse height spectrum like the one shown above.

■ Problem 15.11. Show the angle between scattered neutron and reoil proton must be 90 degrees in lab frame.

This is easiest to prove using vector notation. Writing conservation of momentum in vector notation: $\overline{p}_p + \overline{p}_{n'} = \overline{p}_{n_o}$.

Square this equation (i.e., dot this equation with itself) and then using $m_n = m_p = m$, and $p^2 = 2$ mE, and applying conservation of energy yields:

$\overline{p}_p \bullet \overline{p}_{n'} = 0.$

This can only be always true if the angle between these two vectors is 90 degrees.

■ Problem 15.12. Fast neutron pulse height in gas proportional recoil detector

We recall that the multiplication is given by Diethorne's model, and we calculate M using the problem's parameters for methane [Table 6-1 of text].

$$M = e^{\dfrac{V \ln(2) \ln\left(\dfrac{V}{K\, p\, a \ln\left(\frac{b}{a}\right)}\right)}{\ln\left(\frac{b}{a}\right) \Delta V}}$$

Substituting V=2000, b=2, a=0.005, ΔV=36.5, K=6.9×10^{4}and p=.75 into the above equation yields:

$$M = 5.02589$$

The pulse height is given by $n_0 \frac{eM}{C}$or:

$$\text{pulse height} = \frac{E_{\text{deposit}}\ (\text{Electron Charge})\ M}{29\ \text{eV} \times\ 60\ \text{pF}}$$

pulse height = 0.463 millivolts

■ Problem 15.13. Si recoil energies from 1 MeV neutrons

The recoil nucleus energy can vary from 0 (for a glancing 90° scatter), up to a direct hit (0°) yielding a maximum Si recoil energy of:

$$\text{recoil energy} = \left(\frac{4A}{(A+1)^2}\right) \text{KE}_n$$ where we substitute $A = 28$ and $\text{KE}_n = 1$ MeV

recoil energy = 133 keV

Note that this is very small -- only 133 keV -- and it won't travel very far at all since it is massive and highly charged, but it is large enough to dislodge the atom from its place in the lattice. These crystal defects can lead to degraded energy resolution over time, and is the reason that semiconductor detectors are often avoided when fast neutrons must be measured. This is a particular problem for long space flights, for example, and detectors are sometimes annealed during the transit.

■ Problem 15.14. Energy distribution from neutron scattering on H or He?

Neutron scattering is isotropic (when viewed in the center-of-mass system) off of hydrogen, meaning that probability to scatter into any particular angle is the same as all others. Since the energy distribution is related to the angle of scattering, this means that all deposited energies are also equally probabile. For helium, the scattering is not isotropic, instead larger and smaller scattering angles are more likely. This can be seen in the plot of σ(E) for He in the textbook. Since the scattering angles are not equally probable, and there is a direct relationship between energy and scattering angle, the detector deposited energy will not be uniform.

■ Problem 15.15. Efficiency of proton radiator in proton recoil spectrometer.

The proton recoil spectrometer can provide an accurate measurement of the fast neutron spectrum, but is usually quite inefficient. The efficiency can be improved if the hydrogenated scattering material is made thicker, but only up to a point. The detection reaction is (n,p), and it is the proton which must escape from the radiator with a neglible loss of energy and then its energy must be accurately detected separately. If the scattering material becomes too thick, the proton will not be able to escape without losing a significant fraction of its kinetic energy.

■ Problem 15.16. Capture Gated Neutron Spectrometer

Using BC-454 as an example, calculate the time of thermal diffusion before absorption. From the text, we expect a number on the order of 10-20 microseconds.

We first find the mean free path of a thermal neutron in the material. The material is 5% by weight of natural boron, which is 19.8 atomic percent of B^{10}, and has an atomic weight of 10.82 gm/mole. The density of the material, using table 8.1, is 1.026 $\mathrm{gm}/\mathrm{cm}^3$. So the weight density of B^{10}atoms is just:

$$\textbf{weight density} = \frac{1.026\ g}{\mathrm{cm}^3} \times 5\,\% \times \frac{(10)\,(19.8\,\%)}{(10)\,(19.8\,\%) + (11)\,(80.2\,\%)} = \frac{0.00940\ g}{\mathrm{cm}^3}$$

Converting this to a number density of B-10 atoms:

$$N_{B-10} = \frac{0.00940326\ g}{\mathrm{cm}^3} \times \frac{(\text{Avogadro's Constant})}{10\ g/\text{mole}} = 5.663 \times 10^{20}\ \mathrm{cm}^{-3}$$

Multiplying this number density of B-10 atoms by the microscopic thermal cross section for absorption of 3840 barns gives the macroscopic cross section. Inverting this value gives the mean free path:

$$\text{mfp} = \frac{1}{3840\ \text{barns}\left(5.66277 \times 10^{20}\ \mathrm{cm}^{-3}\right)} = 0.460\ \text{cm}$$

The diffusion time will be this distance divided by the velocity of a thermal neutron (2200 m/s):

$$t_{\text{diffusion}} = \frac{\text{mpf}}{2200\ m/s} = 2.09\ \text{micro seconds}$$

The observed times (~ 10's of microseconds) are longer because the calculation above ignores the time **to** thermalization.

- **Supplemental Problem: Fast neutron recoil spectrometer. Statistics of Distributions**

Two hydrogenous scintillators of identical dimensions are fabricated for use as neutron recoil spectrometers. While the first detector is composed of only ordinary protons, the second detector has only deuterons. Assuming that the scattering cross sections are identical for protons and deuterons,
(a). graph the spectra expected from monoenergetic fast neutrons of En = 2.5 MeV using: (i) the protonated scintillator, (ii) the deuterated scintillator, and (iii) the subtracted spectrum, i.e., (spectrum (i) - spectrum (ii))

(b). Suppose that instead of waiting for the two spectra to be completed and then subtracting them, we measure an energy from the protonated scintillator, subtract the energy of the next event measured by the deuterated scintillator, and then record the net energy measured. In this case, what does the expected differential pulse height spectrum look like?

Pulse processing and shaping

■ Problem 16.1. Pulse travel time

For the cable RG-59U, Table 16.1 lists the pulse speed as .659c, so to travel d=15 meters would take:

$$t = \frac{d}{v} = 7.59 \times 10^{-8} \text{ seconds}$$

■ Problem 16.2. Characteristic Impedance of a cable

Using an oscilloscope, vary the termination resistance until the observed reflections disappear. When the impedance of the termination resistance matches the characteristic impedance of the cable, the signal will not be reflected -- the cable will appear to the signal as if it is infinite in length. The characteristic impedance of the cable is that value of the termination resistance for which there is no signal reflected.

■ Problem 16.3. Situations for which cable termination is needed

Terminating cables in their characteristic impedance is important to avoid reflections of pulses and to transmit the full pulse amplitude to subsequent signal processing elements. This is most critical for pulses whose rise time is short compared to the pulse transit time.

(a). Signals in RG-59/U cables travel at 0.659c, so in 20 m, the pulse transit time is:

$$t_{\text{transit}} = \frac{20}{0.659 \times 3 \times 10^8} = 101 \text{ ns}$$

If the pulse has a rise time of 500 ns, then because this is long compared to the transit time (i.e., the pulse is "slow"), proper cable termination is less important.

(b). Signals in RG-62U cables travel at 0.84c, so in 10 m, the pulse transit time is:

$$t_{\text{transit}} = \frac{10}{0.840 \times 3 \times 10^8} = 39.7 \text{ ns}$$

If the pulse has a rise time of 10 ns, then because the rise time is short compared to the transit time (i.e., the pulse is "fast"), matching the input impedance of the next device in the signal chain through the use of a terminating resistance is important.

■ Problem 16.4*. Impedance matching for fast pulses.

(a). We choose RG-58C/U with $Z_{\text{ch}} = 50\,\Omega$ -- this matches the impedance at the output end of the first device. In order to prevent reflections the pulse must see no discontinuity in impedance. This means that at the input of the device, the incoming pulse must see an effective 50 Ω resistance. We need to add a shunt resistor in parallel with the input impedance so that the combined impedance matches that of the cable:

$$R_{\text{cable}} == \frac{1}{\frac{1}{R_{\text{term}}} + \frac{1}{R_{\text{in}}}}$$

We solve the above equation for R_{term} and substitute R_{in} = 1000 and R_{cable}= 50 to get the termination resistance.

$$\boldsymbol{R_{term} = 52.6316\ \Omega}$$

This is saying that for a high input impedance device, provide a termination equivalent to the cable impedance (which was chosen to match the (much lower) output impedance of the previous device in the chain).

(b). Now find the voltage across the 1 kΩ resistance if the output voltage is 5 V from the previous device in the chain when it has no load on it. This is the value of the voltage source. When current flows through the output impedance, the actual output voltage will drop. Since $R_{term} << R_{input}$, the current flows primarily through the shunt resistor. The circuit equations which apply are:

$$\boldsymbol{V_{in} = V_s - i\,(R_{cable} + R_{out})}$$

$$\boldsymbol{V_{in} = i\,R_{term}}$$

$$\boldsymbol{V_{in} = \frac{R_{term}\,V_s}{R_{cable} + R_{out} + R_{term}}}$$

Recognizing that the terminating (shunt) resistor, the output impedance, and the cable impedance are about the same value of 50 Ω, the voltage at the input is given by:

$$\boldsymbol{V_{in} = \frac{V_s}{3}}$$

Substituting in the numerical value of the output voltage (V_s = 5) from the previous device yields:

$$\boldsymbol{V_{in} = 1.67\ \text{Volts}}$$

This says that the next device in the chain sees an input voltage of about a third of the output voltage of the previous device in the chain.

■ Problem 16.5. Pulse attenuator

We use a pulse attenuator shown in Figure 16.8(b). The key to this design is that the input impedance must be $R_0$and the load also has an equivalent resistance of R_0. The equations shown in the Figure are derived below, where the first equation states that the equivalent input impedance (shown on the right side of the equation) is to be R_0.

$$\boldsymbol{R_0 = R_1 + \frac{R_2\,(R_1 + R_0)}{(R_1 + R_2 + R_0)}} \qquad \boldsymbol{\alpha = \frac{V_{in}}{V_{out}}} \qquad \boldsymbol{i_1 = i_2 + i_3}$$

$$\boldsymbol{V_{in} = i_2\,R_2 + i_1\,R_1} \qquad \boldsymbol{V_{in} = i_3\,R_1 + i_3\,R_0 + i_1\,R_1} \qquad \boldsymbol{V_{out} = i_3\,R_0}$$

Solving the above equations for R_1 and $R_2$by eliminating V_{in}, V_{out}, i_1, i_2 and i_3 yields:

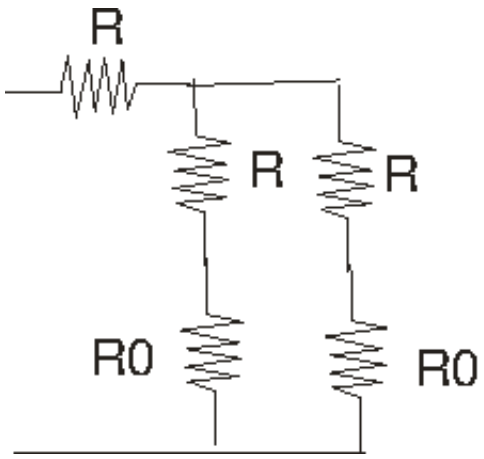
$$\boldsymbol{R_1 = \frac{R_0\,(\alpha - 1)}{\alpha + 1}} \quad \textbf{and} \quad \boldsymbol{R_2 = \frac{2\,R_0\,\alpha}{(\alpha - 1)\,(\alpha + 1)}}$$

For the problem being asked, we substitute α=10 and R_0 = 50 Ω to get our answers:

$$\boldsymbol{R_1 = 40.9\ \Omega} \quad \textbf{and} \quad \boldsymbol{R_2 = 10.1\ \Omega}$$

Problem 16.6. Pulse splitter

We note that the circuit can be drawn more easily as:



We want this to have the equivalent resistance of R_0. Deriving the equivalent resistance $R_{eq} = (3/2)\,R + (1/2)\,R_0$ and setting this equal to R_0 yields the result that R= (1/3) R_0. For R_0= 50 Ω for this problem, yields:

$$R = \frac{50}{3} = 16.7\ \Omega$$

Problem 16.7. Decay time of exponential tail.

We are looking for the time at which the decayed pulse reaches only 1% of its original amplitude. At that time, another pulse riding on it will be measured to be too large by 1%. We use time units of microseconds and solve:

$$e^{-t/\tau} = .01$$

We solve the above equation for t and substitute τ=50 to find the time at which the decayed pulse reaches only 1% of its original amplitude.

$$t = 230\ \mu s$$

Problem 16.8. Voltage sensitive pre-amplifier derivation.

The equations of interest are (balancing current flow and voltage drops):

$$V_{in} = i\,(R_1 + R_2) + V_{out}$$
$$V_{out} = -A\,(V_{in} - i\,R_1)$$

We solve the above equations to find an expression for V_{out} that is independent of i.

$$V_{out} = -\frac{A\,R_2\,V_{in}}{R_1 + A\,R_1 + R_2}$$

Let's look at the numerator and denominator separately after dividing each by AR_2:

$$\textbf{Numerator :} \qquad -V_{in}$$

$$\textbf{Denominator :} \qquad \frac{1}{A} + \frac{R_1}{R_2} + \frac{R_1}{A\,R_2}$$

Look at the denominator and collect terms with A:

$$\frac{1 + \frac{R_1}{R_2}}{A} + \frac{R_1}{R_2}$$

This first term will be negligible if A >> $\frac{R_1+R_2}{R_2}$, in which case $V_{out} \sim \frac{R_2}{R_1} V_{in}$. q.e.d.

■ Problem 16.9. Charge sensitive pre-amplifier derivation.

The circuit equations of interest are (balancing charge flow and voltage drops):

$$V_{in} = \frac{Q_f}{C_f} + V_{out} \qquad V_{in} = \frac{Q_{in}}{C_{in}} \qquad V_{out} = -A\,V_{in} \qquad Q = Q_f + Q_{in}$$

Solving the system of equations above for V_{out} gives us :

$$V_{out} = \frac{-A\,Q}{C_f + A\,C_f + C_{in}}$$

Note that the feedback resistor doesn't play a role here. All of the initial charge flow goes into charging the capacitors. Let's look at the solution more carefully by multiplying the numerator and denominator by $\frac{1}{A\,C_f}$ and expanding it out:

$$\textbf{Numerator :} \qquad -\frac{Q}{C_f}$$

$$\textbf{Denominator :} \qquad 1 + \frac{1}{A} + \frac{C_{in}}{A\,C_f}$$

If the denominator ~1, then $V_{out} \sim \frac{Q}{C_f}$. So we need A >> $\frac{(C_{in}+C_f)}{C_f}$ for this condition to be true. (To prove this, simplify the denominator:

$$1 + \frac{1 + \frac{C_{in}}{C_f}}{A}$$

For this denominator to be ~1, the condition above for A >> ... must be true).

■ Problem 16.10. Preamp location important.

The preamplifier is normally kept close to the detector in order to keep the connecting cable as short as possible. This cable acts like a capacitor with a capacitance that increases linearly with length. Since increased capacitance loading is associated with increased noise, it is beneficial to keep this capacitance as small as possible. After the preamplifier has amplified the signal (with minimal noise from the cable capacitance hopefully), then any additional cable noise will be relatively small. In fact, the common terminology is that the preamplifier terminates the capacitance.

Pulse Shaping, Counting, and Timing

■ Problem 17.1. Find output form of large-RC integrating circuit with input voltage form 1-$e^{-t/k}$.

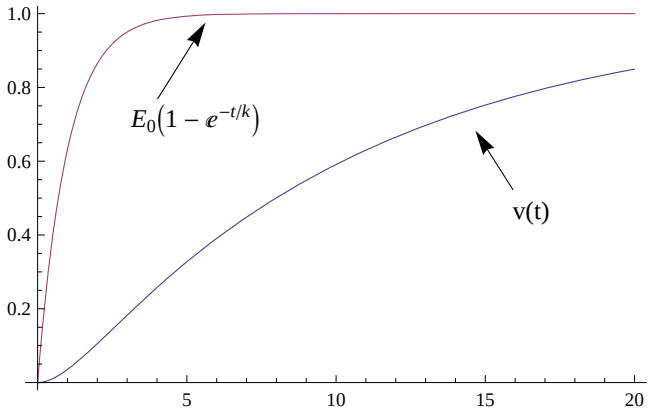
We write down the differential equation governing the time dependence of the output voltage pulse given an input shape of 1-$e^{-t/k}$, and solve it for the boundary condition:

$$v'(t) + \frac{v(t)}{\tau} = E_0\left(1 - e^{-t/k}\right) \qquad \text{where } v(0) = 0$$

which has the solution:

$$v(t) = \frac{E_0\,\tau}{(k-\tau)\,e^{t/\tau}} + \frac{E_0\left(\tau\, e^{t/k} + k\left(-e^{t/k}\right) + k\right)}{e^{t/k}\,(\tau - k)} = E_0\left(1 - e^{-\frac{t}{k}}\right)$$

This is the answer for the voltage as a function of time. To see what this looks like, let's choose E_0=1, τ=10 and k=1. In this case RC is large, and we see integration and the passing of the low-frequency component. Try the example below with RC (i.e., τ) small, and the output is the same shape as the input.



■ Problem 17.2. CR circuit response to a sinusoidal input.

$$\tau = 500\ \text{pF} \times 500\ \Omega = 250\,000\ \text{ps} = 0.25\ \mu\text{s}$$

The amplitude of the output/input pulses is given as 0.5, so we need to solve for the frequency from:

$$\left(1 + \frac{1}{(2\,\pi\tau f)^2}\right)^{-0.5} = 0.5$$

$$f = 368\ \text{Kilo Hertz}$$

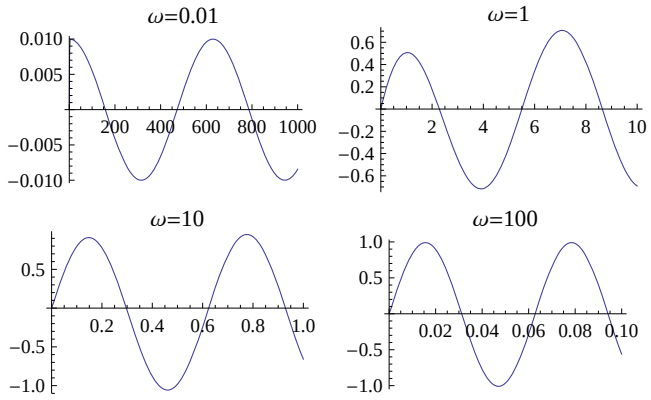
Let's look at this circuit more carefully. We write the differential equation governing the response:

$$v'(t) + \frac{v(t)}{\tau} = \frac{d}{dt}(\sin(\omega t)) \qquad \text{where } v(0) = 0$$

which has a solution:

$$v(t) = \frac{\tau\,\omega\left(\tau\,\omega\,\sin(t\,\omega) - e^{-\frac{t}{\tau}} + \cos(t\,\omega)\right)}{\tau^2\,\omega^2 + 1}$$

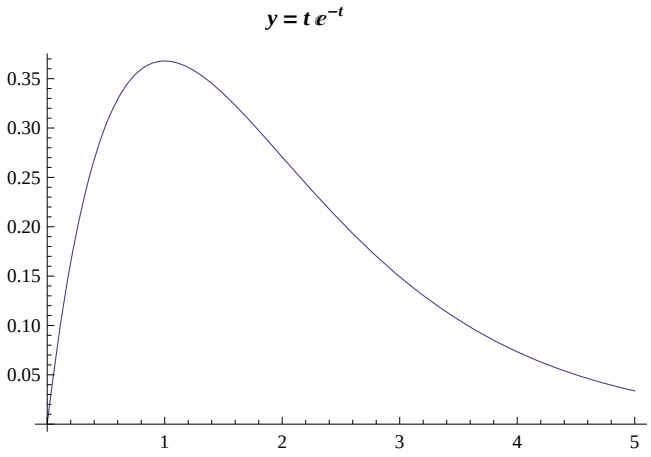
Now we'll fix τ and vary ω and watch how the output changes. We expect that the CR circuit is a high pass filter, so lower frequencies will be attenuated. Choose ω values of {0.1,1.0, 10, 100}. Watch how the amplitude changes on the ordinate.



Experiment with different input functions and watch the output. You'll see that the CR circuit differentiates (if τ is small) and acts as a high pass filter.

■ Problem 17.3 CR-RC network. Maximum amplitude of shaped pulse.

The pulse shape is shown below, where we measure time in units of τ:



The maximum of the pulse occurs at the time where the derivative is zero:

$$\frac{d}{dt}\left(t\,e^{-t}\right) = 0$$

Taking the derivative, setting it equal to zero, and solving for t yields:

$$t_{max} = 1$$

and the value of the function there is:

$$y(1) = \frac{1}{e} = 0.368$$

so the maximum value is 37% of the input voltage and occurs at a time t=τ.

■ Problem 17.4. Derive the CR-RC shaping circuit output for a step input.

Let's start this problem with the CR circuit first. The differential equation which needs to be solved is:

$$v'(t) + \frac{v(t)}{\tau} = 0 \qquad \text{where } v(0) = v_0$$

which has a solution:

$$v(t) = \frac{v_0}{e^{t/\tau}}$$

Note that the CR circuit works by charging up the capacitor to a high enough voltage so no current needs to flow through the resistor.

The RC circuit, which will use this previous v(t) as its input, must satisfy the differential equation:

$$v'(t) + \frac{v(t)}{\tau_2} = \frac{1}{\tau_2}\left(\frac{v_0}{e^{t/\tau}}\right) \qquad \text{where } v(0) = 0$$

This has a solution:

$$v(t) = \frac{\tau v_0\left(e^{-\frac{t}{\tau_2}} - e^{-\frac{t}{\tau}}\right)}{\tau_2 - \tau}\Bigg\}$$

The key to this CR-RC problem is to recognize that the initial voltage across the capacitor has to be 0! At the instant that $E_0$is applied, all voltage is dropped across the resistor until the capacitor begin to charge. So the initial output voltage from the circuit always has to be 0.

As an aside let's solve the same problem with equal time constants, τ.

$$v'(t) + \frac{v(t)}{\tau} = \frac{1}{\tau}\left(\frac{v_0}{e^{t/\tau}}\right)$$

The solution to this equation (the clever student can verify this) is:

$$v(t) = \frac{t\,v_0}{\tau\, e^{t/\tau}}$$

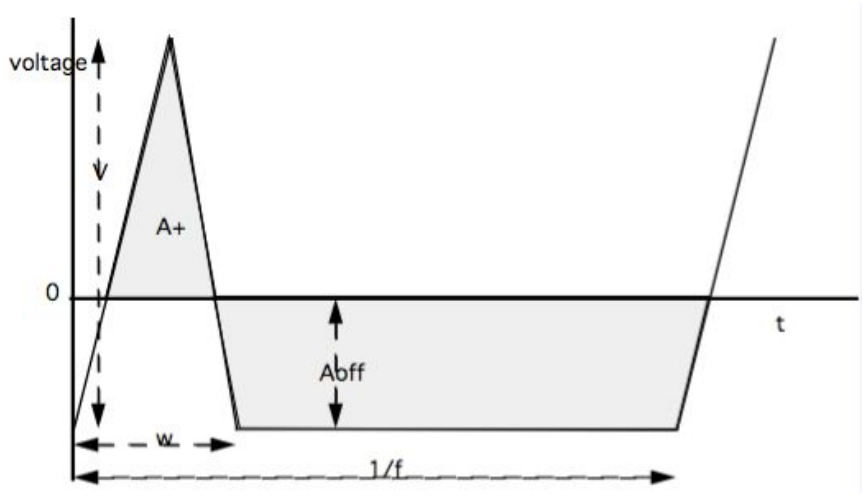
which agrees with the equation given in the text.

■ Problem 17.5. Advantage of bipolar pulses

The bipolar pulse, because it has essentially 0 net voltage over its duration, is less susceptible to baseline shift. At high count rates, this is particularly advantageous.

Problem 17.6. Baseline shift.

The origin of the problem is that the capacitor cannot have any non-zero dc current component. Because of this, any component past the capacitor has zero average current and thus, zero average voltage. On a plot of voltage versus time, the positive area is given by $\frac{(V-A_{off})^2\,w}{2\,V}$, and the negative area is given by $\frac{A_{off}{}^2\,w}{2\,V} + A_{off}\left(\frac{1}{f} - w\right)$, where f is the frequency of the pulses, w is the base width of the pulse, V is total pulse height, and A_{off} is the voltage offset below zero between pulses. Equating positive and negative areas and solving for A_{off} yields:



$$A_{off}\left(\frac{1}{f} - w\right) + \frac{w\,A_{off}^2}{2\,V} = \frac{w\,(V - A_{off})^2}{2\,V}$$

which yields:

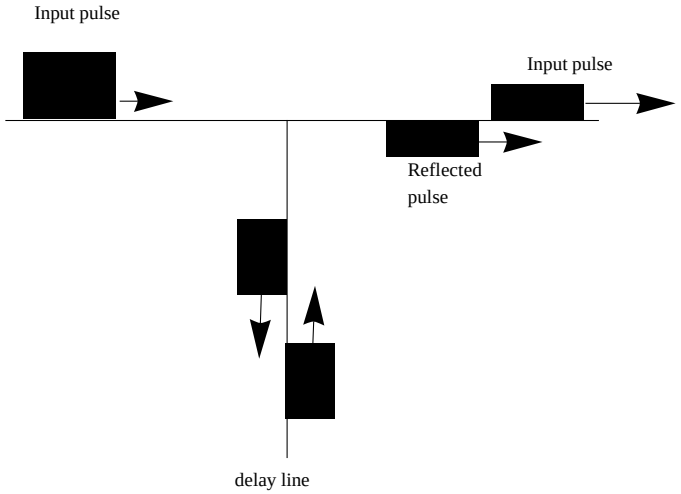
$$A_{off} = \frac{f\,V\,w}{2}$$

We substitute f=100 and 50,000,V=10 and w=$5\times10^{-6}$to get the offset voltages (in volts).

A_{off} = 0.025 volts and 1.25 volts

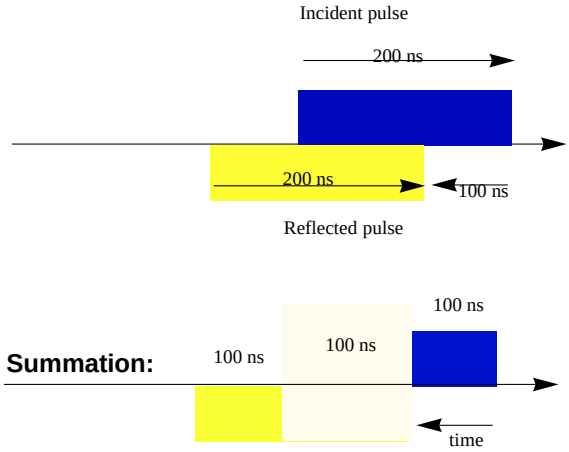
■ Problem 17.7. Pulse shaping using a shorted delay line.

The idea is to take a monopolar pulse and combine it with the reflection from delay line to form a bipolar pulse. A diagram is useful.



The time delay between the start of the transmitted input pulse and the start of the reflected pulse is just the transit time down and back the delay line. This time is t = d/v= $\frac{20}{0.659 \times 3 \times 10^8}$ = 101 ns.

Since the input pulse is 200 ns wide, the reflected pulse meets up with it and cancels it after 101 ns (~100 ns). After 200 ns, the incident pulse has ended and the reflected pulse continues for 100 ns longer. The result is shown in the summation below:



Problem 17.8. Logic Function. Replace an SCA by two integral discriminators and anti-coincidence unit.

The SCA logic function is:

$$(V < ULD) \cap (V > LLD) = True$$

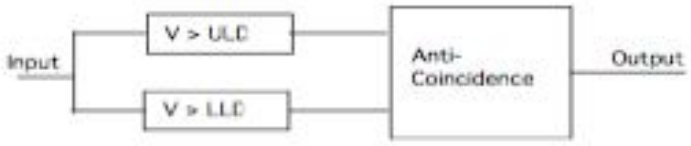
This can be rewritten as (where ¬ is defined as "not"):

$$\neg(V > ULD) \cap (V > LLD) = True$$

or:

$$(V > LLD) \cap \neg (V > ULD) = True$$

The (V >) test is operationally accomplished by using an integral discriminator and the $\cap\neg$ operation is accomplished by an anticoincidence unit. Thus, the same logic function as the SCA is accomplished using:



- **Problem 17.9. Rate Meter Time Constant Selection**

See text Chapter 17.II.E (p. 646). A longer time constant, RC, will smooth the output from random pulses so that the measured count rate is not jumpy. The disadvantage is the inability to catch rapid fluctuations in the count rate.

- **Problem 17.10. Best choice of shaping parameters for spectroscopy**

A longer time constant allows a longer integration time to accumulate all of the charge carriers, although too long of a time constant will add noise to the measurement. There is normally an optimal shaping time constant, perhaps a factor of twice the maximum charge collection time.

At low rates, monopolar shaping is used since the additional steps to produce bipolar pulses can add noise to the signal.

Baseline restoration is useful at high rates, but also introduces electronic noise. At low count rates, one normally will not engage this feature.

■ Problem 17.11. Pileup losses.

There is frequent confusion on the difference between deadtime losses and pileup losses. Deadtime losses are associated with pulse counting systems in which the system is incapable of counting the next event for a time τ (or longer) after the event. If τ is extendable, the system is termed paralyzable. For deadtime losses, we are concerned with the number of counts we get from the incoming events. The probability that an event will yield a count is $\frac{m}{n}$. Chapter 4 discusses this topic.

Having registered a count, we might also ask whether that count is pileup free. That is, that nothing occurs for a time τ after the arrival of the event being counted so that the measured amplitude is representative of a single event. Pileup losses are of concern with spectroscopy systems which must measure single event pulse amplitudes. In the case of multiple events following a count, we map the count to an improper channel. For a ***count*** to be pileup free, we must have no events arriving for a period τ after the count. So the ***fraction of events*** which give pileup free counts is given by:

Probability that an event yields a count • Probability that a count is pileup free

$$= \frac{m}{n} e^{-n\tau}.$$

Substituting for m/n for the nonparalyzable deadtime expression yields for this problem, the fraction of events yielding pileup free counts:

$$\textbf{pileup free counts} = \frac{e^{-n\tau}}{1 + n\tau}$$ where we substitute $n =$ 25 000 sec^{-1} and $\tau = 4 \times 10^{-6}$ seconds to get our answer :

pileup free counts = 0.823

■ Problem 17.12. Best timing method.

See text, p. 685. Leading edge timing is the simplest time and most accurate pick off method if rise time and amplitude walk is not an issue since it does not require additional electronics that can add noise to the measurement.

■ Problem 17.13. Bipolar shaping methods

The double dealy line method and CR-RC-CR double differentiating methods both produce bipolar pulses. Review your textbook for the details of how these work.

■ Problem 17.14. Time anolog to SCA

The Coincidence Unit performs the analogous function to the SCA, but in time unts. If two pulses arrive within the coincidence resolving time τ, the output is a logic pulse indicating the condition has been met.

■ Problem 17.15. Output rate for an anticoincidence unit with coincident window τ.

The anticoincident unit will output a pulse only if there are NOT two pulses which arrive within a τ time of each other. The probability that a count will not arrive during time τ is given by $e^{-r\tau}$ so our answer is:

probability $= r_1 e^{-r_2 \tau} + r_2 e^{-r_1 \tau}$

For small values of rτ, we expand the answer in a Taylor series around 0 to order 1:

$$\textbf{probability} = r_1 + r_2 - 2\,\tau\, r_1\, r_2$$

This makes sense. We count $r_1 + r_2$ minus the chance coincident rate.

■ Problem 17.16. Measuring the chance coincidence contribution.

The simplest method to measure the chance coincidence contribution is to deliberately prevent the true concidences from registering by choosing a value for the delay that ensures that no true coincidences can be counted. Once can also use the calculated value of $2\,\tau\, r_1\, r_2$ if the individual singles rates (r_1 and r_2) are known along with τ.

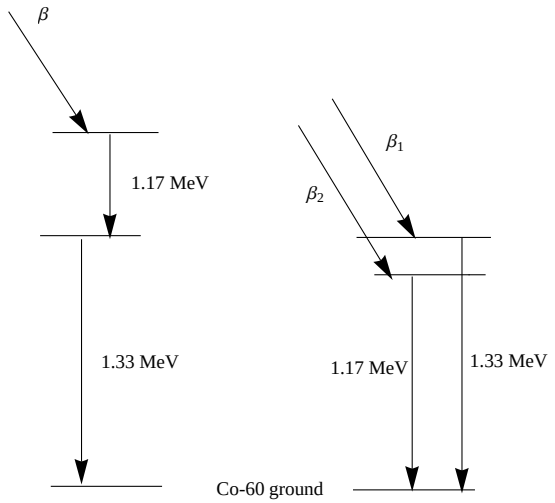
■ Problem 17.17. Length of cable needed for 100 ns delay

For RG- 59cable, the signal velocity is 0.659c or 1.98 x 10^8m/sec or, inverting this, 5.06 ns/m. So for a delay of 100 ns, we require a length of cable, L:

$$L = \frac{\textbf{100 ns}}{\textbf{5.06 (ns/}m\textbf{)}} = \textbf{19.76 }m$$

■ Problem 17.18. ^{60}Co decay scheme

Two possibilities to explain the emissions are:



These two possibilities could be resolved by either β spectroscopy or by a coincidence measurement to see whether the two gamma rays are emitted in a cascade, i.e., in coincidence.

■ Problem 17.19. Improving true to chance coincidence ratio.

In answering these questions, recall that the chance coincidence rate is $r_{true} = S\,\epsilon_1\,\epsilon_2$ and $r_{chance} = (2\,r_1\,r_2\,\tau) = 2\,(S\,\epsilon_1\,S\,\epsilon_2\,\tau)$ so

$$\frac{r_{tr}}{r_{ch}} = \frac{1}{2\,S\tau}$$

a. Changing the solid angle does not change S or τ so there is no effect.

b. Increasing S reduces the true to chance coincidence ratio as shown in the equation above.

c. Increasing τ reduces the ratio above.

d. Increasing the energy window slightly only changes the efficiency by which the pulses are selected. If one opens this window too large so that pulses other than the desired energy are included, then additional pulses that are not coincidence pulses are allowed, thereby effecively increasing the number of non-true coincidence pulses and thereby decreasing the ratio.

■ Problem 17.20. Interpretting a Coincidence Delay curve experiment

(a). The width of the step is 2τ, so $\tau = \frac{125-25}{2} = 50\,\text{ns}$

(b). The full width at the base of the prompt coincidence peak in the time spectrum is what gives rise to the sloped edges on the coincidence delay curve, and is 140-110=40-10= 30 ns.

(c). Assuming the singles rates are the same, i.e., $r_1 = r_2 = r_s$, then using $r_{chance} = 2\,r_s^2\,\tau$, and noting from the curve that the background is the chance rate $r_{ch} = 5\,\text{cps}$, we solve for r_s :

$$r_s = \sqrt{\frac{r_{chance}}{2\,\tau}} = 7071/s$$

■ Problem 17.21. Subranging ADC.

A 12 bit-subranging ADC can consist of 3 stages of 4 bits or 4 stages of 3 bits. Which has the smallest (a) number of comparators, and (b) latency (assuming the same clock frequency)?

The *number of comparitors needed is related to the number of bits of information N according to* 2^N . So the 3-stages of four bit ADCs use $3*2^4 = 48$ comparators, whereas 4 stages of three bit ADCs use $4*2^3 = 32$ comparators. While the 3 stage ADC needs more comparators, the latency depends on the number of stages, however, so the *4-stage ADC can be expected to have a longer latency*. However, since the architecture involves serial*pipelining, the throughput should be proportional to the clock speed -- which we assumed was constant -- and should be the same for both cases.

Looking at the two extremes of the subranging ADC, an N-bit subranging ADC which has:
(a). 1 stage is known as a Flash ADC, and
(b). N stages is a successive approximation ADC.

■ Problem 17.22. Digital Filtering

For a step function with V_i=1 for i ≥ 0, and 0 elsewhere, apply four different digital filters. We will use Matematica for this problem.

We define our input function: :

```
V[i_] := 1 for i ≥ 1
V[i_] := 0 for i < 1
```

Our first filter is:

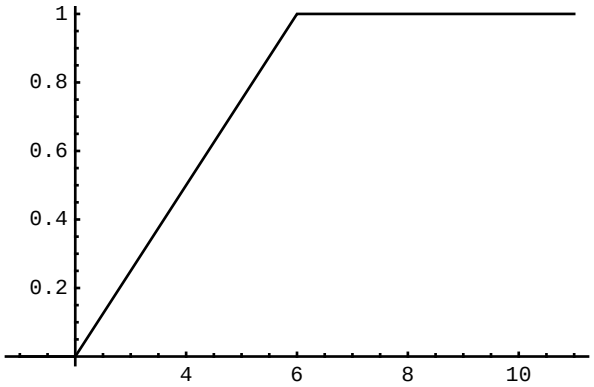
```
h = {0.25, 0.25, 0.25, 0.25}
```

The filter output is just the dot product of the filter and the input vectors over the length of the filter.

The output looks like :

```
{0., 0., 0.25, 0.5, 0.75, 1., 1., 1., 1., 1., 1.}
```

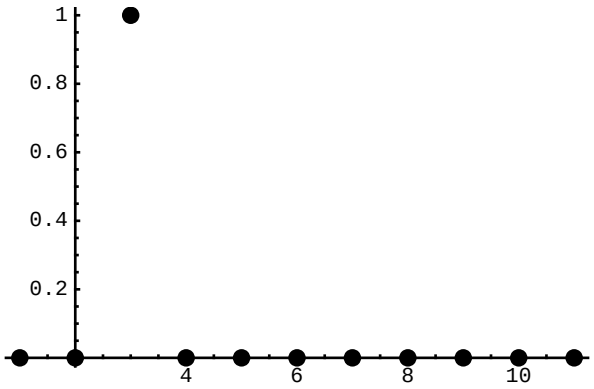
which, when plotted, looks like:



Changing the filter code to that of a "derivative" and reversing the code order:

```
h = {1, -1, 0, 0}
```

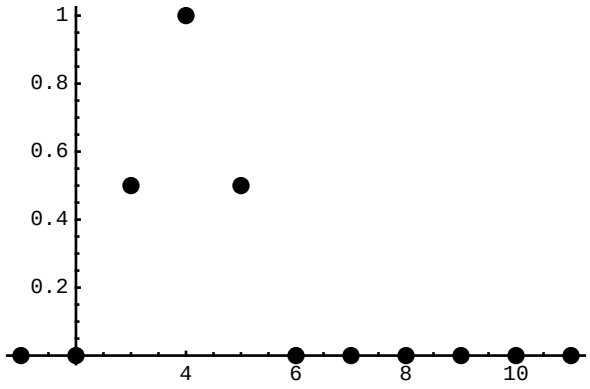
yields a filtered output of:



(c). Here is a "differentiator" filter with memory:

```
h = {0.5, 0.5, -0.5, -0.5}
```

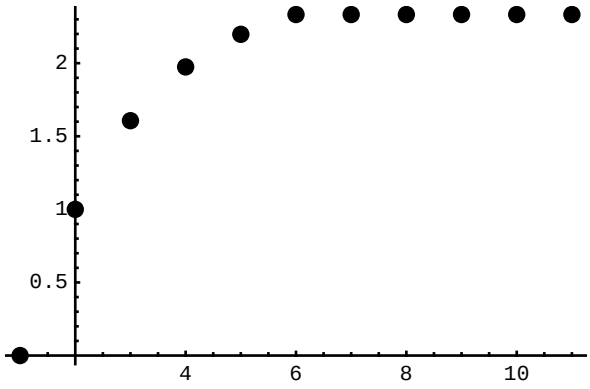
and this is the output of the filter applied to the input:



(d). Finally, here is the exponential integrating filter:

$$h = \left\{1, \frac{1}{\sqrt{e}}, \frac{1}{e}, \frac{1}{e^{3/2}}, \frac{1}{e^2}\right\}$$

which produces the outcome:



Multichannel Pulse Analysis

■ Problem 18.1. Channels needed for fixed detector resolution

For a fixed resolution, we want the peak FWHM to contain 5 channels, so

FWHM = 5 channels = (Resolution)*(Peak centroid channel).

The ADC must have at least the Peak centroid channel number of channels, so the ADC must have:

$$\textbf{ADC} = \frac{\textbf{5}}{\textbf{.003}} = \textbf{1667 channels}$$

■ Problem 18.2. Channels versus gain.

We note that the channel a pulse amplitude is mapped to will be linearly related to the gain, where k is the proportionality constant:

Channel = k (Pulse Amplitude) (Gain) + Zero Offset

The separation between two centroids is thus given by:

ΔChannel = k (Δ Pulse Amplitude) (Gain)

If the detector pulse amplitudes are kept constant, but the gain is changed, we find that the separation is changed as:

$$\frac{\delta(\Delta\text{Channel})}{\Delta\text{Channel}} = \frac{\delta(\text{Gain})}{\text{Gain}}$$

$$\textbf{change in separation} = -250 \times \frac{\textbf{24}}{\textbf{1000}} = \textbf{-6 channels}$$

So the centroid channel separation decreases by 6 channels.

■ Problem 18.3. Wilkinson ADC Oscillator Frequency.

We have to be able to resolve the maximum amplitude into 2048 parts in 25 μs. Each click of the clock corresponds to one channel number. The conversion time is given by $t_c = \frac{\text{PH}}{f}$ where the Pulse Height PH refers to the pulse height channel number. If the offset is zero, then:

$$f = \frac{\text{PH}}{t_c} = \frac{2048}{25\,\mu\text{s}} = 81.9\ \text{MHz}.$$

■ Problem 18.4. Successive Approx ADC steps for N channels.

Recall that each ADC comparison gives us 1 bit of information. Suppose our spectrum contains M channels. That means we must have a measurement accuracy equal to 1/M. Letting N be the number of ADC steps (comparisons), we must have $2^N = M$, so N= Log[M]/Log[2]:

$$N = \frac{\ln(M)}{\ln(2)} = 12 \text{ ADC steps} \qquad \text{where we substituted } M = 4096$$

■ Problem 18.5. Deadtime losses in Wilkinson ADC

If the ADC uses a linear ramp, then the deadtime is proportional to the pulse-height and not a fixed value. The expressions derived for the deadtime losses in Chapter 4 assumed that the deadtime was a fixed value, regardless of the pulse height. Note that other ADC designs, such as the Flash ADC or the Successive Approximation ADC have a fixed deadtime per conversion.

■ Problem 18.6. Wilkinson ADC.

The ADC operates at 80 MHz and has a storage time of 2.5 μs. Since each clock cycle corresponds to one channel:
(a). for channel 300, deadtime will be N/f + B

$$\text{dead time} = \frac{300}{80 \times 10^6} + \frac{2.5}{10^6} = 6.25 \times 10^{-6} \text{ seconds}$$

(b). The fractional dead time for an event rate of $r_0$is given by mτ where we assume a nonparalyzable model for m = $\frac{n}{1+n\tau}$. Here we substitute $n = \frac{5000}{\text{second}}$ and $\tau = 2.5\,\mu\text{s} + \frac{220 \text{ seconds}}{80\times10^6}$ to get the fractional dead time as thus:

$$f_{\text{dead}} = \frac{n\tau}{n\tau + 1} = 0.0256$$

(c). Real time for a live time of 10 minutes in (b) above. Here we substitute $f_{\text{dead}} = 0.025579$ and live-time = 10 minutes to get the real time as thus:

$$\text{real time} = \frac{\text{live} - \text{time}}{1 - f_{\text{dead}}} = 10.3 \text{ minutes}$$

■ Problem 18.7. Periodic pulser dead time.

The key to this problem is to note that the pulser pulses are periodic. Therefore, they will each be counted if the time between pulses is longer than the dead time per pulse (t_{dead}). Noting the time between the pulses of frequency f is $\Delta t = 1/f$, the count rate CR is less than f if $\Delta t > t_{\text{dead}}$ or $f < 1/t_{\text{dead}}$. If the frequency is high enough, multiple pulses may be included in the dead time. You can show that the count rate CR is given by:

CR= f/(n+1) if $f > (n/t_{\text{dead}})$ for n= {0,1,2, ...}

A plot of CR versus f (in units of $1/t_{\text{dead}}$) thus looks like a set of steps starting at height f and going down by half its height each step along the frequency axis.

Problem 18.8. Multiparameter spectra

The number of memory channels required for two detector spectra will be $D_1 \, x \, D_2$where the number of channels for either is given from Problem 18.1 as D_i= (5 channels)/(Resolution$_i$). So the number of memory locations will be given by:

$$\text{locations} = \frac{5}{R_1} \times \frac{5}{R_2}$$

We substitute R_1=.005 and R_2 = .025 to get the number of memory locations.

locations = 200 000 memory locations

■ Problem 18.9. Channel counting statistics.

The average count in a channel should be zero, but this will fluctuate due to counting statistics. Indeed, half of the channels will have negative values. Since: N = (S+B)-(B) = B - B, then by the propagation of error:
σ_N= Sqrt [2B] =

$$\sigma_N = \sqrt{600} = 24.5 \text{ counts}$$

So the average channel will have a value of 0±24.5 counts.

■ Problem 18.10. Finding the Centroid.

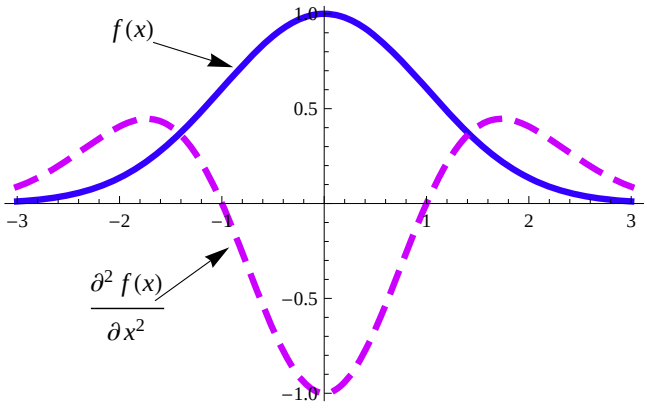
Using the Gaussian function, let's look at the second derivative behavior. Our function is:

$$f(x) = y_0 \, e^{\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)}$$

so the second derivative is given by (we substitute $x_0 = 0$, $\sigma = 1$ and $y_0 = 1$ since these factors don't affect the results):

$$\frac{\partial^2 f(x)}{\partial x^2} = e^{-\frac{x^2}{2}} x^2 - e^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}} \left(x^2 - 1\right)$$

We plot the function and its second derivative:



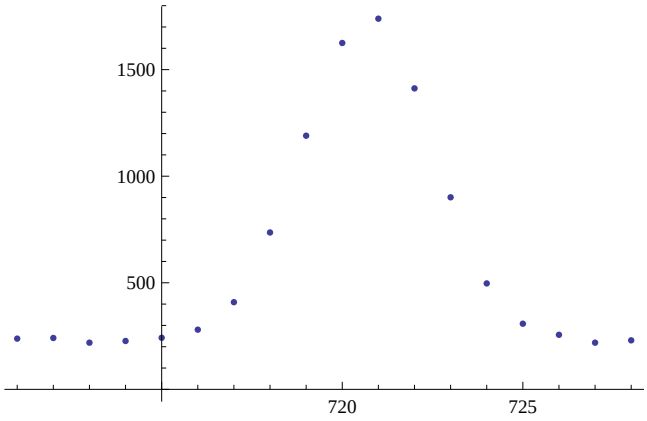
Note that the peak of the function is characterized by a large negative excursion in second derivative. Hence, this is often used as the indicator of a peak in automated peak finding.

■ Problem 18.11*. Peak Fitting of Data.

We are given an MCA data set consisting of a peak and constant background. We first try to visually estimate the centroid, area, and FWHM. The data in the list below corresponds to a channel number starting with channel 711, increasing by one, and ending with channel 728. Below we plot the data versus the channel number.

data = {238, 241, 219, 227, 242, 280, 409, 736, 1190, 1625, 1739, 1412, 901, 497, 308, 256, 219, 230}

channel = {711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 721, 722, 723, 724, 725, 726, 727, 728}



From the graph, we estimate a constant background of about 235 and FWHM of 4 channels, with peak channel at 721. We estimate the area by summing the data values and then subtracting an assumed constant background over the 5th through 17th data points:

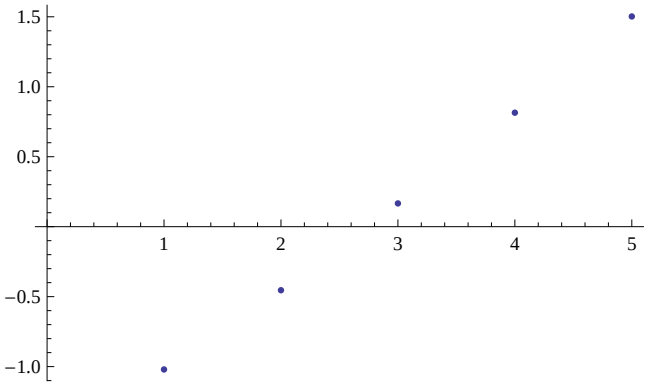
area = (242 + 280 + 409 + 736 + 1190 + 1625 + 1739 + 1412 + 901 + 497 + 308 + 256 + 219) − (235 × 13)

area = 6759

(b). Now, we try the linearized method described in the text. We define ln[Q(x)] = ln[$\frac{y(x-1)}{y(x+1)}$], do a linear least squares fit of ln[Q(x)] versus x over 5 channels, and plot this versus x. (In this problem the 5 channels we will use are channels 719 through 723) ***We first subtract off the background*** of 235 counts, and then proceed:

data = raw counts − 235
= {3, 6, −16, −8, 7, 45, 174, 501, 955, 1390, 1504, 1177, 666, 262, 73, 21, −16, −5}

We now plot x versus $\ln\left(\frac{\text{data}[\![i-1]\!]}{\text{data}[\![i+1]\!]}\right)$ where i is the element number of the data list, starting with 9 and ending at 13.



Now, let's do a linear least squares fit over data points (9 → 13). We'll skip the weighting factor (data[[i-1]] data[[i+1]]/(data[[i-1-]]+data[[i+1]])) here. The (x, y) coordinates and the linear least squares fit line are shown below.

$$\begin{pmatrix} 719 & -\log\left(\frac{1390}{501}\right) \\ 720 & -\log\left(\frac{1504}{955}\right) \\ 721 & \log\left(\frac{1390}{1177}\right) \\ 722 & \log\left(\frac{752}{333}\right) \\ 723 & \log\left(\frac{1177}{262}\right) \end{pmatrix} \qquad y = 0.631443\, x - 455.069$$

We now find the centroid, FWHM and σ.

$$x_0 = \frac{-\text{intercept}}{\text{slope}} = -\frac{-455.069}{0.631443} = 721$$

$$\sigma = \sqrt{\frac{2}{\text{slope}}} = \sqrt{\frac{2}{0.631443}} = 1.78$$

Recognizing that the FWHM of a Gaussian is 2.35 time the standard deviation of the distribution:

$$\text{FWHM} = 2.35\,\sigma = 2.35\sqrt{\frac{2}{0.631443}} = 4.18 \text{ channels}$$

Now let's get the area, assuming no uncertainties in the centroid and σ:

$$\text{area} = \sqrt{2\pi}\ \sigma\, e^{\left(\frac{\sum_{i=9}^{13} \text{data}[i]\left(\frac{(\text{channel}[i]-x_0)^2}{2\sigma^2}+\ln(\text{data}[i])\right)}{\sum_{i=9}^{13}\text{data}[i]}\right)} = 6788$$

This is quite close to our original estimate of the net area.

Extra: Note that we could also apply a nonlinear fit to the data using the three fitting parameters y_0, x_0, and σ.

We find that a nonlinear fit for the data to the function $y(x) = y_0\, e^{\left(\frac{-(x-x_0)^2}{2\sigma^2}\right)}$ yields:

$$y_0 = 1518, \quad x_0 = 721 \text{ and } \sigma = 1.78$$

and the net area is then:

$$\textbf{area} = \sigma\, y_0 \sqrt{2\pi} \;=\; \textbf{6774}$$

The nonlinear fit is generally regarded as more accurate since it avoids the problem of a potential bias associated with biased estimators.

Miscellaneous Detectors

■ Problem 19.1. Cherenkov emission by electron

(a). We use the energy threshold relationship for Cherenkov radiation:

$$\textbf{Energy} = m_e c^2 \left(\sqrt{1 + \frac{1}{n^2 - 1}} - 1 \right) \qquad \text{where we substitute } n = 1.47$$

Energy = 0.186 MeV

(b). For this amount of energy to be given to a Compton electron, the minimum gamma ray energy corresponds to a 180° scattering angle, with $E_e = \left(2 E_\gamma{}^2\right) / \left(m_e * c^2 + 2 * E_\gamma\right)$. Solving this equation for E_γ and substituting E_e= 0.186173 MeV gives us:

E_γ **= −0.144 MeV** **and** E_γ **= 0.330 MeV**

Clearly, the negative energy is not physical, so our solution is 330 keV.

■ Problem 19.2. Fewer Cherenkov photons/MeV than in NaI

Figure 19.2 shows about 600 photons/(2MeV e^-), or about 300 photons/MeV for Cherenkov emission in water. In NaI, we expect about 38000 photons/MeV (see textbook) -- about 100 times larger light yield.

■ Problem 19.3. Liquid Xe attenuation at 30 keV.

Figure 6.18 tells us that 2 inches of gaseous Xe at 1 atm will attenuate ~20% of 30 keV x-rays. But the density of Xe gas at STP is 5.85 g/liter while the density of liquid Xe is 3.52 kg/liter. Recall:
$I(t)/I_0 = e^{-(\mu/\rho)(\rho t)} = 80\ \%$ so the mass attenuation constant μ/ρ is

$$\mu_\rho = \frac{\mu}{\rho} = \frac{-\ln(.8)}{(\rho\, t)} \qquad \text{and when we substitute } \rho = 5.85 \text{ gram/Liter and } t = 5.08 \text{ cm}$$

$$\mu_\rho = \frac{\textbf{0.0075087 Liter}}{\textbf{gram–cm}}$$

Using the previous expression, $t = \left(1/\rho\, \mu_\rho\right) \text{Log}[I_0/I(t)]$

$$t = \frac{1}{(\rho\, \mu_\rho)} \ln\left[\frac{1}{f}\right] \qquad \text{where we substitute } f = 0.5 \text{ and } \rho = 3.52 \text{ kg/Liter}$$

t **= 262** μ**m**

■ Problem 19.4. Intensifier screens for radiology

X-rays have only a small probbilitiy of directly interacting in the radiographic film, so a high-Z scintillating material placed adjacent to the film to produce the light that exposes the emulsion. In place of the film, one can also view the intensifier screen emissions by a CCD for real-time radiographic applications.

■ Problem 19.5. Electron traps: Advantage or disadvantage?

Radiation interactions in a crystaline material produce electrons and holes. If the application requires an immediate readout of the interaction, then the electron traps are undesirable because the desired signal is held up by the electron being trapped at the impurity site. For most scintillators used in spectroscopy, this is a disadvantage. However, for dosimeters, where one wishes to integrate the total number of events over a period of time, then trapping the elctrons at an impurity site is an advantage. The electrons are later released under thermal or optical stimulation, and the resulting emission measures the integrated number of events.

■ Problem 19.6. Track detectors

We are looking for a track etch detector that will be sensitive to fission products but not alpha particles. Fission products are high Z (~100 amu) and high kinetic energy (~100 MeV). Alpha particles have A=4 and a typical kinetic energy of several MeV. Looking at the referenced figure for He, any of the track materials whose horizontal lines are above that of Lexan will not produce a track density sufficient to register the alpha particle. For the fission products, Cronar offers the highest sensitivty (lowest required ionization density to register a track) while still being insensitive to the alpha particles.

■ Problem 19.7. Activation and counting

(a). We need to find the induced activity for a 10 minute and 20 minute irradiation since the count rate is proportional to the induced activity. Since the induced activity is given by $A(t) = A_\infty\left(1 - e^{-\lambda t}\right)$, the ratio we want is given by:

$$A = \frac{1 - e^{-\lambda t1}}{1 - e^{-\lambda t2}} \qquad \text{where we substitute } \lambda = \frac{\ln(2)}{2.3},\ t_1 = 10 \text{ and } t_2 = 20 \text{ (all time units are in minutes)}$$

A = 0.953

This says that a 10 minute irradiation gives 95% of the counts of a 20 minute irradiation.

(b). Suppose we counted forever (instead of just 10 minutes). What would the gain be in the number of net counts? Recall that Counts = k(1 - $e^{-\lambda tC}$) and assume the irradiation and wait time remain constant. So the factor increase in counts is given by:

$$\text{counts} = \frac{1}{1 - e^{-\lambda t}} \qquad \text{where we substitute } \lambda = \frac{\ln(2)}{2.3} \text{ and } t = 10$$

counts = 1.05

Thus, only a 5% increase in counts would be obtained by increasing the counting time from 10 minutes to infinity. This is because of the short 2.3 minute half life of this isotope -- it's mostly gone after 10 minutes.

Problem 19.8 Role of Frisch Grid in HP Xe detectors

A Frisch grid is employed to permit sensitivity to only one carrier type. Since ions move so slowly in a gas and often don't make it to the cathode in a reasonable time, the Frisch grid permits the generation of the signal solely on the more mobile carrier since the detector signal is induced only when this fast carrier passes the grid and is collected on the anode. Thus, the role of the other carrier plays no role in the signal formation and the output signal is proportional only to the number of fast carriers (electrons) collected.

Background and Detector Shielding

■ Problem 20.1. Background from ^{40}K in NaI

We note that the total number of NaI molecules in the detector will be ρV*(Avogardro's Constant)/(Molecular Weight) where V= $\pi r^2 H$ for a cylinder. Natural potassium is 0.012% ^{40}K, which has a half life of 1.26 x 10^9 years. We are allowed only 1 cps in the detector volume. So the maximum number of K atoms is λ (0.012%)K = 1 cps, or solving for K (where we are using the symbol K also for the number of potassium atoms):

$$K = \frac{(1/\text{second})}{0.012\,\%\left(\frac{\ln 2}{T_{1/2}}\right)} \qquad \text{where we substitute } T_{1/2} = 1.26\times10^{9} \text{ years}$$

$$K = 4.78\times10^{20} \text{ atoms}$$

But the total number of NaI molecules in the detector are:

$$\text{NaI} = \frac{\rho V\,(\text{Avogadro's Constant})}{\text{MolecularWeight}}$$

where we substitute $V = \pi r^2 H$, $\text{MolecularWeight} = \frac{150\,g}{\text{mole}}$, $\rho = \frac{3.67\,g}{\text{cm}^3}$, $r = \frac{7.62\text{ cm}}{2}$ and $H = 7.62$ cm

$$\text{NaI} = 5.12\times10^{24} \text{ molecules}$$

So the atom ppm of K that is allowable is (noting 2 atoms per NaI molecule):

$$\text{ppm} = \frac{K}{2\,\text{NaI}}\times10^{6} = 46.7 \text{ atoms of } K \text{ per million atoms}$$

■ Problem 20.2. ^{14}C source

When cosmic rays interact in the upper atmosphere, they produce a variety of elementary particles, including neutrons. When one of these neutrons interacts with a nitrogen nucleus in the atmosphere, the (n,p) reaction $^{14}N\,(n,\,p)$ produces ^{14}C which then enters the environment and serves as chronometer. When a living organism dies, it no longer ingests fresh carbon, and so the decreased relative abundance of the ^{14}C is a measure of the time since death.

■ Problem 20.3. Cosmic ray pulse rejection

Cosmic rays tend to be high energy charged particles that have a low probability of interacting in the relatively low density gas in the proportional tube (i.e., low $\frac{dE}{dx}$). As a result, the pulses that they produce are small and can be rejected by a discriminator setting. In contrast, scintillators are solids and therefore their higher density (and often higher Z) leads to greater energy deposition and signal size.

■ Problem 20.4. Proportional counting on the α plateau

The alpha plateau corresponds to where the pulse height is large enough to register a count for the alpha particle, but not large enough to record the energy deposition of electrons. Since any background that could enter the gas through the detector wall will be either an electron or photon (which produces an electron in the gas), these pulses will not be seen because they are under the discrimator value. Any background that is registered must be an alpha particle, and these can only reach the gas if they are emitted from the inside of the detector wall. So the source of background will be natural radioactivity either in the gas as an impurity or in the detector wall.

■ Problem 20.5. 2.22 MeV background line

Around concrete shields and other environmental sources that contain large amounts of hydrogen or $H_2 O$, one often sees a 2.22 MeV background line in gamma spectrum. The source of this line is neutron capture on hydrogen, or ${}^1H\,(n,\,\gamma)\,{}^2H$, which has a Q value of 2.22 MeV. The neutrons are primarily produced by cosmic ray interactions in the atmosphere and local environment.

■ Problem 20.6. Coincidence vs. Singles Counting Statistics

We recall that the optimal division of time is given by Eqn. 3.54 and yields a fractional uncertainty in the net source rate S of $\epsilon = \sqrt{(1/T)}\ \frac{\sqrt{(S+B)}+\sqrt{B}}{S}$. For our data, the fractional uncertainty in the first measurements is:

$$\epsilon = \sqrt{(1/T)}\ \frac{\sqrt{(S+B)}+\sqrt{B}}{S} \quad \text{where we substitute } T = 1,\ S = 2 \text{ and } B = 8$$

$$\epsilon = 2.99535$$

We look for the background B that gives the same fractional uncertainty in S:

$$\epsilon = \sqrt{(1/T)}\ \frac{\sqrt{(S+B)}+\sqrt{B}}{S} \qquad \text{where we substitute } T = 1,\ S+B = 3$$

$$\epsilon = 2.99535 = \frac{\sqrt{3}+\sqrt{B}}{3-B}$$

Solving the above equation for B yields:

B = 1.955 and therefore S=1.0045

The problem is also solvable using approximations. Suppose we could assume S << B. Then from Eqn. 3.57, $\epsilon^2 T = \frac{4B}{S^2}$. Since the left-hand side stays constant in this problem, we can solve for the value of B that satisfies this:

$$\frac{4B}{S^2} = 8 \quad \text{and} \quad S + B = 3 \qquad \text{where } S > 0$$

Therefore :

$$S = 1 \quad \text{and} \quad B = 2$$

Thus, our approximate solution is S=1, and B=2. We should be careful of this answer because it doesn't strictly satisfy S << B, and yet it comes quite close to the exact answer above.

What this problem is showing is the benefit of doing coincident counting. The background is usually significantly reduced in coincident measurements, which means the relative uncertainty in the desired source strength can be significantly lower.