Chapter 10

Time-Domain Analysis and Design of Control Systems

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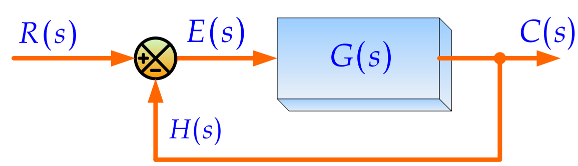
10.5 STEADY STATE ERRORS AND SYSTEM TYPES

Steady-state errors constitute an extremely important aspect of the system performance, for it would be meaningless to design for dynamic accuracy if the steady output differed substantially from the desired value for one reason or another.

The steady state error is a measure of system accuracy. These errors arise from the nature of the inputs, system type and from nonlinearities of system components such as static friction, backlash, etc. These are generally aggravated by amplifiers drifts, aging or deterioration. The steady-state performance of a stable control system is generally judged by its steady state error to step, ramp and parabolic inputs.

Consider a ***unity feedback*** system as shown in the Figure. The input is $R(s)$, the output is $C(s)$, the feedback signal $H(s)$ and the difference between input and output is the error signal $E(s)$.



From the above Figure

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)} \tag{1}$$

On the other hand

$$C(s)=E(s)G(s) \tag{2}$$

Substitution of Equation (2) into (1) yields

$$E(s)=\frac{1}{1+G(s)}R(s) \tag{3}$$

The steady-state error e_{ss} may be found by use of the Final Value Theorem (FVT) as follows:

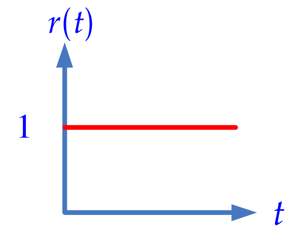
$$\boxed{e_{ss}=\lim_{t\to\infty}e(t)=\lim_{s\to 0}SE(s)=\lim_{s\to 0}\frac{sR(s)}{1+G(s)}} \tag{4}$$

Equation (4) shows that the steady state error depends upon the input $R(s)$ and the forward transfer function $G(s)$. The expression for steady-state errors for various types of standard test signals are derived next.

1. Unit Step (Positional) Input.

Input $r(t)=1(t)$

or $R(s)=L[r(t)]=\dfrac{1}{s}$



From Equation (4)

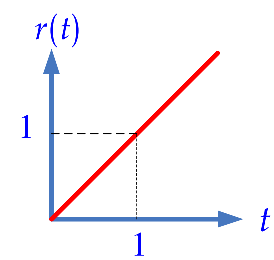
$$e_{ss}=\lim_{s\to 0}\frac{sR(s)}{1+G(s)}=\lim_{s\to 0}\frac{s(1/s)}{1+G(s)}=\lim_{s\to 0}\frac{1}{1+G(s)}=\frac{1}{1+G(0)}=\frac{1}{1+K_p}$$

where $K_p=G(0)$ is defined as the ***position error constant.***

2. Unit Ramp (Velocity) Input.

Input $r(t)=t$ or $\dot{r}(t)=1$

or $R(s)=L[r(t)]=\dfrac{1}{s^2}$



From Equation (4)

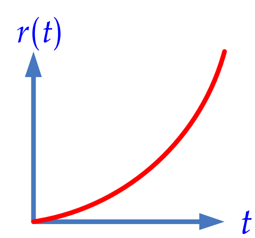
$$e_{ss}=\lim_{s\to 0}\frac{sR(s)}{1+G(s)}=\lim_{s\to 0}\frac{s(1/s^2)}{1+G(s)}=\lim_{s\to 0}\frac{1}{s+sG(s)}=\lim_{s\to 0}\frac{1}{sG(s)}=\frac{1}{K_v}$$

where $K_v=\lim_{s\to 0}sG(s)$ is defined as the ***velocity error constant.***

3. Unit Parabolic (Acceleration) Input.

Input $r(t)=\dfrac{1}{2}t^2$ or $\ddot{r}(t)=1$

or $R(s)=L[r(t)]=\dfrac{1}{s^3}$



From Equation (4)

$$e_{ss}=\lim_{s\to 0}\frac{sR(s)}{1+G(s)}=\lim_{s\to 0}\frac{s(1/s^3)}{1+G(s)}=\lim_{s\to 0}\frac{1}{s^2+s^2G(s)}=\lim_{s\to 0}\frac{1}{s^2G(s)}=\frac{1}{K_a}$$

where $K_a=\lim_{s\to 0}s^2G(s)$ is defined as the ***acceleration error constant.***

10.6 TYPES OF FEEDBACK CONTROL SYSTEMS

The open-loop transfer function of a *unity feedback system* can be written in two standard forms:

- The time constant form

$$\boxed{G(s)=\frac{K(T_{z1}s+1)(T_{z2}s+1)\cdots(T_{zj}s+1)}{s^n(T_{p1}s+1)(T_{z2}s+1)\cdots(T_{zk}s+1)}} \tag{8}$$

where K and T are constants. *The system type refers to the order of the pole of* $G(s)$ *at* $s=0$. *Equation (8) is of type* n.

- The pole-zero form

$$\boxed{G(s)=\frac{K'(s+z_1)(s+z_2)\cdots(s+z_j)}{s^n(s+p_1)(s+p_2)\cdots(s+p_k)}} \tag{9}$$

The gains in the two forms are related by

$$\boxed{K=K'\frac{\prod_j z_j}{\prod_k p_k}} \tag{10}$$

with the gain relation of Equation (10) for the two forms of $G(s)$, it is sufficient to obtain steady state errors in terms of the gains of any one of the forms. We shall use the time constant form in the discussion below.

Equation (8) involves the term s^n in the denominator which corresponds to number of integrations in the system. As $s\to 0$, this term dominates in determining the steady-state error. *Control systems are therefore classified in accordance with the number of integration in the open loop transfer function* $G(s)$ *as described below*.

1. Type-0 System.

If $n=0$, $G(s)=\frac{K}{s^0}=K$ the steady-state errors to various standard inputs, obtained from Equations (5), (6), (7) and (8) are

$$\boxed{\begin{aligned} e_{ss}(\text{Position}) &= \frac{1}{1+G(0)}=\frac{1}{1+K}=\frac{1}{1+K_p} \\ e_{ss}(\text{Velocity}) &= \lim_{s\to 0}\frac{1}{sG(s)}=\lim_{s\to 0}\frac{1}{sK}=\infty \\ e_{ss}(\text{Acceleration}) &= \lim_{s\to 0}\frac{1}{s^2G(s)}=\lim_{s\to 0}\frac{1}{s^2K}=\infty \end{aligned}} \tag{11}$$

Thus a system with $n=0,$ or no integration in $G(s)$ has

- a constant position error,
- infinite velocity error and
- infinite acceleration error

2. Type-1 System.

If $n=1,\ G(s)=\dfrac{K}{s^1}$, the steady-state errors to various standard inputs, obtained from Equations (5), (6), (7) and (8) are

$$
\boxed{\begin{aligned}
e_{ss}(\text{Position}) &= \frac{1}{1+G(0)} = \lim_{s\to 0}\frac{1}{1+\dfrac{K}{s}} = \frac{1}{1+\infty} = 0\\
e_{ss}(\text{Velocity}) &= \lim_{s\to 0}\frac{1}{sG(s)} = \lim_{s\to 0}\frac{1}{s\dfrac{K}{s}} = \frac{1}{K} = \frac{1}{K_v}\\
e_{ss}(\text{Acceleration}) &= \lim_{s\to 0}\frac{1}{s^2G(s)} = \lim_{s\to 0}\frac{1}{s^2\dfrac{K}{s}} = \frac{1}{0} = \infty
\end{aligned}} \tag{12}
$$

Thus a system with $n=1,$ or with one integration in $G(s)$ has

- a zero position error,
- a constant velocity error and
- infinite acceleration error

3. Type-2 System.

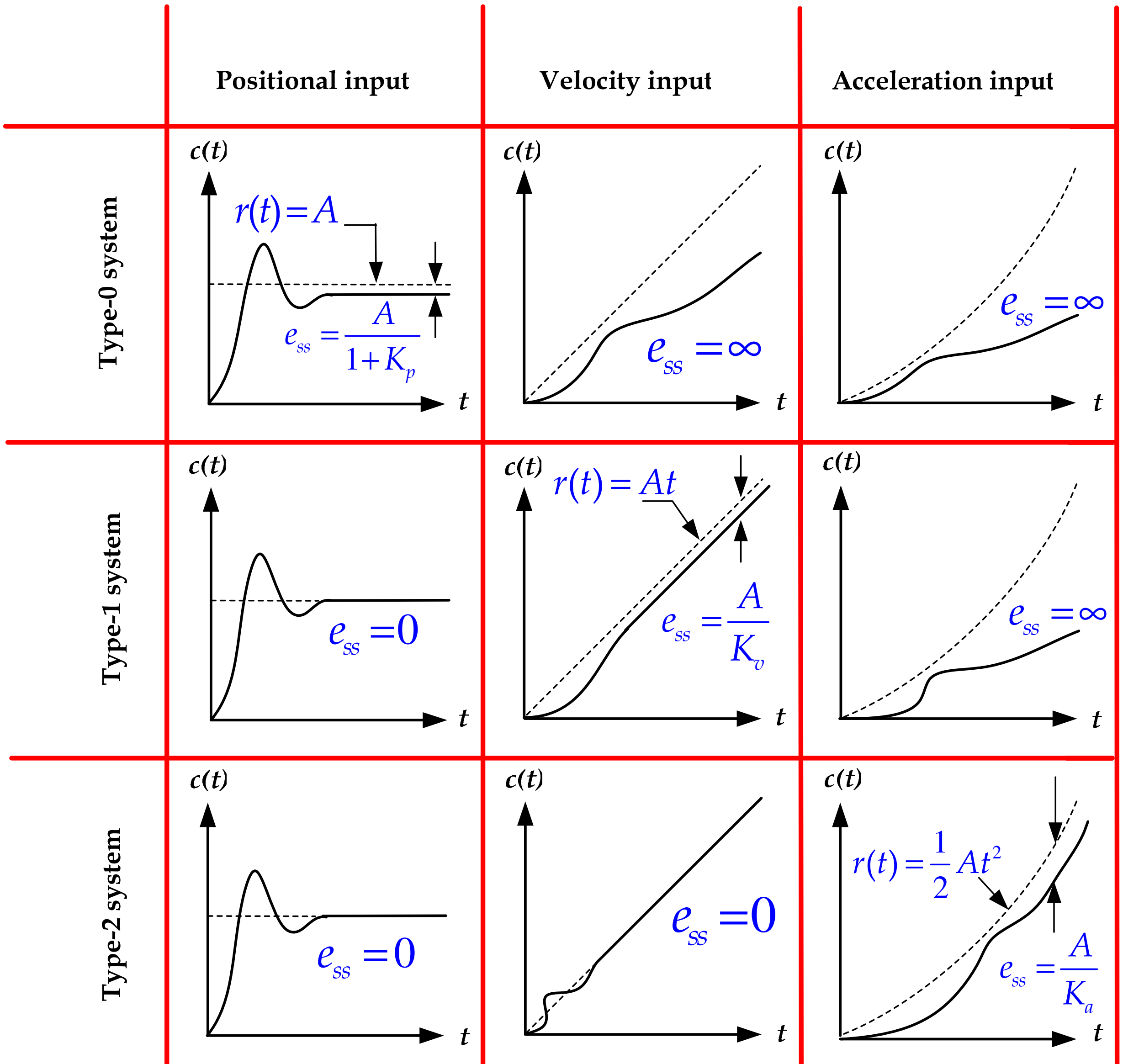
If $n=1,\ G(s)=\dfrac{K}{s^2}$, the steady-state errors to various standard inputs, obtained from Equations (5), (6), (7) and (8) are

$$
\boxed{\begin{aligned}
e_{ss}(\text{Position}) &= \frac{1}{1+G(0)} = \lim_{s\to 0}\frac{1}{1+\dfrac{K}{s^2}} = \frac{1}{1+\infty} = 0\\
e_{ss}(\text{Velocity}) &= \lim_{s\to 0}\frac{1}{sG(s)} = \lim_{s\to 0}\frac{1}{s\dfrac{K}{s^2}} = \frac{1}{\infty} = 0\\
e_{ss}(\text{Acceleration}) &= \lim_{s\to 0}\frac{1}{s^2G(s)} = \lim_{s\to 0}\frac{1}{s^2\dfrac{K}{s^2}} = \frac{1}{K} = \frac{1}{K_a}
\end{aligned}} \tag{13}
$$

Thus a system with $n=2,$ or with one integration in $G(s)$ has

- a zero position error,
- a zero velocity error and
- a constant acceleration error

TABLE 1. Steady-state errors in closed loop systems



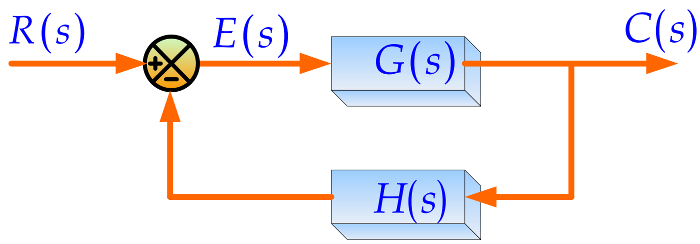
$K_p = \lim\limits_{s\to 0} G(s)$	$K_v = \lim\limits_{s\to 0} sG(s)$	$K_a = \lim\limits_{s\to 0} s^2G(s)$

where $K_p = G(0)$ is defined as the ***position error constant.***

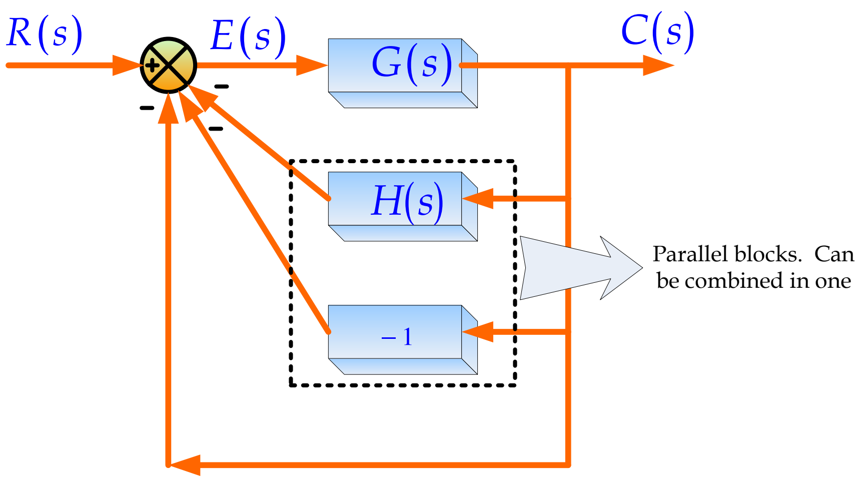
where $K_v = \lim\limits_{s\to 0} sG(s)$ is defined as the ***velocity error constant.***

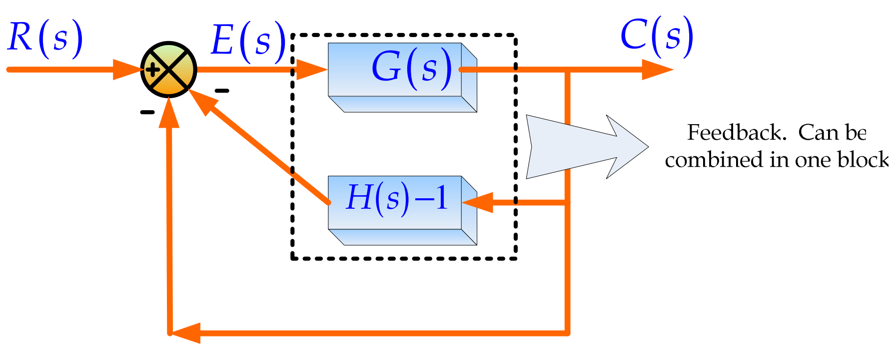
where $K_a = \lim\limits_{s\to 0} s^2G(s)$ is defined as the ***acceleration error constant.***

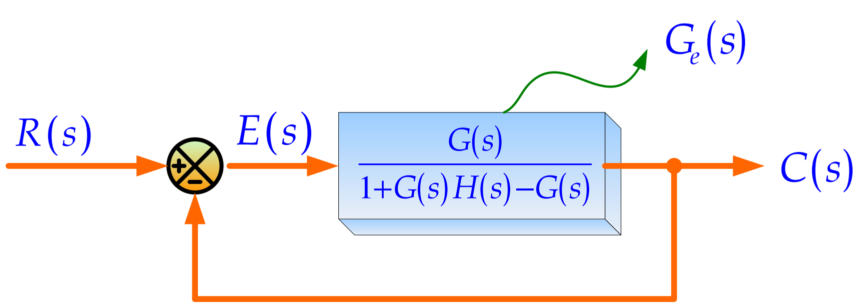
10.7 STEADY STATE ERROR FOR NON-UNITY FEEDBACK SYSTEMS



Add to the previous block two feedback blocks $H_1(s) = -1$ and $H_1(s) = 1$



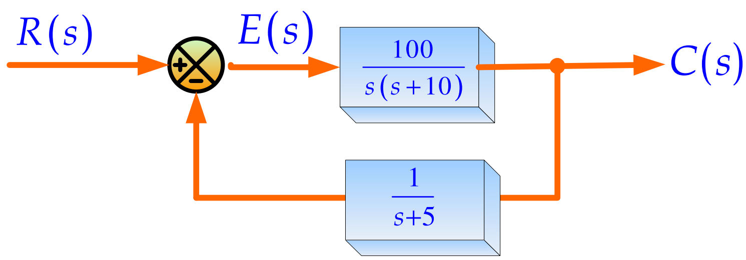




Example 1

For the system shown below, find

- The system type
- Appropriate error constant associated with the system type, and
- The steady state error for unit step input



Solution

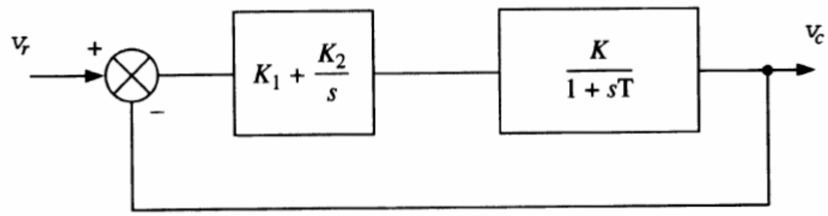


Fig. SP8.1.1

SAMPLE PROBLEM 8.1

An engine speed control system is shown in Fig. SP8.1.1. The engine itself is modeled as a first-order system with time constant T, while the electronic throttle controller may have the constants K_1 and K_2 set to arbitrary values.

1. What is the steady-state error for a step of magnitude A if $K_2 = 0$?
2. What is the steady-state error for a step of magnitude A when $K_2 \neq 0$?
3. Determine the steady-state error when the input is a ramp of slope A and (i) $K_2 = 0$, (ii) $K_2 \neq 0$.
4. Given $K_1 = 1.2$, $K_2 = 8.4$, and $T = 0.5$, what value of K gives a velocity error constant of 6 for a unit ramp input? Find the corresponding steady-state error, and sketch the input and output as functions of time for this case.

Solution

The system has unity feedback; therefore the various error constants and steady-state errors may be determined from Table 8.1. The open-loop transfer function is given by

$$G(s) = \frac{K(K_1 s + K_2)}{s(1 + sT)}$$

1. When $K_2 = 0$, this transfer function reduces to

$$G(s) = \frac{KK_1}{1 + sT}$$

which represents a system of type 0. The position error constant for a step input is obtained by writing the open-loop transfer function in the form of equation 8.8:

$$G(s) = \frac{KK_1/T}{s + 1/T}$$

hence

$$K_p = \lim_{s \to 0} G(s) = KK_1$$

This could also have been obtained directly from

$$G(s) = \frac{KK_1}{1 + sT}$$

without first writing the open-loop transfer function in the form of equation 8.8. The steady-state error for a unit step input, from table 8.1, becomes

$$e_{ss}(t) = \frac{1}{1 + K_p} = \frac{1}{1 + KK_1}$$

For a step of magnitude A, therefore

$$e_{ss}(t) = \frac{A}{1 + KK_1}$$

2. When $K_2 \neq 0$, the open-loop transfer function reverts to the form

$$G(s) = \frac{K(K_1 s + K_2)}{s(1 + sT)}$$

which represents a system of type 1. From Table 8.1, the steady-state error for any magnitude step is zero, i.e.,

$$e_{ss}(t) = 0$$

3. When $K_2 = 0$ and the input is a ramp, Table 8.1 indicates that because the system is of type 0, the steady-state error becomes infinite. Hence

$$e_{ss}(t) = \infty$$

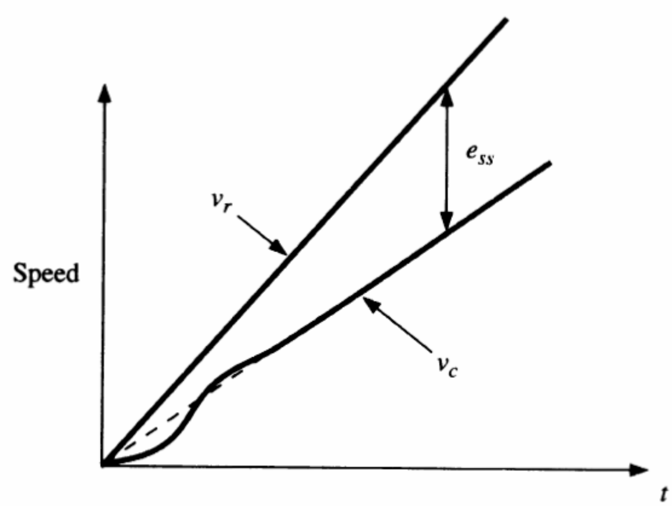
This result does not necessarily imply that the system is unstable, but rather that the output is a ramp like the input but of different slope, as shown in Fig. SP8.1.2, where it is seen that the error becomes infinite as t becomes large.

When $K_2 \neq 0$, the system becomes type 1, and the open-loop transfer function may be written in the form

$$G(s) = \frac{K(K_1 s + K_2)}{s(1 + sT)} = \frac{KK_1(s + K_2/K_1)}{Ts(s + 1/T)}$$

The velocity error coefficient K_v is obtained as

$$K_v = KK_2$$



This result may also be obtained from either form of the open-loop transfer function $G(s)$. The steady-state error becomes

$$e_{ss}(t) = \frac{A}{K_v} = \frac{A}{KK_2}$$

4. Given $K_1 = 1.2$, $K_2 = 8.4$, and $T = 0.5$, it is required that

$$K_v = 6 = 8.4K$$

Hence the result

$$K = 0.714$$

The steady-state error of the closed-loop system becomes

$$e_{ss}(t) = \frac{1}{K_v} = 0.167$$

The output will be a unit ramp similar to the input but will lag behind it in the steady state. In order to sketch both the input and output, the transient portion of the output will also be determined. This may be necessary since the closed-loop transfer function will be of second order and the transient will be either underdamped or overdamped. This will lead to two possible types of responses as indicated in Fig. SP8.1.3. The response may be evaluated from the characteristic equation

$$1 + G(s) = 0$$

which gives

$$1 + \frac{K(K_1 s + K_2)}{s(1 + sT)} = 0$$

Substituting known values yields

$$s^2 + 3.716s + 12 = 0$$

giving the closed-loop poles

$$s = -1.858 \pm 2.94j$$

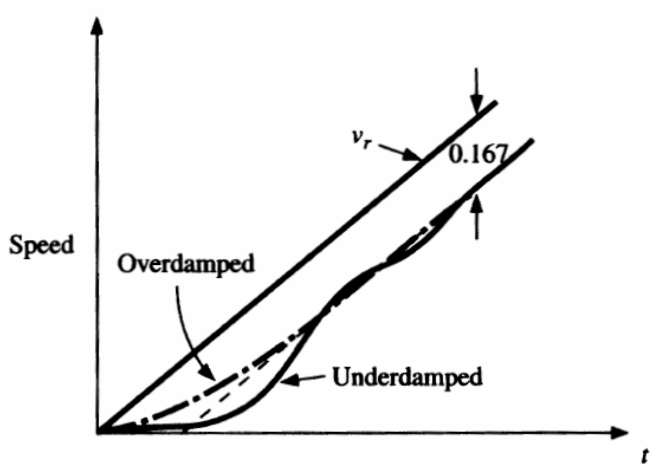


Fig. SP8.1.3

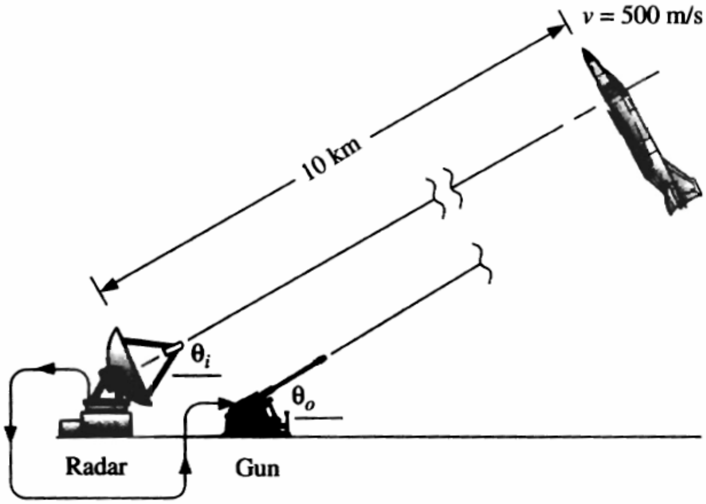


Fig. SP8.2.1

Since both closed-loop poles are complex, the system is underdamped, as indicated in Fig. 8.1.3.

Sample Problem 8.2

Figure SP8.2.1 shows an antiaircraft system consisting of a ground radar that measures the distance along the radar beam to an aircraft, together with the position relative to the tracking system based on the angle of the antenna with the ground. This information is passed to the gun-pointing system that attempts to track the aircraft, as shown in Fig. SP8.2.2, where $K = 240$. If the speed of the aircraft is $v = 500$ m/s at a range of 10 km and may be assumed perpendicular to the line of sight from the radar, by what distance do shells miss the target? If the required miss distance is to be no greater than 1 m, calculate the required value of system gain K.

Solution

Although the radar antenna provides an angular position input, the aircraft motion makes the angle θ change uniformly with time, at least for the instantaneous position shown. This subjects the gun-pointing system to a ramp input. Note that the system has unity feedback; therefore the results summarized in

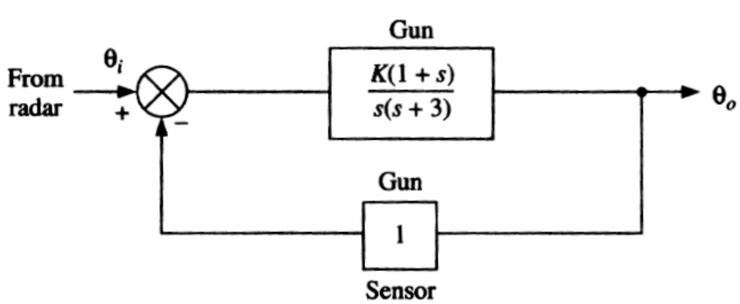


Fig. SP8.2.2

Table 8.1 are valid. The slope of the ramp is given by

$$\frac{d\theta}{dt} = \frac{v}{r} = \frac{500}{10{,}000} = 0.05 \text{ rad/s}$$

The open-loop transfer function of the system is

$$GH(s) = \frac{K(1 + s)}{s(s + 3)} = \frac{240(1 + s)}{s(s + 3)}$$

The system is of type 1 and therefore has a steady-state error for a unit ramp input (obtained from Table 8.1) of

$$e_{ss}(t) = \frac{1}{K_v}$$

where K_v, the velocity error constant, is obtained from the open-loop transfer function as

$$K_v = \frac{K \prod_{i=1}^{m} z_i}{\prod_{k=1}^{q} p_k} = \frac{240}{3}$$

The steady-state error for a ramp of slope 0.05 rad/s will be, therefore,

$$e_{ss}(t) = \frac{0.05}{240/3} = 0.000625 \text{ rad}$$

This represents the angular error between the radar antenna that is pointing directly at the aircraft and the gun. If the shells travel in a straight line, they will miss the target 10 km away by d, where

$$d = r\delta\theta = 10{,}000 \times 0.000625 = 6.25 \text{ m}$$

To reduce the miss distance to no more than 1 m, the value of K has to be increased. Since the miss distance is given by

$$d = r\delta\theta = \frac{0.05r}{K_v} = \frac{0.05r}{K/3}$$

which yields the required value of K to be

$$K = \frac{0.15r}{d} = 1500$$

Sample Problem 8.3

1. Consider the feedback control system shown in Fig. SP8.3.1. For $H(s) = 1$ and a unit ramp input, determine the velocity error constant and the steady-state error. After what approximate time interval does the error achieve its steady-state value?

2. Calculate the steady-state error for the non-unity-feedback case

$$H(s) = \frac{1}{s + 1}$$

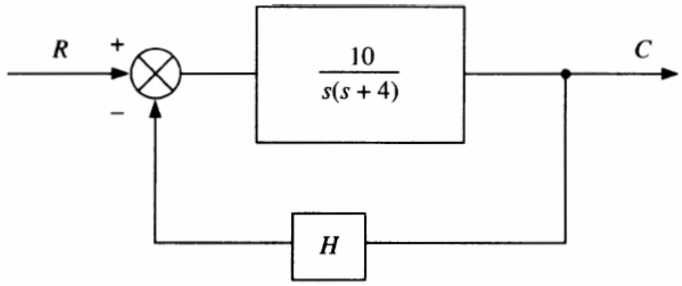


Fig. SP8.3.1

3. What is the steady-state error for $H(s) = 1$ and a time-domain input given by

$$r(t) = \begin{cases} 0 & t \le 0 \\ 2t + 5 & t > 0 \end{cases}$$

Solution

1. For the unity-feedback case, the open-loop transfer function is given by

$$G(s) = \frac{10}{s(s + 4)}$$

and the system is of type 1. The velocity error coefficient is given by

$$K_v = \lim_{s\to 0} G(s) = \tfrac{10}{4} = 2.5$$

The steady-state error is therefore

$$e_{ss}(t) = \frac{1}{K_v} = 0.4$$

To determine the time taken for the transient to disappear, the 2% settling time introduced in Module 5 may be used. Although this was defined for a step-input response, we will assume it is a good measure of the disappearance of the transient for any input. The closed-loop transfer function is

$$\frac{C}{R} = \frac{10}{s(s + 4) + 10} = \frac{10}{s^2 + 4s + 10}$$

yielding

$$\zeta = 0.63 \qquad \omega_n = 3.16 \text{ rad/s}$$

The settling time is

$$T_s = \frac{4}{\zeta\omega_n} = 2 \text{ s}$$

It may be concluded, therefore, that the system achieves its steady-state error 2 s after the input is applied, and the total response will be as shown in Fig. SP8.3.2.

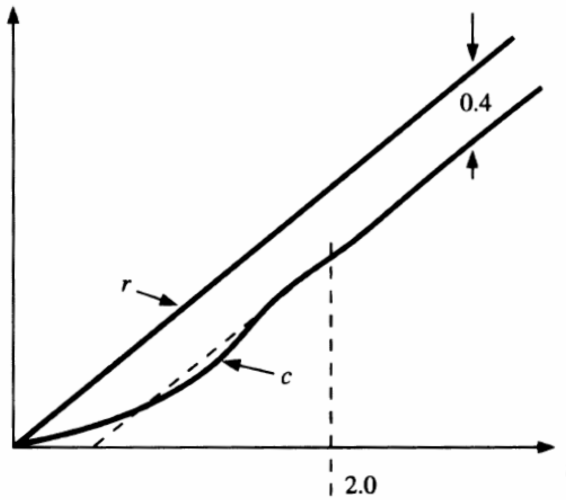


Fig. SP8.3.2

2. Considering the non-unity-feedback case now, the steady-state error will be calculated using equation 8.59. For this system we have

$$\frac{E}{R} = \frac{1 + G(H - 1)}{1 + GH}$$

$$= \frac{s(s + 1)(s + 4) - 10s}{s(s + 1)(s + 4) + 10}$$

Letting the input

$$R(s) = \frac{1}{s^2}$$

the steady-state error becomes

$$e_{ss}(t) = \lim_{s \to 0} sR(s) \frac{s(s + 1)(s + 4) - 10s}{s(s + 1)(s + 4) + 10} = -\frac{6}{10}$$

An alternative approach is to reduce the system to an equivalent one with unity feedback. The resulting forward-path transfer function becomes

$$G' = \frac{1 + G(H - 1)}{1 + GH} = \frac{10(s + 1)}{s(s^2 + 5s - 6)}$$

This system is of type 1 and third order, with a velocity error constant of

$$K_v = \lim_{s \to 0} G(s) = -\frac{10}{6}$$

The velocity error becomes the same as previously calculated:

$$e_{ss}(t) = \frac{1}{K_v} = -\frac{6}{10}$$

The significance of the negative error becomes clear by recalling that

$$e = r - c$$

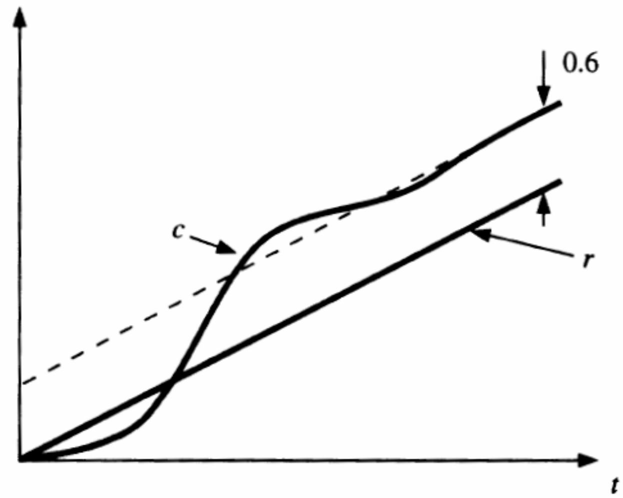


Fig. SP8.3.3

implying that, in this case, the steady-state output is larger than the input, as shown in Fig. SP8.3.3.

3. For the final part of the question, the input takes the form

$$r(t) = \begin{cases} 0 & t \leq 0 \\ 2t + 5 & t > 0 \end{cases}$$

Because the relationship between the input R and the output C is described by a *linear* differential equation, the principle of superposition may be applied. This means that the steady-state error will be given by the sum of the steady-state errors due to the individual inputs

$$r(t) = 2t$$

and

$$r(t) = 5$$

applied separately. Since the system is of order 1 and the feedback element is unity, Table 8.1 may be used to determine the steady-state error to each of these inputs, which may be identified as a ramp of slope 2 and a step of magnitude 5, respectively. As a system of order 1 has zero steady-state error for any step, only the ramp will contribute to the steady-state error. Because the ramp input has a slope twice that of a unit ramp, the steady-state error will be twice that calculated previously:

$$e_{ss}(t) = 2 \times 0.4 = 0.8$$

The total steady-state error is therefore 0.8.