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# Investigation of the constraints on harmonic morphisms of warped product type from Einstein manifolds


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# Contents

## Abstract and Keywords

Harmonic morphisms are maps between Riemannian manifolds which preserve germs of harmonic functions. These can be described as the harmonic maps which are horizontally (weakly) conformal.

A class of harmonic morphisms directly related to a physically significant geometric structure is the class of harmonic morphisms of warped product type. Such maps are characterized as non-constant horizontally homothetic harmonic morphisms with totally geodesic fibres and integrable horizontal distribution.

The principal aim of this project is to investigate constraints on the existence of harmonic morphisms of warped product type from Einstein manifolds. A Bochner type technique is developed which leads to general restrictions on the existence of harmonic morphisms of warped product type. These restrictions are utilized to obtain non-existence results for harmonic morphisms of warped product type from Einstein manifolds.

As an application, the Bochner type technique for harmonic morphisms of warped product type is adapted to address existence questions related to certain Einstein warped products and warped space-times.

KEYWORDS:
Harmonic morphisms, warped products, Bochner technique, Einstein manifolds.

# Chapter 1

# Harmonic morphisms of warped product type from Einstein manifolds

**Summary**

Weitzenböck type identities for harmonic morphisms of warped product type are developed which lead to some necessary conditions for their existence. These necessary conditions are further studied to obtain many non-existence results for harmonic morphisms of warped product type from Einstein manifolds.

## 1.1 Introduction

Harmonic morphisms are maps between Riemannian manifolds which preserve germs of harmonic functions, i.e. these (locally) pull back real-valued harmonic functions to real-valued harmonic functions. Harmonic morphisms are characterized as harmonic maps which are horizontally (weakly) conformal. On the one hand this characterization endows harmonic morphisms with analytic as well as geometric properties. On the other hand, it puts strong restrictions on their existence as solutions of an over-determined system of partial differential equations. This makes the investigation of questions related to their existence, classification and construction of prime interest. Many interesting results in this regard can be found in [1, 2, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26].

A class of harmonic morphisms directly related to a geometric structure of physical interest is the class of harmonic morphisms of warped product type. Such maps have been investigated in [11, 13, 25, 26, 27]. In [25, 26], these have been particularly studied in the context of Einstein manifolds where the constructions involving harmonic morphisms of warped product type are discussed. However, the results have not led to any non-trivial example of harmonic morphisms of warped product type from compact Einstein manifolds; where by a *trivial harmonic morphism of warped*

*product type* we mean a map which is locally the projection of a Riemannian product. The only known result in this context, proved in [9, Proposition 12.7.1], is for one dimensional fibres.
*Let $\phi$: $(M^{n+1}, g) \to (N^n, h) (n \geq 3)$ be a harmonic morphism of warped product type from a compact manifold. If $M$ is Einstein then, up to a homothety, $\phi$ is locally the projection of a Riemannian product.*

Motivated by the above result and the fact that there are natural obstructions to the existence of harmonic morphisms from compact domains, the purpose of this chapter is to investigate constraints on the existence of harmonic morphisms of warped product type (with compact fibres of any dimension) from Einstein manifolds. A Bochner type argument is developed, in Section 3, which leads to general restrictions on the existence of harmonic morphisms of warped product type. These restrictions are applied, in Section 4, to obtain several non-existence results for harmonic morphisms of warped product type from Einstein manifolds.

**Remark 1.1.1** *In this chapter we are interested in restrictions on harmonic morphisms of warped product type from Riemannian manifolds, but the technique can easily be adapted to obtain restrictions on harmonic morphisms of warped product type, with compact Riemannian fibres, from semi-Riemannian manifolds.*

## 1.2 Harmonic morphisms of warped product type

The formal theory of harmonic morphisms between Riemannian manifolds began with the work of Fuglede [12] and Ishihara [19].

**Definition 1.2.1** *A map $\phi : M^m \to N^n$ is called a harmonic morphism if for every open subset $U$ of $N$ (with $\phi^{-1}(U)$ non-empty) and every harmonic function $f : U \to \mathbb{R}$, the composition $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is harmonic.*

Harmonic morphisms are related to horizontally (weakly) conformal maps which can be defined in the following manner.

For a smooth map $\phi : M^m \to N^n$, let $C_\phi = \{x \in M | \mathrm{rank} d\phi_x < n\}$ be its *critical set*. The points of the set $M \setminus C_\phi$ are called *regular points*. For each $x \in M \setminus C_\phi$, the *vertical space* at $x$ is defined by $\mathcal{V}_x = \mathrm{Ker}\ d\phi_x$. The *horizontal space* $\mathcal{H}_x$ at $x$ is given by the orthogonal complement of $\mathcal{V}_x$ in $T_xM$.

**Definition 1.2.2** *A smooth map $\phi : (M^m, \mathbf{g}) \to (N^n, \mathbf{h})$ is called horizontally (weakly) conformal if $d\phi = 0$ on $C_\phi$ and the restriction of $\phi$ to $M \setminus C_\phi$ is a conformal submersion, that is, for each $x \in M \setminus C_\phi$, the differential $d\phi_x : \mathcal{H}_x \to T_{\phi(x)}N$ is conformal and surjective. This means that there exists a function $\lambda : M \setminus C_\phi \to \mathbb{R}^+$ such that*

$$\mathbf{h}(d\phi(X), d\phi(Y)) = \lambda^2 \mathbf{g}(X, Y) \quad \forall X, Y \in \mathcal{H}_x \ \ \textit{and} \ \ x \in M \setminus C_\phi.$$

By setting $\lambda = 0$ on $C_\phi$, we can extend $\lambda : M \to \mathbb{R}_0^+$ to a continuous function on $M$ such that $\lambda^2$ is smooth. The extended function $\lambda : M \to \mathbb{R}_0^+$ is called the *dilation* of the map.

Let $\mathbf{grad}_{\mathcal{H}}\lambda^2$ and $\mathbf{grad}_{\mathcal{V}}\lambda^2$ denote the horizontal and vertical projections of $\mathbf{grad}\lambda^2$.

**Definition 1.2.3** *A smooth map $\phi : M^m \to N^n$ is called horizontally homothetic if it is a horizontally conformal submersion whose dilation is constant along the horizontal curves i.e.* $\mathbf{grad}_{\mathcal{H}}\lambda^2 = 0$.

Recall that a map $\phi : M^m \to N^n$ is said to be *harmonic* if it extremizes the associated energy integral $E(\phi) = \frac{1}{2}\int_\Omega \|\phi_*\|^2 dv^M$ for every compact domain $\Omega \subset M$. It is well-known that a map $\phi$ is harmonic if and only if its tension field $\tau(\phi) = \text{trace}\nabla d\phi$ vanishes.

Harmonic morphisms can be viewed as a subclass of harmonic maps in the light of the following characterization, obtained in [12, 19].
*A smooth map is a harmonic morphism if and only if it is harmonic and horizontally (weakly) conformal.*

The following result of Baird-Eells [3, Riemannian case] and Gudmundsson [16, semi-Riemannian case] reflects a significant geometric feature of harmonic morphisms.

**Theorem 1.2.4** *Let $\phi : M^m \to N^n$ be a horizontally conformal submersion with dilation $\lambda$. If*

1. *$n = 2$, then $\phi$ is a harmonic map if and only if it has minimal fibres.*

2. *$n \geq 3$, then two of the following imply the other,*

   *(a) $\phi$ is a harmonic map*

   *(b) $\phi$ has minimal fibres*

   *(c) $\phi$ is horizontally homothetic.*

Here we deal with a class of harmonic morphisms, closely related to a physically significant geometric structure, namely harmonic morphisms of warped product type which are defined as follows.

**Definition 1.2.5** *[9, 27] A map is called a harmonic morphism of warped product type if it is a non-constant horizontally homothetic map with totally geodesic fibres and integrable horizontal distribution.*

Note that, due to Theorem 1.2.4, these maps are harmonic morphisms and are related to the usual warped product structures through following characterization.

**Proposition 1.2.6** *[9, 27]*

1. *The projection $F \times_{f^2} N \to N$ of a warped product onto its second factor is a horizontally homothetic map with totally geodesic fibres and integrable horizontal distribution.*

2. *Conversely, any horizontally homothetic map $(M, g) \to (N, h)$ with totally geodesic fibres and integrable horizontal distribution is locally the projection of a warped product.*

The reader is referred to [2, 9, 12, 28] for fundamental results and properties of harmonic morphisms and to [11, 13, 25, 26, 27] particularly for constructions and classifications involving harmonic morphisms of warped product type.

## 1.3 Restrictions on harmonic morphisms of warped product type

The Weitzenböck type identities established in the following Theorem are the main tool for drawing results about the non-existence of certain harmonic morphisms of warped product type.

**Theorem 1.3.1** *Let $\phi\colon (M^m, g_{_M}) \to (N^n, g_{_N})$ be a non-constant harmonic morphism of warped product type between Riemannian manifolds. If $\lambda$ denotes the dilation of $\phi$ then*

**(i)** $-n\lambda\overset{\mathcal{V}}{\Delta}\frac{1}{\lambda} = \sum_{r=n+1}^{m}\mathrm{Ric}^{\mathrm{M}}(e_r, e_r) - \mathrm{Scal}^{\mathcal{V}}$

**(ii)** $\mathrm{Ric}^{\mathrm{M}}(X, Y) = \mathrm{Ric}^{\mathrm{N}}(d\phi\cdot X, d\phi\cdot Y) + g_{_M}(X, Y)\overset{M}{\Delta}\ln\lambda$

*where $X$, $Y$ are horizontal vectors, $(e_r)_{r=n+1}^{m}$ is a local orthonormal frame for vertical distribution,* $\mathrm{Scal}^{\mathcal{V}}$ *is the scalar curvature of fibres of $\phi$ and $\overset{\mathcal{V}}{\Delta}$ is the Laplacian on fibres defined as $\overset{\mathcal{V}}{\Delta}f = \overset{F}{\Delta}(f|_F)$ for the fibre $F = \phi^{-1}(\phi(x))$ with $\overset{F}{\Delta}$ denoting the Laplacian on $F$.*

**Proof.** We start with a curvature identity for submersive harmonic morphisms, proved in [9, Theorem 11.5.1(i)], which relates the Ricci curvatures of $M$ and fibres of $\phi$.

$$\begin{aligned}\mathrm{Ric}^{\mathrm{M}}(U, V) &= \mathrm{Ric}^{\mathcal{V}}(U, V) + \sum_{a=1}^{n}\langle(\nabla_{e_a}B^*)_U e_a, V\rangle + 2(n-1)d\ln\lambda(B_U V)\\ &+n\nabla d\ln\lambda(U, V) - nU(\ln\lambda)V(\ln\lambda) + \frac{1}{4}\sum_{a,b=1}^{n}\langle U, I(e_a, e_b)\rangle\langle V, I(e_a, e_b)\rangle\end{aligned} \tag{1.3.1}$$

where $U$, $V$ are vertical vectors and $(e_a)_{a=1}^{n}$ is a local orthonormal basis for the horizontal distribution. Since the fibres of $\phi$ are totally geodesic, the horizontal distribution is integrable and $\phi$ is horizontally homothetic we have

$$d\ln\lambda(B_U V) = 0, \tag{1.3.2}$$

$$\sum_{a,b=1}^{n}\langle U, I(e_a, e_b)\rangle\langle V, I(e_a, e_b)\rangle = 0, \tag{1.3.3}$$

and

$$\sum_{r=n+1}^{m}\langle(\nabla_{e_a}B^*)_{e_r}e_a, e_r\rangle = 0. \tag{1.3.4}$$

Taking trace over vertical vectors in Equation 1.3.1 and using Equations 1.3.2, 1.3.3, 1.3.4 gives

$$\sum_{r=n+1}^{m} \mathrm{Ric}^{\mathrm{M}}(e_r, e_r) = \mathrm{Scal}^{\mathcal{V}} + n \sum_{r=n+1}^{m} \nabla d\ln\lambda(e_r, e_r) - n \sum_{r=n+1}^{m} [e_r(\ln\lambda)]^2 . \tag{1.3.5}$$

Because of totally geodesic fibres we can write

$$\nabla d\ln\lambda(e_r, e_r) = e_r\left(e_r(\ln\lambda)\right) - \left(\nabla^M_{e_r} e_r\right)(\ln\lambda) = e_r\left(e_r(\ln\lambda)\right) - \left(\nabla^{\mathcal{V}}_{e_r} e_r\right)(\ln\lambda),$$

therefore, Equation 1.3.5 implies

$$\sum_{r=n+1}^{m} \mathrm{Ric}^{\mathrm{M}}(e_r, e_r) = \mathrm{Scal}^{\mathcal{V}} + n\overset{\mathcal{V}}{\Delta}\ln\lambda - n \sum_{r=n+1}^{m} [e_r(\ln\lambda)]^2 .$$

Formula (i) now follows by using the relation

$$\overset{\mathcal{V}}{\Delta}\ln\lambda - \sum_{r=n+1}^{m} [e_r(\ln\lambda)]^2 = -\lambda\overset{\mathcal{V}}{\Delta}\frac{1}{\lambda},$$

which can be established from

$$e_r\left(e_r(\ln\lambda)\right) = [e_r(\ln\lambda)]^2 - \lambda e_r\left(e_r(\frac{1}{\lambda})\right).$$

Formula (ii) follows directly from [9, Theorem 11.5.1 (iii)] and hypothesis. □

**Lemma 1.3.2** *A harmonic morphism of warped product type is totally geodesic iff it has constant dilation.*

As an immediate consequence of Theorem 1.3.1, we have

**Corollary 1.3.3** *Let $\phi\colon M^m \to N^n$ be a (non-constant) harmonic morphism of warped product type with compact fibres. If*

1. *either* $\mathrm{Ric}^{\mathrm{M}} \geq 0$ *and the fibres have scalar curvature* $\mathrm{Scal}^{\mathcal{V}} \leq 0$
2. *or* $\mathrm{Ric}^{\mathrm{M}} \leq 0$ *and the fibres have scalar curvature* $\mathrm{Scal}^{\mathcal{V}} \geq 0$

*then* $\mathrm{Scal}^{\mathcal{V}} \equiv 0$ *and, up to a homothety, $\phi$ is a totally geodesic Riemannian submersion.*

**Proof.** $\lambda$ is constant from Theorem 1.3.1(i), hypothesis and compactness of fibres. Rest follows by using Theorem 1.3.1(i) and Lemma 1.3.2. □

On rewriting the Weitzenböck type identities, we obtain applications involving only the curvature of domain manifolds.

**Corollary 1.3.4** *Every (non-constant) harmonic morphism $\phi\colon M^m \to N^n$ of warped product type, with compact fibres, from a Riemannian manifold of non-negative sectional curvature or non-positive sectional curvature is, up to a homothety, a totally geodesic Riemannian submersion.*

**Proof.** Since the fibres of $\phi$ are totally geodesic, the Riemannian curvature tensor $R^{\mathcal{V}}$ of fibres agrees with the Riemannian curvature tensor $R^M$ of $M$ on vertical vectors. Hence

$$\sum_{r=n+1}^{m} \mathrm{Ric}^{\mathrm{M}}(e_r, e_r) - \mathrm{Scal}^{\mathcal{V}} = \sum_{r=n+1}^{m}\sum_{a=1}^{n} g_{_M}(R^M(e_a, e_r)e_a, e_r)$$

where $\mathrm{Scal}^{\mathcal{V}}$ is the scalar curvature of fibres, $(e_r)_{a=1}^{n}$ and $(e_r)_{r=n+1}^{m}$ are local orthonormal frames for horizontal and vertical distributions respectively.
Using above in Theorem 1.3.1(i) gives

$$-n\lambda\overset{\mathcal{V}}{\Delta}\frac{1}{\lambda} = \sum_{r=n+1}^{m}\sum_{a=1}^{n} g_{_M}(R^M(e_a, e_r)e_a, e_r).$$

The proof then follows from the hypothesis and compactness of fibres. □

## 1.4 Applications to harmonic morphisms of warped product type from Einstein manifolds

By using the Einstein metric in Theorem 1.3.1 we have the following Weitzenböck type identity for harmonic morphisms of warped product type from Einstein manifolds.

**Proposition 1.4.1** *Let $\phi\colon M^m \to N^n$ be a (non-constant) harmonic morphism of warped product type with dilation $\lambda$. If $M$ is Einstein with Einstein constant $c^M$ then*

$$-n\lambda\overset{\mathcal{V}}{\Delta}\frac{1}{\lambda} = (m-n)c^M - \mathrm{Scal}^{\mathcal{V}} \tag{1.4.1}$$

*and*

$$\frac{\mathrm{Scal}^{\mathrm{N}}}{n} = \frac{c^M - \overset{M}{\Delta}\ln\lambda}{\lambda^2}.$$

*For $n \geq 3$, $N$ is Einstein with Einstein constant $c^N$ satisfying*

$$c^N = \frac{c^M - \overset{M}{\Delta}\ln\lambda}{\lambda^2}. \tag{1.4.2}$$

In order to obtain applications we first find some necessary conditions for the existence of non-trivial harmonic morphisms of warped product type from Einstein manifolds.

**Theorem 1.4.2** *Let $\phi\colon M^m \to N^n$ be a harmonic morphism of warped product type with non-constant dilation $\lambda$. If $M$ is Einstein with Einstein constant $c^M$ and the fibres of $\phi$ are compact then*

**(a)** $\inf(\mathrm{Scal}^{\mathcal{V}}) < (m-n)c^M < \sup(\mathrm{Scal}^{\mathcal{V}})$,

**(b)** *the total scalar curvature $S^{\mathcal{V}} = \int \mathrm{Scal}^{\mathcal{V}} \upsilon^F$ of fibres satisfies $S^{\mathcal{V}} > 0$.*

*Furthermore if $M$ is compact then, for $n \geq 3$,*

**(c)** $c^M > 0$ *and hence the Einstein constant $c^N$ of $N$ satisfies $c^N > 0$,*

**(d)** $\lambda^2$ *is neither bounded below nor bounded above by $\frac{c^M}{c^N}$.*

**Proof.**

**(a)** $\mathrm{Scal}^{\mathcal{V}} \geq (m-n)c^M$ or $\mathrm{Scal}^{\mathcal{V}} \leq (m-n)c^M$ makes $\frac{1}{\lambda}$ a subharmonic or superharmonic function. Since fibres are compact $\lambda$ must be constant; a contradiction.

**(b)** Integrating Equation 1.4.1 and using Green's formula gives

$$S^{\mathcal{V}} = \int \mathrm{Scal}^{\mathcal{V}} \upsilon^F = n \int \lambda^2 \|\mathrm{grad}\frac{1}{\lambda}\|^2 \upsilon^F + (m-n)c^M \mathrm{Vol}(F) > 0$$

where $\mathrm{Vol}(F)$ is the volume of fibre.

**(c)** Assume $c^M \leq 0$. Equation 1.4.2 gives

$$\frac{c^N}{\mathrm{Vol}(M)} \int \lambda^2 \upsilon^M = c^M, \tag{1.4.3}$$

hence $c^N \leq 0$.
Since $M$ is compact, $\frac{1}{\lambda}$ assumes its minimum on $M$. Let $p_0$ be minimum point of $\frac{1}{\lambda}$ on $M$ then

$$\frac{1}{\lambda(p_0)} > 0, \qquad \mathrm{grad}\frac{1}{\lambda}(p_0) = 0, \qquad \text{and} \qquad \overset{M}{\Delta}\frac{1}{\lambda}(p_0) \geq 0.$$

On the other hand, using

$$\lambda \overset{M}{\Delta}\frac{1}{\lambda} = \lambda^2 \|\mathrm{grad}\frac{1}{\lambda}\|^2 - \overset{M}{\Delta}\ln\lambda$$

we have

$$\begin{aligned} \lambda(p_0)\overset{M}{\Delta}\frac{1}{\lambda}(p_0) &= -\overset{M}{\Delta}\ln\lambda(p_0) \\ &= c^N\lambda^2(p_0) - c^M & \{\text{From Equation 1.4.2}\} \\ &= \frac{c^N}{\mathrm{Vol}(M)}\int \left(\lambda^2(p_0) - \lambda^2\right)\upsilon^M & \{\text{Using Equation 1.4.3}\} \\ &\leq 0. \end{aligned}$$

Hence $\frac{1}{\lambda}$ must be constant, which contradicts the hypothesis.

**(d)** If $\lambda^2 \leq \frac{c^M}{c^N}$ or $\lambda^2 \geq \frac{c^M}{c^N}$ then from Equation 1.4.2, $\ln\lambda$ is a subharmonic or superharmonic function. Since fibres are compact, this gives a contradiction.

□

The above result obviously eliminates, for instance, the possibility of (non-trivial) harmonic morphisms of warped product type from Einstein manifolds to have compact fibres which

- are Einstein (or have constant scalar curvature),
- compact locally symmetric spaces of non-compact type (or spaces of negative scalar curvature)

**Theorem 1.4.3** *Let $(M^m, g)$ $(m > n \geq 3)$ be a compact manifold conformally equivalent to a manifold with non-positive scalar curvature. If $M$ is Einstein then there are no harmonic morphisms $\phi\colon M^m \to N^n$ of warped product type, with non-constant dilation.*

**Proof.** Let $g^1$ be the metric conformal to $g$ and set $g^1 = \psi^{\frac{4}{m-2}} g$ for a function $\psi > 0$ on $M$. If $\mathrm{Scal}^{g^1}$, $\mathrm{Scal}^M$ denote the scalar curvatures of $g^1$, $g$, respectively, then by standard computations cf. [3, Page 59]

$$\psi^{\frac{m+2}{m-2}} \mathrm{Scal}^{g^1} = 4\frac{m-1}{m-2}\Delta\psi + \mathrm{Scal}^M \psi.$$

Therefore, by hypothesis we must have

$$\begin{aligned} & 4\frac{m-1}{m-2}\Delta\psi + \mathrm{Scal}^M \psi \leq 0 \\ \textit{or} \quad & mc^M \int_M \psi v^M \leq 0 \end{aligned}$$

where $c^M$ is the Einstein constant of $M$. This contradicts Theorem 1.4.2 if there exists a harmonic morphism $\phi\colon M^m \to N^n$ of warped product type, with non-constant dilation. □

Proposition 1.4.1 and Theorem 1.4.2 yield the following non-existence result for harmonic morphisms of warped product type to surfaces.

**Corollary 1.4.4** *There are no harmonic morphisms of warped product type, with non-constant dilation, from a compact Einstein manifold to a Riemann surface $N^2$ of genus $g \geq 1$.*

**Proof.** The notion of harmonic morphisms to a Riemann surface does not depend on any specific Hermitian metric on $N^2$. The proof then follows from above and the fact that every compact Riemann surface of genus $g \geq 2$ and genus $g = 1$ has a Hermitian metric of constant negative and zero curvature, respectively. □

In case of symmetric domains we have

**Corollary 1.4.5** *There exist no harmonic morphisms $\phi\colon M^m \to N^n$ of warped product type, with non-constant dilation, in each of the following case:*

**(i)** *$M$ is an irreducible symmetric space of compact type,*

**(ii)** *$M$ is a compact locally symmetric space of non-compact type,*

**(iii)** *$M$ is an irreducible symmetric space of non-compact type and $\phi$ has compact fibres.*

**Proof.** Follows from Corollary 1.3.4 by using the facts about the curvatures of symmetric spaces of compact and non-compact type. □

A nonexistence result for harmonic morphisms of warped product type, with 1-dimensional fibres, from compact manifolds is obtained in [9, Proposition 12.7.1]. Corollary 1.4.6 relaxes the hypothesis by replacing the compactness of domain with the compactness of fibres.

**Corollary 1.4.6** *Let $M^{n+1}$ be an Einstein manifold. Then there are no harmonic morphisms $\phi\colon M^{n+1} \to N^n$ of warped product type, with non-constant dilation and compact fibres.*

**Proof.** Follows directly from Theorem 1.4.2(a). □

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# Chapter 2

# A non-existence result for compact Einstein warped products

**Summary**

Warped products provide a rich class of physically significant geometric objects. The existence of compact Einstein warped products was questioned in [3, Section 9.103]. Using the methods of Chapter 1, it is shown that there exists a metric on every compact manifold $B$ such that (non-trivial) Einstein warped products, with base $B$, cannot be constructed.

## 2.1 Introduction

Warped product construction is a construction in the class of Riemannian manifolds that generalizes direct product. This construction was introduced in [3] where it was used to construct a variety of complete Riemannian manifolds with negative sectional curvature. Warped products have significant applications, in general relativity, in the studies related to solutions of Einstein's equations [1, 2]. Besides general relativity, warped product structures have also generated interest in many areas of geometry, especially due to their role in construction of new examples with interesting curvature and symmetry properties cf. [3, 5, 6, 5].

**Definition 2.1.1** *Let $(B, g_B)$ and $(F, g_F)$ be Riemannian manifolds with $f : B \to (0, \infty)$ a smooth function on $B$. The warped product $M = B \times_f F$ is the product manifold $B \times F$ equipped with the metric*

$$g = \pi^*(g_B) \oplus (f \circ \pi)^2 \sigma^*(g_F),$$

*where $\pi : B \times F \to B$, $\sigma : B \times F \to F$ are usual projections and $*$ denotes pullback. $(B, g_B)$ is called the base, $(F, g_F)$ is called the fiber and $f$ the warping function of the warped product.*

If the warping function '$f$' is constant then the warped product $B\times_f F$ (up to a change of scale) is a (global) Riemannian product, which we call as trivial warped product.

The reader is referred to [3, 5] for the fundamental results and properties of warped products.

A Riemannian manifold $(M^m, g)$ is said to be Einstein if its Ricci curvature is a constant multiple of $g$. The notion of Einstein manifolds coincides with manifolds of constant curvature for $m \leq 3$ but Einstein manifolds constitute quite a large class in higher dimensions. Many new examples of Einstein manifolds have been obtained using warped products, cf. [3]. Einstein warped products, due to their useful curvature and symmetry properties, provide a rich class of examples of practical interest in Riemannian as well as semi-Riemannian geometry. Yet there are no known examples of (non-trivial) compact Einstein warped products. This is what was questioned in [3, Section 9.103]: *Can a (non-trivial) compact Einstein warped product be constructed?*

The purpose of this chapter is to study this conjecture about non-existence of (non-trivial) compact Einstein warped products and to show that there exists a metric on every compact manifold $B$ such that a (non-trivial) Einstein warped product $M = B\times_f F$ cannot be constructed.

## 2.2 Main result

We begin by proving some necessary conditions for the existence of (non-trivial)Einstein warped products with compact base.

**Proposition 2.2.1** *Let $M^m = B\times_f F$ be a warped product of an (m-n)-dimensional compact Riemannian manifold $B$ and an $n$-dimensional Riemannian manifold $F$. Let* $\mathrm{Scal}^B$ *denote the scalar curvature of $B$. If $M$ is Einstein with Einstein constant $c^M$ and*

$$\text{either} \quad \mathrm{Scal}^B \leq (m-n)c^M \qquad \text{or} \quad \mathrm{Scal}^B \geq (m-n)c^M$$

*then the warping function $f$ is constant and, up to a scale, $M$ is a Riemannian product.*

**Proof.** Using Theorem 1.3.1 or the well-known curvature identity [5, Page 211] relating the Ricci curvatures of the warped product $M$ and the base $B$, we have

$$-\frac{n}{f}\Delta^B f = \sum_{i=n+1}^{m} \mathrm{Ric}^M(e_i, e_i) - \mathrm{Scal}^B$$

where Ric denotes the Ricci curvature of $M$ and $\Delta^B$ is the Laplacian on $B$.
For the Einstein metric, the above equation becomes

$$-\frac{n}{f}\Delta^B f = (m-n)c^M - \mathrm{Scal}^B.$$

Now the conditions

$$\mathrm{Scal}^B \leq (m-n)c^M \qquad \text{or} \quad \mathrm{Scal}^B \geq (m-n)c^M$$

make $f$ a subharmonic or superharmonic function. Since $B$ is compact, $f$ must be constant. □

The above necessary conditions yield the following non-existence result for compact Einstein warped products.

**Theorem 2.2.2** *There exists a metric on every compact manifold $B$ such that there are no (non-trivial) Einstein warped products $M^m = B \times_f F$ with base $B$.*

**Proof.** For 1-dimensional base with any metric, $\mathrm{Scal}^B \equiv 0$. Every 2-dimensional manifold admits a metric of constant curvature. Every compact manifold of dimension at least 3 carries a metric of constant negative curvature [4]. Using Proposition 2.2.1 and the above facts completes the proof. □

The question of existence of compact Einstein manifolds has also been addressed recently in [7] where it is shown necessary for these manifolds to have positive scalar curvature.

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# Chapter 3

# Restrictions on warped space-times with Riemannian base

**Summary**

Warped space-times provide several well-known examples of exact solutions to Einstein's field equations. Yet there are no known examples of (non-trivial) warped space-times with compact Riemannian base. Using the methods of Chapter 1, it is shown that the Ricci curvature imposes restrictions on the existence of such space-times. As a consequence, non-existence results are established for Einstein warped space-times with compact Riemannian base.

## 3.1 Introduction

Warped product construction is a construction in the class of Riemannian manifolds that generalizes direct product. This construction was introduced in [3] where it was used to construct a variety of complete Riemannian manifolds with negative sectional curvature. The connection with general relativity was established due to their role in the studies related to solutions of the Einstein's equations [1, 2, 5]. In fact, many of the well-known solutions of the Einstein's equations are semi-Riemannian warped products.

**Definition 3.1.1** *Let $(B, g_B)$ and $(F, g_F)$ be semi-Riemannian manifolds with $f : B \to (0, \infty)$ a smooth function on $B$. The warped product $M = B \times_f F$ is the product manifold $B \times F$ equipped with the metric*

$$g = \pi^*(g_B) \oplus (f \circ \pi)^2 \sigma^*(g_F),$$

*where $\pi : B \times F \to B$, $\sigma : B \times F \to F$ are usual projections and $*$ denotes pullback operator. $(B, g_B)$ is called the base, $(F, g_F)$ is called the fiber and $f$ the warping function of the warped product.*

If the warping function '$f$' is constant then the warped product $B\times_f F$ (up to a change of scale) is a semi-Riemannian product, which we call as trivial warped product.

**Definition 3.1.2** *A warped space-time is a 4-dimensional warped product equipped with a Lorentzian metric.*

Due to their construction, warped space-times exhibit useful curvature and symmetry properties, and hence provide a rich class of examples of solutions to Einstein field equations. For instance Schwarzchild space-time, Robertson-Walker space-times, de-Sitter space-time, generalized Robertson-Walker space-times and standard static space-times are all examples of warped space-times.

The class of warped space-time of practical interest mostly consists of warped products of the form $B\times_f F$ where $B$ is a Lorentzian manifold and $F$ is a Riemannian manifold i.e. the warping function is defined on a Lorentzian manifold and acts on the positive definite metric on Riemannian manifold $F$. On the other hand it is possible to construct warped space-times by considering warped products of the form $B\times_f F$ where $B$ has a positive definite metric and $F$ is a Lorentzian manifold i.e. the warping function is defined on a Riemannian factor and acts on the Lorentzian metric on $F$ (See [2]). Well known examples of such warped products are

- the Minkowski space-time,
- the Einstein static universe which is the trivial warped product $\mathbb{R} \times \mathbb{S}^3$ with the metric $g = -dt^2 + (dr^2 + sin^2 r d\theta^2 + sin^2 r sin^2\theta d\phi^2)$
- and the universal covering space of anti de-Sitter space-time which is the warped product $\mathbb{H}^3\times_f\mathbb{R}$ where the warping function is defined on the hyperbolic 3-space with a Riemannian metric of constant negative sectional curvature $-1$.

Though the class of standard static space-times provides further examples of such warped space-times, yet there are no known examples of non-trivial warped space-times where the warping function is defined on compact Riemannian factor. We treat such warped space-times in this chapter and investigate the restrictions on the existence of non-trivial warped space-times with the warping function defined on compact Riemannian factor.

## 3.2 Constraints on warped space-times with Riemannian base

In this section we consider warped space-times of the form $B\times_f F$ where $B$ is a manifold with positive definite metric, and study obstructions to their existence.

**Proposition 3.2.1** *Let $M = B^{4-n}\times_f F^n$ $(n = 1, 2, 3)$ be a warped space-time. Then*

$$-\frac{n}{f}\Delta^B f = \sum_{r=n+1}^{4} \mathrm{Ric}^M(e_r, e_r) - \mathrm{Scal}^B$$

*where* $\mathrm{Ric}^M$, $\mathrm{Scal}^B$ *denote the Ricci curvature and scalar curvature of* $M$ *and* $B$ *respectively, and* $\Delta^B$ *is the Laplacian on* $B$.

**Proof.** The proof follows from the well-known curvature identity [2, Proposition 3.76(4)] relating the Ricci curvatures of the warped product $M$ and the base $B$. □

As a consequence, we see that the Ricci curvature of the space-times places restrictions on the existence of warped space-times under consideration.

**Theorem 3.2.2** *Let* $M = B^{4-n}\times_f F^n$ $(n = 1, 2, 3)$ *be a warped space-time, with* $B$ *a compact manifold with a positive definite metric.*

**(a)** *For* $n = 3$, *if either* $\mathrm{Ric}^M \geq 0$ *or* $\mathrm{Ric}^M \leq 0$ *then the warping function* $f$ *is constant and up to a scale* $M$ *is a Lorentzian product.*

**(b)** *For* $n < 3$, *if*

- *either* $\mathrm{Ric}^M \geq 0$ *and* $\mathrm{Scal}^B \leq 0$
- *or* $\mathrm{Ric}^M \leq 0$ *and* $\mathrm{Scal}^B \geq 0$

*then the warping function* $f$ *is constant and up to a scale* $M$ *is a Lorentzian product.*

**Proof.** From the hypothesis and Proposition 3.2.1, $f$ is a subharmonic or superharmonic function on $B$. Since $B$ is compact, $f$ must be constant. □

**Corollary 3.2.3** *Let* $M = B^{4-n}\times_f F^n$ $(n = 1, 2, 3)$ *be a warped space-time, with* $B$ *a compact manifold with a positive definite metric. If* $M$ *is Einstein with Einstein constant* $c^M$ *then the warping function* $f$ *is constant in each of the following case.*

**(i)** $\mathrm{Scal}^B \leq (4-n)c^M$ *or* $\mathrm{Scal}^B \geq (4-n)c^M$.

**(ii)** *If* $c^M \geq 0$ *and the total scalar curvature* $S^B = \int \mathrm{Scal}^B \upsilon^B$ *of* $B$ *is non-positive.*

**Proof.** If $c^M$ is the Einstein constant of $M$ then from Proposition 3.2.1

$$-\frac{n}{f}\Delta^B f = (4-n)c^M - \mathrm{Scal}^B. \tag{3.2.1}$$

**(i)** If $\mathrm{Scal}^B \leq (4-n)c^M$ or $\mathrm{Scal}^B \geq (4-n)c^M$ then $f$ is constant due to Equation 3.2.1 and compactness of $B$.

**(ii)** Integrating Equation 3.2.1 and using Green's identity gives

$$S^B = \int \mathrm{Scal}^B \upsilon^B = n\int \frac{1}{f^2}\|\mathrm{grad} f\|^2 \upsilon^B + (4-n)c^M Vol(B)$$

If $S^B \leq 0$ the $f$ must be constant.

□

The above result obviously eliminates the possibility of constructing non-trivial Einstein warped space-times $M = B^{4-n} \times_f F^n$ with $B$ a compact Riemannian manifold with Einstein metric (or with a metric of constant scalar curvature).

Next we present a non-existence result for non-trivial Einstein warped space-times with 1-dimensional base.

**Corollary 3.2.4** *Let $M = B \times_f F$ be an Einstein warped space-time, with $B$ a compact manifold with a positive definite metric. If $B$ is 1-dimensional then $f$ must be constant.*

**Proof.** Follows from Theorem 3.2.2. □

For Einstein warped space-times with 2 or 3 dimensional base, we have
**Corollary 3.2.5** *Let $M = B^{4-n} \times_f F^n$ $(n = 1, 2)$ be an Einstein warped space-time, with $B$ a compact manifold with a positive definite metric. Then there exists a metric on the base $B$ such that the warping function $f$ cannot be non-constant.*

**Proof.** Every 2-dimensional Riemannian manifold admits a metric of constant curvature. Every compact Riemannian manifold of dimension at least 3 carries a metric of constant negative scalar curvature [4, Corollary 5.4]. Using the above facts in Corollary 3.2.3 completes the proof. □

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