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REGULARIZATION OF AN INITIAL INVERSE PROBLEM BY USING SMOOTHING SPLINES AND HYPERBOLIC HEAT EQUATION WITH BESSEL OPERATOR

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Abstract

A smoothing splines method and a hyperbolic heat conduction model is applied to regularize the recovery of the initial profile from a parabolic heat conduction model with Bessel operator. The recovery of initial profile with parabolic model is extremely ill-posed. Instead of the parabolic heat conduction model, we considered a hyperbolic heat conduction model and then a smoothing splines method is applied to filter out the noise from a corrupted signal. It is demonstrated in this paper that the recovery of the initial profile by a parabolic model is impossible in case of corrupted data, however, it is possible to recover the initial profile in a stable way by reformulating the problem by a hyperbolic model together with a smoothing splines method. One example is used for comparison of parabolic model and the proposed model for different modes and time displacements. The recovered initial profile by parabolic model and the exact initial profile are compared to see the unstable behavior of the model. The regularized results and the exact results are plotted on the same graph to see the accuracy and validity of the proposed method.

1. Introduction

The classical direct problem in heat conduction is to determine the temperature distribution of a body as the time progresses. The task of determining the initial temperature distribution from the final distribution is distinct from the direct problem and is identified as the initial inverse heat conduction problem. This type of inverse problem is extremely ill-posed, see e.g. Engle [1]. There is an alternative approach which consists of a reformulation of the classical heat equation by a hyperbolic heat equation, see Weber [2], Elden [3], Yang [4], and Masood et al. [5, 6]. This alternative approach is more attractive in practical engineering problems such as the non-homomogeneous solids conduction process, the slow conduction process, and the short-pulse laser applications; see Vedavarz et al. [7] and Gratzke et al. [8] among others. The initial inverse problem in the hyperbolic heat equation is stable and well posed. Moreover, numerical methods for hyperbolic problems are efficient and accurate. We will utilize the small value of the parameter and apply the WKBJ (Wentzel, Kramers, Brillouin and Jeffreys) method to solve the initial inverse problem, see Bender and Orszag [9].

In this paper a smoothing spline method, see de Boor [10], combined with the hyperbolic heat conduction model is proposed to solve the ill-posed inverse problem of determining the initial temperature distributions from the observations of the final temperature distributions. The problem of recovering the initial profile by using a hyperbolic model was investigated by Masood et al. [5, 6], but the results were not very accurate and may not be suitable for a particular application. Therefore, it is desirable to find a suitable method to solve the inverse problem in a stable and accurate way. In this proposed work there is a visible improvement, as opposed to the previous work, in the recovered initial profile. In the second section, initial inverse problem in the classical parabolic heat equation with Bessel operator is presented.

The inverse problem by an alternate approach of hyperbolic heat equation with Bessel operator is considered in the third section. An outline of the smoothing splines method is given in the fourth section. In the fifth section, some numerical experiments are performed to compare the recovered profile and the exact profile by the parabolic model and by the proposed method. Finally, in the sixth section the results are summarized.

2. Initial inverse problem in the parabolic heat equation with Bessel operator

Supposing we have a cylindrical rod, which for the sake of convenience we take to extend over the interval $0 \leq x \leq 1$, whose temperature distribution at the point x and at time t is given by the function $u\,(x,t)$. Then, for an appropriate choice of units, $u\,(x,t)$ satisfies the usual parabolic heat equation with Bessel operator

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x}, \qquad 0 < x < 1, \tag{1}$$

with zero temperature at one boundary

$$u(1,t) = 0, \tag{2}$$

and assume the final temperature distribution

$$f(x) = u(x,T)\,. \tag{3}$$

We want to recover the initial temperature profile

$$g(x) = u(x,0)\,. \tag{4}$$

The boundary condition (2) can be replaced by an insulated boundary, i.e. $u_x(1,t) = 0$, since it is important in some applications, see for example Beck et. al. [11] and

Al-Khalidy [12].

The direct problem can be solved by separation of variables by assuming solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} v_n(t)\,\phi_n(x)\,, \tag{5}$$

The corresponding eigenvalue problem is given by

$$\frac{d}{dx}\left[x\frac{d\phi(x)}{dx}\right] + \lambda x\phi(x) = 0, \qquad 0 < x < 1, \tag{6}$$

together with

$$\phi(1) = 0. \tag{7}$$

At the singular end point $x = 0$, from the application point of view, we impose a boundary condition of the form

$$\lim_{x\to 0} x\frac{d\phi(x)}{dx} = 0. \tag{8}$$

In application to heat conduction problems, x is the radial cylindrical coordinate and condition (8) states that total heat flux through a small circle surrounding the origin vanishes, that is, there is no heat source at the origin. The independent solutions of (6) for $\lambda \neq 0$ are $J_0\left(\sqrt{\lambda}x\right)$ and $N_0\left(\sqrt{\lambda}x\right)$. But from these two independent solutions only $J_0\left(\sqrt{\lambda}x\right)$ satisfies (8). It is usual to replace (8) by the condition that the solution be finite at the origin, which has the same effect as far as selecting $J_0\left(\sqrt{\lambda}x\right)$ as the only admissible solution. Now we apply condition (7), which gives rise to a sequence $\{\lambda_n\}$ of positive eigenvalues and the corresponding eigenfunctions are $J_0\left(\sqrt{\lambda_n}x\right)$. These eigenfunctions are orthogonal in the Hilbert space $H_x[0,1]$, where x is the weight function. These eigenfunctions can be normalized to give

$$\phi_n(x) = \frac{\sqrt{2}}{J_0'\left(\sqrt{\lambda_n}\right)} J_0\left(\sqrt{\lambda_n} x\right), \tag{9}$$

where $'$ denotes derivative with respect to x.

The eigenfunctions given by (9) are complete in $H_x[0,1]$, and therefore $g(x) \in H_x[0,1]$ can be expanded as

$$g(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \tag{10}$$

where

$$c_n = \int_0^1 \zeta \phi_n(\zeta)\, g(\zeta)\, d\zeta. \tag{11}$$

Now by using (5), $v_n(t)$ satisfies the following initial value problem

$$\frac{dv_n(t)}{dt} = -\lambda_n v_n(t), \tag{12}$$

$$v_n(0) = c_n, \tag{13}$$

where we have used the following relation

$$\int_0^1 x \phi_n'(x)\, \phi_m'(x)\, dx = \begin{cases} 0 & m \neq n \\ \lambda_n & m = n \end{cases}. \tag{14}$$

So, we can write the solution of the direct problem (1) in the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp[-\lambda_n t]\, \phi_n(x). \tag{15}$$

This represents an analytical solution to the heat equation in cylindrical coordinates. Now by applying (3), we can write

$$f(x) = \int_0^1 K(x,\zeta)\, g(\zeta)\, d\zeta, \tag{16}$$

where

$$K(x,\zeta)=\sum_{n=1}^{\infty}\zeta\exp[-\lambda_n T]\,\phi_n(\zeta)\,\phi_n(x)\,. \tag{17}$$

Thus the inverse problem is reduced to solving integral equation of the first kind. The singular system of the integral operator in (16) is

$$\{\exp[-\lambda_n T]\,;\ \phi_n(x)\,,\ \phi_n(x)\} \tag{18}$$

Now by application of Picard's theorem (see Engl [1]) the inverse problem is solvable if and only if

$$\sum_{n=1}^{\infty}\exp[2\lambda_n T]\,|f_n|^2<\infty, \tag{19}$$

where

$$f_n=\int_0^1\zeta\phi_n(\zeta)\,f(\zeta)\,d\zeta, \tag{20}$$

are the classical Fourier coefficients of f. Now again by Picard's theorem, we can recover the initial profile by the following expression

$$g\,(x)=\sum_{n=1}^{\infty}\exp\,[\lambda_n T]\,f_n\phi_n\,(x)\,. \tag{21}$$

Picard's theorem demonstrates the ill-posed nature of the problem considered. If we perturb the data by setting $f^\delta=f+\delta\phi_n$ we obtain a perturbed solution $g^\delta=g+\delta\phi_n\exp[\lambda_n T]$. Hence the ratio $\left\|g^\delta-g\right\|/\left\|f^\delta-f\right\|=\exp[\lambda_n T]$ can be made arbitrarily large due to the fact that the singular values $\exp[-\lambda_n T]$ decay exponentially. The influence of errors in the data f is obviously controlled by the rate of this decay. This error can also be controlled further by choosing a small value of T, for example for $T=1$, a small error in the n-th Fourier coefficient is amplified by the factor $\exp[\lambda_n]$. So in regularizing, we will confine ourselves to lower modes by only retaining the first few terms in the series (20). This technique of truncating the series is known as truncated singular value decomposition (TSVD), see Hansen [13].

3. The Hyperbolic model

The method we apply is similar to the quasi-inverse method of Lions [14]. The Lions' method is based on replacing the problem $(1)-(4)$ by a problem for equation of higher order with a small parameter. There are several methods for solving ill-posed problems. The quasi-solution method to solve the equation of the first kind was introduced by Ivanov [15]. The essence of this method is to change the notion of solution of an ill-posed problem so that, for certain conditions, the problem of its determination will be well-posed. Tikhonov's regularization method is widely used for solving linear and nonlinear operator equations of the first kind, see Tikhonov and Arsenin [16]. Iterative methods are applied to solve different problems and particularly these methods can also be applied to solve operator equations of the first kind. Moultanovsky [17] applied such an iterative method to solve an initial inverse heat transfer problem. The projective methods for solving various ill-posed problems are based on the representation of the approximate solution as a finite linear combination of a certain functional system, see e.g. Vasin and Ageev [18].

The methods mentioned in the previous paragraph may be applied for solving the extensive class of inverse problems. These methods do not take into account the specific character of concrete inverse problems. The Lion's method and the method we present in this paper take into account peculiarities of the inverse problem. There is an alternative approach to the inverse heat conduction problem $[2-6]$, which consists of introducing non-Fourier effect in the classical heat conduction model. In other words, heat propagates at a finite speed in a hyperbolic model instead of infinite speed in a parabolic model. The mathematical representation for the non-Fourier law is a hyperbolic heat conduction equation which includes a wave propagation term, that is, a small damping parameter ϵ multiplied with the term $\dfrac{\partial^2 u}{\partial t^2}$ is added in the parabolic heat equation. So, let us consider the following hyperbolic heat equation with Bessel operator

$$\epsilon\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{1}{x}\frac{\partial u}{\partial x}, \qquad 0<x<1, \tag{22}$$

together with $(2)-(4)$ and (8) and $\epsilon \to 0^+$. We impose one more condition as follows:

$$\frac{\partial u}{\partial t}(x,0)=0. \tag{23}$$

As before, by separation of variables, we assume solution of the form (5). In this case $v(t)$ has to solve the following initial value problem

$$\epsilon\frac{d^2 v_n(t)}{dt^2}+\frac{dv_n(t)}{dt}=-\lambda_n v_n(t)\,, \tag{24}$$

$$v_n(0)=c_n, \tag{25}$$

$$\frac{dv_n(0)}{dt}=0. \tag{26}$$

Since $\epsilon \to 0^+$, this is a singular perturbation problem. We apply the WKBJ method to obtain an asymptotic representation for the solution of (24) containing parameter ϵ; the representation is to be valid for small values of the parameter. It is demonstrated in [8] that the solution stays closer to the exact solution for large values such as $\epsilon = 0.5$. The solution of (24) is given by

$$v_n(t)=\left(\frac{\epsilon\lambda_n-1}{2\epsilon\lambda_n-1}\right)c_n\exp[-\lambda_n t]+\left(\frac{\epsilon\lambda_n c_n}{2\epsilon\lambda_n-1}\right)\exp\left[\lambda_n t-\frac{t}{\epsilon}\right]. \tag{27}$$

As before, we can use this solution in (5) to arrive at an integral equation of the form (16). The singular system in this case is

$$\left\{ \left(\frac{\epsilon\lambda_n - 1}{2\epsilon\lambda_n - 1} \right) \exp[-\lambda_n T] + \left(\frac{\epsilon\lambda_n}{2\epsilon\lambda_n - 1} \right) \exp\left[\lambda_n T - \frac{T}{\epsilon} \right] ; \; \phi_n(x) \, , \; \phi_n(x) \right\} \tag{28}$$

Now by Picard's theorem (see Engl[1]) the solution exists if and only if

$$\sum_{n=1}^{\infty} \frac{|f_n|^2}{\left[\left(\frac{\epsilon\lambda_n - 1}{2\epsilon\lambda_n - 1} \right) \exp[-\lambda_n T] + \left(\frac{\epsilon\lambda_n}{2\epsilon\lambda_n - 1} \right) \exp\left[\lambda_n T - \frac{T}{\epsilon} \right] \right]^2} < \infty, \tag{29}$$

where the Fourier coefficients f_n are given by (20). The initial profile can be recovered by the following expression

$$g(x) = \sum_{n=1}^{\infty} \frac{f_n \phi_n(x)}{\left[\left(\frac{\epsilon\lambda_n - 1}{2\epsilon\lambda_n - 1} \right) \exp[-\lambda_n T] + \left(\frac{\epsilon\lambda_n}{2\epsilon\lambda_n - 1} \right) \exp\left[\lambda_n T - \frac{T}{\epsilon} \right] \right]}. \tag{30}$$

The solution given by (19) and (21) can be recovered by letting $\epsilon \to 0^+$ in equation (29) and (30) respectively.

Example: Let us consider an initial temperature distribution of the form

$$g(x) = \frac{\sqrt{2}}{J_0'\left(\sqrt{\lambda_m}\right)} J_0\left(\sqrt{\lambda_m} x\right). \tag{31}$$

First we solve the direct problems (1) and (22) together with conditions (2) – (4), (8) and (23), to find the final profiles . The Fourier coefficients corresponding to the final profiles of the parabolic and hyperbolic models respectively are

$$f_m = \exp[-\lambda_m T] \, , \tag{32}$$

$$f_m = \left(\frac{\epsilon\lambda_m - 1}{2\epsilon\lambda_m - 1} \right) \exp[-\lambda_m T] + \left(\frac{\epsilon\lambda_m}{2\epsilon\lambda_m - 1} \right) \exp\left[\lambda_m T - \frac{T}{\epsilon} \right]. \tag{33}$$

Now we use the Fourier coefficients given by (32) and (33) in (21) and (30) respectively to recover the initial profile. It is clear that in both cases the recovered initial profile is (31).

4. Noise removal using smoothing splines

In the process of finding the inverse solution of recovering the initial temperature distribution by using final data which is contaminated with noise, we employ a de-noising method based on smoothing the noisy solution by using smoothing splines.

Let $(x_i; g(x_i))_{i=1}^n$ denote the given noisy data. An efficient approach for smoothing $(x_i; g(x_i))_{i=1}^n$ is to find the function $u(x)$ that minimizes

$$\sum_{i=1}^{n} \left[g(x_i) - u\left(x_i\right)\right]^2 + \lambda \int_{x_1}^{x_n} \left[u''\left(x\right)\right]^2 dx. \tag{34}$$

This is a well posed least square problem and its solution is known as cubic smoothing spline. For the general treatment of the notion of smoothing splines, the reader is referred to [10, 19].

The first term in (34) forces the approximation $u(x)$ to get closer to given noisy data whereas the term $\int_{x_1}^{x_n} \left[u''\left(x\right)\right]^2 dx$ contributes to the smoothness of fitting. The amount of smoothness is controlled by the parameter $\lambda \geq 0$; the approximation corresponds to linear least square approximation for $\lambda \to \infty$ and it corresponds to the exact interpolation of data for $\lambda \to 0$.

The computer implementation of smoothing by cubic smoothing splines, for different cases in Section 5, is carried out using the routine spaps of MATLAB which is based on the algorithm of Reinsch [19]. The smoothing parameter λ in each case is interactively tuned by visual inspection of the results.

5. Numerical experiments

In this section we present numerical examples intended to demonstrate the usefulness of reformulating parabolic model into a hyperbolic model and then applying the smoothing splines method to remove the noise. In all the figures, the thick solid line represents the exact initial profile, the thin solid line represents the recovered initial profile by hyperbolic model together with the smoothing splines method and the dashed line represents the recovered initial profile by the parabolic model. The parabolic model is highly unstable to the contaminated signal, contamination is more if the signal to noise ratio (SNR) is low, so we choose $SNR = 30db$ in all the experiments. The parameters ϵ and λ are selected by repeated experiments in such a way that the recovered initial profile is free from noise and appears smooth.

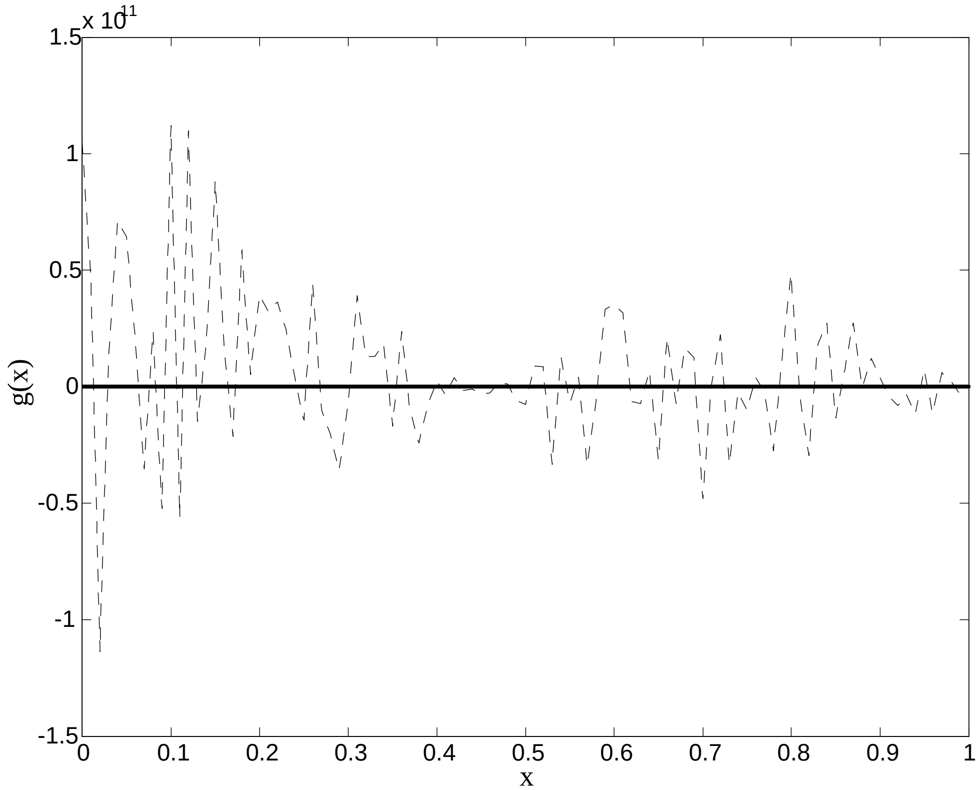

Fig. 1 The recovered initial profile when the noisy signal is used in the parabolic model in the case $m = 2$ and $T = 1$.

We consider the initial profile given by (31) for different values of the mode m, and time parameter T. The initial profile given by (31) leads to the final data given by (32) which is free from noise and this final data obviously leads to the exact initial profile by the parabolic model as well as by the hyperbolic model as $\epsilon \to 0^+$. In any real world experiment the recorded final profile cannot be free from noise, so we add white Gaussian noise in the final data given by (32). We use this corrupted data in the parabolic model (21) and compare it with the exact initial profile. The parabolic model as discussed in Section 3 will give unstable results.

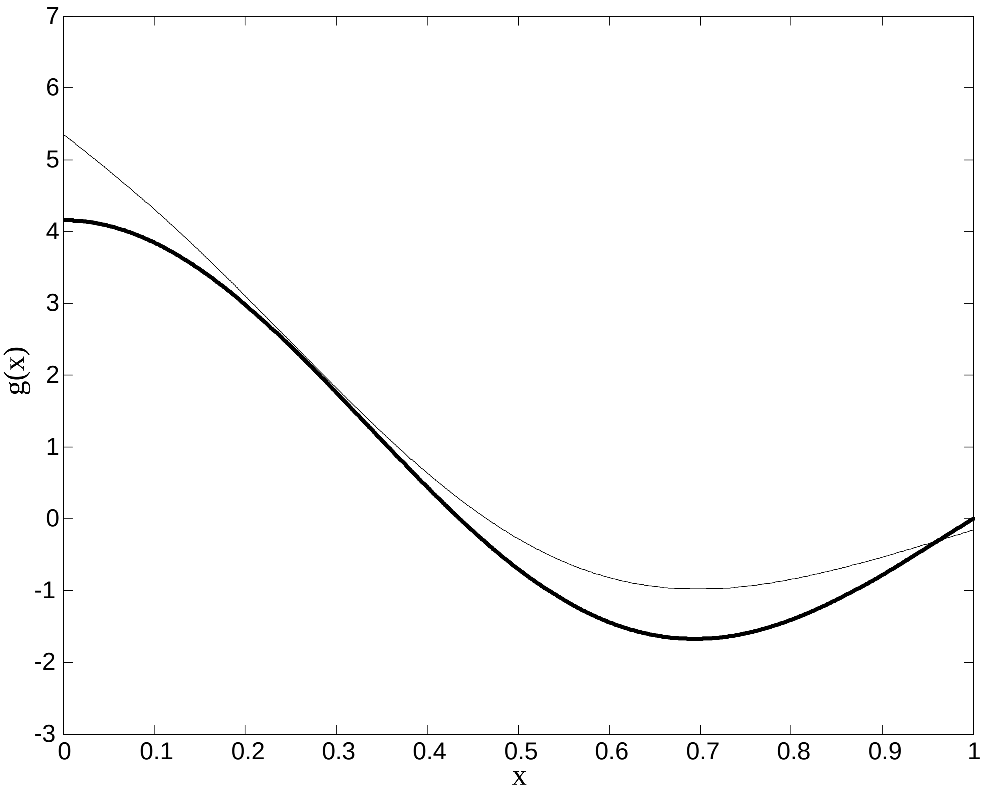

Fig. 2 The recovered initial profile when TV mehod is applied to the noisy signal and is used in the hyperbolic model for the case $m = 2$ and $T = 1$.

Next we apply the smoothing spline method to the noisy signal and obtain a smoothed signal by choosing an appropriate value of the parameter λ. Then we use the smoothed signal so obtained in the hyperbolic model (30) to recover the initial profile for an appropriate value of the parameter ϵ. We also consider further smoothness by applying smoothing spline method again to the initial profile obtained by the hyperbolic model.

The inherent instability of the parabolic model is clear from Fig. 1. The exact initial profile appears as a straight line and the recovered initial profile oscillates with amplitude being a multiple of 10^{11}. Therefore the parabolic model does not give any information about the initial profile even for the lower modes. In contrast, from Fig. 2, the hyperbolic model together with the smoothing spline method recovers the initial profile in good agreement with the exact initial profile.

In Fig. 3 and Fig. 5, the inherent instability of the parabolic model is even more evident as the vertical axes is a multiple of 10^{30} and 10^{58} respectively. It is also clear from Figures 3 and 5 that as the size of mode m increases by only one the recovered initial profile deteriorates exponentially. On the other hand, from Fig. 4 and Fig. 6, the hyperbolic model together with the smoothing splines method is stable. Furthermore, the recovered initial profile shows the same behavior as the exact initial profile and it does not deteriorate much as the size of m increases.

In the preceding paragraph, we discussed the recovery of initial profile for different modes. Now we discuss the recovery of the initial profile as the size of the time parameter T increases for different modes. Comparing Fig. 3 with Fig. 7 ($m = 3$), the vertical axes is a multiple of 10^{30} and 10^{96} respectively. So the change in the recovered profile by parabolic model is huge, that is, the power of 10 is multiplied by 3 for changing the value of time parameter T from 1 to 3. But on the other hand if we compare Fig. 4 and Fig. 8, there is almost no deterioration in the recovered initial profile as the time parameter is increased by two. We have same observations of highly unstable behavior of the parabolic model from the comparison of Fig. 5 and Fig. 9, and stable behavior of the hyperbolic model together with the smoothing splines method from the comparison of Fig. 6 and Fig. 10.

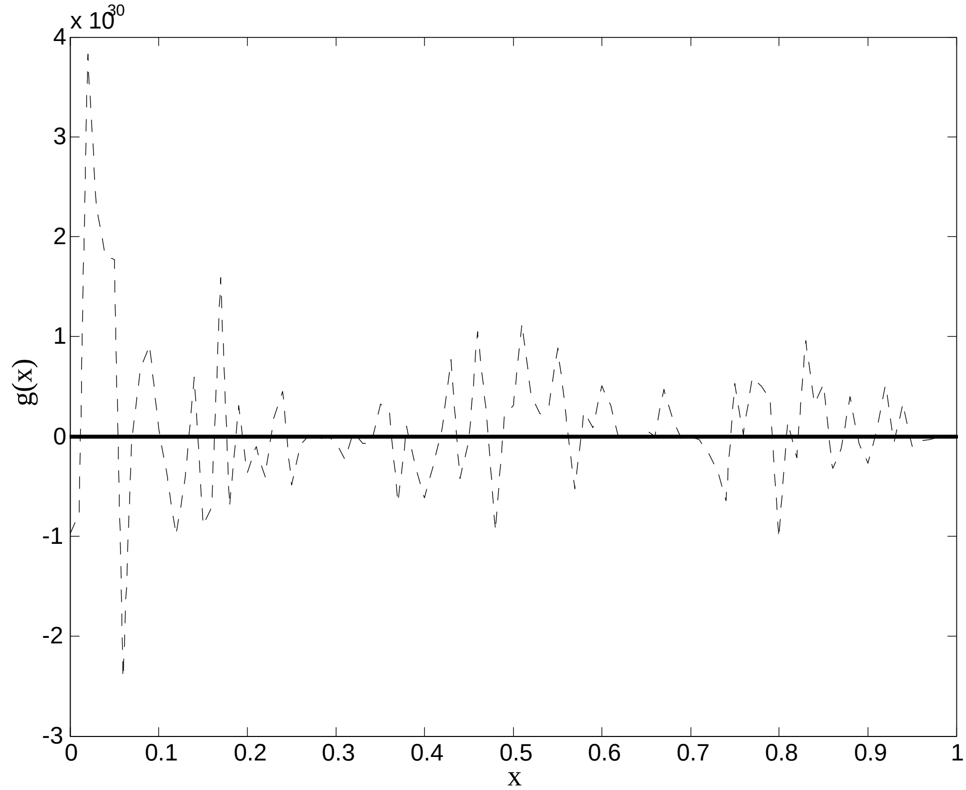

Fig. 3 The recovered initial profile when the noisy signal is used in the parabolic model in the case $m = 3$ and $T = 1$.

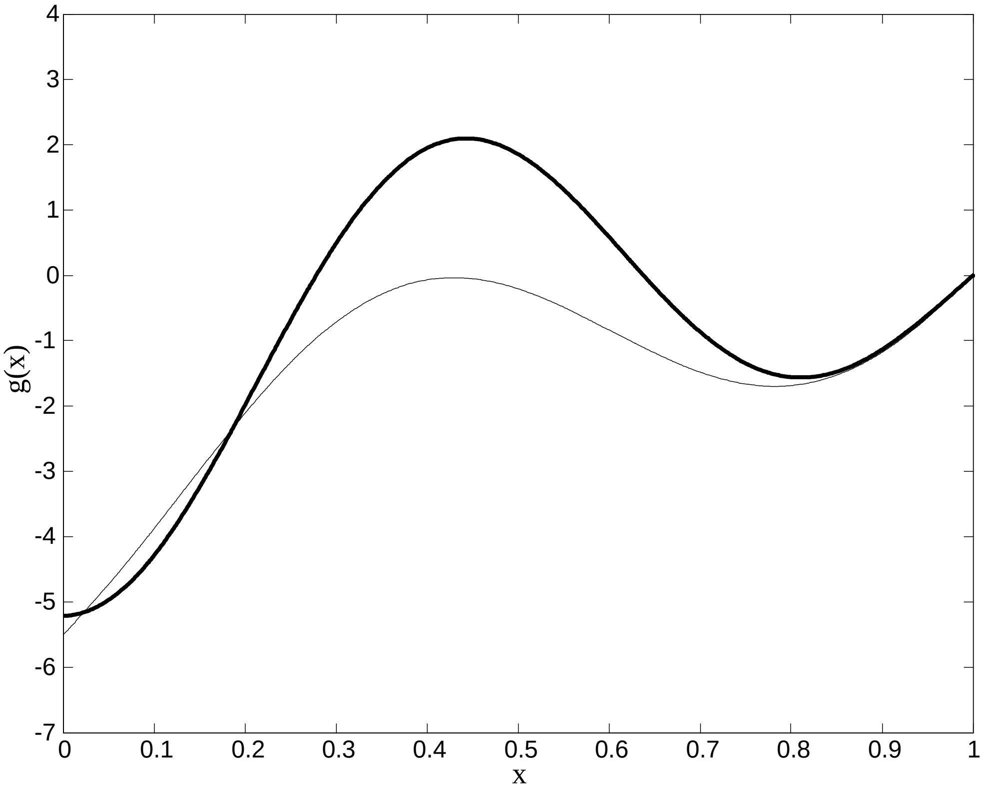

Fig. 4 The recovered initial profile when TV mehod is applied to the noisy signal and is used in the hyperbolic model for the case $m = 3$ and $T = 1$.

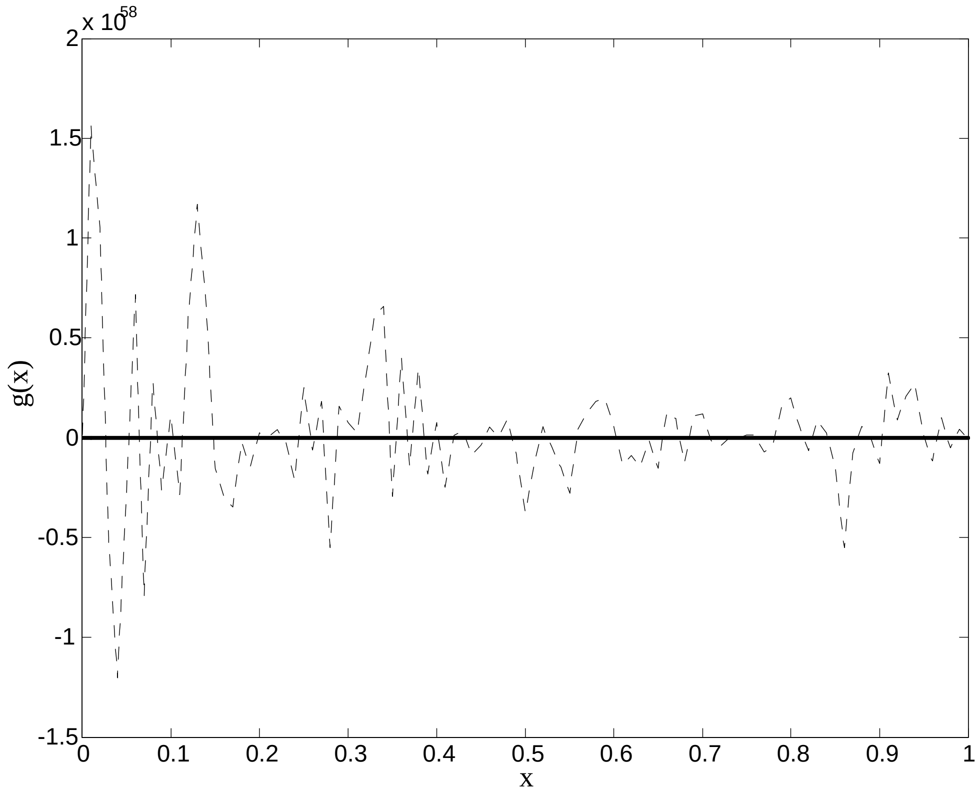

Fig. 5 The recovered initial profile when the noisy signal is used in the parabolic model in the case $m = 4$ and $T = 1$.

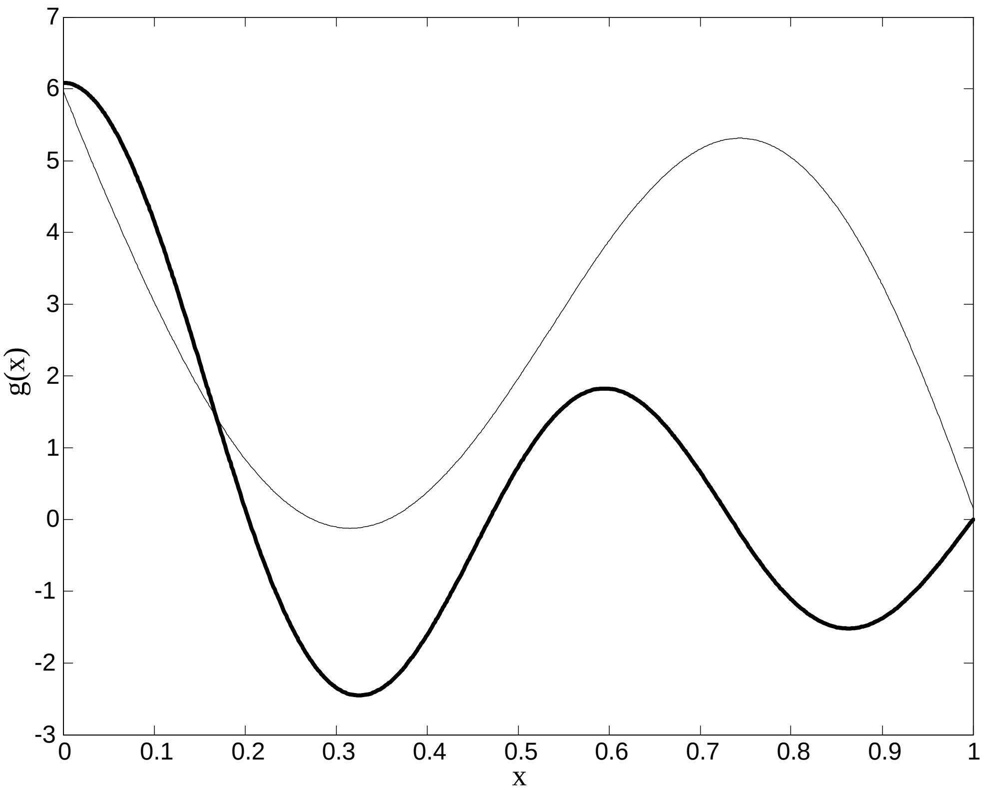

Fig. 6 The recovered initial profile when TV mehod is applied to the noisy signal and is used in the hyperbolic model for the case $m = 4$ and $T = 1$.

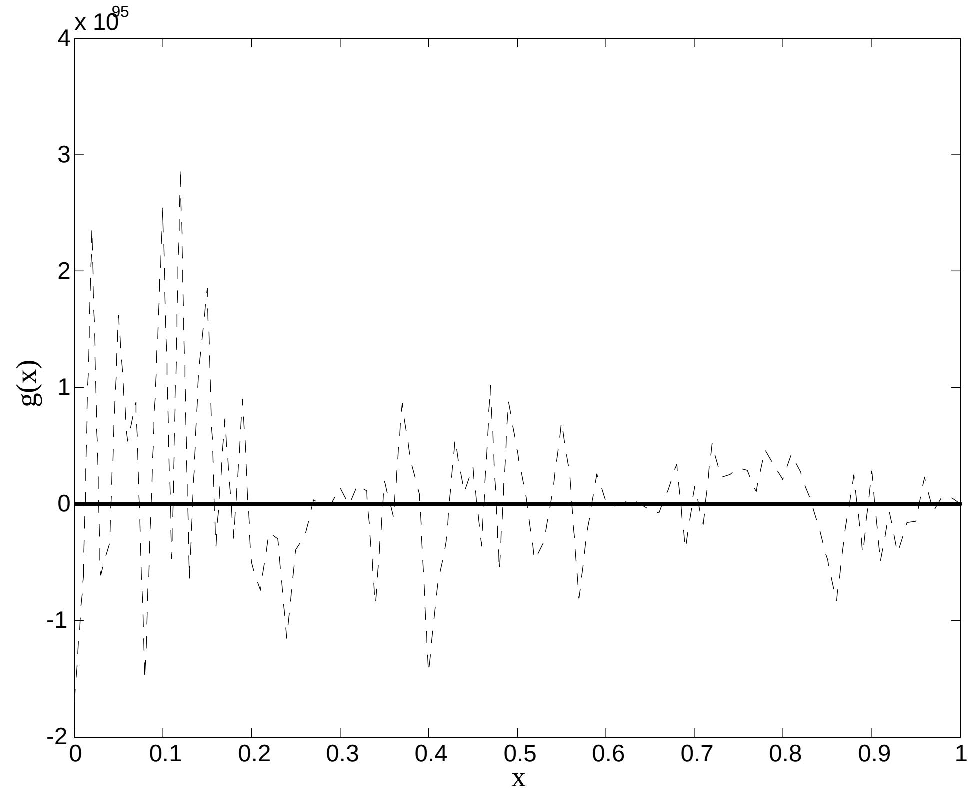

Fig. 7 The recovered initial profile when the noisy signal is used in the parabolic model in the case $m = 3$ and $T = 3$.

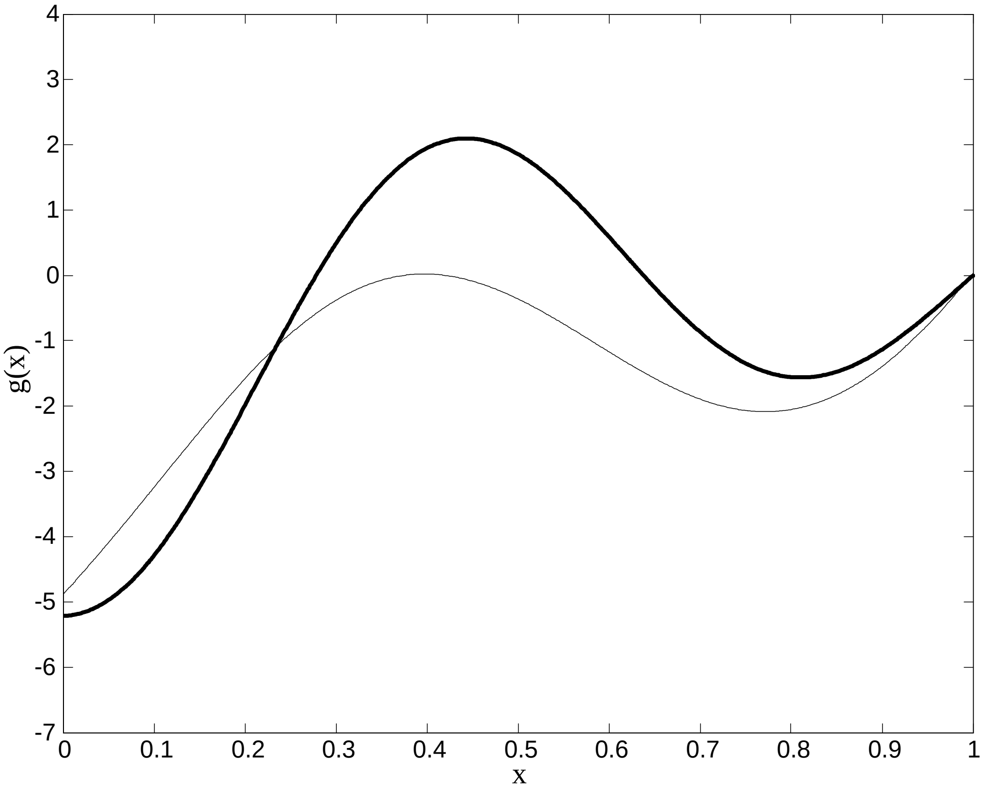

Fig. 8 The recovered initial profile when TV mehod is applied to the noisy signal and is used in the hyperbolic model for the case $m = 3$ and $T = 3$.

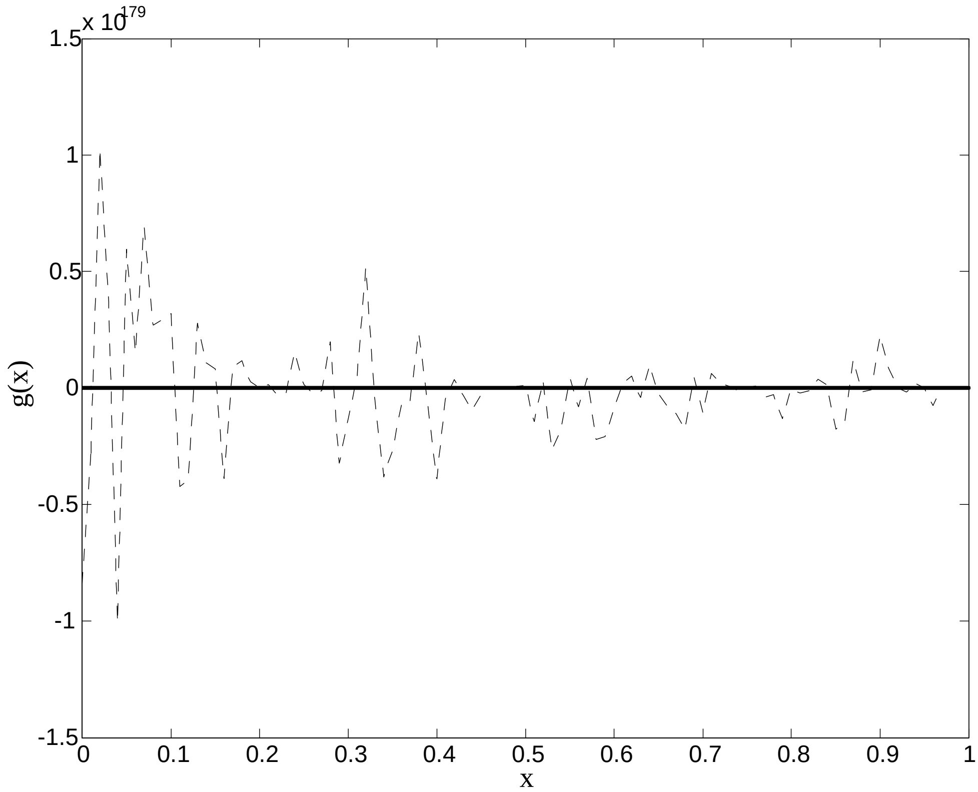

Fig. 9 The recovered initial profile when the noisy signal is used in the parabolic model in the case $m = 4$ and $T = 3$.

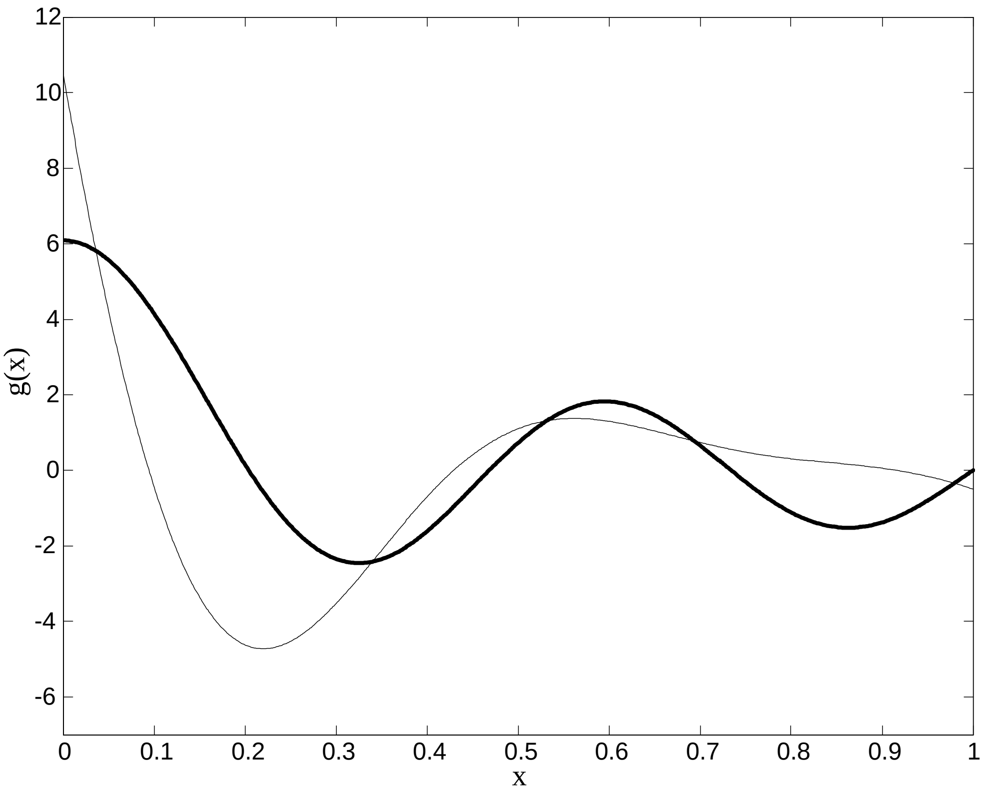

Fig. 10 The recovered initial profile when TV mehod is applied to the noisy signal and is used in the hyperbolic model for the case $m = 4$ and $T = 3$.

Conclusions

In this paper, we have proposed a robust method to solve an extremely ill-posed inverse problem to recover the initial profile from the corrupted final data. It is shown that the recovered initial profile from the noisy final data is reasonably close to the exact initial profile by our proposed method. The close agreement between the exact initial profile and the estimated initial profile is used to confirm the validity and accuracy of the proposed method. The inverse solution of the parabolic heat conduction model is characterized by discontinuous dependence on the data. It is shown that in case of noisy data, the parabolic model does not give any information about the nature of the initial profile, on the other hand the hyperbolic model together with the smoothing splines method recovers the initial profile in close agreement with the exact initial profile. Furthermore, in the case of noisy data, the recovered initial profile by parabolic model is more unstable for higher modes and time displacements but by the hyperbolic model together with the smoothing splines method the recovered initial profile shows the same behavior as the exact initial profile and it does not deteriorate much for higher modes and time displacements.

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