An Existence Result for a Nonlocal Fractional Differential Problem

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Abstract

We are concerned with the following family of fractional differential problems of weighted Cauchy-type

$$\begin{cases} D^{\alpha}u(t) = f(t,u) + \int_0^t g(t,s,u(s))ds, \ t>0 \\ t^{1-\alpha}u(t)\,|_{t=0} = b, \end{cases}$$

where $0<\alpha<1$ and $b\in\mathbf{R}$. Using the Schauder fixed point Theorem, we prove a local existence result for some classes of nonlinear functions $f(t,u)$ and $g(t,s,u(s))$ involving power functions in t, s and u. In particular, we allow for fractional integrals of order less than one in our nonlinearities.

Key Words and Phrases: Fractional differential equation, Riemann-Liouville integral, singular kernel, weighted Cauchy-type problem

AMS subject Classification: 26A33, 34A12, 45D05

1 Introduction

In this work we are interested in the weighted Cauchy-type problem

$$\begin{cases} D^{\alpha}u(t) = f(t,u) + \int_0^t g(t,s,u(s))ds, \ t>0 \\ t^{1-\alpha}u(t)\,|_{t=0} = b, \end{cases} \tag{1}$$

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where $0 < \alpha < 1$, $b \in \mathbf{R}$, f and g are continuous functions. D^α denotes the fractional derivative of order α. Our nonlinearities $f(t,u)$ and $g(t,s,u(s))$ will involve power functions in t, s and u.

This problem with $g \equiv 0$ has been considered by Delbosco and Rodino [3]. The authors proved several local existence and uniqueness results in a weighted space of continuous functions in the case where $f(t,u)$ is Lipschitzian in its second argument u. They also studied the not-globally-Lipschitzian case $f(t,u) = f(u) = u^s$ with $0 < s < \frac{1}{1-\alpha}$. The same problem (with $g \equiv 0$) has also been investigated by Kilbas, Bonilla and Trujillo [5]. Local existence and uniqueness theorems were proved in the space of summable functions. Again they assumed $f(t,u)$ to be Lipschitz continuous in u. Their proof was based on the successive approximation method.

In this context, it seems that the most studied problem is when the initial condition is given by $(I^{1-\alpha}u)(0) = b$ (or $(D^{\alpha-1}u)(0) = b$) see [1,2,4,5-8,14,15], see also the references in the survey paper by Kilbas and Trujillo [8]. We mention here that the equation in (1) (i.e. with $g \not\equiv 0$) subject to the latter initial condition has been studied recently by Momani [10].

Here in this work the nonlinearities we consider are bounded by polynomials in t, s and u. In addition, we allow g to contain unbounded, singular and non-integrable kernels such as $(t-s)^{-\beta}$, $0 < \beta < 1$. In particular, fractional integrals of order between 0 and 1 may be present in the form of g.

The local existence is proved on a weighted space of continuous functions where the equivalence between the differential problem (1) and its associated integral equation (see (2) below) has been shown.

The plan of our paper is as follows: in Section 2, we prepare some material needed to prove our main result. Section 3 contains the statement and proof of the local existence result.

2 Preliminaries

In this section we present some definitions and lemmas which will be used in the sequel.

Definition 1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a Lebesgue-measurable function $f : \mathbf{R}^+ \to \mathbf{R}$ is defined by (the Abel-integral*

operator)

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int\limits_{0}^{t}(t-s)^{\alpha-1}f(s)ds$$

provided that the integral exist.

Definition 2. *The fractional derivative (in the sense of Riemann-Liouville) of order* $0<\alpha<1$ *of a continuous function* $f:\mathbf{R}^{+}\rightarrow\mathbf{R}$ *is defined as the left inverse of the fractional integral of* f

$$D^{\alpha}f(t)=\frac{d}{dt}(I^{1-\alpha}f)(t).$$

That is

$$D^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_{0}^{t}(t-s)^{-\alpha}f(s)ds,$$

provided that the right side exists.

For the theory and applications of fractional integrals and fractional derivatives we refer the reader to [9,11-13].

For $h>0$,we define the space

$$C_{r}^{0}([0,h]):=\left\{v\in C^{0}((0,h]):\ \lim_{t\rightarrow0^{+}}t^{r}v(t)\ exists\ and\ is\ finite\right\}.$$

Here $C^{0}((0,h])$ is the usual space of continuous functions on $(0,h]$. The space $C_{r}^{0}([0,h])$ endowed with the norm

$$\left\|v\right\|_{r}:=\max_{0\leq t\leq h}t^{r}\left|v(t)\right|$$

is a Banach space.

If we look for solutions in $C_{1-\alpha}^{0}([0,h])$ and the right hand side of the equation in (1) is in $C^{0}((0,h))\cap L^{1}((0,h))$ then problem (1) is equivalent to the integral equation

$$u(t)=bt^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int\limits_{0}^{t}(t-s)^{\alpha-1}\left(f(s,u(s))+\int\limits_{0}^{s}g(s,z,u(z))dz\right)ds. \quad (2)$$

See Delbosco and Rodino [3].

By a solution of problem (1) we mean a real valued function $u \in C^0((0,h)) \cap L^1((0,h))$ with continuous fractional derivative $D^\alpha u$, $0 < \alpha < 1$ on $(0,h)$ satisfying the equation in (1) pointwise and the weighted initial condition. From this definition of a solution it is clear that even if we look for solutions in $C^0_{1-\alpha}([0,h])$, in fact they are a little bit smoother than continuous.

The following well known lemmas will also be needed.

Lemma 1. *For all $\alpha > 0$ and $\mu > -1$, we have*

$$\int_0^t (t-s)^{\alpha-1} s^\mu ds = \frac{\Gamma(\alpha)\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} t^{\alpha+\mu},\ t \geq 0.$$

Lemma 2. *For $0 < \sigma \leq 1$ and $0 < a \leq b$, we have*

$$|a^\sigma - b^\sigma| \leq (b-a)^\sigma.$$

3 Local existence

In this section we prove local existence of solutions for (1) with the nonlinearities f and g such that

(F) $t^{1-\alpha} f(t,u)$ is continuous on $\mathbf{R}^+ \times C^0_{1-\alpha}(\mathbf{R}^+)$ and

$$|f(t,u)| \leq t^\mu \varphi(t) \left|u\right|^{m_1},\ \mu \geq 0,\ m_1 > 1 \tag{3}$$

(G) $s^{1-\alpha} g(t,s,u(s))$ is continuous on $D_{\mathbf{R}^+} \times C^0_{1-\alpha}(\mathbf{R}^+)$ where

$$D_{\mathbf{R}^+} = \left\{(t,s) \in \mathbf{R}^+ \times \mathbf{R}^+, 0 \leq s \leq t\right\}$$

and

$$|g(t,s,u(s))| \leq (t-s)^{\beta-1} s^\sigma \psi(s) \left|u\right|^{m_2},\ 0 < \beta < 1,\ \sigma \geq 0,\ m_2 > 1 \tag{4}$$

where $\varphi(t)$ and $\psi(s)$ are continuous functions on $\mathbf{R}^+$.

(Φ) $\varphi(t)$ is continuous and $t^{\mu-(1-\alpha)m_1}\varphi(t)$ is continuous on $\mathbf{R}^+$ in case

$$\mu - (1-\alpha)m_1 < 0.$$

(Ψ) $\psi(t)$ is continuous and $t^{\sigma-(1-\alpha)m_2}\psi(t)$ is continuous on $\mathbf{R}^+$ in case

$$\sigma - (1-\alpha)m_2 < 0.$$

Theorem 1. *Assume that the above hypotheses* ***(F)***, ***(G)***, ***(Φ)*** *and* ***(Ψ)*** *on f, g, φ and ψ hold. Suppose further that $1+\mu-(1-\alpha)m_1>0$ and $1+\sigma-(1-\alpha)m_2>0$. Then problem (1) admits at least one solution on a sufficiently small interval $[0,\tilde{h}]$ with $\tilde{h}<h$.*

Proof: Let us consider the operator

$$(Tu)(t)=bt^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(f(s,u(s))+\int_0^s g(s,z,u(z))dz\right)ds. \quad (5)$$

In view of (2), it suffices to prove the existence of a fixed point for the operator T. This will be established by the Schauder fixed-point theorem. Let us define the ball

$$S_r=\left\{u\in C^0_{1-\alpha}([0,h]):\left\|u-bt^{\alpha-1}\right\|_{1-\alpha}\leq r\right\}.$$

Then,

$$\begin{aligned}&\left\|Tu-bt^{\alpha-1}\right\|_{1-\alpha}\\&=\max_{t\in[0,h]}\frac{t^{1-\alpha}}{\Gamma(\alpha)}\left|\int_0^t(t-s)^{\alpha-1}\left(f(s,u(s))+\int_0^s g(s,z,u(z))dz\right)ds\right|.\end{aligned}$$

By our assumptions on f and g, we have

$$\begin{aligned}\left\|Tu-bt^{\alpha-1}\right\|_{1-\alpha}&\leq\max_{t\in[0,h]}\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}s^{\mu}\varphi(s)\left|u(s)\right|^{m_1}ds\\&+\max_{t\in[0,h]}\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\int_0^s(s-z)^{\beta-1}z^{\sigma}\psi(z)\left|u(z)\right|^{m_2}dzds.\end{aligned}$$

By the definition of $\|.\|_{1-\alpha}$, we may write

$$\begin{aligned}\left\|Tu-bt^{\alpha-1}\right\|_{1-\alpha}&\leq\|\varphi\|_0\max_{t\in[0,h]}\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}s^{\mu-m_1(1-\alpha)}\left\|u\right\|^{m_1}_{1-\alpha}ds\\&+\|\psi\|_0\max_{t\in[0,h]}\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(\int_0^s(s-z)^{\beta-1}z^{\sigma-m_2(1-\alpha)}\left\|u\right\|^{m_2}_{1-\alpha}dz\right)ds,\end{aligned}$$

where $\|.\|_0$ is the sup norm in $C([0,h])$. If $u\in S_r$ then it follows that

$$\begin{aligned}\left\|Tu-bt^{\alpha-1}\right\|_{1-\alpha}&\leq\|\varphi\|_0\frac{(r+|b|)^{m_1}}{\Gamma(\alpha)}\max_{t\in[0,h]}t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}s^{\mu-m_1(1-\alpha)}ds\\&+\|\psi\|_0\frac{(r+|b|)^{m_2}}{\Gamma(\alpha)}\max_{t\in[0,h]}t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\left(\int_0^s(s-z)^{\beta-1}z^{\sigma-m_2(1-\alpha)}dz\right)ds.\end{aligned}$$

Using Lemma 1, we find

$$\left\|Tu-bt^{\alpha-1}\right\|_{1-\alpha}\leq\frac{\Gamma(\mu-m_1(1-\alpha)+1)\|\varphi\|_0(r+|b|)^{m_1}}{\Gamma(\mu-m_1(1-\alpha)+\alpha+1)}h^{\mu-m_1(1-\alpha)+1}$$
$$+\frac{\Gamma(\beta)\Gamma(\sigma-m_2(1-\alpha)+1)\|\psi\|_0(r+|b|)^{m_2}}{\Gamma(\alpha)\Gamma(\sigma-m_2(1-\alpha)+1+\beta)}\max_{t\in[0,h]}t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}s^{\sigma+\beta-m_2(1-\alpha)}ds$$

or

$$\left\|Tu-bt^{\alpha-1}\right\|_{1-\alpha}\leq\frac{\Gamma(\mu-m_1(1-\alpha)+1)\|\varphi\|_0(r+|b|)^{m_1}}{\Gamma(\mu-m_1(1-\alpha)+\alpha+1)}h^{\mu-m_1(1-\alpha)+1}$$
$$+\frac{\Gamma(\sigma-m_2(1-\alpha)+1)\Gamma(\beta)\|\psi\|_0(r+|b|)^{m_2}}{\Gamma(\sigma+\beta+\alpha-m_2(1-\alpha)+1)}h^{\sigma+\beta-m_2(1-\alpha)+1}.$$

Therefore,

$$\left\|Tu-bt^{\alpha-1}\right\|_{1-\alpha}\leq C_1(r+|b|)^{m_1}h^{\gamma}+C_2(r+|b|)^{m_2}h^{\delta},\tag{6}$$

where $C_1=\frac{\|\varphi\|_0\Gamma(\mu-m_1(1-\alpha)+1)}{\Gamma(\mu-m_1(1-\alpha)+\alpha+1)}$, $C_2=\frac{\Gamma(\beta)\|\psi\|_0\Gamma(\sigma-m_2(1-\alpha)+1)}{\Gamma(\sigma+\beta+\alpha-m_2(1-\alpha)+1)}$, $\gamma=\mu-m_1(1-\alpha)+1>0$ and $\delta=\sigma+\beta-m_2(1-\alpha)+1>0$.

Clearly, we can make the expression in the right hand side of (6) less than r by an appropriate choice of r (for instance $r=|b|$) and h sufficiently small. Consequently $T(S_r)\subset S_r$.

Now we prove that T is continuous on S_r. Let $u_1,u_2\in S_r$, from (5) we get

$$\left\|Tu_1-Tu_2\right\|_{1-\alpha}\leq\frac{1}{\Gamma(\alpha)}\max_{t\in[0,h]}t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\left(\left|f(s,u_1(s)-f(s,u_2(s))\right|\right.$$
$$\left.+\int_0^s\left|g(s,z,u_1(z))-g(s,z,u_2(z))\right|dz\right)ds.$$

By the continuity of f and g with respect to $u\in C^0_{1-\alpha}([0,h])$ we can write that for a given $\varepsilon>0$ there exists a $\delta_1>0$ such that for all $(s,u_1),(s,u_2)\in[0,h]\times C^0_{1-\alpha}([0,h])$ we have

$$s^{1-\alpha}\left|f(s,u_1)-f(s,u_2)\right|<\varepsilon\ \textit{provided that}\ \left\|u_1-u_2\right\|_{1-\alpha}<\delta_1.$$

For the same $\varepsilon>0$, there exists a $\delta_2>0$ such that for all $(s,z,u_1),(s,z,u_2)\in D_{[0,h]}\times C^0_{1-\alpha}([0,h])$ we have

$$z^{1-\alpha}\left|g(s,z,u_1)-g(s,z,u_2)\right|<\varepsilon\ \textit{provided that}\ \left\|u_1-u_2\right\|_{1-\alpha}<\delta_2.$$

For that given $\varepsilon > 0$, taking $\delta = \min(\delta_1, \delta_2)$ we obtain

$$\begin{aligned} \|Tu_1 - Tu_2\|_{1-\alpha} &\leq \tfrac{\varepsilon}{\Gamma(\alpha)} \max_{t\in[0,h]} t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} ds \\ &\quad + \tfrac{\varepsilon}{\Gamma(\alpha)} \max_{t\in[0,h]} t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} \left(\int_0^s z^{\alpha-1} dz \right) ds \\ &\leq \varepsilon C \left(h^\alpha + \tfrac{h^{\alpha+1}}{\alpha} \right), \end{aligned} \tag{7}$$

where $C = \max\left\{ \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}, \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \right\}$. We infer from (7) that T is continuous on S_r. Next, we prove that $T(S_r)$ is equicontinuous. Suppose that $t_1, t_2 \in [0,h]$, $t_1 < t_2$ and $t_2 - t_1 < \delta$ for some δ to be determined. For a given $\varepsilon > 0$ and $u \in S_r$, we have

$$\begin{aligned} &\left| t_1^{1-\alpha} Tu(t_1) - t_2^{1-\alpha} Tu(t_2) \right| \\ &\leq \left| t_1^{1-\alpha} \int_0^{t_1} (t_1-\tau)^{\alpha-1} \left(f(\tau, u(\tau)) + \int_0^\tau g(\tau, z, u(z)) dz \right) d\tau \right. \\ &\left. - t_2^{1-\alpha} \int_0^{t_2} (t_2-\tau)^{\alpha-1} \left(f(\tau, u(\tau)) + \int_0^\tau g(\tau, z, u(z)) dz \right) d\tau \right|. \end{aligned} \tag{8}$$

Let us add and substract the term

$$t_2^{1-\alpha} \int_0^{t_2} (t_2-\tau)^{\alpha-1} \left(f(\tau, u(\tau) + \int_0^\tau g(\tau, z, u(z)) dz \right) d\tau$$

inside the absolute value of the right hand side of (8). We obtain

$$\begin{aligned} &\left| t_1^{1-\alpha} Tu(t_1) - t_2^{1-\alpha} Tu(t_2) \right| \\ &\leq \left| t_1^{1-\alpha} \int_0^{t_1} (t_1-\tau)^{\alpha-1} \left(f(\tau, u(\tau)) + \int_0^\tau g(\tau, z, u(z)) dz \right) d\tau \right. \\ &\left. - t_2^{1-\alpha} \int_0^{t_1} (t_1-\tau)^{\alpha-1} \left(f(\tau, u(\tau)) + \int_0^\tau g(\tau, z, u(z)) dz \right) d\tau \right| \\ &+ \left| t_2^{1-\alpha} \int_0^{t_1} (t_1-\tau)^{\alpha-1} \left(f(\tau, u(\tau)) + \int_0^\tau g(\tau, z, u(z)) dz \right) d\tau \right. \\ &\left. - t_2^{1-\alpha} \int_0^{t_2} (t_2-\tau)^{\alpha-1} \left(f(\tau, u(\tau)) + \int_0^\tau g(t, z, u(z)) dz \right) d\tau \right|. \end{aligned} \tag{9}$$

Next, we add and substract the expression

$$t_2^{1-\alpha}\int_0^{t_1}(t_2-\tau)^{\alpha-1}\left(f(\tau,u(\tau))+\int_0^{\tau}g(\tau,z,u(z))dz\right)d\tau$$

inside the second absolute value of the right hand side of (9)

$$\begin{aligned}
&\left|t_1^{1-\alpha}Tu(t_1)-t_2^{1-\alpha}Tu(t_2)\right| \\
&\leq \left|t_1^{1-\alpha}-t_2^{1-\alpha}\right|\int_0^{t_1}(t_1-\tau)^{\alpha-1}\left|f(\tau,u(\tau))+\int_0^{\tau}g(\tau,z,u(z))dz\right|d\tau \\
&+t_2^{1-\alpha}\left|\int_0^{t_1}\left((t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1}\right)\left(f(\tau,u(\tau))+\int_0^{\tau}g(\tau,z,u(z))dz\right)d\tau\right. \\
&\left.+\int_0^{t_2}(t_2-\tau)^{\alpha-1}\left(f(\tau,u(\tau))+\int_0^{\tau}g(\tau,z,u(z))dz\right)d\tau\right|.
\end{aligned} \tag{10}$$

As $\varphi(t)$ and $\psi(t)$ are continuous functions on $[0,h]$, we have

$$\begin{aligned}
|f(\tau,u(\tau))| &\leq \tau^{\mu}\varphi(\tau)\left|u(\tau)\right|^{m_1} \leq \tau^{\mu-(1-\alpha)m_1}\varphi(\tau)\tau^{(1-\alpha)m_1}\left|u^{m_1}(\tau)\right| \\
&\leq \tau^{\mu-(1-\alpha)m_1}\varphi(\tau)\left\|u\right\|_{1-\alpha}^{m_1} \leq C_1(h)
\end{aligned}$$

provided that $\mu-(1-\alpha)m_1\geq 0$ or that $\mu-(1-\alpha)m_1<0$ and $t^{\mu-(1-\alpha)m_1}\varphi(t)$ is a continuous function and

$$\begin{aligned}
\int_0^{\tau}\left|g(\tau,z,u(z))dz\right| &\leq \int_0^{\tau}(\tau-z)^{\beta-1}z^{\sigma-(1-\alpha)m_2}\psi(z)\left(z^{(1-\alpha)}\left|u(z)\right|\right)^{m_2}dz \\
&\leq \max_{z\in[0,h]}\psi(z)\left\|u\right\|_{1-\alpha}^{m_2}C\tau^{\beta+\sigma-(1-\alpha)m_2} \leq C_2(h)
\end{aligned}$$

provided that $\sigma-(1-\alpha)m_2\geq 0$ or that $\sigma-(1-\alpha)m_2<0$ and $z^{\sigma-(1-\alpha)m_2}\psi(z)$ continuous. Here C comes from the application of Lemma 1.

Taking this into account in (10) we obtain

$$\begin{aligned}
\left|t_1^{1-\alpha}Tu(t_1)-t_2^{1-\alpha}Tu(t_2)\right| &\leq C_3(h)\left|t_1^{1-\alpha}-t_2^{1-\alpha}\right|\int_0^{t_1}(t_1-\tau)^{\alpha-1}d\tau \\
&+C_3(h)t_2^{1-\alpha}\int_0^{t_1}\left|(t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1}\right|d\tau \\
&+C_3(h)t_2^{1-\alpha}\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}d\tau
\end{aligned}$$

where $C_3(h)=C_1(h)+C_2(h)$. Computing the integrals and making use of

the inequality in Lemma 2 it appears that

$$\begin{gathered}\left|t_1^{1-\alpha}Tu(t_1)-t_2^{1-\alpha}Tu(t_2)\right| \\ \leq C_5\left(t_2-t_1\right)^{1-\alpha}+C_6t_2^{1-\alpha}\left(t_2-t_1\right)^{\alpha}+C_7t_2^{1-\alpha}\left(t_2-t_1\right)^{\alpha} \\ \leq C_8\left(\left(t_2-t_1\right)^{1-\alpha}+\left(t_2-t_1\right)^{\alpha}\right).\end{gathered}$$

At this point we may conclude that $t^{1-\alpha}T(S_r)$ and hence $T(S_r)$ is equicontinuous. From the uniform boundedness and the equicontinuity we deduce by the Ascoli-Arzela Lemma that $T(S_r)$ is relatively compact. The Schauder fixed point theorem assures the existence of at least one fixed point in S_r for the operator T. ■

Remark 1. *The continuity of $t^{1-\alpha}f(t,u)$ and $s^{1-\alpha}g(t,s,u(s))$ on $C_{1-\alpha}^0(\mathbf{R}^+)$ assumed here is what is called the S_2-continuity in Tazali [14].*

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