BEHAVIOR OF SOLUTIONS FOR A WEIGHTED CAUCHY-TYPE FRACTIONAL DIFFERENTIAL PROBLEM*

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Abstract

In this paper we shall investigate the behavior of solutions of an ordinary fractional differential problem. Namely, we consider a weighted Cauchy-type problem involving a fractional derivative in the sense of Riemann-Liouville and a non-local term in the second member of the equation. We show that, for certain nonlinearities, solutions decay polynomially on their interval of existence.

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1 Introduction

In this work we are concerned by the following ordinary differential problem of fractional type

$$\begin{cases} D^{\alpha}u(t) = f(t,u) + \int\limits_0^t g(t,s,u(s))ds, \ t > 0 \\ t^{1-\alpha}u(t)\mid_{t=0} = b, \end{cases} \tag{1}$$

where $0 < \alpha < 1$, $b \in \mathbf{R}$, f and g are continuous functions. D^{α} denotes the fractional derivative of order α. This problem with $g \equiv 0$ has been considered by Delbosco and Rodino [10]. The authors proved several local existence and uniqueness results in a weighted space of continuous functions in the case where $f(t,u)$ is Lipschitzian in its second argument u. They also studied the not-globally-Lipschitzian case $f(t,u) = f(u) = u^s$ with $0 < s < \frac{1}{1-\alpha}$. The same problem (with $g \equiv 0$) has also been investigated by Kilbas, Bonilla and Trujillo [16]. Local existence and uniqueness theorems were proved in the space of summable functions. Again they assumed $f(t,u)$ to be Lipschitz continuous in u. Their proof was based on the successive approximation method. They also considered problem (1) with the initial condition

$$(I^{1-\alpha}u)(0) = b \tag{2}$$

This problem (with $g \not\equiv 0$ and condition (2)) has been considered by many other authors in the past (see references in [16,17,19]). We mention here in particular the work of Tazali [38] where the author proved local existence results in a space of weighted continuous functions using the Picard method and the Schauder fixed-point theorem. In [39], Zhang proved the existence and uniqueness of positive solutions for some monotone functions $f(t,u)$.

Many authors have also contributed in investigating the linear (homogeneous and nonhomogeneous) case. We refer the reader to Miller and Ross [29], Compos [8], Barett [1], Kilbas and Saigo [18] and references therein. In general explicit solutions are found by means of Laplace Transforms, operational calculus and several other approaches which proved to be efficient for ordinary differential equations. Let us also mention the recent work of Diethelem and Ford [12] where the authors studied a similar problem (allowing also for higher order fractional derivatives) with the left hand side modified (according to an idea of Caputo [9]) so as to specify classical initial data like $u(0) = u_0$, $u_t(0) = u_1$, ...

For the numerical treatment of Abel-integral equations and fractional differential equations, we refer the reader to Lubich [22-24], Brunner [5,6], Brunner and Van der Houven [7], Blank [2-4], Shkhamkov [36], Diethelem [11], Diethelem and Walz [13], Diethelem and Ford [12] and Podlubny [31,32]. From the literature, it is evident that there is a large gap between the numerical treatment and the theoretical analysis of integral equations and differential equations. The numerical approach remains an open field.

Fractional derivatives and fractional differential equations arise naturally in various fields and have several applications. Indeed, many phenomena which exhibit memory effects can be found in viscoelasticity, electromagnetics, radiation physics, seismic analysis, control, robotics, acoustics, electrochemistry (see for instance references in [3,19]). As a result we are witnessing a growing interest in developing the theory of fractional calculus.

We note here that the question of geometric and physical interpretation of the fractional integration and differentiation remained open for more than 300 years (since 1965). We refer the reader to Podlubny [34] for a suggestion in this sense.

In contrast to the (local) existence, uniqueness and finding the explicit solutions in a finite time interval, the asymptotic behavior question has not been well explored. Indeed, according to Kilbas and Trujillo [19] there is only one known result by Fujiwara [14] in this regard. We also cite the paper by Hadid and Alshammari [15]. There the authors discussed the growth of solutions and some stability criteria.

In the present paper we would like to make a contribution by proving some properties for solutions of problem (1). We will establish sufficient conditions under which solutions of (1) can be extended and decay in a polynomial manner in a weighted space of continuous functions. We mention here that in our nonlinearities we allow for non-local terms to be included. In particular, the right hand side of (1) may contain "fractional integrals". These "fractional integrals" involve naturally (by definition) kernels of singular type.

The main difficulties encountered here are due to the presence of a singular (and nonintegrable) kernel in the definition of the fractional derivative in addition to the non-local character of the problem. Unlike derivatives of integer order, derivatives of fractional order ($0 < \alpha < 1$) takes into account the whole history of the function from 0 up to time t. They are defined by a time convolution of the function (solution) in question (or its time derivative in case of the Caputo fractional derivative) with the weight $t^{-\alpha}$. Dealing with such singular kernels leads to considerable difficulties when trying to apply the existing standard methods. In our study of the weighted Cauchy-type problem (1) we allow for polynomials in t and polynomials in u in the nonlinearities f and g. Moreover, the function g may include singular kernels of the type $(t-s)^{-\beta}$, $0<\beta<1$. Our proofs are mainly based on some "desingularization" techniques. These techniques are due to Medved [26,27], Kirane and Tatar [20,21], Tatar [37] and Mazouzi and Tatar [25].

Our paper is organized as follows. In the next section we prepare some material needed to show our theorems. Section 3 is devoted to the statements and proofs of our main results concerning the polynomial decay of solutions. In the last section we discuss some possible extensions and generalizations of our results.

2 Preliminaries

In this section we present some definitions, lemmas and notation which will be used in our theorems.

Definition 1. *The Riemann-Liouville fractional integral of order $\alpha>0$ of a Lebesgue-measurable function $f:\mathbf{R}^+\to\mathbf{R}$ is defined by (the Abel-integral operator)*

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds$$

provided that the integral exists.

Definition 2. *The fractional derivative (in the sense of Riemann-Liouville) of order $0<\alpha<1$ of a continuous function $f:\mathbf{R}^+\to\mathbf{R}$ is defined as the left inverse of the fractional integral of f*

$$D^\alpha f(t)=\frac{d}{dt}(I^{1-\alpha}f)(t).$$

That is

$$D^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}f(s)ds,$$

provided that the right side exists.

See [29], [30], [33] and [35] for more on fractional integrals and fractional derivatives.

For $h > 0$, we define the space

$$C_r^0([0,h]) := \left\{ v \in C^0((0,h]) : \lim_{t \to 0^+} t^r v(t) \ exists \ and \ is \ finite \right\}.$$

Here $C^0((0,h])$ is the usual space of continuous functions on $[0,h]$. The space $C_r^0([0,h])$ endowed with the norm

$$\|v\|_r := \max_{0 \le t \le h} t^r |v(t)|$$

is a Banach space.

If we look for solutions in $C_{1-\alpha}^0([0,h])$ and the right hand side of the equation in (1) is in $C^0((0,h)) \cap L^1((0,h))$ then problem (1) is equivalent to the integral equation

$$u(t) = bt^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(f(s, u(s) + \int_0^s g(s,z,u(z)) dz \right) ds. \quad (3)$$

See Delbosco and Rodino [10].

We will also need the following well known lemmas.

Lemma 1. *For all $\alpha > 0$ and $\mu > -1$, we have*

$$\int_0^t (t-s)^{\alpha-1} s^\mu ds = D t^{\alpha+\mu}, \ t \ge 0,$$

where $D = D(\alpha-1, \mu) = \frac{\Gamma(\alpha)\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}$.

Lemma 2. *If λ, ν, $\omega > 0$, then for any $t > 0$ we have*

$$t^{1-\nu} \int_0^t (t-s)^{\nu-1} s^{\lambda-1} e^{-\omega s} ds \le C,$$

where $C = C(\nu-1, \lambda-1, \omega)$ is a positive constant independent of t. In fact,

$$C = \max\left\{1, 2^{1-\nu}\right\} \Gamma(\lambda)(1+\lambda/\nu)\omega^{-\lambda}.$$

For the proof of this lemma we refer to [21] or [29].

Lemma 3. *For $a, b, c \geq 0$ and $\gamma > 1$, we have*

$$(a+b+c)^\gamma \leq 3^{\gamma-1}(a^\gamma + b^\gamma + c^\gamma).$$

Lemma 4. *For $a, b \geq 0$ and $\gamma > 0$, we have*

$$(a+b)^\gamma \leq 2^{\max(\gamma,1)}(a^\gamma + b^\gamma).$$

We will assume the following hypotheses on the nonlinearities f and g.

(F) $t^{1-\alpha}f(t,u)$ is continuous on $\mathbf{R}^+ \times C^0_{1-\alpha}(\mathbf{R}^+)$ and

$$|f(t,u)| \leq t^\mu \varphi(t) |u|^{m_1}, \ \mu \geq 0, \ m_1 > 1 \tag{4}$$

(G) $s^{1-\alpha}g(t,s,u(s))$ is continuous on $D_{\mathbf{R}+} \times C^0_{1-\alpha}(\mathbf{R}^+)$ where

$$D_{\mathbf{R}+} = \left\{(t,s) \in \mathbf{R}^+ \times \mathbf{R}^+, 0 \leq s \leq t\right\}$$

and

$$|g(t,s,u(s))| \leq (t-s)^{\beta-1} s^\sigma \psi(s) |u|^{m_2}, \ 0 < \beta < 1, \ \sigma \geq 0, \ m_2 > 1 \tag{5}$$

where $\varphi(t)$ and $\psi(s)$ are such that

(Φ) $\varphi(t)$ is continuous and $t^{\mu-(1-\alpha)m_1}\varphi(t)$ is continuous in case

$$\mu - (1-\alpha)m_1 < 0.$$

(Ψ) $\psi(t)$ is continuous and $t^{\sigma-(1-\alpha)m_2}\psi(t)$ is continuous in case

$$\sigma - (1-\alpha)m_2 < 0.$$

Under these assumptions there exists at least one local solution to problem (1) in the space $C^0_{1-\alpha}([0,h])$.

Theorem 1. *(see [15]) Assume that the above hypotheses **(F)**, **(G)**, **(Φ)** and **(Ψ)** on f, g, φ and ψ hold on $[0,h]$, $h > 0$. Suppose further that $1 + \mu - (1-\alpha)m_1 > 0$ and $1 + \sigma - (1-\alpha)m_2 > 0$. Then problem (1) admits at least one solution on a sufficiently small interval $[0,\tilde{h}]$ with $\tilde{h} < h$.*

3 Polynomial decay

In this section we state and prove some results on the behavior of solutions to problem (1). In particular we show that solutions decay to zero at the rate of a polynomial. In order to lighten the exposition of our results further, let us prepare some notation. Let $q = \alpha + 1$, $p = \beta + 1$ and $q^\star$ and $p^\star$ their conjugate exponents, respectively. We denote by

$$K_1 := C^{\frac{q^\star}{q}} \left(q(\alpha - 1), q[\mu - (1-\alpha)m_1], \varepsilon q \right) / \Gamma^{q^\star}(\alpha),$$

$$K_2 := C^{\frac{q^\star}{p}} \left(p(\beta - 1), p[\sigma - (1-\alpha)m_2], \varepsilon p \right)$$
$$\times C^{\frac{q^\star}{q}} \left(q(\alpha - 1), q(\beta - 1), \varepsilon q \right) / \Gamma^{q^\star}(\alpha),$$

$$Q(t) := 2^{\max(1, q^\star/p^\star)} 3^{q^\star - 1} e^{\varepsilon q^\star t} \max\left\{ K_1 \varphi^{q^\star}(t), K_2 \right\},$$

$$a_1 := 3^{q^\star - 1} |b|^{q^\star} \ \textit{and} \ l := a_1^{\frac{m_1 p^\star}{q^\star} - 1},$$

$$R(t) := \max\left\{ Q(t), \frac{q^\star}{m_1 p^\star} \psi^{q^\star}(t) e^{\varepsilon p^\star t} \right\}$$

and

$$\tilde{R}(t) := \max\left\{ l Q(t), \frac{q^\star}{m_1 p^\star} \psi^{q^\star}(t) e^{\varepsilon p^\star t} \right\}$$

for some $\varepsilon > 0$.

Theorem 2. *Assume that the hypotheses **(F)**, **(G)**, **(**Φ**)** and **(**Ψ**)** hold. Assume also that* $1 + q[\mu - (1-\alpha)m_1] > 0$ *and* $1 + p[\sigma - (1-\alpha)m_2] > 0$. *Then*

***(1)** the solutions of problem (1) decay polynomially as* $t^{\alpha-1}$*. In fact,*

$$|u(t)| \le C t^{\alpha - 1} \tag{6}$$

for some positive constant C *as long as*

(a) $m_1 p^\star \ge q^\star$ *and*

$$\int_0^t R(s) ds < \int_{a_1}^{\infty} \frac{dz}{z^{m_1} + z^{1 + (m_2 - m_1)p^\star/q^\star}},$$

(b) $m_1 p^\star < q^\star$ *and*

$$\int_0^t \tilde{R}(s)ds < \int_{a_1}^{\infty} \frac{dz}{z^{1-m_1p^\star/q^\star}\left(z^{m_1} + z^{m_2p^\star/q^\star}\right)}.$$

(2) *Suppose further that* φ *and* ψ *are bounded on* $\mathbf{R}^+$, *then there exist two positive constants* T *and* $T^\star$ *such that*

(a) *if* $T \le 1$, *then (6) holds on* $(0, T)$,

(b) *if* $T > 1$ *and* $T^\star > 1$, *then* $u(t)$ *can be extended beyond* 1 *and (6) holds on* $(0, T^\star)$.

Proof: Multiplying the integral equation (3)

$$u(t) = bt^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(f(s,u(s)) + \int_0^s g(s,z,u(z))dz \right) ds$$

by $t^{1-\alpha}$ and using the assumptions (4) and (5) on f and g, we find

$$\begin{array}{c} t^{1-\alpha}\left|u(t)\right| \le |b| + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\mu} \varphi(s) \left|u(s)\right|^{m_1} ds \\ + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\int_0^s (s-z)^{\beta-1} z^{\sigma} \psi(z) \left|u(s)\right|^{m_2} dz \right) ds. \end{array} \quad (7)$$

Put $v(t) = t^{1-\alpha}\left|u(t)\right|$, then (7) can be rewritten as

$$\begin{array}{c} v(t) \le |b| + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\mu-(1-\alpha)m_1} \varphi(s) v^{m_1}(s) ds \\ + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\int_0^s (s-z)^{\beta-1} z^{\sigma-(1-\alpha)m_2} \psi(z) v^{m_2}(z)\, dz \right) ds. \end{array} \quad (8)$$

This will be our reference inequality.

(1) We estimate the integral terms in the right hand side of (8) separately. Multiplying by $e^{-\varepsilon s} e^{\varepsilon s}$, for $\varepsilon > 0$ (as in the statement of the theorem), inside the integral and using the Hölder inequality with $q = \alpha + 1$ and $q^\star = \frac{\alpha+1}{\alpha}$, we get

$$\begin{aligned}
&\tfrac{t^{1-\alpha}}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1} s^{\mu-(1-\alpha)m_1} e^{-\varepsilon s} e^{\varepsilon s} \varphi(s) v^{m_1}(s) ds \\
&\leq \tfrac{t^{1-\alpha}}{\Gamma(\alpha)} \left(\int\limits_0^t (t-s)^{q(\alpha-1)} s^{q[\mu-(1-\alpha)m_1]} e^{-q\varepsilon s} ds \right)^{\frac{1}{q}} \\
&\qquad \times \left(\int\limits_0^t e^{q^\star \varepsilon s} \varphi^{q^\star}(s) v^{q^\star m_1}(s) ds \right)^{\frac{1}{q^\star}} .
\end{aligned}$$

Since $1+q\left[\mu-(1-\alpha)\, m_1\right]>0$ and $q\,(\alpha-1)>-1$ we can apply Lemma 2. We obtain

$$\begin{aligned}
&\tfrac{t^{1-\alpha}}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1} s^{\mu-(1-\alpha)m_1} \varphi(s) v^{m_1}(s) ds \\
&\leq C_1 \left(\int\limits_0^t e^{q^\star \varepsilon s} \varphi^{q^\star}(s) v^{q^\star m_1}(s) ds \right)^{\frac{1}{q^\star}} ,
\end{aligned} \tag{9}$$

where C_1 is the constant which arises from the application of Lemma 2

$$C_1 = C^{\frac{1}{q}}\left(q(\alpha-1), q[\mu-(1-\alpha)m_1], \varepsilon q\right)/\Gamma(\alpha).$$

Similarly, since $1+p\left[\sigma-(1-\alpha)\, m_2\right]>0$ it follows that

$$\begin{aligned}
&\int\limits_0^s (s-z)^{\beta-1} z^{\sigma-(1-\alpha)m_2} \psi(z) v^{m_2}(z) dz \\
&\leq C_2 s^{\beta-1} \left(\int\limits_0^s e^{\varepsilon p^\star z} \psi^{p^\star}(z) v^{m_2 p^\star}(z) dz \right)^{\frac{1}{p^\star}}
\end{aligned}$$

where C_2 is the constant from the inequality in Lemma 2

$$C_2 = C^{\frac{1}{p}}\left(p(\beta-1), p[\sigma-(1-\alpha)m_2], \varepsilon p\right)$$

Therefore,

$$\begin{aligned}
&\tfrac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1} \left(\int\limits_0^s (s-z)^{\beta-1} z^{\sigma-(1-\alpha)m_2} \psi(z) v^{m_2}(z) dz \right) ds \\
&\leq C_3 \int\limits_0^t (t-s)^{\alpha-1} s^{\beta-1} \left(\int\limits_0^s e^{\varepsilon p^\star z} \psi^{p^\star}(z) v^{p^\star m_2}(z) dz \right)^{\frac{1}{p^\star}} ds
\end{aligned}$$

with $C_3 = C_2/\Gamma(\alpha)$. We multiply by $e^{-\varepsilon s}.e^{\varepsilon s}$ and use the Hölder inequality once again, we obtain

$$\begin{aligned}
&\int_0^t (t-s)^{\alpha-1} s^{\beta-1} e^{-\varepsilon s} e^{\varepsilon s} \left(\int_0^s e^{\varepsilon p^\star z} \psi^{p^\star}(z) v^{p^\star m_2}(z) dz \right)^{\frac{1}{p^\star}} ds. \\
&\leq \left(\int_0^t (t-s)^{q(\alpha-1)} s^{q(\beta-1)} e^{-q\varepsilon s} ds \right)^{\frac{1}{q}} \\
&\times \left(\int_0^t e^{q^\star \varepsilon s} \left(\int_0^s e^{\varepsilon p^\star z} \psi^{p^\star}(z) v^{p^\star m_2}(z) dz \right)^{\frac{q^\star}{p^\star}} ds \right)^{\frac{1}{q^\star}} \\
&\leq C_4 t^{\alpha-1} \left(\int_0^t e^{q^\star \varepsilon s} \left(\int_0^s e^{\varepsilon p^\star z} \psi^{p^\star}(z) v^{p^\star m_2}(z) dz \right)^{\frac{q^\star}{p^\star}} ds \right)^{\frac{1}{q^\star}}
\end{aligned} \tag{10}$$

provided that $1 + q(\alpha - 1) > 0$ and $1 + q(\beta - 1) > 0$. Notice that in the last estimation we have used Lemma 2 and C_4 comes from this lemma

$$C_4 = C^{\frac{1}{q}} \left(q(\alpha-1), q(\beta-1), \varepsilon q \right).$$

Using the estimations (9) and (10) in (8) we find

$$\begin{aligned}
v(t) &\leq |b| + C_1 \left(\int_0^t e^{\varepsilon q^\star s} \varphi^{q^\star}(s) v^{q^\star m_1}(s) ds \right)^{\frac{1}{q^\star}} \\
&+ C_5 \left(\int_0^t e^{q^\star \varepsilon s} \left(\int_0^s e^{\varepsilon p^\star z} \psi^{p^\star}(z) v^{p^\star m_2}(z) dz \right)^{\frac{q^\star}{p^\star}} ds \right)^{\frac{1}{q^\star}}
\end{aligned}$$

where $C_5 = C_3 C_4$. Now by Lemma 3, we entail that

$$\begin{aligned}
v^{q^\star}(t) \leq 3^{q^\star-1} \Bigg[&|b|^{q^\star} + C_1^{q^\star} \int_0^t e^{\varepsilon q^\star s} \varphi^{q^\star}(s) v^{q^\star m_1}(s) ds \\
&+ C_5^{q^\star} \int_0^t e^{\varepsilon q^\star s} \left(\int_0^s e^{\varepsilon p^\star z} \psi^{p^\star}(z) v^{p^\star m_2}(z) dz \right)^{\frac{q^\star}{p^\star}} ds \Bigg].
\end{aligned} \tag{11}$$

Let us denote the right hand side of (11) by $w(t)$. Clearly,

$$w(0) = 3^{q^\star-1} |b|^{q^\star}$$

and

$$\begin{aligned} w'(t) &= 3^{q^\star-1}C_1^{q^\star}e^{\varepsilon q^\star t}\varphi^{q^\star}(t)v^{q^\star m_1}(t) \\ &+3^{q^\star-1}C_5^{q^\star}e^{\varepsilon q^\star t}\left(\int_0^t e^{\varepsilon p^\star s}\psi^{p^\star}(s)v^{p^\star m_2}(s)ds\right)^{\frac{q^\star}{p^\star}} \\ &\leq 3^{q^\star-1}e^{\varepsilon q^\star t}\max\left\{C_1^{q^\star}\varphi^{q^\star}(t),C_5^{q^\star}\right\} \\ &\times\left[w^{m_1}(t)+\left(\int_0^t e^{\varepsilon p^\star s}\psi^{p^\star}(s)w^{m_2p^\star/q^\star}(s)ds\right)^{\frac{q^\star}{p^\star}}\right]. \end{aligned} \tag{12}$$

Now we define $x(t)$ by

$$x^{m_1}(t) := w^{m_1}(t)+\left(\int_0^t e^{\varepsilon p^\star s}\psi^{p^\star}(s)w^{m_2p^\star/q^\star}(s)ds\right)^{\frac{q^\star}{p^\star}}. \tag{13}$$

Raising both sides of (13) to the power $\frac{p^\star}{q^\star}$ and using Lemma 4 we find

$$x^{\frac{m_1p^\star}{q^\star}}(t)\leq 2^{\max(p^\star/q^\star,1)}\left(w^{\frac{m_1p^\star}{q^\star}}(t)+\int_0^t e^{\varepsilon p^\star s}\psi^{p^\star}(s)w^{m_2p^\star/q^\star}(s)ds\right).$$

Let us denote by $y^{\frac{m_1p^\star}{q^\star}}(t)$ the expression

$$y^{\frac{m_1p^\star}{q^\star}}(t)=w^{\frac{m_1p^\star}{q^\star}}(t)+\int_0^t e^{\varepsilon p^\star s}\psi^{p^\star}(s)w^{m_2p^\star/q^\star}(s)ds. \tag{14}$$

Then $y(0)=w(0)=3^{q^\star-1}\left|b\right|^{q^\star}$ and $w(t)\leq y(t)$.

Differentiating the last expression (14) and using (12), we find

$$\begin{aligned} \frac{m_1p^\star}{q^\star}y^{\frac{m_1p^\star}{q^\star}-1}(t)y'(t) &= \frac{m_1p^\star}{q^\star}w^{\frac{m_1p^\star}{q^\star}-1}(t)w'(t)+e^{\varepsilon p^\star t}\psi^{p^\star}(t)w^{m_2p^\star/q^\star}(t) \\ &\leq \frac{m_1p^\star}{q^\star}w^{\frac{m_1p^\star}{q^\star}-1}(t)Q(t)y^{m_1}(t)+e^{\varepsilon p^\star t}\psi^{p^\star}(t)y^{m_2p^\star/q^\star}(t) \end{aligned} \tag{15}$$

where $Q(t)=3^{q^\star-1}2^{\max(1,q^\star/p^\star)}e^{\varepsilon q^\star t}\max\left\{C_1^{q^\star}\varphi^{q^\star}(t),C_5^{q^\star}\right\}$.

(a) If $\frac{m_1p^\star}{q^\star}-1\geq 0$, then we get

$$\frac{m_1p^\star}{q^\star}y^{\frac{m_1p^\star}{q^\star}-1}(t)y'(t)\leq\frac{m_1p^\star}{q^\star}y^{\frac{m_1p^\star}{q^\star}-1}(t)Q(t)y^{m_1}(t)+e^{\varepsilon p^\star t}\psi^{p^\star}(t)y^{m_2p^\star/q^\star}(t) \tag{16}$$

If we divide both sides of the last inequality (16) by $\frac{m_1 p^\star}{q^\star} y^{\frac{m_1 p^\star}{q^\star}-1}(t)$, it appears that

$$\begin{aligned} y'(t) &\leq Q(t) y^{m_1}(t) + \tfrac{q^\star}{m_1 p^\star} e^{\varepsilon p^\star t} \psi^{p^\star}(t) y^{1+(m_2-m_1)p^\star/q^\star} \\ &\leq R(t) \left(y^{m_1}(t) + y^{1+(m_2-m_1)p^\star/q^\star}(t) \right) \end{aligned}$$

with $R(t) := \max \left(Q(t), \frac{q^\star}{m_1 p^\star} e^{\varepsilon p^\star t} \psi^{p^\star}(t) \right)$.

Clearly, if $a_1 = 3^{q^\star - 1} |b|^{q^\star}$ we have

$$\int_{a_1}^{y(t)} \frac{dz}{z^{m_1} + z^{1+(m_2-m_1)p^\star/q^\star}} \leq \int_0^t R(s) ds < \int_{a_1}^{\infty} \frac{dz}{z^{m_1} + z^{1+(m_2-m_1)p^\star/q^\star}}.$$

The last relation is from the hypotheses in part (a). Consequently, $y(t)$ is bounded as long as the relation

$$\int_0^t R(s) ds < \int_{a_1}^{\infty} \frac{dz}{z^{m_1} + z^{1+(m_2-m_1)p^\star/q^\star}} \tag{17}$$

holds. It follows that $w(t)$ and then $v(t)$ are also bounded. Hence $t^{1-\alpha} |u(t)|$ is bounded as long as (17) is valid.

(b) If $\frac{m_1 p^\star}{q^\star} - 1 < 0$, then $w(t) \geq w(0) = 3^{q^\star - 1} |b|^{q^\star}$ and hence $w^{\frac{m_1 p^\star}{q^\star}-1}(t) \leq w^{\frac{m_1 p^\star}{q^\star}-1}(0) = \left(3^{q^\star-1} |b|^{q^\star} \right)^{\frac{m_1 p^\star}{q^\star}-1} =: l$. From (15), we obtain

$$\begin{aligned} y'(t) &\leq l Q(t) y^{1+m_1-m_1 p^\star/q^\star}(t) + \tfrac{q^\star}{m_1 p^\star} e^{\varepsilon p^\star t} \psi^{p^\star}(t) y^{1+(m_2-m_1)p^\star/q^\star} \\ &\leq \tilde{R}(t) \left(y^{1+m_1-m_1 p^\star/q^\star}(t) + y^{1+(m_2-m_1)p^\star/q^\star}(t) \right) \end{aligned}$$

where $\tilde{R}(t) = \max \left(l Q(t), \frac{q^\star}{m_1 p^\star} e^{\varepsilon p^\star t} \psi^{p^\star}(t) \right)$. This relation together with the hypothesis on $\tilde{R}(t)$ imply that

$$\int_{a_1}^{y(t)} \frac{dz}{z^{1-m_1 p^\star/q^\star} \left(z^{m_1} + z^{m_2 p^\star/q^\star} \right)} \leq \int_0^t \tilde{R}(s) ds < \int_{a_1}^{\infty} \frac{dz}{z^{1-m_1 p^\star/q^\star} \left(z^{m_1} + z^{m_2 p^\star/q^\star} \right)}.$$

(2) We now estimate the integrals with the help of Lemma 1. By the Hölder inequality, we see that the first integral is less than or equal to

$$\left(\int_0^t (t-s)^{q(\alpha-1)} s^{q[\mu-(1-\alpha)m_1]} ds\right)^{\frac{1}{q}} \left(\int_0^t \varphi^{q^\star}(s) v^{q^\star m_1}(s) ds\right)^{\frac{1}{q^\star}}.$$

By the Lemma 1 we have

$$\begin{aligned} &\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\mu-(1-\alpha)m_1} \varphi(s) v^{m_1}(s) ds \\ &\leq C_6 t^{\frac{1}{q}+\alpha-1+\mu-(1-\alpha)m_1} \left(\int_0^t \varphi^{q^\star}(s) v^{q^\star m_1}(s) ds\right)^{\frac{1}{q^\star}} \end{aligned} \tag{18}$$

where

$$C_6 = \frac{1}{\Gamma(\alpha)} D^{\frac{1}{q}} \left(q(\alpha-1), q[\mu-(1-\alpha)m_1]\right)$$

because $1+q\left[\mu-(1-\alpha)m_1\right] > 0$ and $1+q(\alpha-1) > 0$.

Similarly, we obtain

$$\begin{aligned} &\tfrac{1}{\Gamma(\alpha)} \int_0^s (s-z)^{\beta-1} z^{\sigma-(1-\alpha)m_2} \psi(z) v^{m_2}(z) dz \\ &\leq \tfrac{1}{\Gamma(\alpha)} \left(\int_0^s (s-z)^{p(\beta-1)} z^{p[\sigma-(1-\alpha)m_2]} dz\right)^{\frac{1}{p}} \left(\int_0^s \psi(z)^{p^\star} v^{p^\star m_2}(z) dz\right)^{\frac{1}{p^\star}} \\ &\leq C_7 s^{\frac{1}{p}+\beta-1+\sigma-(1-\alpha)m_2} \left(\int_0^s \psi(z)^{p^\star} v^{p^\star m_2}(z) dz\right)^{\frac{1}{p^\star}} \end{aligned}$$

with

$$C_7 = \frac{1}{\Gamma(\alpha)} D^{\frac{1}{p}} \left(p(\beta-1), p[\sigma-(1-\alpha)m_2]\right)$$

since $1+p\left[\sigma-(1-\alpha)m_2\right] > 0$ and $1+p(\beta-1) > 0$. Therefore,

$$\begin{aligned} &\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\int_0^s (s-z)^{\beta-1} z^{\sigma-(1-\alpha)m_2} .\psi(z) v^{m_2}(z) dz\right) ds \\ &\leq C_7 \int_0^t (t-s)^{\alpha-1} s^{\frac{1}{p}+\beta-1+\sigma-(1-\alpha)m_2} \left(\int_0^s \psi(z)^{p^\star} v^{p^\star m_2}(z) dz\right)^{\frac{1}{p^\star}} ds \end{aligned}$$

Using the Hölder inequality again with $\frac{1}{q}+\frac{1}{q^\star}=1$ and Lemma 1, we find

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s (s-z)^{\beta-1} z^{\sigma-(1-\alpha)m_2}\psi(z)v^{m_2}(z)dz\right)ds \\
&\qquad \le C_7\left(\int_0^t (t-s)^{q(\alpha-1)} s^{\frac{q}{p}+q[\beta-1+\sigma-(1-\alpha)m_2]}ds\right)^{\frac{1}{q}} \\
&\qquad \times\left(\int_0^t\left(\int_0^s \psi(z)^{p^\star} v^{p^\star m_2}(z)dz\right)^{\frac{q^\star}{p^\star}} ds\right)^{\frac{1}{q^\star}} \\
&\le C_8 t^{\frac{1}{q}+\frac{1}{p}+\alpha+\beta+\sigma-2-(1-\alpha)m_2}\left(\int_0^t\left(\int_0^s \psi(z)^{p^\star} v^{p^\star m_2}(z)dz\right)^{\frac{q^\star}{p^\star}} ds\right)^{\frac{1}{q^\star}},
\end{aligned} \tag{19}$$

where

$$C_8 = C_7 D^{\frac{1}{q}}\left(q(\alpha-1), \frac{q}{p}+q[\beta-1+\sigma-(1-\alpha)m_2]\right).$$

Taking (18) and (19) into account in (8), we get

$$\begin{aligned}
v(t) &\le |b| + C_6 t^{\frac{1}{q}+\mu-(1-\alpha)m_1}\left(\int_0^t \varphi^{q^\star}(s)v^{q^\star m_1}(s)ds\right)^{\frac{1}{q^\star}} \\
&+C_8 t^{\frac{1}{q}+\frac{\beta}{p^\star}+\sigma-(1-\alpha)m_2}\left(\int_0^t\left(\int_0^s \psi(z)^{p^\star} v^{p^\star m_2}(z)dz\right)^{\frac{q^\star}{p^\star}} ds\right)^{\frac{1}{q^\star}}.
\end{aligned} \tag{20}$$

(a) Suppose that $0 < t \le 1$. Then by Lemma 3 (and the fact that the exponents of t are positive) we infer that

$$\begin{aligned}
v^{q^\star}(t) &\le 3^{q^\star-1}|b|^{q^\star} + 3^{q^\star-1}\Big[C_6^{q^\star}\max\nolimits_{0\le t\le 1}\left\{\varphi^{q^\star}(t)\right\}\int_0^t v^{q^\star m_1}(s)ds \\
&+C_8^{q^\star}\max\nolimits_{0\le t\le 1}\left\{\psi^{q^\star}(t)\right\}\int_0^t\left(\int_0^s v^{p^\star m_2}(\tau)d\tau\right)^{\frac{q^\star}{p^\star}} ds\Big].
\end{aligned} \tag{21}$$

We designate the right hand side of (21) by $w(t)$. Then, $w(0) = 3^{q^\star-1}|b|^{q^\star}$ and

$$\begin{aligned}
w'(t) &= 3^{q^\star-1}\Big[C_6^{q^\star}\max\nolimits_{0\le t\le 1}\left\{\varphi^{q^\star}(t)\right\}v^{q^\star m_1}(t) \\
&+C_8^{q^\star}\max\nolimits_{0\le t\le 1}\left\{\psi^{q^\star}(t)\right\}\left(\int_0^t v^{p^\star m_2}(\tau)d\tau\right)^{\frac{q^\star}{p^\star}}\Big].
\end{aligned}$$

Since $v^{q^\star}(t) \le w(t)$, we find

$$w^{'}(t) \le C_9 w^{m_1}(t) + C_{10}\left(\int\limits_0^t w^{\frac{p^\star m_2}{q^\star}}(\tau)d\tau\right)^{\frac{q^\star}{p^\star}}, \qquad (22)$$

where $C_9 = 3^{q^\star-1}C_6^{q^\star}\max_{0\le t\le 1}\left\{\varphi^{q^\star}(t)\right\}$ and $C_{10} = 3^{q^\star-1}C_8^{q^\star}\max_{0\le t\le 1}\left\{\psi^{q^\star}(t)\right\}$. Let us now denote by $\xi(t)$ the right hand side of (22). Then, $\xi(0) = C_9 w^{m_1}(0) = C_9\left(3^{q^\star-1}|b|^{q^\star}\right)^{m_1}$ and from Lemma 4

$$\xi^{\frac{p^\star}{q^\star}}(t) \le 2^{\max(1,p^\star/q^\star)}\left(C_9^{\frac{p^\star}{q^\star}} w^{\frac{p^\star m_1}{q^\star}}(t) + C_{10}^{\frac{p^\star}{q^\star}}\int\limits_0^t w^{\frac{p^\star m_2}{q^\star}}(\tau)d\tau\right).$$

Next, denoting by $\lambda^{\frac{p^\star m_1}{q^\star}}(t)$ the expression

$$C_9^{\frac{p^\star}{q^\star}} w^{\frac{p^\star m_1}{q^\star}}(t) + C_{10}^{\frac{p^\star}{q^\star}}\int\limits_0^t w^{\frac{p^\star m_2}{q^\star}}(\tau)d\tau,$$

we see that $\lambda(0) = C_9^{\frac{1}{m_1}} w(0)$, $w(t) \le C_9^{-\frac{1}{m_1}}\lambda(t)$ and

$$\frac{p^\star m_1}{q^\star}\lambda^{\frac{p^\star m_1}{q^\star}-1}(t)\lambda'(t) = \frac{p^\star m_1}{q^\star} C_9^{\frac{p^\star}{q^\star}} w^{\frac{p^\star m_1}{q^\star}-1}(t)w'(t) + C_{10}^{\frac{p^\star}{q^\star}} w^{\frac{p^\star m_2}{q^\star}}(t).$$

(i) If $\frac{p^\star m_1}{q^\star} - 1 \ge 0$, it is clear that $w^{\frac{p^\star m_1}{q^\star}-1}(t) \le \left(C_9^{-\frac{1}{m_1}}\lambda(t)\right)^{\frac{p^\star m_1}{q^\star}-1}$ and using the fact that

$$w^{'}(t) \le \xi(t) \le 2^{\max(1,q^\star/p^\star)}\lambda^{m_1}(t)$$

we obtain

$$\frac{p^\star m_1}{q^\star}\lambda^{\frac{p^\star m_1}{q^\star}-1}(t)\lambda'(t) \le 2^{\max(1,q^\star/p^\star)}\frac{p^\star m_1}{q^\star} C_9^{\frac{p^\star}{q^\star}}\left(C_9^{-\frac{1}{m_1}}\lambda(t)\right)^{\frac{p^\star m_1}{q^\star}-1}\lambda^{m_1}(t)$$
$$+C_{10}^{\frac{p^\star}{q^\star}}\left(C_9^{-\frac{1}{m_1}}\lambda(t)\right)^{\frac{p^\star m_2}{q^\star}}.$$

Dividing both terms by $\frac{p^\star m_1}{q^\star}\lambda^{\frac{p^\star m_1}{q^\star}-1}(t)$, we get

$$\lambda'(t) \le 2^{\max(1,q^\star/p^\star)} C_9^{\frac{1}{m_1}} \lambda^{m_1}(t) + \frac{q^\star}{p^\star m_1}\left(C_{10} C_9^{-\frac{m_2}{m_1}}\right)^{\frac{p^\star}{q^\star}} \lambda^{1+p^\star(m_2-m_1)/q^\star}(t).$$

or simply

$$\lambda'(t) \le C_{11}\left[\lambda^{m_1}(t) + \lambda^{1+p^\star(m_2-m_1)/q^\star}(t)\right]$$

where $C_{11} = \max\left\{2^{\max(1,q^\star/p^\star)} C_9^{\frac{1}{m_1}}, \frac{q^\star}{p^\star m_1}\left(C_{10} C_9^{-\frac{m_2}{m_1}}\right)^{\frac{p^\star}{q^\star}}\right\}$. It is apparent that, if $a_2 = \lambda(0) = C_9^{\frac{1}{m_1}} w(0) = C_9^{\frac{1}{m_1}} 3^{q^\star-1}|b|^{q^\star}$ then

$$\int_{a_2}^{\lambda(t)} \frac{dz}{z^{m_1} + z^{1+p^\star(m_2-m_1)/q^\star}} \le \int_0^t C_{11} ds = C_{11} t.$$

Suppose that

$$T = C_{11}^{-1} \int_{a_2}^{+\infty} \frac{dz}{z^{m_1} + z^{1+p^\star(m_2-m_1)/q^\star}},$$

(note that all the constants are known) then for any $t < T$

$$\int_{a_2}^{\lambda(t)} \frac{dz}{z^{m_1} + z^{1+p^\star(m_2-m_1)/q^\star}} < \int_{a_2}^{+\infty} \frac{dz}{z^{m_1} + z^{1+p^\star(m_2-m_1)/q^\star}}.$$

Therefore $\lambda(t)$ is bounded in t. It follows that $v(t)$ is bounded which means that $|u(t)| \le C t^{\alpha-1}$ as long as $t < T$.

(ii) if $\frac{p^\star m_1}{q^\star} - 1 < 0$, we estimate $w^{\frac{p^\star m_1}{q^\star}-1}(t)$ by using the fact that $w(t) \ge w(0)$. We arrive at

$$\frac{p^\star m_1}{q^\star}\lambda^{\frac{p^\star m_1}{q^\star}-1}(t)\lambda'(t) \le \frac{p^\star m_1}{q^\star} C_9^{\frac{p^\star}{q^\star}} 2^{\frac{q^\star}{p^\star}} \left(3^{q^\star-1}|b|^{q^\star}\right)^{\frac{p^\star m_1}{q^\star}-1} \lambda^{m_1}(t)$$
$$+ C_{10}^{\frac{p^\star}{q^\star}}\left(C_9^{-\frac{1}{m_1}}\lambda(t)\right)^{\frac{p^\star m_2}{q^\star}}.$$

Therefore,

$$\lambda'(t) \leq 2^{\frac{q^\star}{p^\star}} C_9^{\frac{p^\star}{q^\star}} \left(3^{q^\star-1} |b|^{q^\star}\right)^{\frac{p^\star m_1}{q^\star}-1} \lambda^{1+m_1(q^\star-p^\star)/q^\star}(t)$$
$$+\frac{q^\star}{p^\star m_1} \left(C_{10} C_9^{-m_2/m_1}\right)^{p^\star/q^\star} \lambda^{1+p^\star(m_2-m_1)/q^\star}(t)$$

or

$$\lambda'(t) \leq C_{12} \left[\lambda^{1+m_1(q^\star-p^\star)/q^\star}(t) + \lambda^{1+p^\star(m_2-m_1)/q^\star}(t)\right]$$

where C_{12} is equal to

$$\max\left\{2^{\frac{q^\star}{p^\star}} C_9^{\frac{p^\star}{q^\star}} \left(3^{q^\star-1} |b|^{q^\star}\right)^{\frac{p^\star m_1}{q^\star}-1}, \frac{q^\star}{p^\star m_1} \left(C_{10} C_9^{-m_2/m_1}\right)^{p^\star/q^\star}\right\}.$$

As in the first case (i) we infer that $\lambda(t)$ is bounded as long as

$$C_{12} t < \int_{a_2}^{+\infty} \frac{dz}{z^{1-m_1 p^\star/q^\star}\left(z^{m_1} + z^{p^\star m_2/q^\star}\right)}.$$

(b) Suppose now that $t > 1$. If $M_f = \max_{t\in\mathbf{R}^+}\{f(t)\}$, then we have from (20) that

$$v(t)^{q^\star} \leq 3^{q^\star-1} |b|^{q^\star} + 3^{q^\star-1} t^\gamma \left[C_6^{q^\star} M_\varphi^{q^\star} \int_0^t v^{q^\star m_1}(s) ds\right.$$
$$\left. + C_8^{q^\star} M_\psi^{q^\star} \int_0^t \left(\int_0^s v^{p^\star m_2}(\tau) d\tau\right)^{\frac{q^\star}{p^\star}} ds\right], \tag{23}$$

where $\gamma = q^\star \max\left\{\frac{1}{q} + \mu - (1-\alpha) m_1, \frac{1}{q} + \frac{\beta}{p^\star} + \sigma - (1-\alpha) m_2\right\}$.

If we denote by $w(t)$ the right hand side of (23), we can see that $w(0) = 3^{q^\star-1} |b|^{q^\star}$ and

$$w'(t) = 3^{q^\star-1} \gamma t^{\gamma-1}$$
$$\times \left[C_6^{q^\star} M_\varphi^{q^\star} \int_0^t v^{q^\star m_1}(s) ds + C_8^{q^\star} M_\psi^{q^\star} \int_0^t \left(\int_0^s v^{p^\star m_2}(\tau) d\tau\right)^{\frac{q^\star}{p^\star}} ds\right]$$
$$+ 3^{q^\star-1} t^\gamma \left[C_6^{q^\star} M_\varphi^{q^\star} v^{q^\star m_1}(t) + C_8^{q^\star} M_\psi^{q^\star} \left(\int_0^t v^{p^\star m_2}(\tau) d\tau\right)^{\frac{q^\star}{p^\star}}\right]. \tag{24}$$

At this point we distinguish two cases:

(*i*) If $\gamma - 1 > 0$, then from (24) we have

$$w'(t) \leq 3^{q^\star - 1} t^\gamma \left[\gamma C_6^{q^\star} M_\varphi^{q^\star} \int_0^t w^{m_1}(s) ds + \gamma C_8^{q^\star} M_\psi^{q^\star} \int_0^t \left(\int_0^s w^{m_2 p^\star / q^\star}(\tau) d\tau \right)^{\frac{q^\star}{p^\star}} ds \right.$$
$$\left. + C_6^{q^\star} M_\varphi^{q^\star} w^{m_1}(t) + C_8^{q^\star} M_\psi^{q^\star} \left(\int_0^t w^{m_2 p^\star / q^\star}(\tau) d\tau \right)^{\frac{q^\star}{p^\star}} \right] \leq 3^{q^\star - 1} t^\gamma \xi(t) \tag{25}$$

where $\xi(t)$ is the expression between brackets. By definition we see that

$$\xi(0) = C_6^{q^\star} M_\varphi^{q^\star} w^{m_1}(0) = C_6^{q^\star} M_\varphi^{q^\star} \left(3^{q^\star - 1} |b|^{q^\star} \right)^{m_1}$$

and

$$C_6^{q^\star} M_\varphi^{q^\star} w^{m_1}(t) \leq \xi(t) \text{ or } w(t) \leq \left(\frac{\xi(t)}{C_6^{q^\star} M_\varphi^{q^\star}} \right)^{\frac{1}{m_1}}. \tag{26}$$

Differentiating $\xi(t)$, we obtain

$$\xi'(t) = \gamma C_6^{q^\star} M_\varphi^{q^\star} w^{m_1}(t) + m_1 C_6^{q^\star} M_\varphi^{q^\star} w^{m_1 - 1}(t) w'(t)$$
$$+ \tfrac{q^\star}{p^\star} C_8^{q^\star} M_\psi^{q^\star} w^{m_2 p^\star / q^\star}(t) \left(\int_0^t w^{m_2 p^\star / q^\star}(\tau) d\tau \right)^{\frac{q^\star}{p^\star} - 1}$$
$$+ \gamma C_8^{q^\star} M_\psi^{q^\star} \left(\int_0^t w^{m_2 p^\star / q^\star}(\tau) d\tau \right)^{\frac{q^\star}{p^\star}}.$$

By our last observation (26) we may write

$$\xi'(t) \leq \gamma C_6^{q^\star} M_\varphi^{q^\star} \tfrac{\xi(t)}{C_6^{q^\star} M_\varphi^{q^\star}}$$
$$+ 3^{q^\star - 1} m_1 C_6^{q^\star} M_\varphi^{q^\star} \left(\tfrac{\xi(t)}{C_6^{q^\star} M_\varphi^{q^\star}} \right)^{\frac{m_1 - 1}{m_1}} t^\gamma \xi(t)$$
$$+ \tfrac{q^\star}{p^\star} \left(C_8^{q^\star} M_\psi^{q^\star} \right)^{\frac{p^\star}{q^\star}} \left(\tfrac{\xi(t)}{C_6^{q^\star} M_\varphi^{q^\star}} \right)^{\frac{m_2}{m_1}} \xi^{1 - \frac{p^\star}{q^\star}}(t) + \gamma \xi(t).$$

In the last inequality we also have used the fact that from (25) and the definition of $\xi(t)$ we have

$$C_8^{q^\star} M_\psi^{q^\star} \left(\int_0^t w^{m_2}(\tau) d\tau \right)^{\frac{q^\star}{p^\star}} \leq \xi(t).$$

Arranging terms we see that

$$\xi'(t) \le 2\gamma\xi(t) + 3^{q^\star-1}m_1\left(C_6^{q^\star}M_\varphi^{q^\star}\right)^{\frac{1}{m_1}}t^\gamma\xi^{2-\frac{1}{m_1}}(t)$$
$$+\tfrac{q^\star}{p^\star}\tfrac{C_8^{p^\star}M_\psi^{p^\star}}{(C_6M_\varphi)^{q^\star m_2/m_1}}\xi^{1+\frac{m_2}{m_1}-\frac{p^\star}{q^\star}}(t).$$

We can write this expression as

$$\xi'(t) \le 2\gamma\xi(t) + C_{13}t^\gamma\xi^{2-\frac{1}{m_1}}(t) + C_{14}\xi^{1+\frac{m_2}{m_1}-\frac{p^\star}{q^\star}}(t),$$

where $C_{13} = 3^{q^\star-1}m_1\left(C_6^{q^\star}M_\varphi^{q^\star}\right)^{\frac{1}{m_1}}$ and

$$C_{14} = q^\star C_8^{p^\star}M_\psi^{p^\star-q^\star m_2/m_1}/p^\star(C_6M_\varphi)^{q^\star m_2/m_1}.$$

We can also write

$$\xi'(t) \le \max\{2\gamma, C_{13}t^\gamma, C_{14}\}\left[\xi(t) + \xi^{2-\frac{1}{m_1}}(t) + \xi^{1+\frac{m_2}{m_1}-\frac{p^\star}{q^\star}}(t)\right].$$

If $T^\star$ is any value of t such that

$$\int_0^t \max\{2\gamma, C_{13}s^\gamma, C_{14}\}\,ds < \int_{\xi(0)}^\infty \frac{dz}{z\left(1+z^{1-\frac{1}{m_1}}+z^{\frac{m_2}{m_1}-\frac{p^\star}{q^\star}}\right)}$$

then for any $t < T^\star$ we have

$$\int_{\xi(0)}^{\xi(t)} \frac{dz}{z\left(1+z^{1-\frac{1}{m_1}}+z^{\frac{m_2}{m_1}-\frac{p^\star}{q^\star}}\right)} \le \int_0^t \max\{2\gamma, C_{13}s^\gamma, C_{14}\}\,ds$$
$$< \int_{\xi(0)}^\infty \frac{dz}{z\left(1+z^{1-\frac{1}{m_1}}+z^{\frac{m_2}{m_1}-\frac{p^\star}{q^\star}}\right)}.$$

This implies that $\xi(t)$ is bounded. Thus $|u(t)| \le Ct^{\alpha-1}$ on $(0, T^\star)$.

(ii) If $\gamma - 1 \le 0$, then from (24) it appears that

$$w'(t) \le 3^{q^\star-1}\gamma \left[C_6^{q^\star} M_\varphi^{q^\star} \int_0^t v^{q^\star m_1}(s)ds \right]$$

$$+ C_8^{q^\star} M_\psi^{q^\star} \int_0^t \left(\int_0^s v^{p^\star m_2}(\tau)d\tau \right)^{\frac{q^\star}{p^\star}} ds \Bigg]$$

$$+3^{q^\star-1}t^\gamma \left[C_6^{q^\star} M_\varphi^{q^\star} v^{q^\star m_1}(t) + C_8^{q^\star} M_\psi^{q^\star} \left(\int_0^t v^{p^\star m_2}(\tau)d\tau \right)^{\frac{q^\star}{p^\star}} \right]$$

or

$$w'(t) \le 3^{q^\star-1}t^\gamma \Bigg[\gamma C_6^{q^\star} M_\varphi^{q^\star} \int_0^t w^{m_1}(s)ds$$

$$+ \gamma C_8^{q^\star} M_\psi^{q^\star} \int_0^t \left(\int_0^s w^{m_2 p^\star/q^\star}(\tau)d\tau \right)^{\frac{q^\star}{p^\star}} ds \Bigg]$$

$$+ C_6^{q^\star} M_\varphi^{q^\star} w^{m_1}(t) + C_8^{q^\star} M_\psi^{q^\star} \left(\int_0^t w^{m_2 p^\star/q^\star}(\tau)d\tau \right)^{\frac{q^\star}{p^\star}} \Bigg].$$

As this is the same inequality as (25) we may proceed as in the case (i). The proof is complete■

Remark 1. *The proof in Theorem 2 works in fact for any* q *and* p *provided that* $1 + q(\alpha - 1) > 0$, $1 + p(\beta - 1) > 0$, $1 + q[\mu - (1-\alpha)m_1] > 0$ *and* $1 + p[\sigma - (1-\alpha)m_2] > 0$.

Remark 2. *If* $1 > \beta > \frac{\alpha}{\alpha+1}$*, it may be interesting to use the Hölder inequality with the same exponents* q *and* $q^\star$ *instead of* p *and* $p^\star$ *in (10). Note that in this case* $1 + q(\beta - 1) > 0$ *and Lemma 1 and Lemma 2 are applicable. One has to compare then the different constants, functions and integrals.*

Remark 3. *The assumption* $1+q[\mu - (1-\alpha)m_1] > 0$ *is equivalent to* $m_1\alpha^2 + \mu\alpha + (\mu - m_1 + 1) > 0$*. Apart from the following two cases it is easy to see that this condition is valid for any* α *such that* $0 < \alpha < 1$

(a) $\alpha \ne -\frac{\mu}{2m_1}$ *in case* $\mu = 2(m_1 - \sqrt{m_1})$ *or* $\mu = 2(m_1 + \sqrt{m_1})$.

(b) In case $\mu < m_1 - 1$*, then we must have* $\alpha > \alpha_0$ *where* α_0 *is the positive root of the equation* $m_1\alpha^2 + \mu\alpha + (\mu - m_1 + 1) = 0$.

We have the same remark concerning the exponent β.

The next theorem treats some other situations which are not in Remark 3. That is in case the condition $1 + q\left[\mu - (1-\alpha)m_1\right] > 0$ is not fulfilled. In fact, even the conditions in Remark 1 may be relaxed somewhat with a trade-off on the rate of decay. Indeed, if we require only that $1 + q\left[\mu - \delta m_1\right] > 0$ and $1 + q\left[\sigma - \delta m_2\right] > 0$ for some $\delta < 1 - \alpha$, then we can have polynomial decay but of order $t^{-\delta}$. This is shown in the next result.

Theorem 3. *Assume that the hypotheses of Theorem 1 hold. Then, the conclusions of Theorem 2 hold with the assumptions:* $1 + q(\alpha - 1) > 0$, $1 + q\left[\mu - \delta m_1\right] > 0$ *and* $1 + q\left[\sigma - \delta m_2\right] > 0$ *for some* $\delta < 1 - \alpha$. *The solutions decay as* $t^{-\delta}$ *away from zero.*

Proof:

Let us put $v(t) = t^{\delta}\left|u(t)\right|$ with $\delta < 1 - \alpha$. Then, multiplying the integral equation by t^{δ} we arrive at the new reference inequality

$$\begin{gathered} v(t) \leq \tfrac{|b|}{t^{1-\alpha}}t^{\delta} + \tfrac{t^{\delta}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\mu-\delta m_1}\varphi(s)v^{m_1}(s)ds \\ +\tfrac{t^{\delta}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s (s-z)^{\beta-1}z^{\sigma-\delta m_2}\psi(z)v^{m_2}(z)dz\right)ds. \end{gathered}$$

Next, we proceed as in the proof of Theorem 2. When we need to apply Lemma 1 or Lemma 2 we use the above conditions stated in the theorem. We find that solutions decay polynomially as $t^{-\delta}$ away from zero■

Remark 4. *The forms of the conditions in Theorem 2 may sometimes be simplified. For instance in (17), if* $1 + (m_2 - m_1)p^{\star}/q^{\star} < 0$ *then it suffices to require that*

$$\int_0^t R(s)ds < \int_{a_1}^{\infty} \frac{dz}{z^{m_1} + a_1^{1+(m_2-m_1)p^{\star}/q^{\star}}}.$$

In case $\max\left\{\frac{1}{2}, 1 - \frac{1}{m_1}\left(\mu + \frac{1}{2}\right), 1 - \frac{1}{m_2}\left(\sigma + \frac{1}{2}\right)\right\} < \alpha < 1$ and $\frac{1}{2} < \beta < 1$, we can improve the last theorem. In fact, the interval on which the polynomial decay holds may be extended to a larger one as $q^{\star} = \frac{1+\alpha}{\alpha}$ and $p^{\star} = \frac{1+\beta}{\beta}$ may be replaced by $q^{\star} = p^{\star} = 2$. Indeed, since $e^{2q^{\star}t} \leq e^{2\varepsilon t}$, $e^{2p^{\star}t} \leq e^{2\varepsilon t}$ and if for instance $|\alpha - \beta|$ is small enough so that $p^{\star}/q^{\star}$ is almost equal to 1, then the intervals in part (1) are larger. This is the goal of the next result.

Theorem 4. *Suppose that* $\max\left\{\frac{1}{2}, 1-\frac{1}{m_1}\left(\mu+\frac{1}{2}\right), 1-\frac{1}{m_2}\left(\sigma+\frac{1}{2}\right)\right\} < \alpha < 1$ *and* $\frac{1}{2} < \beta < 1$, *then the the results in Theorem 8 are valid with* $p = q = 2$ *everywhere in the statements.*

Proof: It suffices to observe that with the above conditions on α and β we have $1 + 2(\alpha - 1) > 0$, $1 + 2[\mu - (1-\alpha)m_1] > 0$, $1 + 2(\beta - 1) > 0$ and $1 + 2[\sigma - (1-\alpha)m_2] > 0$. Consequently, we can use in every occasion in the proof of Theorem 2 the Cauchy-Schwarz inequality instead of the Hölder inequality. We will find similar results as in Theorem 2 with $p = q = 2$■

Remark 5. *If* $m_1 > 2\mu + 1$, *then the interval* $\left(\frac{1}{2}, 1-\frac{1}{m_1}\left(\mu+\frac{1}{2}\right)\right)$ *is not empty and this case is not covered in Theorem 4. Same remark if* $m_2 > 2\sigma+1$. *We would have* $1+2[\mu-(1-\alpha)m_1] \leq 0$ *and* $1+2[\sigma-(1-\alpha)m_2] \leq 0$. *The problem is that Lemma 1 and Lemma 2 will no longer be applicable. Nevertheless, in case* $\frac{1}{2} < \alpha \leq 1-\frac{1}{m_1}\left(\mu+\frac{1}{2}\right)$ *and/or* $\frac{1}{2} < \alpha \leq 1-\frac{1}{m_2}\left(\sigma+\frac{1}{2}\right)$, *we can still prove polynomial decay. For the case **(1)** we multiply by* $s^d e^{\varepsilon s}.s^{-d}e^{-\varepsilon s}$ *for some* $d > -1/2$. *We get polynomial decay as long as, the equivalent of (17) in the proof of Theorem 4, holds with* $K_1' t^{2[\mu-(1-\alpha)m_1-d]}\varphi^2(t)$ *and/or* $\frac{1}{m_1}t^{2[\sigma-(1-\alpha)m_2-d]}\psi^2(t)$ *in the definition of* $R(t)$. *In case **(2)** we will have to assume the boundedness of these last two functions. One has then to compare the obtained interval with the one found in case* $q = \alpha + 1$.

The next theorem considers the case where we have a weak singular kernel of the form $(t-s)^{\beta-1}e^{-\delta(t-s)}$ with $\delta > 0$ in (G).

Theorem 5. *Assume that the hypotheses in Theorem 2 hold with (G) replaced by*

(G)′ $|g(t,s,u(s))| \leq (t-s)^{\beta-1}e^{-\delta(t-s)}s^{\sigma}\psi(s)|u|^{m_2}$, $\delta > 0$.

Then the conclusions of Theorem 8 remain true.

Proof: The proof is similar to the proof of Theorem 2. It is in fact simpler. In (9) we will not have to multiply by $e^{\varepsilon t}e^{-\varepsilon t}$ to desingularize the integral. Indeed, the presence of $e^{-\delta t}$ allows us to apply Lemma 2. We use the estimate

$$\int_0^t (t-s)^{\alpha-1}\left(\int_0^s (s-z)^{\beta-1}e^{-\delta(s-z)}z^{\sigma-(1-\alpha)m_2}\psi(z)v^{m_2}(z)dz\right)ds$$

$$\leq \int_0^t (t-s)^{\alpha-1}e^{-\delta s}\left(\int_0^s (s-z)^{q(\beta-1)}e^{-\varepsilon qz}z^{q[\sigma-(1-\alpha)m_2]}dz\right)^{\frac{1}{q}}$$

$$\times\left(\int_0^s e^{(\varepsilon+\delta)q^\star z}\psi^{q^\star}(z)v^{q^\star m_2}(z)dz\right)^{\frac{1}{q^\star}} ds$$

and by Lemma 2

$$\int_0^t (t-s)^{\alpha-1}\left(\int_0^s (s-z)^{\beta-1} e^{-\delta(s-z)} z^{\sigma-(1-\alpha)m_2}\psi(z)v^{m_2}(z)dz\right) ds$$
$$\leq C_2 \int_0^t (t-s)^{\alpha-1} s^{\beta-1} e^{-\delta s}\left(\int_0^s e^{(\varepsilon+\delta)q^\star z}\psi^{q^\star}(z)v^{q^\star m_2}(z)dz\right)^{\frac{1}{q^\star}} ds.$$

Again by Hölder inequality

$$\int_0^t (t-s)^{\alpha-1} s^{\beta-1} e^{-\delta s}\left(\int_0^s e^{(\varepsilon+\delta)q^\star z}\psi^{q^\star}(z)v^{q^\star m_2}(z)dz\right)^{\frac{1}{q^\star}} ds$$
$$\leq \left(\int_0^t (t-s)^{q(\alpha-1)} s^{q(\beta-1)} e^{-q\delta s} ds\right)^{\frac{1}{q}}$$
$$\times \left(\int_0^t \int_0^s e^{(\varepsilon+\delta)q^\star z}\psi^{q^\star}(z)v^{q^\star m_2}(z)dzds\right)^{\frac{1}{q^\star}}$$

and by Lemma 2, we have

$$\int_0^t (t-s)^{\alpha-1} s^{\beta-1} e^{-\delta s}\left(\int_0^s e^{(\varepsilon+\delta)q^\star z}\psi^{q^\star}(z)v^{q^\star m_2}(z)dz\right)^{\frac{1}{q^\star}} ds$$
$$\leq C_3 t^{\alpha-1}\left(\int_0^t \int_0^s e^{(\varepsilon+\delta)q^\star z}\psi^{q^\star}(z)v^{q^\star m_2}(z)dzds\right)^{\frac{1}{q^\star}}.$$

Whereas in the proof of part (2) of the theorem we use

$$\frac{1}{\Gamma(\alpha)}\int_0^s (s-z)^{\beta-1} e^{-\delta(s-z)} z^{\sigma-(1-\alpha)m_2}\psi(z)v^{m_2}(z)dz$$
$$\leq \int_0^t (t-s)^{\alpha-1} e^{-\delta s}\left(\int_0^s (s-z)^{q(\beta-1)} z^{q[\sigma-(1-\alpha)m_2]}dz\right)^{\frac{1}{q}}$$
$$\times \left(\int_0^s e^{\delta q^\star z}\psi^{q^\star}(z)v^{q^\star m_2}(z)dz\right)^{\frac{1}{q^\star}} ds.$$

By the Lemma 1 we see that

$$\frac{1}{\Gamma(\alpha)}\int_0^s (s-z)^{\beta-1} e^{-\delta(s-z)} z^{\sigma-(1-\alpha)m_2}\psi(z)v^{m_2}(z)dz$$
$$\leq \int_0^t (t-s)^{\alpha-1} e^{-\delta s} s^{\frac{1}{q}+\beta-1+\sigma-(1-\alpha)m_2}$$
$$\times \left(\int_0^s e^{\delta q^\star z}\psi^{q^\star}(z)v^{q^\star m_2}(z)dz\right)^{\frac{1}{q^\star}} ds.$$

Then we apply Lemma 2 and proceed as in the proof of Theorem 2. We will find that

$$R(t) = \max\left\{\tilde{K}_1 e^{\varepsilon q^\star t} \varphi^{q^\star}(t), \tilde{K}_2, \tilde{K}_3 e^{\varepsilon q^\star t} \psi^{q^\star}(t)\right\}$$

■

Remark 6. *As our proofs are based on some Gronwall type inequalities, it is therefore clear that the present results may be generalized to nonlinearities $h(u)$ (non-decreasing) other than polynomials (of the form $|u|^m$).*

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