JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 129, No. 3, pp. 349–372, June 2006 (© 2006)
DOI: 10.1007/s10957-006-9070-3

Enumeration of All the Extreme Equilibria in Game Theory: Bimatrix and Polymatrix Games[1]

C. AUDET,[2] S. BELHAIZA,[3] AND P. HANSEN[4]

Communicated by P. M. Pardalos

Published Online: 29 November 2006

Abstract. Bimatrix and polymatrix games are expressed as parametric linear 0–1 programs. This leads to an algorithm for the complete enumeration of their extreme equilibria, which is the first one proposed for polymatrix games. The algorithm computational experience is reported for two and three players on randomly generated games for sizes up to 14×14 and $13 \times 13 \times 13$.

Key Words. Bimatrix games, polymatrix games, Nash equilibria, extreme equilibria, enumeration.

1. Introduction

Bimatrix and polymatrix games are normal-form, nonzero-sum games with 2 and $n \geq 2$ players respectively. In the former case, two payoff matrices are given, one for each player; in the latter case, there are $n(n-1)/2$ pairs of matrices, one for each pair of players. In contrast with extensive-form or dynamic games, decision-making is static, simultaneous, and unique.

It follows from Nash's basic result (Ref. 1) that any bimatrix or polymatrix game has at least one equilibrium in mixed strategies, i.e., a solution such that no player has any advantage in changing his strategy without some other one doing so. But a matrix or polymatrix game may have infinitely many equilibria consisting of not necessarily disjoint polyhedra (Millham, Ref. 2). This suggests the enumeration of all the extreme equilibria, or in other words all the equilibria which correspond to the vertices of such polyhedra. In addition to listing and comparing them, this would help in the study of various refinements, a topic to be pursued in further research.

---

[1] The authors thank Bernhard von Stengel for constructive comments on the contents of this paper.

[2] Professor, GERAD and École Polytechnique de Montréal, Montréal, Québec, Canada.
[3] PhD Student, École Polytechnique de Montréal, Montréal, Québec, Canada.
[4] Professor, GERAD and HEC Montréal, Montréal, Québec, Canada.

The purpose of this paper is to present a mixed 0-1 linear programming formulation of bimatrix and polymatrix games and an algorithm that enumerates all their extreme equilibria. The paper is organized as follows.

Previous, new formulations and basic proprieties of bimatrix and polymatrix games are recalled in Sections 2 and 3. The topic of elimination of dominated strategies is addressed in Section 4. In Section 5, the $E\chi$-MIP algorithm enumerating all the extreme equilibria is presented and illustrated on some examples. Computational results on the $E\chi$-MIP algorithm and the EEE algorithm of Audet et al. (Ref. 3) on randomly generated bimatrix games are compared and discussed; also, $E\chi$-MIP results on randomly generated polymatrix games are presented in Section 6.

2. Bimatrix Games

2.1. Literature Review. A bimatrix game is a strategic confrontation of 2 players I and II. Both players can be political, social, or economic agents or institutions. Each player has a finite number of strategies, commonly called pure strategies. Player I has to choose between n pure strategies, while player II has to choose between m pure strategies. A bimatrix game is described through a pair of $n \times m$ payoff matrices A and B. The elements a_{ij} and b_{ij} of the matrices A and B are respectively the immediate payoffs of player I and player II when the first plays his ith strategy while the second simultaneously plays his jth strategy.

Each player attempts to maximize his own payoff by selecting a probability vector over his set of pure strategies. These vectors are combinations of pure strategies, called mixed strategies and represented by probability vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Hence, player I's payoff is $x^t Ay$ and player II's payoff is $x^t By$.

An equilibrium is defined as a situation where simultaneously player I maximizes his payoff given the strategic choice of player II and player II maximizes his payoff given the strategic choice of player I. An equilibrium point is a situation where neither player has an interest to unilaterally change his strategic choice.

As shown by Nash (Ref. 1), a bimatrix game possesses always at least one equilibrium point. Formally, an equilibrium point is a pair of strategies $(\hat{x}, \hat{y})$ such that both

$$\hat{x} \in X(\hat{y}) = \arg\max_x \{x^t A\hat{y} : x^t e_n = 1, x \geq 0\},$$
$$\hat{y} \in Y(\hat{x}) = \arg\max_y \left\{\hat{x}^t By : e_m^t y = 1, y \geq 0\right\},$$

where e_n and e_m are $n \times 1$ and $m \times 1$ column vectors with all elements equal to 1. Clearly, both $X(\hat{y})$ for fixed $\hat{y}$ and $Y(\hat{x})$ for fixed $\hat{x}$ are polytopes.

Mills (Ref. 4) and Mangasarian and Stone (Ref. 5) studied the optimality conditions of the preceding system to establish necessary and sufficient conditions of equilibrium. Introducing real-valued variables α and β, the duals of the above linear programs are

$$\min_{\alpha}\{\alpha : e_n\alpha \geq A\hat{y}\}, \quad \min_{\beta}\left\{\beta : \beta e_m^t \geq \hat{x}^t B\right\}.$$

Primal and dual feasibility conditions yield that a pair of strategies $(\hat{x}, \hat{y})$ is an equilibrium if there exist two scalars $\hat{\alpha}$ and $\hat{\beta}$ satisfying

$$(\hat{x}, \hat{\beta}) \in X \equiv \left\{(x, \beta) \in \mathbb{R}^{n+1} : x^t B \leq \beta e_m^t, x^t e_n = 1, x \geq 0\right\}, \tag{1}$$

$$(\hat{y}, \hat{\alpha}) \in Y \equiv \left\{(y, \alpha) \in \mathbb{R}^{m+1} : Ay \leq e_n\alpha, e_m^t y = 1, y \geq 0\right\}. \tag{2}$$

Moreover, from the duality theory of linear programming, the dual objective values $\hat{\alpha}$ and $\hat{\beta}$ are respectively equal to the primal payoffs of players I and II,

$$\hat{x}^t A\hat{y} = \hat{\alpha}, \quad \hat{x}^t B\hat{y} = \hat{\beta}.$$

The set E of all the equilibrium points is the union of a finite number of polytopes called maximal Nash subsets (Millham, Ref. 2). The set of extreme equilibria is the set of vertices of the maximal Nash subsets. As each equilibrium can be obtained by the convex combination of some extreme equilibria (Mangasarian, Ref. 6), the complete enumeration of the extreme equilibria leads to the complete identification of the set E (Vorobev, Ref. 7).

Keiding (Ref. 8) showed that a nondegenerate $n \times m$ bimatrix game has at most

$$K = \min\{\Phi(m, n+m), \Phi(n, m+n)\} - 1$$

extreme equilibria, where $\Phi(d, h)$ denotes the maximum number of vertices of a d-dimensional polytope with h facets,

$$\Phi(d, h) = \binom{h - \lfloor (d-1)/2 \rfloor - 1}{\lfloor d/2 \rfloor} + \binom{h - \lfloor d/2 \rfloor - 1}{\lfloor (d-1)/2 \rfloor}.$$

This upper bound on the maximum number of extreme equilibria is probably not tight. For the particular case of a nondegenerate $n \times n$ bimatrix game, Von Stengel (Ref. 9) shows that the maximum number of extreme equilibria has a lower bound equal to $\psi(n) - 1$ and an upper bound equal to $\omega(n) - 1$, where

$$\psi(n) = 0.949(2.414^n)/\sqrt{n}, \quad \omega(n) = 0.921(2.5981^n)/\sqrt{n}.$$

Few algorithms have been proposed to enumerate the extreme equilibria of bimatrix games. Early methods were designed to compute one equilibrium (Refs. 7, 10, 11). Others are based on the enumeration of the supports of strategies (Refs. 12, 13), where a support is a set of pure strategies that are assigned a positive probability. Yet others (Refs. 6, 14) enumerate all the extreme vertices of X and Y, then check for all pairs if the complementarity slackness conditions (1) and (2) hold. The state-of-the-art algorithm EEE (Ref. 3) is a selective enumeration method focusing on only vertices that satisfy all the complementarity conditions. This algorithm uses two linear programs with parametrized objective functions to explore a tree where each node corresponds to a pair of subproblems and a number of satisfied complementarity conditions. EEE has been tested on randomly generated bimatrix games of size up to 29×29, when both dimensions are equal, and of size up to 700×5, when the second dimension is fixed (Refs. 3, 15).

2.2. Mixed 0–1 Linear Formulation of a Bimatrix Game. Consider the primal feasible and bounded parametric linear problems

$$\max_{x \geq 0}\{x^t A y : x^t e_n = 1\},$$
$$\max_{y \geq 0}\left\{x^t B y : e_m^t y = 1\right\},$$

and their dual problems

$$\min_{\alpha}\{\alpha : e_n \alpha \geq A y\},$$
$$\min_{\beta}\left\{\beta : \beta e_m^t \geq x^t B\right\}.$$

For any primal and dual feasible solutions, the weak duality theorem yields

$$\alpha \geq x^t A y \quad \text{and} \quad \beta \geq x^t B y,$$

while for any primal and dual optimal solutions, the strong duality theorem ensures that

$$\alpha = x^t A y \quad \text{and} \quad \beta = x^t B y.$$

The complementary slackness conditions can be stated as

$$x_i(\alpha - A_{i\cdot} y) = 0, \qquad \text{for } i \in N = \{1, 2, \ldots, n\},$$
$$(\beta - x^t B_{\cdot j}) y_j = 0, \qquad \text{for } j \in M = \{1, 2, \ldots, m\}.$$

or in matrix form

$$x^t(e_n \alpha - A y) = 0, \tag{3a}$$
$$\left(\beta e_m^t - x^t B\right) y = 0. \tag{3b}$$

Pairs of solutions (x, α) or (y, β), feasible for the primal and the dual and satisfying the complementarity slackness conditions, are optimal. If this holds simultaneously for the programs of both players, the solution (x, y) is a Nash equilibrium.

Linearization of these complementary slackness conditions is made possible through the use of 0–1 variables [Júdice and Mitra (Ref. 16) and Audet et al.

(Ref. 17)],

$$(e_n\alpha - Ay) \le L_1 u, \tag{4a}$$
$$(\beta e_m - B^t x) \le L_2 v, \tag{4b}$$

$$x + u \le e_n, \tag{5a}$$
$$y + v \le e_m, \tag{5b}$$

$$u \in \{0, 1\}^n, \tag{6a}$$
$$u \in \{0, 1\}^m, \tag{6b}$$

where L_1 and L_2 are some large constants. Making sure that the constants L_1 and L_2 are large enough is often problematic. Fortunately, in our case, the following result shows how to obtain easily some valid values.

Proposition 2.1. Let

$$L_1 = \big(\max_{i\in N, j\in M} a_{ij}\big) - \big(\min_{i\in N, j\in M} a_{ij}\big),$$
$$L_2 = \big(\max_{i\in N, j\in M} b_{ij}\big) - \big(\min_{i\in N, j\in M} b_{ij}\big),$$

and at $\hat{x}, \hat{y}, \hat{\alpha}, \hat{\beta}$ be solutions of (4), (5), (6). Then, these solutions satisfy the complementary slackness conditions (3).

Proof. One can observe that

$$\hat{\alpha} = \hat{x}^t A\hat{y} \le \max_{i\in N} A_{i\cdot}\hat{y} \le \max_{i,j} a_{ij}$$
$$\min_{i\in N} A_{i\cdot} y \ge \min_{i,j} a_{ij}$$
$$\hat{\beta} = \hat{x}^t B\hat{y} \le \max_{j\in M} \hat{x} B_{\cdot j} \le \max_{i,j} b_{ij}$$
$$\min_{j\in M} \hat{x} B_{\cdot j} \ge \min_{i,j} b_{ij}.$$

where A_i and $(B_{\cdot j})^t$ are respectively the ith and the jth rows of A and B^t. Thus, the proof follows from these observations:

(i) if $u_i = 0$, then $x_i \le 1$ and $\alpha - A_{i\cdot}y = 0$, so $x_i(\alpha - A_{i\cdot}y) = 0$;
(ii) if $v_j = 0$, then $y_j \le 1$ and $\beta - x^t B_{\cdot j} = 0$, so $(\beta - x^t B_{\cdot j})y_j = 0$;
(iii) if $u_i = 1$, then $x_i = 0$ and $\alpha - A_{i\cdot}y \le L_1$, so $x_i(\alpha - A_{i\cdot}y) = 0$;
(iv) if $v_j = 1$, then $y_j = 0$ and $\beta - x^t B_{\cdot j} \le L_2$, so $(\beta - x^t B_{\cdot j})y_j = 0$.

□

Equations (4)–(6) allow us to model the question of finding the extreme equilibria through mixed integer programming.

Proposition 2.2. The set of bimatrix game equilibria is the set of pairs of mixed strategies $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ for which there exists vectors $(u, v) \in \{0, 1\}^n \times \{0, 1\}^m$ satisfying

$$\begin{aligned}
&x^t e_n = 1,\\
&e_m^t y = 1,\\
&x^t B - \beta e_n^t \leq 0,\\
&Ay - \alpha e_m \leq 0,\\
&x + u \leq e_n,\\
&y + v \leq e_m,\\
&(e_n\alpha - Ay) - L_1 u \leq 0,\\
&(\beta e_m - B^t x) - L_2 v \leq 0,\\
&x \geq 0, \quad y \geq 0,\\
&u \in \{0, 1\}^n, \quad v \in \{0, 1\}^m.
\end{aligned}$$

Proof. Any equilibrium $(\hat{x}, \hat{y})$ satisfies $(\hat{x}, \hat{\beta}) \in X$ and $(\hat{y}, \hat{\alpha}) \in Y$, as well as all the complementary slackness conditions. Hence, a point (x, y, α, β) has to fulfil only the conditions of Proposition 2.2 to be an equilibrium. □

Therefore, if one wishes to use the tools from mixed integer programming, one has the flexibility in selecting an objective function. Furthermore, solving a linear program by a simplex algorithm necessarily produces an extreme point of the domain.

Corollary 2.1. The complete enumeration of all the extreme equilibria of a bimatrix game can be done through the complete enumeration of all the extreme feasible solutions of a mixed 0–1 linear problem (i.e., extreme feasible solutions for each feasible 0–1 vector), defined by the constraints of Proposition 3.2, with any linear objective function.

For example, $f(\alpha, \beta) = \alpha + \beta$ or $f(\alpha, \beta) = 0$ could be used.

Any bimatrix game can then be expressed as a mixed 0–1 linear program with $2 + 3(n + m)$ constraints, $2 + n + m$ continuous variables, and $n + m$ binary variables. For any objective function, the $E\chi$-MIP algorithm presented in Section 6 will enumerate all the extreme equilibria through the complete enumeration of the extreme feasible solutions for feasible 0–1 vectors.

Example 2.1. Let A and B be the payoff matrices of a bimatrix game,

$$A = \begin{pmatrix} 3 & 2.5 & 5 \\ 4 & 0 & 2 \\ 2 & 3.5 & 1.5 \\ 4.5 & 0.5 & 5.5 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 4.5 & 2.5 \\ 2 & 3 & 1 \\ 1 & 1.5 & 3.5 \\ 2.5 & 3.5 & 4 \end{pmatrix}.$$

Using Proposition 2.2 and Corollary 2.1, this bimatrix game can be written as follows

$$\begin{aligned}
\min_{\alpha,\beta,x,y,u,v} \quad & f(\alpha, \beta), \\
\text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 = 1, \\
& y_1 + y_2 + y_3 = 1, \\
& x_i + u_i \leq 1, \quad i = 1, 2, 3, 4, \\
& y_j + v_j \leq 1, \quad j = 1, 2, 3, \\
& -\alpha + 3y_1 + 2.5y_2 + 5y_3 \leq 0, \\
& -\alpha + 4y_1 + 2y_3 \leq 0, \\
& -\alpha + 2y_1 + 3.5y_2 + 1.5y_3 \leq 0, \\
& -\alpha + 4.5y_1 + 0.5y_2 + 5.5y_3 \leq 0, \\
& -\beta + 5x_1 + 2x_2 + x_3 + 2.5x_4 \leq 0, \\
& -\beta + 4.5x_1 + 3x_2 + 1.5x_3 + 3.5x_4 \leq 0, \\
& -\beta + 2.5x_1 + x_2 + 3.5x_3 + 4x_4 \leq 0, \\
& \alpha - 3y_1 - 2.5y_2 - 5y_3 - 5.5u_1 \leq 0, \\
& \alpha - 4y_1 - 2y_3 - 5.5u_2 \leq 0, \\
& \alpha - 2y_1 - 3.5y_2 - 1.5y_3 - 5.5u_3 \leq 0, \\
& \alpha - 4.5y_1 - 0.5y_2 - 5.5y_3 - 5.5u_4 \leq 0, \\
& \beta - 5x_1 - 2x_2 - x_3 - 2.5x_4 - 4v_1 \leq 0, \\
& \beta - 4.5x_1 - 3x_2 - 1.5x_3 - 3.5x_4 - 4v_2 \leq 0, \\
& \beta - 2.5x_1 - x_2 - 3.5x_3 - 4x_4 - 4v_3 \leq 0, \\
& x \geq 0, \quad y \geq 0, \\
& u \in \{0, 1\}^4, \quad v \in \{0, 1\}^3.
\end{aligned}$$

3. Polymatrix Games

3.1. Literature Review. The strategic confrontation of n players, $n \geq 2$, in a normal and noncooperative context is a polymatrix game if the payoffs are sums of the values for each player and all other ones pairwise. Let $N = \{1, \ldots, n\}$ be the set of all players and set each player $i \in N$ have a finite set of pure strategies $S_i = \{s_i^1, \ldots, s_i^{m_i}\}$ with $|S_i| = m_i$.

If player i chooses his strategy s_i^k and if player j chooses his strategy s_j^l a partial payoff $a_{ij}(s_i^k, s_j^l)$ is assigned for player i. So, for any pure strategic choice $(s_1^k, \ldots, s_n^l)$ of the n players, the overall payoff of player i at the end of the game is

$$A_i\left(s_1^k, \ldots, s_n^l\right) = \sum_{j \neq i} a_{ij}\left(s_i^k, s_j^l\right).$$

The $m_i \times m_j$ matrix $A_{ij} = (a_{ij}^{kl})$ is defined as player i's partial payoff matrix relative to player j's strategic decisions. Thus, player i's payoff relative to player j's decisions is not correlated with any of the remaining players' choices.

As in bimatrix games, in a polymatrix game each player i attempts to maximize his own overall payoff by selecting a probability vector X_i over his set of pure strategies. The mixed strategy vector X_i is such that

$$(X_i)^T = \left(x_i^1, \ldots, x_i^{m_i}\right),$$

where for all $k \in \{1, \ldots, m_i\}$, x_i^k is the relative frequency or probability with which player i plays his strategy $s_i^k \in S_i$. So, player i's mixed strategies belong to the set

$$\tilde{S}_i = \{X_i : \; e^t X_i = 1, \; X_i \geq 0\}.$$

The overall payoff of player i at the end of a polymatrix game is

$$\begin{aligned} R_i(X) &= (X_i)^T \sum_{j \neq i} A_{ij} X_j \\ &= \sum_{j \neq i} \sum_{k=1}^{m_i} \sum_{l=1}^{m_j} a_{ij}^{kl} x_i^k x_j^l. \end{aligned}$$

A n-tuple $X^* = (X_1^*, \ldots, X_n^*)$ of mixed strategies is called a Nash equilibrium in a polymatrix game if and only if, for any other n-tuple $X = (X_1^*, \ldots, X_{i-1}^*, X_i, X_{i+1}^*, \ldots, X_n^*)$, the following inequality is satisfied:

$$(X_i^*)^T \sum_{j \neq i} A_{ij} X_j^* \geq (X_i)^T \sum_{j \neq i} A_{ij} X_j^*, \quad \text{for } i \in N; \tag{7}$$

i.e., player i's payoff relative to all other players is simultaneously maximized for $i \in N$. Again, it follows that a polymatrix game has at least one equilibrium (Ref. 1). For a set of mixed strategies $X_1, \ldots, X_n$ and for $i \in N$, let

$$\alpha_i = (X_i^*)^T \sum_{j \neq i} A_{ij} X_j. \tag{8}$$

Consider an $m_i \times 1$ column vector e_r^i with its rth element equal to 1 and all other elements equal to 0 and use the fact that inequality (7) holds for all X_i, even for $X_i = e_r^i$, $r = 1, \ldots, m_i$. Howson (Ref. 18) showed that (8) holds only if

$$\alpha_i e^i \geq \sum_{j \neq i} A_{ij} X_j, \quad \text{for } i \in N, \tag{9}$$

where e^i is an $m_i \times 1$ column vector with all elements equal to 1. This leads to the statement

$$(X_i^*)^T \alpha_i e \geq (X_i^*)^T \sum_{j \neq i} A_{ij} X_j \Rightarrow (X_i^*)^T \left(\sum_{j \neq i} A_{ij} X_j^* - \alpha_i e \right) = 0. \tag{10}$$

This last result, due to Quintas (Ref. 19), implies that each α_i, $i \in N$, corresponds to the overall payoff of player i at an equilibrium. The relation (10) is a first complementarity condition. Similarly, define

$$Y_i = \alpha_i e - \sum_{j \neq i} A_{ij} X_j, \quad \mu_i = e^T X_i^* - 1, \quad \text{for } i \in N.$$

Using (8)–(10) and the fact that X_i^* is a probability vector, the following conditions can be stated:

$$X_i \geq 0, \quad Y_i \geq 0, \quad (X_i^*)^T Y_i = 0, \quad \text{for } i \in N, \tag{11}$$

$$\mu_i \geq 0, \quad \alpha_i \geq 0, \quad \mu_i \alpha_i = 0, \quad \text{for } i \in N. \tag{12}$$

Then, computing polymatrix game equilibria is equivalent to seeking solutions for the linear complementarity problem (Refs. 20, 21)

$$\text{(LCP)} \quad Z \geq 0, \quad W = Q + MZ \geq 0, \quad Z^T W = 0,$$

where Q and M are well chosen, while Z and W are the decision variables.

Again, the set E of all the equilibrium points is the union of a finite number of polytopes called maximal Nash subsets and the set of the extreme equilibria is the set of vertices of these maximal Nash subsets.

For some class of matrices Q and M, the linear complementarity problem (LCP) has been solved by Cottle and Dantzig (Refs. 20, 22) and by Lemke and Howson (Refs. 11, 23). However, polymatrix games data do not belong to this class of matrices. Yanovskaya (Ref. 21) was the first author to compute polymatrix equilibria by solving a LCP. The problem was solved using the complementary pivoting method. Howson (Ref. 18), Eaves (Ref. 24), and Howson and Rosenthal (Ref. 25) adopted the same approach to solve the LCP. However, the enumeration of all the polymatrix game extreme equilibria does not appear to have been done yet for a number of players n exceeding 2.

3.2. Mixed 0–1 Linear Formulation of a Polymatrix Game. This section presents a mixed 0–1 linear formulation to be used for the complete enumeration of the extreme equilibria of bimatrix and polymatrix games. The order of the presentation is very similar to that of Section 2.2 for bimatrix games.

Considering player i's primal multiparametric linear program in a polymatrix game,

$$\begin{aligned} \max_{X_i} \quad & X_i \sum_{j \neq i} A_{ij} X_j, \\ \text{s.t} \quad & e_{m_i}^t X_i = 1, \\ & X_i \geq 0, \end{aligned}$$

and player i's dual problem

$$\begin{aligned} \min_{\alpha_i} \quad & \alpha_i, \\ \text{s.t} \quad & \alpha_i e^i \geq \sum_{j \neq i} A_{ij} X_j, \end{aligned}$$

the linearization of all the complementary slackness conditions (10) can be done again using binary variables. This leads to a mixed 0–1 linear formulation of a polymatrix game.

For $i \in N$, the complementary slackness conditions are written as

$$(X_i)^T \left(\alpha_i e^i - \sum_{j=1, j \neq i}^{n} A_{ij} X_j \right) = 0, \tag{13}$$

$$\Longleftrightarrow \begin{cases} \alpha_i e^i - \displaystyle\sum_{j=1, j \neq i}^{n} A_{ij} X_j - L_i U_i \leq 0, & \text{(14a)} \\ X_i + U_i \leq e. & \text{(14b)} \end{cases}$$

Selection of L_i can be done again by simple arithmetic. L_i is of the same order of magnitude as the input.

Proposition 3.1. Let $L_i = \sum_{j=1, j \neq i}^{n} \Gamma_{ij}$, for $i \in N$, where $\Gamma_{ij} = a_{ij}^{\max} - a_{ij}^{\min}$ is the difference between the largest and the smallest elements of A_{ij}, i.e. $a_{ij}^{\max} = \max_{k \in m_i, l \in m_j} a_{ij}^{kl}$ and $a_{ij}^{\min} = \min_{k \in m_i, l \in m_j} a_{ij}^{kl}$. The solutions $\hat{X} = (\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n)$ and $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$ of (14) satisfy the complementary slackness conditions (13).

Proof. For $i \in N$, one can observe that

$$\sum_{j=1, j\neq i}^{n} a_{i,j}^{\min} \leq \alpha_i \leq \sum_{j=1, j\neq i}^{n} a_{i,j}^{\max},$$
$$\sum_{j=1, j\neq i}^{n} a_{i,j}^{\min} \leq \sum_{j=1, j\neq i}^{n} A_{ij} X_j \leq \sum_{j=1, j\neq i}^{n} a_{i,j}^{\max}.$$

Therefore,

$$\left(\alpha_i e^i - \sum_{j=1, j\neq i}^{n} A_{ij} X_j\right) \leq \left(\sum_{j=1, j\neq i}^{n} \Gamma_{i,j}\right) U_i, \quad \text{for } i \in N.$$

Thus, choosing L_i as in Proposition 3.1 makes (13) hold. $\square$

The question of finding the extreme equilibria for a polymatrix game can be stated through mixed integer programming.

Proposition 3.2. The set of polymatrix game equilibria is the set of mixed strategies vectors $(X_1, X_2, \ldots, X_n) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_n}$ such that, for $i \in N$,

$$X_i^t e = 1,$$
$$X_i + U_i \leq e,$$
$$0 \leq \alpha_i e^i - \sum_{j=1, j\neq i}^{n} A_{ij} X_j \leq L_i U_i,$$
$$X_i \geq 0, \quad U_i \in \{0, 1\}^{m_i}.$$

Proof. Any polymatrix game equilibrium (X, α) is such that, for each player $i \in N$,

$$(X_i, \alpha_i) \in (X, \alpha) \equiv \left\{(X_i, \alpha_i) \in \mathbb{R}^{m_i+1} : X_i^t e = 1, \alpha_i e^i \geq \sum_{j=1, j\neq i}^{n} A_{ij} X_j, X_i \geq 0\right\},$$

and with all complementary slackness conditions fulfilled,

$$\alpha_i e^i - \sum_{j=1, j\neq i}^{n} A_{ij} X_j - L_i U_i \leq 0, \quad X_i + U_i \leq e.$$

$\square$

Moreover, the use of mixed integer programming allows flexibility in the selection of an objective function. Again, one can observe that solving a linear program by a simplex algorithm necessarily produces an extreme point of the domain.

Corollary 3.1. The enumeration of all the extreme equilibria of a polymatrix game can be done through the complete enumeration of all the extreme feasible solutions of a mixed 0–1 linear program, subject to the constraints appearing in Proposition 3.2, with a linear objective function.

For example,

$$f(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{i=1}^{n} \alpha_i \quad \text{or} \quad f(\alpha_1, \alpha_2 \ldots, \alpha_n) = 0$$

can be used. Hence, to each player i, a binary column U_i and a mixed strategy vector X_i are associated, both with the same dimension m_i, recalling that m_i is the number of pure strategies of player i.

For a polymatrix game, where N is the number of players and $M = \sum_{i=1}^{n} m_i$ is the overall number of pure strategies, the preceding mixed 0–1 linear formulation has $N + 3M$ constraints.

Example 3.1. Let A, B, C be the payoff matrices of a 3-player polymatrix game,

$$A = \left(\begin{array}{cc|cc} 2 & 0 & 3 & 5 \\ 3 & 4 & 1 & 4 \end{array}\right), \quad B = \left(\begin{array}{cc|cc} 0 & 5 & 3 & 2 \\ 1 & 4 & 6 & 7 \end{array}\right), \quad C = \left(\begin{array}{cc|cc} 4 & 1 & 6 & 0 \\ 1 & 2 & 5 & 3 \end{array}\right).$$

Using Proposition 3.2 and Corollary 3.1, this polymatrix game could be written as

$$\begin{aligned}
\min_{\alpha,\beta,\gamma,x,y,z,u,v,w} \quad & f(\alpha, \beta, \gamma), \\
\text{s.t.} \quad & x_1 + x_2 = 1, \\
& y_1 + y_2 = 1, \\
& z_1 + z_2 = 1, \\
& x_i + u_i \leq 1, \quad i = 1, 2, \\
& y_j + v_j \leq 1, \quad j = 1, 2, \\
& z_k + w_k \leq 1, \quad k = 1, 2, \\
& -\alpha + 2y_1 + 3z_1 + 5z_2 \leq 0, \\
& -\alpha + 3y_1 + 4y_2 + z_1 + 4z_2 \leq 0, \\
& -\beta + 5x_2 + 3z_1 + 2z_2 \leq 0,
\end{aligned}$$

$$
\begin{aligned}
&-\beta + x_1 + 4x_2 + 6z_1 + 7z_2 \le 0,\\
&-\gamma + 4x_1 + x_2 + 6y_1 \le 0,\\
&-\gamma + x_1 + 2x_2 + 5y_1 + 3y_2 \le 0,\\
&\alpha - 2y_1 - 3z_1 - 5z_2 - 8u_1 \le 0,\\
&\alpha - 3y_1 - 4y_2 - z_1 - 4z_2 - 8u_2 \le 0,\\
&\beta - 5x_2 - 3z_1 - 2z_2 - 10v_1 \le 0,\\
&\beta - x_1 - 4x_2 - 6z_1 - 7z_2 - 10v_2 \le 0,\\
&\gamma - 4x_1 - x_2 - 6y_1 - 9w_1 \le 0,\\
&\gamma - x_1 - 2x_2 - 5y_1 - 3y_2 - 9w_2 \le 0,\\
&x \ge 0,\ y \ge 0,\ z \ge 0,\\
&u,\ v,\ w \text{ binary vectors.}
\end{aligned}
$$

4. Elimination of Dominated Strategies

A polymatrix game strategy is dominated, for a given player, if its payoff is less than or equal to the payoff of a convex combination of all his remaining strategies. However, two kinds of dominated strategies are observed; strongly dominated strategies and weakly dominated strategies.

4.1. Strongly Dominated Strategies. Elimination of strongly dominated strategies offers the incentive of reducing the polymatrix game size before any process of enumeration of the extreme equilibria is executed.

Definition 4.1. For a given player i, a strategy is strongly dominated if and only if it does not belong to the set of best responses to all the remaining players strategic choices.

In a polymatrix game, if the ℓth strategy of player i is strongly dominated, there exists a vector of positive scalars

$$\lambda^t = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_{m_i})^t,$$

such that, for any combination of all the remaining players strategic choices $S - \{S_i\}$, the following relations hold:

$$\sum_{q=1,\ q\neq\ell}^{m_i} \lambda_q = 1,$$

$$\sum_{q=1,\ q\neq\ell}^{m_i}\left(\sum_{j=1,\ j\neq i}^{n}\lambda_q a_{ij}^{qh}\right) > \sum_{j=1,\ j\neq i}^{n} a_{ij}^{\ell h},\quad \forall h\in S_j,\quad \forall S_j\in S\backslash\{S_i\}.$$

Example 4.1. Consider the following 3-player polymatrix game:

$$A=\left(\begin{array}{cc|cc}2&0&3&5\\3&4&1&4\end{array}\right),\quad B=\left(\begin{array}{cc|cc}0&5&3&2\\1&4&6&7\end{array}\right),\quad C=\left(\begin{array}{cc|cc}4&1&6&0\\1&2&5&3\end{array}\right).$$

The first strategy of player II is strongly dominated by his second strategy, where if $\lambda_2=1$ for player II, then

$$\begin{array}{ll}\lambda_2\times(1+6)=7>0+3, & \lambda_2\times(1+7)=8>0+2,\\ \lambda_2\times(4+6)=10>5+3, & \lambda_2\times(4+7)=11>5+2.\end{array}$$

Elimination of this strategy yields a $2\times 1\times 2$ polymatrix game,

$$A'=\left(\begin{array}{c|cc}0&3&5\\4&1&4\end{array}\right),\quad B'=\left(\begin{array}{cc|cc}1&4&6&7\end{array}\right),\quad C'=\left(\begin{array}{cc|c}4&1&0\\1&2&3\end{array}\right).$$

The first strategy of player I is also a strongly dominated strategy. Recursively, let $\lambda_2'=1$ for player I. Then,

$$\lambda_2'\times(4+1)=5>0+3,\qquad \lambda_2'\times(4+4)=8>0+5.$$

Elimination of this strategy yields a $1\times 1\times 2$ polymatrix game,

$$A''=\left(\begin{array}{c|cc}4&1&4\end{array}\right),\quad B''=\left(\begin{array}{c|cc}4&6&7\end{array}\right),\quad C''=\left(\begin{array}{c|c}1&0\\2&3\end{array}\right).$$

Finally, let $\lambda_2''=1$ for player III, then

$$\lambda_2'\times(2+3)=5>1+0.$$

Even player III's first strategy is then a strongly dominated strategy. Elimination of this strategy yields a $1\times 1\times 1$ polymatrix game,

$$A'''=(4\,|\,4),\quad B'''=(4\,|\,7),\quad C'''=(2\,|\,3).$$

Consequently, only one extreme equilibrium can be found for this polymatrix game. It is important to notice that the second-time and third-time eliminated strategies were not strongly dominated in the original game.

Elimination of strongly dominated strategies reduces the size of the matrices. This reduces the computational work for the extreme equilibria enumeration of a polymatrix game.

Proposition 4.1. For a polymatrix game, recursive identification of all strongly dominated strategies requires at most $M^2 = (\sum_{i=1}^{n} m_i)^2$ iterations.

Proof. The maximum number of iterations required for the complete identification of a polymatrix game's strongly dominated strategies is equal to the sum of all the iterations if only one strongly dominated strategy is identified at each iteration. This number of iterations is the sum of arithmetic series terms,

$$M + (M-1) + (M-2) + \cdots + N \simeq M(M+1)/2.$$

Therefore, the maximum number of iterations required for the complete identification of all strongly dominated strategies is in the range

$$M^2 = \left(\sum_{i=1}^{n} m_i\right)^2.$$

□

All the extreme equilibria of the original game will be enumerated in the residual game (i.e., without strongly dominated strategies), because a rational player will never have an incentive to play a strongly dominated strategy (Myerson, Ref. 26).

4.2. Weakly Dominated Strategies.

Definition 4.2. Considering a polymatrix game, the ℓth strategy of a given player i is weakly dominated if there exists a vector of positive scalars $\lambda^t = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots \lambda_{m_i})^t$, such that, for any combination of all the remaining players strategic choices $S - \{S_i\}$, the following relations hold:

$$\sum_{q=1, q\neq l}^{m_i} \lambda_q = 1,$$

$$\sum_{q=1, q\neq \ell}^{m_i} \left(\sum_{j=1, j\neq i}^{n} \lambda_q a_{ij}^{qh} \right) \geq \sum_{j=1, j\neq i}^{n} a_{ij}^{\ell h}, \quad \text{for each } h \in S_j \text{ and } S_j \in S \backslash \{S_i\}.$$

A rational player would be indifferent between choosing a weakly dominated strategy or a convex combination of all his remaining strategies (Myerson, Ref. 26). Thus, elimination of weakly dominated strategies is risky and offers no warranties on the complete enumeration of the extreme equilibria of the original game.

5. Algorithm $E\chi$-MIP

Algorithm $E\chi$-MIP for the enumeration of all the extreme equilibria in bimatrix and polymatrix games is presented next, its rules are stated, and a proof of its validity is given. The $E\chi$-MIP algorithm is designed to generate a branching tree where, at each node, a dichotomous branching over one of the binary variables is done. The tree generated is composed of a principal tree and many secondary trees.

The principal tree is designed to detect all binary variables combinations involved in one or more extreme equilibria. A secondary tree is generated from every principal tree node offering a feasible solution, i.e. an extreme equilibrium. Hence, each binary variable combination involved in an extreme equilibrium is completely explored in order to find all the extreme equilibria that could be obtained from this combination. To avoid the repetitive exploration of the same binary variables combination, an eliminating constraint is added once the secondary exploration is achieved. These constraints are often redundant and could be eliminated in part to reduce the problem size (see the example below).

At each node of the principal tree, a mixed 0–1 linear program is solved. This program is composed of the original program with some binary variables fixing constraints and some combination of binary variables eliminating constraints.

Each secondary tree node represents the solution of a linear problem, composed of the original problem with some binary variable fixing constraints, some combination of binary variables eliminating constraints, and some continuous variables fixing constraints. The algorithm can now be formally stated.

Algorithm $E\chi$-MIP.

Step 1. Initialization. Let
$P =$ initial mixed 0–1 linear problem,
$X =$ set of P's continuous variables,
$U =$ set of P's binary variables,
$E = \emptyset$, set of extreme equilibria,
$N = 0$, depth level in the principal tree,
R, principal tree root node,
C, current node,
x_i^q, continuous variable associated to player $i, i = 1, \ldots, n$, qth strategy,
u_i^q, binary variable associated to player $i, i = 1, \ldots, n$, qth strategy.
Take $C = R$ and go to Step 2.

Step 2. Solving and Memorizing. If $N \leq |X|$, solve the current node problem. If the problem is infeasible go to Step 4. Else, let $\hat{e}$ be the solution obtained; if $\hat{e} \neq E$, add $\hat{e}$ to E. Go to Step 3.

Step 3. Secondary Branching. If the current node C belongs to the principal tree, fix the binary variables vector $\hat{u}$, $\hat{u} \in U$, at its value in $\hat{e}$, $\forall x_i^q \in X$ such that $\hat{x}_i^q > 0$: Add the branch $x_i^q = 0$ and go to Step 2.

Step 4. Principal Branching. If the current node belongs to the principal tree, no extreme equilibria can be found from this node or its sons. Else, return to the father node in the principal tree and add a constraint to eliminate the combination of binary variables $\hat{u}$ found in $\hat{e}$.

Choose a binary variable $u_i^q \in U$, on which no branching was done in the preceding nodes and such that its continuous variable x_i^q is the closest to 0.5 (choose arbitrarily in case of equality).

Let $p = N + 1$. If $p \leq |X|$, set $N = p$. Then, add the branch $u_i^q = 0$; if in $\hat{u}$, $\hat{u}_i^q = 1$, delete $\hat{u}$'s eliminating constraint, go to Step 2; add the branch $u_i^q = 1$; if in $\hat{u}$, $\hat{u}_i^q = 0$, delete $\hat{u}$'s eliminating constraint, go to Step 2. Else, go to Step 5.

Step 5. End. The set $|E|$ contains all the extreme of the game.

Theorem 5.1. Algorithm $E\chi$-MIP enumerates in finite time all the extreme equilibrium points of a bimatrix or a polymatrix game.

Proof. By principal branching, $E\chi$-MIP explores all the binary variables combinations involved in one or more extreme equilibria, due to Propositions 3.2 and 5.2, and by

(i) adding eliminating constraints for already explored combinations,
(ii) branching on binary variables till the maximum depth equals the overall number of strategies involved in the game.

By secondary branching, $E\chi$-MIP enumerates all the extreme equilibria that can be obtained from a binary variables combination $\hat{u}$ by

(iii) fixing the combination of binary variables $\hat{u}$,
(iv) adding branches $x_i^q = 0$.

Therefore, this branching enumerates from $\hat{u}$ all the extreme equilibria where some complementary slackness conditions are satisfied from both sides: $x_i^q > 0$ and $u_i^q = 0$. Thus, after branching,

$$x_i^q = 0, \quad u_i^q = 0, \quad \left(\alpha_i - \sum_{j=1, j\neq i}^{n} A_{ij}^q X_j\right) = 0$$

$$\Rightarrow x_i^q \left(\alpha_i - \sum_{j=1, j\neq i}^{n} A_{ij}^q X_j\right) = 0.$$

The algorithm explores all the possible ways to satisfy the complementary slackness conditions and, if $\hat{e}$ is an extreme equilibrium, there exists necessarily a path in the tree generated by $E\chi$-MIP leading to $\hat{e}$. $\square$

Example 5.1. Consider a 3-player $3 \times 3 \times 3$ polymatrix game, where A, B, C are the payoff matrices of players I, II, III,

$$A = \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 2 & 3 & 2.5 \\ 2 & -1 & -1 & -1 & 0 & 2 \\ 3 & -1.5 & -1 & -1 & 2 & 1 \end{array}\right),$$

$$B = \left(\begin{array}{ccc|ccc} -1 & 2 & 1 & -2 & 3 & 1 \\ 3 & 0 & 3.5 & 1 & -1 & -1 \\ 3 & 3.5 & 3 & 2 & 1 & -2 \end{array}\right),$$

$$C = \left(\begin{array}{ccc|ccc} -3 & -1 & 1 & 1 & 2 & -3 \\ 4 & 1 & 4 & 2 & 1 & 2 \\ 1 & 2 & 2.2 & 3 & 2 & 4 \end{array}\right).$$

Algorithm $E\chi$-MIP enumerates 7 extreme equilibria for this game. Figure 1 illustrates a subset of the 107 nodes generated by $E\chi$-MIP.

Solving the original mixed 0–1 linear problem gives the first extreme equilibrium at the root node,

$$\begin{aligned} &\alpha = 5, && \beta = 4, && \gamma = 6, \\ &X^t = (0, 0, 1), && Y^t = (1, 0, 0), && Z = (0, 1, 0), \\ &U^t = (1, 1, 0), && V^t = (0, 1, 1), && W = (1, 0, 1). \end{aligned}$$

Fixing the binary variables combination creates a secondary branching tree over nonequal to zero continuous variables. No feasible solution is found on this tree and the algorithm returns to the principal tree to add a constraint in order to eliminate the binary variables combination found in Equilibrium 1,

$$\begin{aligned} &U^t = (1, 1, 0), \quad V^t = (0, 1, 1), \quad W^t = (1, 0, 1), \\ &u_1 + u_2 + (1 - u_3) + (1 - v_1) + v_2 + v_3 + w_1 + (1 - w_2) + w_3 \leq 8 \\ &\Longleftrightarrow u_1 + u_2 - u_3 - v_1 + v_2 + v_3 + w_1 - w_2 + w_3 \leq 5. \end{aligned}$$

However, this constraint is already satisfied at the node created by the $w_3 = 0$ branching. Equilibrium 2 is found at this node:

$$\begin{aligned} &\alpha = 4, && \beta = 3, && \gamma = 5, \\ &X^t = (0, 1, 0), && Y^t = (1, 0, 0), && Z = (0, 0, 1), \\ &U^t = (1, 0, 0), && V^t = (0, 1, 1), && W = (1, 1, 0). \end{aligned}$$

Table 1 summarizes the 7 extreme equilibria enumerated in this game.

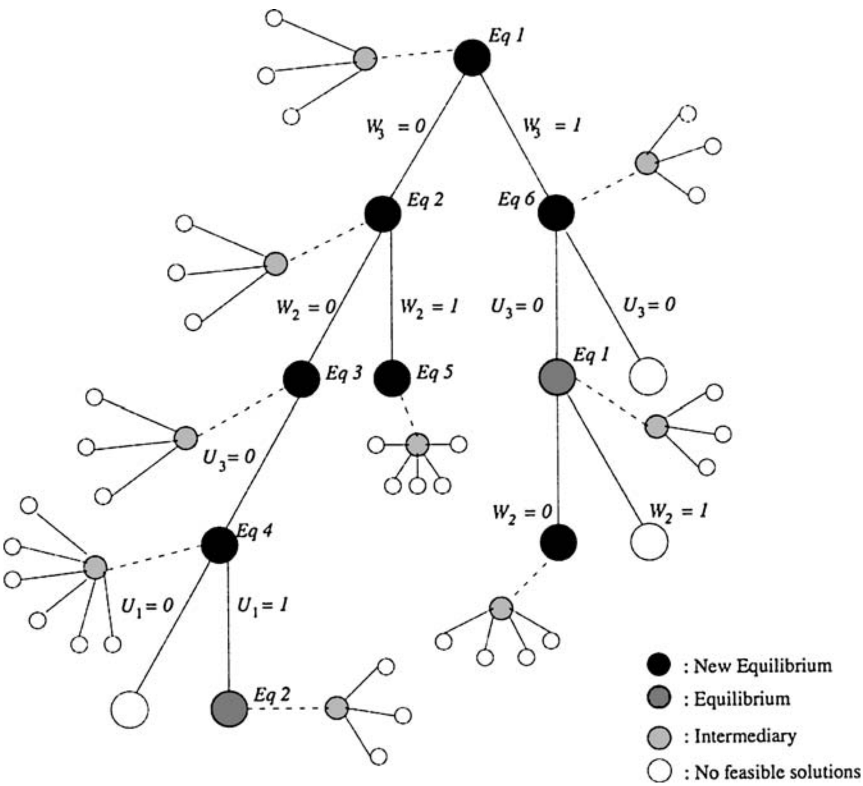

Fig. 1. Example 5.1, size 3 × 3 × 3.

6. Numerical Results

Two versions of the $E\chi$-MIP algorithm were implemented, one for bimatrix games and one for 3-player polymatrix games. The algorithm is coded in C++

Table 1. Example 5.1, size 3 × 3 × 3.

Eq.	α	β	γ	X^t	Y^t	Z^t	U^t	V^t	W^t
1	5	4	6	(0, 0, 1)	(1, 0, 0)	(0, 1, 0)	(1, 1, 0)	(0, 1, 1)	(1, 0, 1)
2	4	3	5	(0, 1, 0)	(1, 0, 0)	(0, 0, 1)	(1, 0, 0)	(0, 1, 1)	(1, 1, 0)
3	4	2.286	5.143	(0, 0.29, 0.71)	(1, 0, 0)	(0, 0, 1)	(1, 0, 0)	(0, 1, 1)	(1, 0, 0)
4	3	1.487	4.713	(0.56, 0.43, 0.01)	(0.73, 0, 0.27)	(0, 0.09, 0.91)	(0, 0, 0)	(0, 1, 0)	(1, 0, 0)
5	2.5	1.2	4.9	(0.6, 0.4, 0)	(0.5, 0, 0.5)	(0, 0, 1)	(0, 0, 1)	(0, 1, 0)	(1, 1, 0)
6	2	4	6	(1, 0, 0)	(0, 0, 1)	(0, 1, 0)	(0, 1, 1)	(1, 1, 0)	(1, 0, 1)
7	3	4	6	(0, 0, 1)	(0.5, 0, 0.5)	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)	(1, 0, 1)

Table 2. E_χ-MIP Algorithm vs EEE, $n = m$.

		d = 0.12						d = 0.25					
$n = m$		E_χ-MIP			EEE			E_χ-MIP			EEE		
n		NDS	Time(sec)	E	NDS	Time(sec)	E	NDS	Time(sec)	E	NDS	Time(sec)	E
5	μ	1432.6	1.93	18.2	8037.2	2.22	18.2	655.6	1.07	13.2	2981	0.93	13.2
5	σ	647.1	0.69	4.9	10617.4	1.24	4.9	434.7	0.64	8.4	1894	0.62	8.4
7	μ	11156	19.42	56.4	185544	59	56.4	1984	4.69	41	55883	18.4	41
7	σ	12353	6.9	29.5	13483	26	29.5	2029	4.63	36.1	10213	21	36.1
10	μ	176921	543	271.7	2.2×10^7	8311	271.7	12380	55.8	247	1.9×10^7	7109	247
10	σ	134527	292.3	166.8	14073	6201	166.8	7184	39.7	160.1	13700	6614	160.1
14	μ	1.5×10^6	20893	2881		$> 4 \times 10^5$	2881	228498	9736	2387	1.3×10^8	151843	2387
14	σ	.5443	16518	2190			2190	18369	7012	3801	1.5×10^8	236496	3801
		d = 0.50						d = 1.00					
5	μ	114.6	0.24	6.4	730.8	0.24	6.4	22.6	0.09	3.4	86.3	0.04	3.4
5	σ	162.7	0.27	7.0	1049.9	0.34	7.0	11	0.05	1.2	38.8	0.02	1.2
7	μ	27.8	0.22	3.8	168.5	0.08	3.8	32	0.33	4.4	197.7	0.09	4.4
7	σ	24.8	0.18	3.4	118.3	0.04	3.4	16.1	0.17	2	97.6	0.05	2
10	μ	33	0.98	5.6	409.4	0.23	5.6	50.6	3.18	10.4	900.9	0.54	10.4
10	σ	18.1	0.52	2.9	196.6	0.11	2.9	21.1	1.33	4.7	436.2	0.27	4.7
14	μ	107.4	16.77	23.4	2715	1.98	23.4	119.6	41.12	25.2	4166.8	3.5	25.2
14	σ	40.2	10.25	9.2	1121.2	0.87	9.2	46.6	18.85	9.7	1577.4	1.52	9.7

Table 3. E_χ-MIP Algorithm, $m_1 = m_2 = m_3$.

d = 0.12					d = 0.25				
$m_1 = m_2 = m_3$		NDS	Time(sec)	E	$m_1 = m_2 = m_3$		NDS	Time(sec)	E
3	μ	281.7	1.365	10.7	3	μ	40.9	0.188	4.6
3	σ	230.1	1.272	3.6	3	σ	36.9	0.142	2.4
5	μ	6532.3	15.981	29.8	5	μ	408.5	1.719	12.1
5	σ	7981.8	9.138	13.7	5	σ	679.3	2.681	12
7	μ	46476.9	197.443	129.8	7	μ	419.4	4.472	21.6
7	σ	40377.4	179.128	79.9	7	σ	1006.1	5.569	35.4
9	μ	223409	1736.11	418.9	9	μ	155.4	16.752	10.4
9	σ	9438.3	2273.81	307.9	9	σ	310.9	8.286	5.5
11	μ	528015	5223.02	586.5	11	μ	141.6	93.402	31.2
11	σ	8810	5694.68	738.9	11	σ	116	92.053	24.9
13	μ	570925	18189	2507.9	13	μ	319.8	921.763	75.2
13	σ	643217	22344.1	3345.6	13	σ	213.9	677.844	49.4
d = 0.50					d = 1.00				
3	μ	5.8	0.043	2.4	3	μ	6.2	0.058	2.6
3	σ	4	0.029	2.0	3	σ	3	0.044	1.5
5	μ	18.8	0.225	2.9	5	μ	29	0.539	5.2
5	σ	13.3	0.180	1.9	5	σ	24.6	0.509	4.9
7	μ	44.6	4.510	9.4	7	μ	37.6	4.625	7.8
7	σ	27.1	4.932	5.7	7	σ	23.7	2.278	5.2
9	μ	115.2	46.908	26.4	9	μ	106.6	68.348	23.1
9	σ	35.5	32.283	8.6	9	σ	41.7	26.633	11.0
11	μ	232.4	330.787	54.2	11	μ	336.2	643.011	78.6
11	σ	148.4	209.451	32.2	11	σ	264.9	524.062	65.6
13	μ	468.2	1465.54	113.1	13	μ	572.6	3721.27	137.6
13	σ	349	1109.62	87.8	13	σ	388.1	2259.23	95.5

and CPLEX 6.0 library is used to solve mixed 0–1 linear problems. Computational experiments are made on a SPARC station ULTRA 2 under Solaris 2.4-27.

In the following tables, the entries are the mean value μ and standard deviation σ of the number of nodes NDS, computing CPU times in seconds (Time), and the number of extreme equilibria $|E|$ obtained on 10 randomly generated problems, where the coefficients of the payoff matrices are drawn from a uniform distribution over the real interval [0, 10]. Hence, 10 randomly generated problems are solved after elimination of the strongly dominated strategies for each value of the payoff matrices density parameter d: 1 and 0.5.

6.1. Bimatrix Games. The $E\chi$-MIP algorithm results are compared with those of the EEE algorithm (Ref. 3). It is worth noticing that the Audet et al. (Ref. 3) results are all for problems with density equal to 1.

Table 2 shows that $E\chi$-MIP performs better than EEE on low-density problems. This appears to be due to the large number of nodes generaged by EEE when the density decreases and the number of weakly dominated strategies increases. However, on larger bimatrix games, with size from 5×5 to 14×14 and density equal to 1 or 0.5, EEE is on the average faster than $E\chi$-MIP, the difference getting more important when the problem size increases. This appears to be due to the explosion of the computational time required to solve mixed integer programs.

6.2. Polymatrix Games. Tables 3 shows that, on 3-player polymatrix games, such that $m_1 = m_2 = m_3$, with size from $3 \times 3 \times 3$ up to $13 \times 13 \times 13$ and density equal to 0.12, 0.25, 0.50, 1.00, the computing time of $E\chi$-MIP increases exponentially.

Problems with density $d = 0.12$ appear to be harder to solve than those with density $d = 0.25$; they have a larger number of extreme equilibria on the average. Problems with density $d = 0.5$ appear to be easier to solve than those with density $d = 1$; they have slightly less equilibria on the average.

7. Discussion

The algorithm $E\chi$-MIP proposed in this paper allows the complete enumeration of the extreme equilibria of bimatrix and polymatrix games, using a mixed 0–1 linear programming formulation. Compared to the state-of-the-art algorithm EEE (Ref. 3) on bimatrix games, $E\chi$-MIP finds the same set of equilibria, but suffers from larger solution times. However, while $E\chi$-MIP permits, for the first time, the enumeration of all the extreme equilibria of a polymatrix game with a number of players $n > 2$, EEE cannot be extended directly to polymatrix games because of its bilinear programming approach.

References

1. Nash, J. F., *Equilibrium Points in n-Person Games*, Proceedings of the National Academy of Sciences, Vol. 36, pp. 48–49, 1950.
2. Millham, C. B., *On Nash Subsets of Bimatrix Games*, Naval Research Logistics Quarterly, Vol. 74, pp. 307–317, 1974.
3. Audet, C., Hansen, P., Jaumard, B., and Savard, G., *Enumeration of All Extreme Equilibrium Strategies of Bimatrix Games*, SIAM Journal on Statistical and Scientific Computing, Vol. 23, pp. 323–338, 2001.
4. Mills, H., *Equilibrium Points in Finite Games*, SIAM Journal on Applied Mathematics, Vol. 8, pp. 397–402, 1960.
5. Mangasarian, O. L., and Stone, H., *Two-Person Nonzero-Sum Games and Quadratic Programming*, Journal of Mathematical Analysis and Applications, Vol. 9, pp. 348–355, 1964.

6. MANGASARIAN, O. L., *Equilibrium Points of Bimatrix Games*, SIAM Journal on Applied Mathematics, Vol. 12, pp. 778–780, 1964.
7. VOROBEV, N. N., *Equilibrium Points in Bimatrix Games*, Theoriya Veroyatnostej i ee Primwneniya, Vol. 3, pp. 318–331, 1958 [English Version: Theory of Probability and Its Applications, Vol. 3, pp. 297–309, 1958].
8. KEIDING, H., *On the Maximal Number of Nash Equilibria in a Bimatrix Game*, Games and Economic Behavior, Vol. 21, pp. 148–160, 1997.
9. VON STENGEL, B., *New Maximal Numbers of Equilibria in Bimatrix Games*, Discrete and Computational Geometry, Vol. 21, pp. 557–568, 1998.
10. KUHN, H. W., *An Algorithm for Equilibrium Points in Bimatrix Games*, Proceedings of the National Academy of Sciences, Vol. 47, pp. 1657–1662, 1961.
11. LEMKE, C. E., and HOWSON, T. T., *Equilibrium Points of Bimatrix Games*, SIAM Journal on Applied Mathematics, Vol. 12, pp. 413–423, 1961.
12. DICKHAUT, J., and KAPLAN, T., *A Program for Finding Nash Equilibria*, Mathematica Journal, Vol. 1, pp. 87–93, 1991.
13. MCKELVEY, R. D., and MCLENNAN, A., *Computation of Equilibria in Finite Games*, Handbook of Computational Economics, Edited by H. M. Amman, D. A. Kendrick, and J. Rust, Elsevier, Amsterdam, Holland, Vol. 1, pp. 87–142, 1996.
14. WINKELS, R., *An Algorithm to Determine all Equilibrium Points of a Bimatrix Games*, Game Theory and Related Topics, Edited by O. Moeschlin and D. Pallaschke, North Holland Publishing Company, Amsterdam, Holland, 1972.
15. AUDET, C., *Optimisation Globale Structurée: Propriétés, Equivalences et Résolution*, PhD Thesis, École Polytechnique de Montréal, pp. 91–109, 1997.
16. JÚDICE, J., and MITRA, G., *Reformulations of Mathematical Programming Problems as Linear Complementarity Problems and Investigation of Their Solution Methods*, Journal of Optimization Theory and Applications, Vol. 57, pp. 123–149, 1988.
17. AUDET, C., HANSEN, P., JAUMARD, B., and SAVARD, G., *Links between Linear Bilevel and Mixed 0-1 Programming Problems*, Journal of Optimization Theory and Applications, Vol. 93, pp. 273–300, 1997.
18. HOWSON, J. T., *Equilibria of Polymatrix Games*, Management Science, Vol. 18, pp. 312–318, 1972.
19. QUINTAS, L. G., *A Note on Polymatrix Games*, International Journal of Game Theory, Vols. 18–19, pp. 261–272, 1989.
20. COTTLE, R. W., and DANTZIG, G. B., *Complementary Pivot Theory of Mathematical Programming*, Mathematics of the Decision Scienes, Part 1, Vol. 11, pp. xxx–xxx, 1968.
21. YANOVSKAYA, E. B., *Equilibrium Points in Polymatrix Games*, Latvian Mathematical Collection, Vol. 8, pp. 381–384, 1968.
22. COTTLE, R. W., and DANTZIG, G. B., *Linear Programming Extensions*, Princeton University Press, Princeton, New Jersey, 1963.
23. LEMKE, C. E., *Bimatrix Games Equilibrium Points and Mathematical Programming*, Management Science, Vol. 11, pp.xxx-xxx, 1965.
24. EAVES, C. B., *Polymatrix Games with Joint Constraints*, SIAM Journal on Applied Mathematics, Vol. 24, pp. 418–423, 1973.

25. HOWSON, J. T., and ROSENTHAL, R. W., *Bayesian Equilibria of Finite Two-Person Games with Incomplete Information*, Management Science, Vol. 21, pp. 313–315, 1974.
26. MYERSON, R. B., *Game Theory: Analysis of Conflict*, Harvard University Press, Cambridge, Massachusetts, 1997.